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Results for a turbulent system with unbounded viscosities: Weak formulations, existence of solutions, boundedness and smoothness

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Abstract

We consider a circulation system arising in turbulence modelling in fluid dynamics with unbounded eddy viscosities. Various notions of weak solution are considered and compared. We establish existence and regularity results. In particular we study the boundedness of weak solutions. We also establish an existence result for a classical solution.

MSC: 35J70; 35D05; 35D10; 86A05

Keywords: Degenerate elliptic system; Weak solution; Regularity; Oceanography

1. Introduction

Let Ω be an open bounded set in $\mathbb{R}^3$, with a Lipschitz boundary. We consider the following turbulent circulation model:

$$
(\mathrm{P}) \qquad \begin{cases} -\mathrm{div}(\nu(k)\nabla u) = f \quad \text{in } \Omega \\ -\mathrm{div}(a(k)\nabla k) = \nu(k)|\nabla u|^2 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \\ k = 0 \quad \text{on } \partial\Omega. \end{cases}
$$

Here f, a and ν are given, and the functions $u, k : \Omega \to \mathbb{R}$ are the unknowns.

We study Problem (P) under the following main assumption:

$$
(\mathrm{H}_0) \qquad \begin{cases} f \in L^r(\Omega), \quad \text{with } r > \dfrac{3}{2} \\ a, \nu : \mathbb{R}^+ \to \mathbb{R}^+ \text{ are continuous} \\ \exists \delta > 0 : a(s), \nu(s) \geq \delta \quad \forall s \in \mathbb{R}^+. \end{cases}
$$

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Problem (P) is a simplified scalar version of the RANS model arising in oceanography (see [9,10,1]): the function u is an idealization of the mean velocity of the fluid and k is the turbulent kinetic energy. The mathematical analysis of (P) is a step towards better understanding the RANS model. Various studies were made in this direction. Some existence results were established in [9,7].

In this paper we focus on the case where the viscosity functions a and ν are not a priori bounded. In fact (see [10, 7]), in the relevant physical situation, we have

$$(\mathrm{H_p})\quad \begin{cases} a(s) = a_1 + a_2\sqrt{s} \\ \nu(s) = \nu_1 + \nu_2\sqrt{s}. \end{cases}$$

We will establish an existence result for a weak solution for (P) under less restrictive assumptions than in [7]. An important feature is that our assumptions are satisfied under $(\mathrm{H_p})$, contrarily to the assumptions made in [7].

Moreover, we give additional regularity results for the weak solution we obtain. In particular, under $(\mathrm{H_0})$ and the following additional assumption: a is proportional to ν, $\partial\Omega$ is of class $\mathcal{C}^{2,\alpha}$, $f \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ and $\nu \in \mathcal{C}^{1,\alpha}(\mathbb{R}^+)$, we prove the existence of a classical solution for (P).

We also compare our results with the results presented in [9].

Another feature of our work is to consider various notions of weak solution for Problem (P): W-solution, H-solution, distributional solution, renormalized solution, 'energy solution', classical solution. We give some relations between these notions.

1.1. Notions of weak solution for (P)

We can reformulate (P).2 by using the Kirchoff transform. Let

$$A(s) := \int_0^s a(t)\mathrm{d}t.$$

Instead of (P).2, we can consider

$$(\mathrm{P}).2' \quad -\Delta K = \nu \circ A^{-1}(K)|\nabla u|^2 \quad \text{on } \Omega,$$

where $K = A(k)$.

In fact, from every distributional solution $K \in W^1(\Omega)$ of (P).2′ we obtain a distributional solution k of (P).2 by setting $k = A^{-1}(K)$. This property is related to the facts that A is invertible, $A^{-1}(0) = 0$ and $|A^{-1}(s)| \leq C.s$ (this can be seen by using the assumptions made on ν in $(\mathrm{H_0})$).

The situation is more complicated for (P).1, where the a priori unbounded coefficient $\nu(k)$ appears in the principal part of the operator and cannot be removed. Hence we have to restrict u to satisfy the energy condition

$$\int_\Omega \nu(k)|\nabla u|^2 < \infty. \tag{1}$$

Nevertheless, we will see later on that various non equivalent notions of weak solution can be considered for (P).1.

We will introduce the notions of the W-solution and the H-solution. It is also possible to consider the notion of renormalized solution (see [9] Chap. 5). In [7] the authors defined another notion that they call energy solution.

We will give some relations between these notions in Appendix A.

Notice that now, under restriction (1), the right-hand side in (P).2 (or in (P).2′) is only a priori in $L^1(\Omega)$. Hence (see [2]) it is natural to seek k in the space $\cap_{p<3/2} W_0^{1,p}(\Omega)$.

We want to find a function u vanishing on $\partial\Omega$ that satisfies the energy condition (1). This leads to considering the following spaces:

$$W_k = \{v \in H_0^1(\Omega) : [v]_k < \infty\}$$
$$H_k = \text{ closure of } \mathcal{C}_c^\infty(\Omega) \text{ with respect to } [.]_k$$

where we used the notation

$$[v]_k = \left(\int_\Omega \nu(k)|\nabla v|^2\right)^{1/2}.$$

For any measurable function k, the map $[.]_k$ defines a norm on W_k. In the general situation H_k and W_k are not equal. Moreover, W_k is not necessarily complete and a function in H_k does not always have a uniquely defined gradient (see [13]). If we assume that $\nu(k) \in L^1(\Omega)$, then W_k is complete and, in fact, $H_k \subset W_k$ are Hilbert spaces (see [6,13, 12]) when they are equipped with the scalar product

$$(v, w) = \int_\Omega \nu(k)\nabla v \nabla w.$$

Consequently, we will consider the following two distinct notions of solution for (P).1:

u is called a H_k-solution of (P).1 if $u \in H_k$ and

$$\int_\Omega \nu(k)\nabla u \nabla v = \int_\Omega f v \quad \forall v \in H_k$$

u is called a W_k-solution of (P).1 if $u \in W_k$ and

$$\int_\Omega \nu(k)\nabla u \nabla v = \int_\Omega f v \quad \forall v \in W_k.$$

Finally, we define the following notions of weak solution for (P):

(u, k) is called a *H-solution* of (P) if

$$k \in \bigcap_{p<3/2} W_0^{1,p}(\Omega), \quad u \in H_k,$$

k is a distributional solution of (P).2 and u is a H_k-solution of (P).1

(u, k) is called a *W-solution* of (P) if

$$k \in \bigcap_{p<3/2} W_0^{1,p}(\Omega), \quad u \in W_k$$

k is a distributional solution of (P).2 and u is a W_k-solution of (P).1

1.2. Main results

Let (H_1) and (H_2) denote the following conditions:

(H_1) $\exists \gamma > 0 : a(s) \geq \gamma \nu(s) \quad \forall s \in \mathbb{R}^+$

(H_2) $\exists \gamma > 0 : a(s) = \gamma \nu(s) \quad \forall s \in \mathbb{R}^+.$

We will establish:

Theorem 1. *Assume that* (H_0) *and* (H_1) *hold. Then there exists at least one W-solution* (u, k) *for* (P) *such that*

$$u \in L^\infty(\Omega) \quad \text{and} \quad \int_\Omega a(k)|\nabla k|^2 < \infty. \tag{2}$$

Corollary 1. *Assume that in addition to* (H_0) *and* (H_1) *we have*

$$\exists \nu_0 > 0 : \nu(s) \leq \nu_0(1 + s^6), \quad \forall s \in \mathbb{R}^+. \tag{3}$$

Then the W-solution (u, k) *given in Theorem 1 is a distributional solution of* (P).

Theorem 2. *Assume that* (H_0) *and* (H_2) *hold. Then the W-solution* (u, k) *given in Theorem 1 satisfies*

$$u, k \in \mathcal{C}^{0,\alpha}(\overline{\Omega}), \quad \text{for some } \alpha \in (0, 1). \tag{4}$$

Moreover, (u, k) *is also an H-solution of* (P) *(and, in fact, a classical weak solution).*

If, in addition to (H_0) *and* (H_2) *we assume that* $\partial\Omega$ *is of class* $\mathcal{C}^{2,\alpha}$, $f \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ *and* $\nu \in \mathcal{C}^{1,\alpha}(\mathbb{R}^+)$, *then*

$$u, k \in \mathcal{C}^{2,\beta}(\overline{\Omega}), \quad \text{for some } \beta \in (0, 1),$$

and (u, k) *is a classical solution of* (P).

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1.3. Discussion of the results

In Theorem 1 we give an existence result of a W-solution. We next give some regularity results: first, the property (2) and second, (in Theorem 2) the property (4). Finally, in Theorem 2 we give an existence result for a classical solution for (P).

The main previous studies of Problem (P) are presented in [9, Chap. 5] and in [7].

In [9, Chap. 5], the authors prove the existence of a renormalized solution for (P) under the assumptions (H_0) and (H_2). It seems that their proof also works under (H_0) and (H_1). Nevertheless, the notion of renormalized solution is very weak. A renormalized solution (u, k) for (P) is a distributional solution if $\nu(k) \in L^\infty(\Omega)$, whereas a H- or a W-solution is a distributional solution if $\nu(k) \in L^1(\Omega)$ (see Appendix A).

In [7] the authors introduced a notion of solution that they call 'energy solution' (see Appendix A). In fact an 'energy solution' is a W-solution which satisfies an additional property, ensuring that $H_k = W_k$ (the additional property imposed is sufficient but not necessarily having to have this equality). Under this point of view an 'energy solution' is slighty stronger than a W-solution. However, their existence result is obtained by assuming complicated conditions on the coefficients a and ν that are not exactly satisfied in the physically relevant situation (H_p), but only in the following approximate situation:

$$(H'_p) \quad \begin{cases} \text{for some } \epsilon > 0 \text{ we have:} \\ a(s) = a_1 + a_2\sqrt{s+\epsilon} \\ \nu(s) = \nu_1 + \nu_2\sqrt{s+\epsilon}. \end{cases}$$

Conversely, our assumptions in Theorem 1 and Corollary 1 are very simple, and they are satisfied in (H_p).

Note also that we establish the regularity property (2) which is not established in [7] (or in [9]).

In Appendix A we also give a new existence result for an 'energy solution'.

In Theorem 2 we assume that (H_0) and (H_2) hold. These assumptions are fulfilled in the physical situation (H_p) if $a_2\nu_1 = a_1\nu_2$. We then prove that u and k are Hölder continuous. In particular, we give here a positive answer to a central question put in [7]: k is bounded. Note that in this situation we clearly have $W_k = H_k$.

We next establish the existence of a classical solution for Problem (P) by assuming some differentiability properties for a and ν. These properties are fulfilled in the situation (H'_p) if $a_2\nu_1 = a_1\nu_2$.

It seems that this result is completely new: the existence of a classical solution for (P) was not studied in any previous work.

1.4. Organization of the paper

In the sequel n will always denote an arbitrary integer greater or equal to one, and C (possibly with subscript) will denote a positive real that does not depend on n, but that can differ from one part to another.

We always consider the space $H_0^1(\Omega)$ equipped with the gradient norm.

The condition (H_0) is always assumed.

- In Section 2 we introduce an approximate sequence (u_n, k_n) of solutions obtained by truncating the coefficients a and ν.

  We immediately obtain the basic estimates:

$$\int_\Omega \nu_n(k_n)|\nabla u_n|^2 \le C$$
$$\forall p < \frac{3}{2}: \quad \int_\Omega |a_n(k_n)\nabla k_n|^p \le C.$$

  The point is that we establish the following fundamental estimates:

$$\|u_n\|_{L^\infty(\Omega)} \le C$$
$$\int_\Omega a_n(k_n)|\nabla k_n|^2 \le C \qquad (*)$$

  The first estimate above is proved by developing further a technique of Stampacchia's.

The second is obtained under assumption (H_1). The proof is based on the following idea: if (u, k) is a solution of (P), we formally have[1]

$$\nu(k)|\nabla u|^2 = \underbrace{-\text{div}(\nu(k)\nabla u).u}_{=fu} + \text{div}(\nu(k)u\nabla u). \tag{5}$$

In other words, one can hope that the second member in the second equation in (P) is more regular than it seems. In fact, we prove that a similar relation to (5) holds for the approximate sequence. By using next that (u_n) is uniformly bounded in $L^\infty(\Omega)$, we obtain $(*)$ which is the key estimate to prove Theorem 1.

- In Section 3 we extract from (u_n, k_n) a subsequence converging to some element denoted by (u, k). Under assumptions (H_0) and (H_1), we directly obtain that

  $$u \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad k \in H_0^1(\Omega).$$

  We prove, moreover, that we have:

  $$\int_\Omega \nu(k)|\nabla u|^2 < \infty, \qquad \int_\Omega a(k)|\nabla k|^2 < \infty.$$

- In Section 4 we pass to the limit in the approximating problems. In a first step we prove that u is a W_k-solution of (P).1. To do this, we use the test functions $v = h_q(k_n)\varphi$ (where $\varphi \in W_k \cap L^\infty(\Omega)$ and (h_q) is a sequence of functions that cut off the large values), and we pass to the limits $n \to \infty$, $q \to \infty$.
  We next prove that the energies of the approximating sequence converge to the energy $\int_\Omega \nu(k)|\nabla u|^2$.
  Finally, we can pass to the limit in the second equation in order to prove that k is a distributional solution of (P).2. We then obtain Theorem 1 and Corollary 1 follows.
- In Section 5 we assume that (H_0) and (H_2) hold. In a first step we obtain the estimate

  $$\|k_n\|_{L^\infty(\Omega)} \le C.$$

  Hence $k \in L^\infty$ and by using the De Giorgi–Nash Theorem we prove the Hölder continuity of u and k.
  Next, by assuming additional regularity on ν, $\partial\Omega$ and f we can apply Schauder's estimates and we prove Theorem 2.
- In Appendix A we study some relations between the notions of the W-solution, H-solution, distributional solution, renormalized solution and 'energy solution' for Problem (P). We continue the discussion begun in Section 1.3 and we also establish a new existence result for an 'energy solution' for Problem (P).

  In Appendix B we recall some basic properties of Hölder continuous functions.

2. Approximating sequence and estimates

We assume that (H_0) holds and we set

$$\nu_n(s) = T_n(\nu(s)) \tag{6}$$
$$a_n(s) = T_n(a(s)), \tag{7}$$

where T_n is the truncated function defined by $T_n(t) = \min(n, t)$.

We consider the problem of finding $(u_n, k_n) \in (H_0^1(\Omega))^2$ such that

$$(\mathrm{P_n}) \quad \begin{cases} \displaystyle\int_\Omega \nu_n(k_n)\nabla u_n \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega) \\ \displaystyle\int_\Omega a_n(k_n)\nabla k_n \nabla\varphi = \int_\Omega T_n(\nu_n(k_n)|\nabla u_n|^2)\varphi \quad \forall \varphi \in H_0^1(\Omega). \end{cases}$$

For any $n \ge 1$, Problem ($\mathrm{P_n}$) is well posed because $a_n, \nu_n \in L^\infty(\mathbb{R})$ and $a_n^{-1}, \nu_n^{-1} \in L^\infty(\mathbb{R})$ by construction.

[1] We thank Michel Chipot for this remark.

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It is proved in [7] that a solution (u_n, k_n) exists for any$n \geq 1$. Moreover, the following basic properties were established:

$$k_n \geq 0 \tag{8}$$

$$\int_\Omega \nu_n(k_n)|\nabla u_n|^2 \leq C_1 \tag{9}$$

$$\forall p < \frac{3}{2} : \int_\Omega |a_n(k_n)\nabla k_n|^p \leq C_2. \tag{10}$$

We now establish

Lemma 3. *The sequence u_n is uniformly bounded in the $L^\infty(\Omega)$-norm, that is,*

$$\|u_n\|_{L^\infty(\Omega)} \leq C_3. \tag{11}$$

Before proving this lemma we point out that the assumption $f \in L^r(\Omega)$, with $r > \frac{3}{2}$ made in (H_0) implies that

$$f \in W^{-1,\rho}(\Omega), \quad \text{with } \rho = \frac{3r}{3-r} > 3. \tag{12}$$

This last property is easy to prove by using the Sobolev injection Theorem.

Proof. We will obtain the estimate (11) by using the technique presented on p. 108 in [11].

In order to prove that C_3 is independent of n we have to detail the technique of Stampacchia. Let

$$b_n(u, v) := \int_\Omega \nu_n(k_n)\nabla u \nabla v.$$

Recall that f satisfies (12) and then, by using a classical result (see [3]), there exists $g \in (L^\rho(\Omega))^3$ such that $-\text{div}(g) = f$ and $\|g\|_{(L^\rho(\Omega))^3} \leq C\|f\|_{L^r(\Omega)}$.

Hence the sequence u_n satisfies

$$b_n(u_n, v) = \int_\Omega g\nabla v \quad \forall v \in H_0^1(\Omega). \tag{13}$$

For $s \geq 0$, we define the measurable set $A_n(s) \subset \Omega$ by setting

$$A_n(s) = \{x \in \Omega : |u_n(x)| \geq s\}.$$

We also introduce

$$\varphi := \max(|u_n| - s, 0)\,\text{sgn}(u_n). \tag{14}$$

It is proved in [11] that $\varphi \in H_0^1(\Omega)$ and

$$\nabla\varphi = \nabla u_n \quad \text{in } A_n(s)$$
$$\nabla\varphi = 0 \quad \text{in } \Omega \setminus A_n(s).$$

By testing (13) with $v = \varphi$, we obtain

$$b_n(\varphi, \varphi) = b_n(u_n, \varphi) = \int_{A_n(s)} g\nabla\varphi. \tag{15}$$

Remark now that assumption $\nu(s) \geq \delta > 0$ in (H_0) implies that $\nu_n(k_n) \geq \min(\delta, 1)$. Consequently, the bilinear form b_n is uniformly coercive on $H_0^1(\Omega)$. By using this property together with the Hölder inequality, we obtain from (15):

$$\|\varphi\|^2_{H_0^1(\Omega)} \leq \tilde{C}\left(\int_{A_n(s)} |g|^2\right)^{1/2} \|\varphi\|_{H_0^1(\Omega)}.$$

Hence, by using the Cauchy inequality together with the Hölder inequality, we obtain

$$\|\varphi\|^2_{H^1_0(\Omega)^2} \le \tilde{C}_1 \|g\|^2_{L^\rho(\Omega)} |A_n(s)|^{\frac{\rho-2}{\rho}}. \tag{16}$$

On the other hand, the Poincaré–Sobolev inequality gives

$$\left(\int_{A_n(s)} |\varphi|^6\right)^{1/3} \le \tilde{C}_2 \|\varphi\|^2_{H^1_0(\Omega)}. \tag{17}$$

Now let $t > s$. It is clear that $A_n(t) \subset A_n(s)$ and consequently

$$\left(\int_{A_n(s)} |\varphi|^6\right)^{1/3} \ge \left(\int_{A_n(t)} |\varphi|^6\right)^{1/3} \ge \left(\int_{A_n(t)} |t-s|^6\right)^{1/3} \ge |t-s|^2 |A_n(t)|^{1/3}. \tag{18}$$

We set

$$\psi_n(s) := |A_n(s)|, \quad \forall s \ge 0.$$

For fixed n, ψ_n is a decreasing function, and from the estimates (16)–(18), we obtain

$$\psi_n(t) \le \tilde{C}_3 |\psi_n(s)|^\beta (t-s)^{-6} \quad \forall t > s \ge 0,$$

where we have used the notation $\beta := \frac{3(\rho-2)}{\rho} > 1$ and where $\tilde{C}_3 = \tilde{C}_3(\tilde{C}_1, \tilde{C}_2, \|f\|_{L^r})$. Both quantity β and $\tilde{C}_3$ do not depend on n. Hence, by using Lemma 4.1 in [11] it follows:

$$\psi_n(\theta) = 0,$$

where $\theta = 2^{\beta/(\beta-1)} \left(\tilde{C}_3 |\Omega|^{\beta-1}\right)^{1/6} < \infty$ does not depend on n.

This property tells precisely that (11) holds true with $C_3 = \theta$. □

Notice that the bilinear form

$$(u, v) \to \int_\Omega a_n(k_n) \nabla u \nabla v,$$

is also uniformly coercive on $H^1_0(\Omega)$. Moreover, the sequence

$$h_n := T_n(\nu_n(k_n)|\nabla u_n|^2)$$

is imbedded in $L^\infty(\Omega)$. We can then apply again the technique of Stampacchia detailed in the proof of Lemma 3, and obtain:

$$\text{for } n \ge 1: \quad k_n \in L^\infty(\Omega). \tag{19}$$

Nevertheless, the control we have on $\{h_n\}$ is obtained from (9), which gives a uniform bound in the L^1-norm for the sequence. This is not enough to obtain a uniform estimate for $\{k_n\}$ in the L^∞-norm.

However, we can establish:

Lemma 4. *Assume that* (H_0) *and* (H_1) *hold. Then we have*

$$a_n(s) \ge \gamma_1 \nu_n(s), \quad \gamma_1 = \min(1, \gamma) \tag{20}$$

$$\int_\Omega a_n(k_n)|\nabla k_n|^2 \le C_5. \tag{21}$$

Proof. The estimate (20) is easy to obtain. Its verification is left to the reader.

Let (u_n, k_n) be the chosen approximating sequence. We have from (11) and (19) that

$$\forall n \ge 1: \quad u_n, k_n \in H^1_0(\Omega) \cap L^\infty(\Omega).$$

It follows (see [3]) that $v := u_n.k_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is admissible for $(\mathrm{P_n}).1$ and we obtain[2]

$$\int_\Omega \nu_n(k_n)|\nabla u_n|^2 k_n = \int_\Omega f u_n k_n - \int_\Omega \nu_n(k_n) u_n \nabla u_n \nabla k_n. \tag{22}$$

By testing $(\mathrm{P_n}).2$ with $\varphi = k_n$, we obtain:

$$\int_\Omega a_n(k_n)|\nabla k_n|^2 = \int_\Omega T_n(\nu_n(k_n)|\nabla u_n|^2) k_n \le \int_\Omega \nu_n(k_n)|\nabla u_n|^2 k_n, \tag{23}$$

by using the properties $T_n(s) \le s$ and (8).

Hence, by combining (22) with (23) we have:

$$\mathrm{I} := \int_\Omega a_n(k_n)|\nabla k_n|^2 \le \underbrace{\int_\Omega |f u_n k_n|}_{:=\mathrm{II}} + \underbrace{\int_\Omega |\nu_n(k_n) u_n \nabla u_n \nabla k_n|}_{:=\mathrm{III}}. \tag{24}$$

We can estimate the term II as follows:

$$\begin{aligned}
\mathrm{II} &\le C_3 \int_\Omega |f k_n| \overset{\text{Hölder Ineq.}}{\le} C_3 \|f\|_{L^{3/2}} \|k_n\|_{L^3} \\
&\overset{\text{Poincaré–Sobolev Ineq.}}{\le} \tilde{C}_1 \|f\|_{L^{3/2}} \left(\int_\Omega |\nabla k_n|^2 \right)^{1/2} \le \frac{\tilde{C}_1}{\delta} \|f\|_{L^{3/2}} \left(\int_\Omega a_n(k_n)|\nabla k_n|^2 \right)^{1/2} \\
&\overset{\text{Young Ineq.}}{\le} \frac{\tilde{C}_1}{\delta} \left(\frac{1}{\epsilon} \|f\|^2_{L^{3/2}} + \epsilon \int_\Omega a_n(k_n)|\nabla k_n|^2 \right) \quad \text{for any } \epsilon > 0 \text{ given} \\
&\le \frac{1}{3} \int_\Omega a_n(k_n)|\nabla k_n|^2 + \tilde{C}_2 \|f\|^2_{L^{3/2}}
\end{aligned}$$

where $\delta > 0$ is the constant given in $(\mathrm{H_0})$. The last inequality was obtained by choosing $\epsilon = \delta/(3\tilde{C}_1)$, using the estimate (11) and by setting $\tilde{C}_2 = 3\tilde{C}_1^{\,2}/\delta^2$.

We next estimate the term III:

$$\begin{aligned}
\mathrm{III} &= \int_\Omega |u_n \sqrt{\nu_n(k_n)} \nabla u_n \sqrt{\nu_n(k_n)} \nabla k_n| \\
&\le \tilde{C}_3 \int_\Omega |\sqrt{\nu_n(k_n)} \nabla u_n \sqrt{a_n(k_n)} \nabla k_n|, \quad \tilde{C}_3 = C_3 \gamma_1^{-1/2} \\
&\le \frac{1}{3} \int_\Omega a_n(k_n)|\nabla k_n|^2 + \tilde{C}_4 \int_\Omega \nu_n(k_n)|\nabla u_n|^2, \quad \tilde{C}_4 = \tilde{C}_4(\tilde{C}_3)
\end{aligned}$$

where C_3, γ_1 are the constants that appear in (11) and (20). The last inequality follows from the Young inequality.

Recall now the inequality (24) and use the estimates established for the terms II and III. We obtain:

$$\frac{1}{3} \int_\Omega a_n(k_n)|\nabla k_n|^2 \le \tilde{C}_2 \|f\|^2_{L^{3/2}(\Omega)} + \tilde{C}_4 \int_\Omega \nu_n(k_n)|\nabla u_n|^2. \tag{25}$$

By using (25) together with (9) we finally obtain (21). □

3. Basic convergence results for (u_n, k_n)

The estimates established in the previous section allow us to extract a converging subsequence from (u_n, k_n). We have

[2] More generally: $\nu_n(k_n)|\nabla u_n|^2 = \underbrace{-\mathrm{div}(\nu_n(k_n)\nabla u_n).u_n}_{=f u_n} + \mathrm{div}(\nu_n(k_n) u_n \nabla u_n)$ in $\mathcal{D}'(\Omega)$.

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Lemma 5. 1. *Assume that* (H_0) *holds. Then we can extract a subsequence (still denoted by* (u_n, k_n)*) such that*

$$a_n(k_n)\nabla k_n \rightharpoonup a(k)\nabla k \quad \text{in } L^p(\Omega),\ p < \frac{3}{2} \tag{26}$$

$$k_n \to k \quad \text{a.e in } \Omega \tag{27}$$

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega) \tag{28}$$

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(\Omega). \tag{29}$$

2. *If in addition the condition* (H_1) *is fulfilled, then we may assume that*

$$k_n \rightharpoonup k \quad \text{in } H_0^1(\Omega). \tag{30}$$

Proof. 1. The properties (26) and (27) are obtained from (10). Property (28) is obtained by using estimate (9) together with the assumption $\nu(s) \geq \delta > 0$ in (H_0). We establish (29) from estimate (11).

2. By using Lemma 4 together with the assumption $a(s) \geq \delta > 0$ in (H_0) we obtain (30). Notice that the k appearing in (26), (27) and (30) is necessarily the same in the three situations. □

We are able to prove additional regularity results for the element (u, k) introduced in Lemma 5. For technical reasons we introduce the sequence $\{h_q\}_{q\in\mathbb{N}}$ of real functions defined in [9] p. 185. It satisfies:

$$|h_q(s)| \leq 1 \quad \forall (q, s) \in \mathbb{N} \times \mathbb{R} \tag{31}$$

$$h_q(s) = 0 \quad \text{when } |s| > 2q \tag{32}$$

$$|h_q'(s)| \leq \frac{1}{q} \quad \forall q \in \mathbb{N},\ \text{and a.e } s \in \mathbb{R} \tag{33}$$

$$h_q \underset{q\to\infty}{\to} 1 \quad \text{uniformly on the compacts.} \tag{34}$$

Lemma 6. 1. *Assume that* (H_0) *holds. Then the element* (u, k) *given in Lemma 5 satisfies*

$$\int_\Omega \nu(k)|\nabla u|^2 < \infty. \tag{35}$$

2. *Assume that in addition* (H_1) *holds. Then*

$$\int_\Omega a(k)|\nabla k|^2 < \infty. \tag{36}$$

Proof. 1. We take over the arguments presented in [9] p. 192.

For $q \geq 1$, we set

$$\eta_{n,q} := \left(h_q(k_n)\nu_n(k_n)\right)^{1/2} \nabla u_n.$$

Now let q be fixed. The sequence $\{(h_q(k_n)\nu_n(k_n))^{1/2}\}_{n\geq 1}$ is uniformly bounded in $L^\infty(\Omega)$. Consequently, $\{\eta_{n,q}\}_{n\geq 1}$ is bounded in $(L^2(\Omega))^3$ and we can extract a subsequence weakly convergent to some $\eta_q \in (L^2(\Omega))^3$.

On the other hand, we have

$$(h_q(k_n)\nu_n(k_n))^{1/2} \to (h_q(k)\nu_n(k))^{1/2} \quad \text{a.e in } \Omega$$
$$\nabla u_n \rightharpoonup \nabla u \quad \text{in } L^2(\Omega),$$

and thus $\eta_q = \left(h_q(k)\nu(k)\right)^{1/2} \nabla u$.

We now use a classical property of the weak convergence in $L^2(\Omega)$:

$$\|\eta_q\|_{L^2(\Omega)} \leq \liminf_{n\to\infty} \|\eta_{n,q}\|_{L^2(\Omega)} \leq \liminf_{n\to\infty} \left(\int_\Omega \nu_n(k_n)|\nabla u_n|^2\right)^{1/2} \leq C_1^{1/2},$$

where C_1 is a constant independent of q given in (9).

By using properties (34) and (31) we can see that

$$\eta_q^2 \underset{q\to\infty}{\to} \nu(k)|\nabla u|^2 \quad \text{a.e. in } \Omega$$
$$\eta_q^2 \leq \nu(k)|\nabla u|^2.$$

Hence by the Fatou Lemma we finally obtain:

$$\int_\Omega \nu(k)|\nabla u|^2 \leq \liminf_{q\to\infty} \|\eta_q\|_{L^2}^2 \leq C_1.$$

2. If the additional assumption (H_1) holds, then we have estimate (21) and the previous reasoning allows us to obtain (36). □

4. The proof of Theorem 1

In the previous section we have proved that under (H_0) we can extract a converging subsequence of (u_n, k_n). If, moreover, (H_1) holds, then the limit (u, k) obtained satisfies:

$$u \in W_k \cap L^\infty(\Omega) \tag{37}$$
$$k \in H_0^1(\Omega) \quad \text{(and in fact } k \in W_k). \tag{38}$$

4.1. *Passing to the limit in* (P_n).1

We recall that the space W_k was defined by

$$W_k = \{v \in H_0^1(\Omega) : [v]_k < \infty\}.$$

We now establish:

Lemma 7. *Assume that* (H_0) *and* (H_1) *hold. Then the element* (u, k) *given in Lemma 5 satisfies (37) and (38) and:*

$$\int_\Omega \nu(k)\nabla u \nabla v = \int_\Omega f v \quad \forall v \in W_k. \tag{39}$$

Proof. Let $n \geq 1$, $q \in \mathbb{N}$ and $\varphi \in W_k \cap L^\infty(\Omega)$. We consider the function $v := h_q(k_n)\varphi$. By recalling the properties (31)–(34) of h_q, we can verify that $h_q(k_n) \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Consequently $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. By testing ($P_n$).1 with v, we obtain:

$$\mathrm{I} := \int_\Omega \nu_n(k_n)h_q(k_n)\nabla u_n \nabla\varphi + \underbrace{\int_\Omega h_q'(k_n)\nu_n(k_n)\nabla u_n \nabla k_n \varphi}_{:=\mathrm{II}} = \underbrace{\int_\Omega f h_q k_n \varphi}_{:=\mathrm{III}} \tag{40}$$

In a first step we fix q and we study the behaviour of terms I, II and III when n tends to infinity.

By using property (32) we see that

$$|\nu_n(k_n)h_q(k_n)| \leq \max_{s\in[0,2q]} \nu(s) := C_q,$$

and by using (32) together with (27) we obtain

$$\nu_n(k_n)h_q(k_n) \underset{n\to\infty}{\to} \nu(k)h_q(k) \quad \text{a.e in } \Omega.$$

Consequently

$$\nu_n(k_n)h_q(k_n)\nabla\varphi \underset{n\to\infty}{\to} \nu(k)h_q(k)\nabla\varphi \quad \text{in } (L^2(\Omega))^2,$$

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and by also employing (28) we obtain:

$$\mathrm{I} \underset{n\to\infty}{\to} \int_\Omega \nu(k)h_q(k)\nabla u\nabla\varphi. \tag{41}$$

We now estimate II. From (33) we obtain:

$$\begin{aligned}\mathrm{II} &\le \frac{1}{q}\int_{\{q\le k_n\le 2q\}} |\nu_n(k_n)\nabla u_n\nabla k_n\varphi| \\ &\le \|\varphi\|_{L^\infty}\frac{C}{q}\left(\int_\Omega \nu_n(k_n)|\nabla u_n|^2\right)^{1/2}\left(\int_\Omega a_n(k_n)|\nabla k_n|^2\right)^{1/2} \le \frac{C}{q},\end{aligned} \tag{42}$$

where the second inequality is obtained by using (20).

For the last term we get

$$\mathrm{III} \underset{n\to\infty}{\to} \int_\Omega f h_q(k)\varphi. \tag{43}$$

By using estimates (41)–(43) together with (40) we obtain that for any fixed $\varphi \in W_k \cap L^\infty(\Omega)$ the following holds

$$\underbrace{\int_\Omega \nu(k)h_q(k)\nabla u\nabla\varphi}_{:=J_1} = \underbrace{\int_\Omega f h_q(k)\varphi}_{:=J_2} + \mathcal{O}\left(\frac{1}{q}\right). \tag{44}$$

We next note that the integrand in J_1 converges for a.e. $x \in \Omega$ to $\nu(k)\nabla u\nabla\varphi$ when q tends to infinity. Moreover, by using (31) together with the fact that $\varphi \in W_k$ we can see that the integrand in J_1 is dominated by $|\nu(k)\nabla u\nabla\varphi| \in L^1(\Omega)$. Consequently, by the Dominated Convergence Theorem we obtain

$$J_1 \underset{q\to\infty}{\to} \int_\Omega \nu(k)\nabla u\nabla\varphi.$$

Similarly, we can see that

$$J_2 \underset{q\to\infty}{\to} \int_\Omega f\varphi.$$

At this stage we have proved that

$$\int_\Omega \nu(k)\nabla u\nabla\varphi = \int_\Omega f\varphi \quad \forall\varphi \in W_k \cap L^\infty(\Omega), \tag{45}$$

and it remains to show that the condition $\varphi \in L^\infty(\Omega)$ is not necessary.

Let $\varphi \in W_k$ and $i \in \mathbb{N}$. We consider $\varphi_i \in W_k \cap L^\infty(\Omega)$ given by $\varphi_i = T_i(\varphi)$. By using some basic properties of T_i (see [7]), we see that $|\varphi_i| \le |\varphi|$, $|\nabla\varphi_i| \le |\nabla\varphi|$, $\varphi_i \to \varphi$ a.e, and $\nabla\varphi_i \to \nabla\varphi$ a.e in Ω. Consequently, if we take φ_i as test function in (45), we can pass to the limit $i \to \infty$ and we obtain (39). □

In Lemma 7 we have showed that u is a W_k-solution of (P).1. In order to prove Theorem 1 we have to prove that k is a distributional solution of (P).2. We need first to establish:

Lemma 8. *Assume that* (H_0) *and* (H_1) *hold. Then, in addition to the results presented in Lemma 5, we may assume:*

$$\nu_n(k_n)|\nabla u_n|^2 \underset{n\to\infty}{\to} \nu(k)|\nabla u|^2 \quad \text{in } L^1(\Omega). \tag{46}$$

Proof. We test $(\mathrm{P_n}).1$ with the function u_n. By using (28) we obtain:

$$\int_\Omega \nu_n(k_n)|\nabla u_n|^2 \underset{n\to\infty}{\to} \int_\Omega fu = \int_\Omega \nu(k)|\nabla u|^2, \tag{47}$$

where the latter equality is obtained by testing (39) with u.

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We set $\eta_n := \sqrt{\nu_n(k_n)}\nabla u_n$ and $\eta := \sqrt{\nu(k)}\nabla u$. The relation (47) tells us that

$$\|\eta_n\|_{L^2(\Omega)} \underset{n\to\infty}{\to} \|\eta\|_{L^2(\Omega)}. \tag{48}$$

We can next take over the arguments presented in [9] Lemma 5.3.4 in order to obtain:

$$\eta_n \underset{n\to\infty}{\rightharpoonup} \eta \quad \text{in } (L^2(\Omega))^2. \tag{49}$$

Finally, properties (49) and (48) imply that the convergence is strong in (49) and (46) follows. □

4.2. The proofs of Theorem 1 and Corollary 1

Assume that (H_0) and (H_1) hold. In Lemma 5 we have extracted a subsequence (u_n, k_n) which converges in a certain sense to an element (u, k). This element has properties (37) and (38). Next, we have established (39).

Now let $\varphi \in \mathcal{C}_c^\infty(\Omega)$. By using (26) we obtain:

$$\int_\Omega a_n(k_n)\nabla k_n \nabla\varphi \underset{n\to\infty}{\to} \int_\Omega a(k)\nabla k\nabla\varphi. \tag{50}$$

We next remark that the property (46) ensures that

$$\int_\Omega T_n(\nu_n(k_n)|\nabla u_n|^2)\varphi \underset{n\to\infty}{\to} \int_\Omega \nu(k)|\nabla u|^2\varphi. \tag{51}$$

Recall that the sequence (u_n, k_n) satisfies $(\mathrm{P_n})$.2. Then relation (50) together with (51) allows the limit in $(\mathrm{P_n})$.2 to be taken. We obtain:

$$\int_\Omega a(k)\nabla k\nabla\varphi = \int_\Omega \nu(k)|\nabla u|^2\varphi \quad \forall\varphi \in \mathcal{C}_c^\infty(\Omega). \tag{52}$$

Thus (P).2 is fulfilled in the distributional sense.

At this point we have obtained (37)–(39) and (52). The proof of Theorem 1 is complete.

Assume now that condition (3) in Corollary 1 is fulfilled. By using (38) together with the Sobolev Injection Theorem we obtain $k \in L^6(\Omega)$ and thus $\nu(k) \in L^1(\Omega)$. Then we can conclude the proof of Corollary 1 by using Proposition 9 in Appendix A: (u, k) is a distributional solution of (P).

5. The proof of Theorem 2

We assume in this section that (H_0) and (H_2) hold.

In this situation all the results presented in Section 2 and Section 3 are valid. For technical reasons we slightly modify the definition of a_n by setting

$$a_n(s) := \gamma\nu_n(s), \tag{53}$$

where $\gamma > 0$ is the constant appearing in (H_2) and ν_n is defined as before.

We will now consider Problems $(\mathrm{P_n})$ modified by the new definition (53) of a_n. Nevertheless, the modification is very slight, and all the results presented in the previous section can be recovered easily. The verifications are left to the reader.

We now prove that we have the new estimate:

$$\|k_n\|_{L^\infty(\Omega)} \le C_6. \tag{54}$$

In order to prove this result we set

$$\chi_n := k_n + \frac{\gamma}{2}u_n^2, \tag{55}$$

and we note that $(\mathrm{P_n})$.2 leads to

$$\int_\Omega a_n(k_n)\nabla\chi_n\nabla\varphi = \int_\Omega f u_n\varphi \quad \forall\varphi \in H_0^1(\Omega).$$

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Recall that $a_n(k_n) \geq \gamma \min(1, \delta) > 0$, $a_n(k_n) \in L^\infty(\Omega)$ and note that the sequence $f u_n$ is uniformly bounded in $L^r(\Omega)$ with $r > 3/2$. These properties are sufficient (see the proof of Lemma 3) to obtain the estimate

$$\|\chi_n\|_{L^\infty(\Omega)} \leq C, \tag{56}$$

where C does not depend on n.

The estimate (54) is finally obtained by using Lemma 3 together with (56).

Consequently, in addition to the properties in Lemma 5 we may assume that

$$k_n \overset{*}{\rightharpoonup} k \quad \text{in } L^\infty(\Omega). \tag{57}$$

We will now prove that

$$u, k \in \mathcal{C}^{0,\alpha}(\overline{\Omega}) \quad \text{for some } \alpha \in (0, 1). \tag{58}$$

Let $\lambda := \nu(k)$. We have $\lambda, \lambda^{-1} \in L^\infty(\Omega)$ and

$$\int_\Omega \lambda \nabla u \nabla \phi = \int_\Omega f \phi \quad \forall \phi \in H_0^1(\Omega). \tag{59}$$

Recall also that f has the property (12). Hence we can apply the De Giorgi–Nash Theorem (see for instance [5] Prop. 6 p. 683 or [8] Th. 8.22 and Th. 8.29). We obtain that $u \in \mathcal{C}^{0,\alpha_1}(\overline{\Omega})$ for some $\alpha_1 \in (0, 1)$. We next set $\chi := k + (\gamma/2)u^2$. Then $\chi \in H_0^1(\Omega)$ and we have

$$\int_\Omega \frac{\lambda}{\gamma} \nabla \chi \nabla \phi = \int_\Omega f u \phi \quad \forall \phi \in H_0^1(\Omega). \tag{60}$$

By using the fact that $u \in L^\infty(\Omega)$ in (60), we can again apply the De Giorgi–Nash Theorem to obtain $\chi \in \mathcal{C}^{0,\alpha_2}(\overline{\Omega})$ for some $\alpha_2 \in (0, 1)$. Hence also k is Hölder continuous, and (58) follows.

Let $\alpha \in (0, 1)$ be a generic parameter that can differ from one part to another. We assume now that $\partial\Omega$ is of class $\mathcal{C}^{2,\alpha}$, $f \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ and $\nu \in \mathcal{C}^{1,\alpha}(\mathbb{R}^+)$.

We will prove the second part of Theorem 2 by iterating the Schauder estimates.

We have $\lambda = \nu(k) \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ (see Appendix B) and then, by applying the Schauder estimate (see [4] Theorem 2.7 p. 154) on (59) we obtain $u \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$. Similarily, from Eq. (60) we obtain $\chi \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ and thus $k \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$.

Hence (see Appendix B) $\lambda \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$. By iterating again the Schauder estimates (see now Theorem 2.8 p. 154 in [4]) we obtain that u and k are in $\mathcal{C}^{2,\alpha}(\overline{\Omega})$.

Finally, we see that (u, k) is a classical solution of (P). Theorem 2 is proven.

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Appendix A. Some relations between the notions of weak solution

We give here some relations between the various notions of weak solution: W-solution, H-solution, distributional solution, renormalized solution and 'energy solution'.

A.1. Comparison with renormalized solution

We have:

Proposition 9. 1. *Any W- or H-solution (u, k) of Problem* (P) *that satisfies in addition $k \in H_0^1(\Omega)$, is also a renormalized solution.*

2. *If $\nu(k) \in L^1(\Omega)$ then any W- or H-solution of Problem* (P) *is a also a distributional solution of* (P).

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Proof. 1. Let (u, k) be a W-solution of (P). Then the conditions (5.2.1)–(5.2.5) in [9, Chap. 5] are satisfied. We have to prove that (5.2.6) holds.

Let $h \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\phi \in \mathcal{C}_c^\infty(\Omega)$ be arbitrarily chosen. We set $v := h(k)\phi$. Then $v \in L^\infty(\Omega)$ and $\nabla v = h(k)\nabla\phi + h'(k)\nabla k\phi$. Let $M < \infty$ be such that the support of h being included in $[-M, M]$. We have

$$\int_\Omega \nu(k)h^2(k)|\nabla\phi|^2 \leq \max_{[0,M]} \nu \|h\|_{L^\infty}^2 \int_\Omega |\nabla\phi|^2 < \infty$$

$$\int_\Omega \nu(k)(h'(k))^2|\nabla k|^2|\phi|^2 \leq \max_{[0,M]} \nu \|h'\|_{L^\infty}^2 \|\phi\|_{L^\infty}^2 \int_\Omega |\nabla k|^2 < \infty.$$

Hence $v \in W_k$. By testing (39) with v we obtain the relation (5.2.6).a in [9].

We remark that v is also admissible in (52). This allows us to obtain condition (5.2.6).b in [9]. Consequently (u, k) is a renormalized solution of (P).

If we consider an H-solution (u, k) of (P) we can take over the previous argument because the function v is now in H_k.

2. If $\nu(k) \in L^1(\Omega)$ then we have $\mathcal{C}_c^\infty(\Omega) \hookrightarrow H_k \hookrightarrow W_k$. Consequently, a W_k- or a H_k-solution of (P).1 is also a distributional solution of this equation. Hence (h, k) is a distributional solution of (P). □

Remark. 1. The first point in Proposition 9 tells that the notions of an H- or W-solution are stronger that the notion of renormalized solution. This fact is coherent with the second point established in Proposition 9: an H- or W-solution is a distributional solution if $\nu(k) \in L^1(\Omega)$ whereas a renormalized solution is only a priori a distributional solution if $\nu(k) \in L^\infty(\Omega)$ (see [9] p. 185).

2. If we have $k \in H_0^1(\Omega)$ and if ν satisfies the growth condition (3), then $\nu(k) \in L^1(\Omega)$.

A.2. Comparison with 'energy solution'

We have seen that when $\nu(k) \in L^1(\Omega)$ then any W- (or H-) solution is a distributional solution. Moreover, the notion of a W-solution coincides with the notion of a H-solution iff $W_k = H_k$ (see [13]).

Some sufficient conditions to have this last equality were established in [13] and [7], but necessary and sufficient conditions are not known.

Let us consider the following condition:

$$\text{(R)} \qquad \begin{cases} \sqrt{\nu(k)} \in H^1(\Omega) \\ T_n(k) \in H_0^1(\Omega), \quad \forall n \in \mathbb{N}. \end{cases}$$

It was schown in [7] that the first condition in (R) together with the property $\nu^{-1} \in L^\infty(\mathbb{R})$ (which is assumed in (H_0)) implies that $W_k = H_k$.

In [7] the authors introduced the notion of 'energy solution'. They impose (H_0) as the basic assumption. Then an 'energy solution' (u, k) for (P) is in fact a W-solution which satisfies (R). This implies that $W_k = H_k$. The energy solution is also an H-solution, and, moreover, a distributional solution (because the first assumption in (R) implies that $\nu(k) \in L^1(\Omega)$).

We see, then, that the notion of 'energy solution' (in the sense of [7]) has the advantage of unifying various notions by putting us in the situation where $\sqrt{\nu(k)} \in H^1(\Omega)$. The disadvantage is that we have to impose more complicated conditions on the coefficients a and ν, in order to obtain a solution. In particular in [7] Theorem 2.1, the authors prove the existence of an 'energy solution' under assumptions (H_0) and (H_3) (see below).

$$(\mathrm{H}_3) \qquad \begin{cases} \nu \in \mathcal{C}^1(\mathbb{R}^+) \\ \exists\, C > 0 \text{ and } \gamma > 1/2 \text{ such that :} \\ |\nu'(s)| \leq C \quad \forall s \in [0, 1] \\ \dfrac{|\nu'(s)|}{\sqrt{a(s)\nu(s)}} \leq C.s^{-\gamma} \quad \forall s \geq 1. \end{cases}$$

This condition is not verified in the physical situation (H_p), but only in the approximate situation (H_p').

In Theorem 1 we obtain a W-solution under much simpler conditions which are satisfied by (H_p). This solution is a distributional solution under an additionnal simple assumption (see Corollary 1) which is again satisfied in (H_p).

Note also that in the first part of Theorem 2 we prove that under assumptions (H_0) and (H_2) (which are satisfied in (H_p) if $a1\nu_2 = a_2\nu 1$), the functions u and k are Hölder continuous. In particular $\nu(k) \in L^\infty$ which implies that $W_k = H_k$, and the notions of H-solution, W-solution, distributional solution and renormalized solution coincide in this case.

In order to conclude this appendix we give a last existence result. Let (H_4) be the following condition:

$$(H_4) \qquad \begin{cases} \nu \in \mathcal{C}^1(\mathbb{R}^+) \\ \exists\, C > 0 \quad \text{s.t. } \dfrac{|\nu'(s)|}{\nu(s)} \le C\ \forall s \in \mathbb{R}. \end{cases}$$

We have:

Proposition 10. *Assume that* (H_0), (H_1) *and* (H_4) *hold. Then the W-solution given in Theorem 1 is an 'energy solution' (in the sense of* [7]*).*

Proof. We have assumed that (H_0), (H_1) hold and consequently all the results presented in Sections 2–4 can be recovered.

Let (u, k) be the W-solution given by Theorem 1. By using (2) we see that the second condition in (R) is satisfied. Nevertheless, we cannot directly conclude that $\sqrt{\nu(k)} \in H^1(\Omega)$, but we can obtain a new estimate for the approximating sequence (u_n, k_n). More precisely, we have:

$$\|\sqrt{\nu_n(k_n)}\|_{H^1(\Omega)} \le C. \tag{61}$$

In fact, by using the property that $k_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ together with $\nu \in \mathcal{C}^1(\mathbb{R}^+)$ we obtain $\nu(k_n) \in H^1(\Omega) \cap L^\infty(\Omega)$, with $\nabla\nu(k_n) = \nu'(k_n)\nabla k_n$. Recall now that $\nu_n(k_n) = T_n(\nu(k_n))$. Hence we have

$$\nabla\nu_n(k_n) = 1_{\{\nu_n(k_n)<n\}}\nu'(k_n)\nabla k_n.$$

It follows that:

$$\begin{aligned}\nabla\sqrt{\nu_n(k_n)} &= 1_{\{\nu(k_n)<n\}}\frac{\nu'(k_n)\nabla k_n}{2\sqrt{\nu_n(k_n)}} = 1_{\{\nu_n(k_n)<n\}}\frac{\nu'(k_n)}{2\sqrt{\nu_n(k_n)a_n(k_n)}}\sqrt{a_n(k_n)}\nabla k_n \\ &\overset{\text{by (20)}}{\le} C1_{\{\nu_n(k_n)<n\}}\frac{\nu'(k_n)}{\nu_n(k_n)}\sqrt{a_n(k_n)}\nabla k_n = C\frac{\nu'(k_n)}{\nu(k_n)}\sqrt{a_n(k_n)}\nabla k_n.\end{aligned}$$

Hence, by using (21) we obtain

$$\|\nabla\sqrt{\nu_n(k_n)}\|_{L^2(\Omega)} \le C.$$

Moreover $\sqrt{\nu_n(k_n)} = \sqrt{\nu(0)}$ on $\partial\Omega$ and thus we obtain (63) by using a Poincaré inequality. □

Remark. The hypotheses made in Proposition 10 are verified under assumption (H_p'). In the hypotheses, we require only a very weak growth condition at infinity for ν. For instance (contrary to the result presented in [7]) the Proposition 10 works if we have:

$$\nu(s) = \nu_1 + \nu_2 e^{\beta_1 s}, \qquad a(s) = a_1 + a_2 e^{\beta_2 s}, \qquad \beta_1 \le \beta_2.$$

Appendix B. Hölder continuity and composition

Let $\Lambda \subset \mathbb{R}^d$ and $\alpha \in (0, 1)$. We recall that the space $\mathcal{C}^{0,\alpha}(\Lambda)$ of Hölder continuous (with exponent α) functions on Λ is defined by:

$$\mathcal{C}^{0,\alpha}(\Lambda) = \left\{ f : \Lambda \to \mathbb{R} \text{ s.t. } \forall x_0 \in \Lambda : \sup_{x\in\Lambda} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha} < \infty \right\}.$$

More generally, for any integer k, the space $\mathcal{C}^{k,\alpha}(\Lambda)$ is the space of those $f \in \mathcal{C}^k(\Lambda)$ whose kth derivative is in $\mathcal{C}^{0,\alpha}(\Lambda)$.

A first elementary result tells that the product of two Hölder continuous functions is a Hölder continuous function. More precisely we have (see relation (4.7) in [8]):

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Lemma 11. *Assume that* $f_1, f_2 \in \mathcal{C}^{0,\alpha}(\Lambda)$. *Then* $f_1.f_2 \in \mathcal{C}^{0,\alpha}(\Lambda)$

In Section 5 we used a function defined as a composition of two Hölder continuous functions. We needed the following result:

Lemma 12. *Let* $\overline{\Omega}$ *be a compact in* $\mathbb{R}^d$ *and* $\alpha \in (0,1)$. *We consider the following three conditions:*

(A) $\lambda \in \mathcal{C}^1(\mathbb{R})$ *and* $k \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$

(B) $\lambda \in \mathcal{C}^{0,\alpha}(\mathbb{R})$ *and* $k \in \mathcal{C}^1(\overline{\Omega})$

(C) $\lambda \in \mathcal{C}^{1,\alpha}(\mathbb{R})$ *and* $k \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$

We have:

1. *Assume that* (A) *or* (B) *is satisfied. Then* $\lambda(k) \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$.
2. *Assume that* (C) *is satisfied. Then* $\lambda(k) \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$.

Proof. 1. In this situation we clearly have $\lambda(k) \in \mathcal{C}^0(\overline{\Omega})$ and

$$M_1 := \sup_{x\in\overline{\Omega}} |k(x)| < \infty. \tag{62}$$

Let

$$I(x,x_0) := \frac{|\lambda(k(x)) - \lambda(k(x_0))|}{|x-x_0|^\alpha}.$$

We want to prove that

$$\sup_{x,x_0\in\overline{\Omega}} I(x,x_0) < \infty. \tag{63}$$

- Assume that (A) holds. Then, in addition to (62), we have:

$$M_2 := \sup_{t,t_0\in[-M_1,M_1]} \frac{|\lambda(t)-\lambda(t_0)|}{|t-t_0|} < \infty \quad \text{and} \quad M_3 := \sup_{x,x_0\in\overline{\Omega}} \frac{|k(x)-k(x_0)|}{|x-x_0|^\alpha} < \infty.$$

Consequently:

$$I(x,x_0) \le M_2 \frac{|k(x)-k(x_0)|}{|x-x_0|^\alpha} \le M_2.M_3.$$

Hence (63) is satisfyed.

- Assume now that (B) holds. Then, in addition to (62) we have:

$$M_4 := \sup_{x,x_0\in\overline{\Omega}} \frac{|k(x)-k(x_0)|}{|x-x_0|} < \infty \quad \text{and} \quad M_5 := \sup_{t,t_0\in[-M_1,M_1]} \frac{|\lambda(t)-\lambda(t_0)|}{|t-t_0|^\alpha} < \infty.$$

In this situation we can estimate $I(x,x_0)$ as follows:

$$I(x,x_0) \le \frac{|\lambda(k(x)) - \lambda(k(x_0))|}{|k(x)-k(x_0)|^\alpha} . \frac{|k(x)-k(x_0)|^\alpha}{|x-x_0|^\alpha} \le M_5 M_4^\alpha.$$

Hence (63) is again satisfied.

2. Assume that (C) holds and let $\mu := \lambda(k)$. Clearly $\mu \in \mathcal{C}^1(\overline{\Omega})$ and $\nabla\mu = \lambda'(k)\nabla k$.

We remark that $\lambda' \in \mathcal{C}^{0,\alpha}(\mathbb{R})$ and $k \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$. We can then apply the first point of this lemma to obtain: $\lambda'(k) \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$. Moreover $\nabla k \in (\mathcal{C}^{0,\alpha}(\overline{\Omega}))^d$. Hence the product $\lambda'(k)\nabla k$ is Hölder continuous. □

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