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DOI 10.1007/s11118-006-9030-4

Higher Integrability of the Gradient in Degenerate Elliptic Equations

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Received: 16 May 2006 / Accepted: 13 July 2006

Abstract We prove that under some global conditions on the maximum and the minimum eigenvalue of the matrix of the coefficients, the gradient of the (weak) solution of some degenerate elliptic equations has higher integrability than expected. Technically we adapt the Giaquinta–Modica regularity method in some degenerate cases. When the dimension is two, a consequence of our result is a new Hölder continuity result for the weak solution.

Key words degenerate elliptic equations · regularity of solutions · weights

Mathematics Subject Classifications (2000) 35B65 · 35J70

1 Introduction

Let Ω be an open bounded set in $\mathbb{R}^n$ (n equals 2 or 3), with a Lipschitz boundary. We consider a linear, second order, self adjoint, degenerate elliptic equation with a homogeneous Dirichlet boundary condition:

$$\text{(P)} \qquad \begin{cases} \mathcal{L}u := -\sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial u}{\partial x_j}\right) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on} \partial\Omega. \end{cases}$$

Here $f : \Omega \to \mathbb{R}$ is a given function and $\mathcal{A}(x) := [a_{ij}(x)]_{i,j=1,..n}$ is a given symmetric matrix with measurable coefficients. We assume that $\mathcal{A}$ is positive definite almost

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everywhere in Ω, and we denote by $\lambda(x)$, $\Lambda(x)$ its minimum and maximum eigenvalues. It follows that for all $\eta, \theta \in \mathbb{R}^n$ and a.e $x \in \Omega$ we have:

$$\langle \mathcal{A}(x)\eta, \theta\rangle \le \Lambda(x)|\eta||\theta|, \tag{1}$$

$$\langle \mathcal{A}(x)\eta, \eta\rangle \ge \lambda(x)|\eta|^2, \tag{2}$$

where $\langle , \rangle$ denotes the scalar product in $\mathbb{R}^n$.

In this paper, we will study some questions about existence, uniqueness and regularity of weak solution for problem (P). We will also give a regularity result for the weak solution of a class of non-linear degenerate problems which include (P).

1.1 Mathematical Background

When λ may vanish or Λ may be unbounded then $\mathcal{L}$ is called degenerate operator. We will always assume in the following that λ is strictly positive almost everywhere and Λ is finite almost everywhere. These assumptions are not sufficient to analyse problem (P) and therefore we will also assume:

$$\lambda \in L^1(\Omega), \tag{3}$$

$$\lambda^{-1} \in L^1(\Omega), \tag{4}$$

$$\frac{\Lambda}{\lambda} \in L^\infty(\Omega). \tag{5}$$

For $p \ge 1$, we denote by $L^p(\lambda, \Omega)$ the weighted Lebesgue space defined by

$$L^p(\lambda, \Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable and s.t } \int_\Omega \lambda(x)|u(x)|^p dx < \infty \right\},$$

and equipped with the norm

$$\|u\|_{L^p(\lambda,\Omega)} = \left(\int_\Omega \lambda(x)|u(x)|^p dx \right)^{1/p}.$$

Hence $L^p(\lambda, \Omega)$ is a Banach space. Moreover assumption (3) ensures that $\mathcal{D}(\Omega) \subset L^p(\lambda, \Omega)$ with dense injection.

Under the assumptions (3–5), the natural functional setting for problem (P) is given by the following weighted Sobolev spaces:

$$W = \left\{ u \in W^{1,1}(\Omega) : \|u\|_\lambda < \infty \right\}, \tag{6}$$

$$W_0 = \left\{ u \in W_0^{1,1}(\Omega) : \|u\|_\lambda < \infty \right\}, \tag{7}$$

$$H = \overline{\mathcal{C}^\infty(\bar{\Omega})}^W, \tag{8}$$

$$H_0 = \overline{\mathcal{C}_c^\infty(\Omega)}^W, \tag{9}$$

where we have used the notation

$$\|u\|_\lambda^2 := \|u\|^2_{L^2(\lambda,\Omega)} + \|\nabla u\|^2_{L^2(\lambda,\Omega)}. \tag{10}$$

The spaces W, W_0, H and H_0 are Hilbert spaces and we have $H \subset W \subset \mathcal{D}'(\Omega)$ and $H_0 \subset W_0$. Notice that assumption (4) is quite necessary. If we remove it then W_0 need not be complete and the gradient of a function in H_0 need not be uniquely defined (see [40] and [31] Proposition 1.2).

Let $\mathcal{B}$ denote the bilinear form on $\mathcal{D}(\Omega)$ associated to $\mathcal{L}$:

$$\mathcal{B}(u,\varphi) := \int_\Omega \langle A(x)\nabla u(x), \nabla\varphi(x)\rangle dx. \tag{11}$$

By using assumption (1) together with (5) we obtain

$$|\mathcal{B}(u,\varphi)| \le C\|u\|_\lambda \|\varphi\|_\lambda \quad \forall u,\varphi \in W, \tag{12}$$

where $C := \|\frac{\Lambda}{\lambda}\|_{L^\infty(\Omega)}$. The bilinear form $\mathcal{B}$ is then continuous on W and we can consider two natural notions of weak solutions for Problem (P):

W-solution : Let $f \in W_0'$. A function $u \in W_0$ is called a W-solution of (P) if it verifies

$$\mathcal{B}(u,\varphi) = \langle f,\varphi\rangle_{W_0',W_0} \quad \forall\varphi \in W_0.$$

H-solution : Let $f \in H_0'$. A function $u \in H_0$ is called a H-solution of (P) if it verifies

$$\mathcal{B}(u,\varphi) = \langle f,\varphi\rangle_{H_0',H_0} \quad \forall\varphi \in H_0.$$

Note that in general, for $n \ge 2$, $H_0 \ne W_0$ (see [40]), and even for smooth second member, we can obtain a W-solution and a H-solution for Problem (P) that are not equal (see [40] Proposition 1.1, and [41]). When we have the equality $H = W$, which also implies $H_0 = W_0$ (see [31], Remark 1.5), we say that λ is regular. Sufficient conditions ensuring that a weight λ is regular were established in [40] and [15]. An exact characterization of regular weights is not known. In the sequel we will assume that

$$\lambda \ \text{ is a regular weight.} \tag{13}$$

It follows that the two notions of W-solution and H-solution are the same notion and we call it *weak solution*. Clearly in this case we also have $H_0' = W_0'$. Remark that, by definition, $\mathcal{D}(\Omega)$ is dense in H_0 so that H_0' can be identified with a subspace of $\mathcal{D}'(\Omega)$. In fact (see [29] p. 8), a distribution $T \in \mathcal{D}'(\Omega)$ is in H_0' if and only if it can be represented (in general non uniquely) as

$$\langle T,\varphi\rangle = \int_\Omega \big(f_0(x)\lambda(x)\varphi(x) + \lambda(x)\langle f(x),\nabla\varphi(x)\rangle\big)dx \quad \forall\varphi \in \mathcal{D}(\Omega),$$

with $f_0 \in L^2(\lambda^{-1},\Omega)$ and $f \in (L^2(\lambda^{-1},\Omega))^n$.

1.2 Main Results

Instead of Eqs. 3 and 4, we will consider the following stronger assumptions for λ:

$$\lambda^{-1} \in L^\infty(\Omega), \tag{14}$$

$$\lambda \in W^{1,1}(\Omega), \tag{15}$$

$$\text{there exists } \sigma > 2 \text{ s. t. } \int_\Omega \frac{|\nabla\lambda|^\sigma}{\lambda^{\frac{3\sigma}{2}}} < \infty. \tag{16}$$

Notice that condition (16) is equivalent to $\lambda^{-1/2} \in W^{1,\sigma}(\Omega)$. Finally, we will consider λ in the following class $\mathcal{K}$ of weights:

$$\mathcal{K} := \left\{\lambda > 0 \text{ a.e in } \Omega \text{ and it satisfies Eqs. 13–16}\right\}.$$

We will consider problem (P) with a second member satisfying

$$f \in W^{-1,p}(\Omega), \quad \text{with } p > n. \tag{17}$$

Notice that, when λ is in the class $\mathcal{K}$ and $p > n$, we have

$$H_0 = W_0 \subset H_0^1(\Omega) \subset W_0^{1,p'}(\Omega).$$

It follows that if f satisfies Eq. 17 then $f \in H_0'$.

Our main result is the following:

Theorem 1 *Assume that λ is in the class $\mathcal{K}$ and Eq. 5 is fulfilled. Then, for any $f \in H_0'$, there exists a unique weak solution u for problem (P), satisfying:*

$$\|u\|_\lambda \leq C\|f\|_{H_0'}. \tag{18}$$

If in addition f satisfies Eq. 17 then $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{-1,p}. \tag{19}$$

Moreover there exist $\varepsilon > 0$ and $C < \infty$ depending only on Ω, f and λ, such that

$$\|\nabla u\|_{(L^{2+\varepsilon}(\Omega))^n} \leq C. \tag{20}$$

A consequence of Theorem 1 (see [5] Theorem IX.12 p.166) is:

Corollary 2 When the dimension n equals two and f satisfies Eq. 17 then the weak solution u given by Theorem 1 is Hölder continuous.

The result presented in the last part of Theorem 1, and in the Corollary 2 remains valid for a general class of non-linear degenerate problems. In fact, we can consider the following problems:

$$\text{(P')} \quad \begin{cases} -\text{div}\, \mathcal{A}(x, u, \nabla u) = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where, $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Caratheodory functions that satisfy the following growth and coercivity conditions:

$$\left|\mathcal{A}(x, t, \eta)\right| \leq \mu_1\big(\Lambda(x)|\eta| + \varphi_1(x)\big), \tag{21}$$

$$\mathcal{A}(x, t, \eta)\eta \geq \mu_2\big(\lambda(x)|\eta|^2 - \varphi_2(x)\big), \tag{22}$$

$$\left|f(x, t, \eta)\right| \leq \mu_3\big(\Lambda(x)|\eta|^\alpha + \varphi_3(x)\big). \tag{23}$$

Here, $\varphi_i,\ i = 1, 2, 3$, are positive functions satisfying

$$\frac{\varphi_1^2}{\lambda^2}, \frac{\varphi_2^2}{\lambda^2} \in L^s(\Omega) \text{ and } \frac{\varphi_3}{\lambda} \in L^s(\Omega) \quad \text{for some } s > 1,$$

$\alpha < 2$ is a positive number and $\mu_1, \mu_3 \geq 0$, $\mu_2 > 0$ are allowed to depend on t.

We have:

Theorem 3 *Assume that $\lambda \in \mathcal{K}$, Eq. 5 holds true and $\mathcal{A}$, f are Caratheodory functions satisfying Eqs. 21–23. Assume moreover that $u \in H_0 \cap L^\infty(\Omega)$ is a weak solution of problem* (P'). *Then there exists $\varepsilon > 0$ such that $\int_\Omega |\nabla u|^{2+\varepsilon} < \infty$. In particular, for n equals two, the function u is Hölder continuous.*

Remarks

(1) The results in Theorem 1, Corollary 2 and Theorem 3 remain valid if λ is in the following class $\mathcal{K}'$ which includes $\mathcal{K}$:

$$\mathcal{K}' := \left\{ \begin{array}{l} \lambda > 0 \text{ a.e. in } \Omega \text{ and it is regular. Moreover } \lambda = \delta\rho, \text{ with} \\ \delta \in L^\infty(\Omega), \delta^{-1} \in L^\infty(\Omega) \text{ and } \rho \text{ satifies (14)–(16).} \end{array} \right\}$$

(2) The assumption λ regular can be removed in $\mathcal{K}$ or $\mathcal{K}'$. In this situation we have to consider W-solutions instead of weak solutions for problem (P) and we recover the results in Theorem 1, Corollary 2 and Theorem 3.

1.3 Discussion on the Literature

Degenerate problems like (P) have been extensively studied for many years.

In general, the existence of a weak solution is obtained after proving a Poincaré inequality (see [29, 31, 37] and [38]). In particular the first part of Theorem 1, i.e., the existence and uniqueness of the weak solution u satisfying Eq. 18 is an application of [29] Corollary 3.5.

For some studies about the question of the boundedness of u we can consult [12, 29] and [31]. In Theorem 1, the property (19) is an application of [29] Theorem 7.1.

The Hölder continuity of the weak solution, or the higher integrability of its gradient (in the sense of Eq. 20) have also studied been studied for many years. The first situation considered was the case of uniform ellipticity, that is when we have

$$\text{(UE)} \qquad \lambda \in L^\infty(\Omega),\ \lambda^{-1} \in L^\infty(\Omega).$$

In this situation, the Hölder continuity of the weak solution u (there is only one notion because λ is regular, see [40]) was established in the works of DeGiorgi, Nash and Moser (see [9, 27, 28, 30]), without restriction on the dimension.

On the other hand, a result of higher integrability for ∇u was obtained by Boyarski and Meyers (see [3, 24] and [13, 17, 25]).

These results were later generalized in numerous works. The principal generalization of (UE) we want to point out is the following:

$$\text{(M)} \qquad \lambda \in \mathcal{A}_2.$$

Here $\mathcal{A}_2$ is the Muckhenhoupt class of order two, i.e. $\lambda \in \mathcal{A}_2$ means

$$\sup_{\text{balls } B\subset\Omega} \fint_B \lambda \fint_B \lambda^{-1} < \infty,$$

where we have used the notation:

$$\fint_B g(y)dy := |B|^{-1} \int_B g(y)dy.$$

We can see that (UE) implies (M). In this case λ is regular, and Hölder continuity of the weak solution u (again without restriction on the dimension) was established by Fabes, Koenig and Serapioni (see [11] and [1, 39]). Moreover we also have a higher integrability result for ∇u (see [35]).

We will see in Section 2 that $\lambda \in \mathcal{K}$ does not imply (M). It follows that Theorem 1, Corollary 2 and Theorem 3 cannot be deduced from the results in [11] or [35] .

Notice also that, contrary to (M), our assumptions in $\mathcal{K}$ are of global nature. This is an important advantage for some applications of the higher integrability result, as will be explained in the next paragraph.

Under the assumption $\lambda \in \mathcal{K}$, we will obtain the higher integrability result for ∇u by using a method inspired by the works of Giaquinta–Modica (see [17]) and Stredulinsky (see [35]). In the situation they consider (case (UE) or (M)), the following three important properties hold:

(1) The measure λdx is doubling, i.e.

$$\sup_{B(x,2r)\subset\Omega} \frac{\int_{B(x,2r)} \lambda(y)dy}{\int_{B(x,r)} \lambda(y)dy} < \infty.$$

(2) λ is regular.

(3) Uniform Poincaré–Sobolev inequality on the balls, i.e. for all $B(x,r) \subset \Omega$:

$$\left(\int_{B(x,r)} |u(y) - \bar{u}|^2 \lambda(y)dy \right)^2 \leq C \left(\int_{B(x,r)} |\nabla u(y)|^{\frac{2n}{2+n}} \lambda(y)dy \right)^{\frac{2+n}{2n}}, \quad \forall u \in H,$$

where C is a constant, and $\bar{u} := \dfrac{\int_{B(x,r)} u(y)\lambda(y)dy}{\int_{B(x,r)} \lambda(y)dy}$.

The properties (1) and (3) are necessary for their techniques to work. In fact, they employ certain test functions in the weak formulation and, by using (1) and (3) they obtain a weak-reverse Hölder inequality for ∇u. After this, the higher integrability result for ∇u follows from a certain version of the Gehring lemma. The point is that, when $\lambda \in \mathcal{K}$, then the properties (1) and (3) need not hold (see the counterexamples in Section 3). Nevertheless we obtain the higher integrability of the gradient of u by using different test functions in the weak formulation. Notice that there exist relations between the properties (1)–(3) (see [2, 20, 31]).

In some cases, a higher integrability for the gradient can be obtained from interpolation theory (see [6]). Similar results can be established for parabolic equations (see [16, 22]).

1.4 Applications of the Results

Differential problems like (P) arrise in many physical models such as oceanography (see [4, 23]), turbulent fluid flows (see [15]), induction heating (see [8]) and electrochemical problems (see [14]). The knowledge of some regularity results for problem (P) is useful for the analysis of these physical models. In particular, a higher integrability result for the gradient of the weak solution of problem (P) would be useful for the analysis of the models studied in [8, 14, 15]. In fact, in these works, the problem analyzed is to find two scalar functions $u, h : \Omega \to \mathbb{R}$ vanishing on $\partial\Omega$ and such that:

$$-\operatorname{div}(r(h)\nabla u) = f \qquad \text{in } \mathcal{D}'(\Omega), \tag{24}$$

$$-\operatorname{div}(b(h)\nabla h) = r(h)|\nabla u|^2 \quad \text{in } \mathcal{D}'(\Omega), \tag{25}$$

where $f \in L^2(\Omega)$ and $r, b \in \mathcal{C}^1(\mathbb{R})$ are given. One way to solve Problem (24) and (25) is to decuple the two equations. First we solve Eq. 24 with a given $h = \bar{h}$, in some Sobolev space. This subproblem is in fact a particular case of Problem (P) where we have $\mathcal{A}(x) = r(\bar{h})Id$. In a second step, we want to solve Eq. 25 with a second member $r(\bar{h})|\nabla u|^2$ which is known, but only a *priori* to be in $L^1(\Omega)$. This latter fact creates difficulties for the subsequent analysis. The situation would be more favorable if the second member $r(\bar{h})|\nabla u|^2$ would be an element of $L^s(\Omega)$, for some $s > 1$. This property should be obtained in some cases, if we can apply a result like higher integrability for ∇u. For instance, this is the case if we assume that r and r^{-1} are in $L^\infty(\mathbb{R})$. It is then possible to apply the Meyers result (see [7]). However, the assumption $r, r^{-1} \in L^\infty(\mathbb{R})$ doest not always have physical relevance (see [7, 15]). Under more restrictive conditions on r we would apply the Stredulinsky result, but here the difficulty is to find precisely what these conditions are. In fact, we have to ensure that $r(\bar{h}) \in \mathcal{A}_2$, which is not easy if we recall the definition of an $\mathcal{A}_2$-weight. Here our regularity results presented in Theorem 1 and 3 are easier to use.

1.5 Organization of the Paper

In Section 2, we will present the proof of Theorem 1. The higher integrability result for the gradient of the weak solution is obtained from a weak-reverse Hölder inequality. The method is inspirated by the works of Giaquinta–Modica (case (UE), see [17]) and Stredulinsky (case (M), see [35]), but the originality resides in a special choice of test functions. The reason is that, contrary to the case where λ is in the class (M), if $\lambda \in \mathcal{K}$ then the measure λdx need not be doubling (see the counter examples in Section 3) and we do not need to have a uniform Poincaré–Sobolev inequality on the balls in H_0. Theses two properties are necessary in the technique of Giaquinta–Modica and Stredulinsky. With a particular choice of test functions we are able to overcome this difficulty.

Theorem 3 is proved in the same manner. We also give some indications concerning the remarks at the end of paragraph 1.2.

In Section 3, we construct in dimension two and three a weight $\lambda \in \mathcal{K}$ which is not in (M). In dimension three the example presented is particulary instructive. It is apparently close to satisfying the condition (UE) but we will prove that in fact it does not satisfy (M). Moreover this weight does not have bounded mean oscillations.

2 The Proofs

2.1 The First Part of the Proof of Theorem 1

Assume that λ and Λ satisfy the assumptions in Theorem 1. Let $f \in H_0'$. The weak formulation for problem (P) consists in finding $u \in H_0 (= W_0)$ such that:

$$\mathcal{B}(u, \varphi) = \langle f, \varphi \rangle \quad \forall \varphi \in H_0. \tag{26}$$

Recall first that the bilinear form $\mathcal{B}$ is continuous, as seen in Eq. 12.

The first part of Theorem 1 is a consequence of Corrolary 3.5 p. 22 and Theorem 7.1 p. 49 in [29]. In fact the assumption (15) implies that $\lambda \in L^{\frac{n}{n-1}}(\Omega)$, and by using Eq. 14 we can see that the condition (3.2) in [29] p. 21 is fulfilled (for $n = 2$ take $p = 2, s = 2, t = 2$ and for $n = 3$ take $p = 2, s = 3/2, t = \infty$).

Hence, from Corollary 3.5 in [29] we obtain:

$$\|\nabla\varphi\|_{L^2(\lambda,\Omega)} \geq C_1 \|\varphi\|_\lambda \quad \forall \varphi \in H_0, \tag{27}$$

where $C_1 > 0$ is a constant depending only on Ω and λ.

It follows that the bilinear form $\mathcal{B}$ defined by Eq. 11 is coercive on H_0. Then, by the Lax–Milgram theorem, we obtain a unique solution for Eq. 26, with the estimate

$$\|u\|_\lambda \leq C_2 \|f\|_{H_0'}, \quad C_2 = C_2(C_1). \tag{28}$$

Let us now consider f satisfying Eq. 17. By using [5], Proposition IX.20 p.175, we obtain the existence of a function g as follows:

$$g \in (L^p(\Omega))^n \ : \ \langle f, \varphi \rangle_{W^{-1,p}, W^{1,p'}} = \int_\Omega \langle g(x), \nabla\varphi(x) \rangle dx, \qquad \forall \varphi \in W_0^{1,p'}(\Omega). \tag{29}$$

Recall that $\lambda^{-1} \in L^\infty(\Omega)$ and thus $g \in (L^p(\lambda^{-1}, \Omega))^n$. We can now use Theorem 7.1 in [29] p. 49 (take $s = n/(n-1)$, $t = \infty$ and use assumption (17)). We obtain $u \in L^\infty(\Omega)$ and

$$C_4 := \|u\|_{L^\infty(\Omega)} \leq C_3 \|f\|_{-1,p}. \tag{30}$$

Here, $C_3 = C_3(C_2, \Omega, \lambda)$.

This proves the first part of Theorem 1.

For the second part of the theorem, we will use a technique inspired by the works of Giaquinta and Modica (see [17] and [32]). We will obtain the higher integrability of the gradient of u from a weak reverse Hölder inequality. The major tool is the Proposition 1.1 p. 122 in [17]. Notice that this proposition is a refinement of the Gehring lemma (see [18]). Other versions of the Gehring lemma were established in [21, 26, 35].

Let Q_{R_0} denote a n-cube, parallel to the coordinate axis and such that

$$\bar{\Omega} \subset Q_{R_0} \quad \text{and} \quad \operatorname{dist}(\partial Q_{R_0}, \partial\Omega) = R_1 > 0. \tag{31}$$

Let $x \in Q_{R_0}$. For $r > 0$, we denote by $Q_r(x)$ the n-cube centered in x, parallel to the coordinate axis and with side length equal $2r$, that is:

$$Q_r(x) = \left\{ y \in \mathbb{R}^n : |y_i - x_i| < r \right\}.$$

In the sequel we will consider the following bound for r:

$$r < \frac{1}{2}\mathrm{dist}(x, \partial Q_{R_0}). \tag{32}$$

This condition ensures that $Q_{2r}(x) \subset Q_{R_0}$. We next consider the cube $Q_{\frac{3r}{2}}(x)$. We have three possibilities:

1) $Q_{\frac{3r}{2}}(x) \cap \Omega = \varnothing$
2) $Q_{\frac{3r}{2}}(x) \cap \left(Q_{R_0} \setminus \Omega\right) = \varnothing$
3) $Q_{\frac{3r}{2}}(x) \cap \Omega \neq \varnothing$ and $Q_{\frac{3r}{2}}(x) \cap \left(Q_{R_0} \setminus \Omega\right) \neq \varnothing$.

For any function ζ defined on Ω, we denote by $\tilde{\zeta}$ its extension on Q_{R_0} defined by:

$$\tilde{\zeta} = \begin{cases} \zeta(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in Q_{R_0} \setminus \Omega. \end{cases}$$

Let u be the weak solution for problem (P), and $\tilde{u}$ its extension on Q_{R_0}. We set

$$k := \frac{|\nabla u||\nabla \lambda|}{\lambda} + \frac{|\nabla u||g|}{\lambda} + \frac{|\nabla \lambda||g|}{\lambda^2} + \frac{|g|^2}{\lambda^2}, \tag{33}$$

$$q := \frac{2n}{n+2}. \tag{34}$$

In order to apply the Proposition 1.1 in [17], we will prove the following:

Lemma 4 There exists a positive constant M, depending only on f, λ and Ω such that

$$\fint_{Q_r(x)} |\nabla \tilde{u}|^2 dy \leq M\left\{\left(\fint_{Q_{2r}(x)} |\nabla \tilde{u}|^q dy\right)^{2/q} + \fint_{Q_{2r}(x)} \tilde{k} dy\right\}, \tag{35}$$

for each $x \in Q_{R_0}$ and $r \geq 0$ satisfying Eq. 32.

Let $x \in Q_{R_0}$ and $r \geq 0$ satisfy Eq. 32. We will prove that Eq. 35 holds true in each of the cases 1), 2) and 3).

In case 1), we have $Q_{\frac{3r}{2}}(x) \cap \Omega = \varnothing$ and then inequality (35) is trivial since we have $\tilde{u} \equiv 0$ on $Q_r(x)$.

It remains to establish Eq. 35 in the cases 2) and 3). This is the aim of the next two paragraphs.

2.2 The Weak Reverse Hölder Inequality in the Case 2)

Here we have $Q_{\frac{3R}{2}}(x) \cap \left(Q_{R_0} \setminus \Omega\right) = \varnothing$. We will obtain the inequality (35) by using an appropriate test function in the weak formulation (26). Namely, we set

$$\varphi = \frac{u - \bar{u}}{\lambda}\psi^2, \tag{36}$$

where $\bar{u}$ is the mean integral of u over the ball $B_{\frac{3r}{2}}(x)$ (which is contained in $Q_{\frac{3r}{2}}(x)$) and ψ is a cut-off function satisfying:

$$\psi \in \mathcal{C}_c^1(B_{\frac{3r}{2}}(x)), \tag{37}$$

$$0 \le \psi \le 1 \quad \text{and} \quad \psi \equiv 1 \text{ in } Q_r(x), \tag{38}$$

$$|\nabla \psi| \le \frac{C_5}{r}. \tag{39}$$

Here C_5 denotes a constant independent of r.

Remark that we can write $\varphi = \varphi_1\varphi_2$, with $\varphi_1 := (u-\bar{u})\varphi$ and $\varphi_2 := \frac{\psi}{\lambda}$. By using the assumptions (14–16) together with the fact that $u \in W_0 \cap L^\infty(\Omega)$ we can see that each of the two functions φ_1 and φ_2 is an element of the space $H_0^1(\Omega) \cap L^\infty(\Omega)$. It follows that $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (see [5] Proposition IX.4 p. 155). In particular $\varphi \in W_0^{1,1}(\Omega)$ and $\int_\Omega \lambda|\varphi|^2 < \infty$. In order to verify that φ is an admissible test function for Eq. 26, it is sufficient to check that $\int_\Omega \lambda|\nabla\varphi|^2 < \infty$.

To prove this, we first calculate the expression for the gradient of φ. We obtain:

$$\nabla\varphi = \frac{1}{\lambda}\nabla u\psi^2 - (u-\bar{u})\frac{\nabla\lambda}{\lambda^2}\psi^2 + \frac{2}{\lambda}\psi\nabla\psi(u-\bar{u}). \tag{40}$$

By using the assumptions for λ together with the property $u \in W_0 \cap L^\infty(\Omega)$, we can verify that each of the three terms in the right hand side of Eq. 40 are in the space $L^2(\lambda, \Omega)$. Consequently φ is an admissible test function.

We now test the Eq. 26 with φ. By using the expression of $\nabla\varphi$ given in Eq. 40, we obtain:

$$\begin{aligned} I := \int_\Omega \frac{\psi^2}{\lambda}\langle \mathcal{A}(x)\nabla u, \nabla u\rangle &= \underbrace{\int_\Omega \frac{u-\bar{u}}{\lambda^2}\psi^2\langle \mathcal{A}(x)\nabla u, \nabla\lambda\rangle}_{:=II} \\ &- \underbrace{\int_\Omega 2\psi\frac{u-\bar{u}}{\lambda}\langle \mathcal{A}(x)\nabla u, \nabla\psi\rangle}_{:=III} + \underbrace{\int_\Omega \frac{\psi^2}{\lambda}\langle g, \nabla u\rangle}_{:=IV} \\ &- \underbrace{\int_\Omega \frac{u-\bar{u}}{\lambda^2}\psi^2\langle g, \nabla\lambda\rangle}_{:=V} + \underbrace{\int_\Omega 2\psi\frac{u-\bar{u}}{\lambda}\langle g, \nabla\psi\rangle}_{:=VI}. \end{aligned} \tag{41}$$

We estimate the term I by using Eq. 2:

$$I \ge \int_\Omega |\nabla u|^2\psi^2.$$

By using Eq. 1 together with Eqs. 5 and 30, we obtain:

$$|II| \le \int_\Omega \frac{|u-\bar{u}|}{\lambda^2}\psi^2\Lambda|\nabla u\nabla\lambda| \le C_6(\|\frac{\Lambda}{\lambda}\|_{L^\infty}, C_4)\int_{B_{\frac{3r}{2}}(x)} \frac{|\nabla u\nabla\lambda|}{\lambda}.$$

In order to estimate the third term, we use again Eq. 1 together with the Young inequality. We obtain:

$$|III| \leq \frac{1}{2}\int_{\Omega} |\nabla u|^2 \psi^2 + \frac{C_7}{r^2}\int_{B_{\frac{3r}{2}}(x)} |u - \bar{u}|^2.$$

Here $C_7 = C_7(\|\frac{\Lambda}{\lambda}\|_{L^\infty}, C_4, C_5)$.

The terms IV and V can be estimated by employing the property (30). We obtain:

$$|IV| + |V| \leq C_8 \int_{B_{\frac{3r}{2}}(x)} \frac{|\nabla u||g|}{\lambda} + \frac{|\nabla\lambda||g|}{\lambda^2}, \quad C_8 = C_8(C_4).$$

For the last term in Eq. 41 we use the Young inequality and property (39) to obtain:

$$|VI| \leq C_9\left(\frac{1}{r^2}\int_{B_{\frac{3r}{2}}(x)} |u - \bar{u}|^2 + \int_{B_{\frac{3r}{2}}(x)} \frac{|g|^2}{\lambda^2}\right), \quad C_9 = C_9(C_5).$$

At this point, from Eq. 41, we can deduce

$$\frac{1}{2}\int_{\Omega} |\nabla u|^2 \psi^2 \leq C_{10}\left(\frac{1}{r^2}\int_{B_{\frac{3r}{2}}(x)} |u - \bar{u}|^2 + \int_{B_{\frac{3r}{2}}(x)} k\right), \tag{42}$$

with $C_{10} = \mathrm{Max}(C_6, C_7 + C_9, C_8)$.

We next use the property (38), and we divide Eq. 42 by $|B_{\frac{3r}{2}}(x)|$. We have:

$$\fint_{Q_r(x)} |\nabla u|^2 \leq C_{11}\left(\frac{1}{r^2}\fint_{B_{\frac{3r}{2}}(x)} |u - \bar{u}|^2 + \fint_{B_{\frac{3r}{2}}(x)} k\right), \quad C_{11} = C_{11}(C_{10}, n).$$

Recall now that $u \in H_0^1(\Omega)$ and thus (see [10], Theorem 2 p.141) there exists $C_{12} = C_{12}(\Omega)$ such that

$$\frac{1}{r^2}\fint_{B_{\frac{3r}{2}}(x)} |u - \bar{u}|^2 \leq C_{12}\left(\fint_{B_{\frac{3r}{2}}(x)} |\nabla u|^q\right)^{2/q}.$$

Note also that $B_{\frac{3r}{2}}(x)$ is included in $Q_{\frac{3r}{2}}(x)$, and the Lebesgue measure of these sets is comparable. We then get:

$$\fint_{Q_r(x)} |\nabla u|^2 \leq M\left(\left(\fint_{Q_{\frac{3r}{2}}(x)} |\nabla u|^q\right)^{2/q} + \fint_{Q_{\frac{3r}{2}}(x)} k\right), \quad M = M(C_{11}, C_{12}).$$

and inequality (35) follows.

2.3 The Weak Reverse Hölder Inequality in the Case 3)

Here the cube $Q_{\frac{3r}{2}}(x)$ intersects $\partial\Omega$ and we have to slightly modify the technique in order to obtain Eq. 35.

Let

$$\varphi = \frac{u}{\lambda}\psi^2, \tag{43}$$

where ψ is a cut-off function verifying Eqs. 37–39. The expression of $\nabla\varphi$ is now given by:

$$\nabla\varphi = \frac{\nabla u}{\lambda}\psi^2 + \frac{2}{\lambda}\psi\nabla\psi u - u\psi^2\frac{\nabla\lambda}{\lambda^2}. \tag{44}$$

By using the same arguments as in paragraph 2.2, we can verify that $\varphi \in W_0$.

We then test the Eq. 26 with φ, and, instead of Eq. 42, we now obtain:

$$\int\limits_{Q_r(x)\cap\Omega} |\nabla u|^2 \le C'_{10}\left(\frac{1}{r^2}\int\limits_{B_{\frac{3r}{2}}(x)\cap\Omega} |u|^2 + \int\limits_{B_{\frac{3r}{2}}(x)\cap\Omega} k\right). \tag{45}$$

Let us consider the extension $\tilde{u}$ of u, and the cube $Q_{2r}(x)$ (included in Q_{R_0}). We have $\tilde{u} \equiv 0$ in $Q_{2r}(x)\setminus\Omega$. Recall that we have assumed $\partial\Omega$ to be Lipschitz, which implies that $|Q_{2r}(x)\setminus\Omega| \ge \gamma|Q_{2r}(x)|$ for some $\gamma > 0$ independently of r. Moreover, we clearly have $\tilde{u} \in H^1(Q_{2r}(x))$.

It then follows, by using [17] Proposition p. 153 and Eq. 45, that:

$$\int\limits_{Q_{2r}(x)} |\nabla\tilde{u}|^2 \le C'_{11}\left(\frac{1}{r^2}\Bigg(\int\limits_{Q_{2r}(x)} |\nabla\tilde{u}|^q\Bigg)^{2/q} + \int_{Q_{2r}(x)} \tilde{k}\right).$$

By dividing this inequality by $|Q_r(x)|$, we obtain Eq. 35. In fact, $|Q_r(x)|$ is comparable to $Q_{2r}(x)$ and also comparable to $(1/r^2)|Q_r(x)|^{2/q}$.

This ends the proof of Lemma 4.

2.4 The Conclusion of the Proof of Theorem 1

We set

$$h := |\nabla\tilde{u}|^q, \tag{46}$$

$$l := \tilde{k}^{q/2}, \tag{47}$$

where k is the function defined in Eq. 33 and q is the number given in Eq. 34.

With these notations, the inequality (35) can be written as:

$$\fint\limits_{Q_r(x)} h^{2/q} \le M\Bigg(\Bigg(\fint\limits_{Q_{2r}(x)} h\Bigg)^{2/q} + \fint\limits_{Q_{2r}(x)} l^{2/q}\Bigg). \tag{48}$$

Note also that $2/q = (n+2)/n > 1$.

At this point, we can use Proposition 1.1 p. 122 in [17]. We obtain the existence of a constant $C_{13} = C_{13}(M, q, n)$ and of $\varepsilon = \varepsilon(M, q, n) > 0$ such that, for all $x \in Q_{R_0}$ and all $r < (1/2)\mathrm{dist}(x, \partial Q_{R_0})$ the following holds:

$$\left(\fint_{Q_r(x)} h^\chi\right)^{1/\chi} \le C_{13}\left(\left(\fint_{Q_{2r}(x)} h^{2/q}\right)^{q/2} + \left(\fint_{Q_{2r}(x)} l^\chi\right)^{1/\chi}\right), \tag{49}$$

where $\chi := 2/q + \varepsilon$.

Let now $R^* > 0$ be given by

$$R^* := \frac{R_1}{3} = \frac{1}{3}\mathrm{dist}(\partial\Omega, \partial Q_{R_0}). \tag{50}$$

Notice that, for every $x \in \bar{\Omega}$, the number R^* satisfies the condition (32). Since $\bar{\Omega}$ is compact, we have:

$$\bar{\Omega} = \cup_{i=1}^m Q_{R^*}(x_i) \cap \bar{\Omega},$$

where $x_1, x_2, .., x_m$ are some points in $\bar{\Omega}$.

By applying Eq. 49 we obtain:

$$\left(\int_\Omega h^\chi\right)^{1/\chi} \le C_{14}\left(\left(\int_\Omega h^{2/q}\right)^{q/2} + \left(\int_\Omega l^\chi\right)^{1/\chi}\right),$$

where $C_{14} = C_{14}(C_{13}, m, R^*)$.

Recalling the definitions (46) and (47) we have:

$$h^\chi = |\nabla u|^{2+q\varepsilon},\ h^{2/q} = |\nabla u|^2,\ l^\chi = k^{1+q\varepsilon/2} \quad \text{on } \Omega.$$

In order to conclude the proof we have then to show that $k \in L^\beta(\Omega)$ for some $\beta > 1$. We have:

$$k = \underbrace{\frac{|\nabla u||\nabla\lambda|}{\lambda}}_{:=k_1} + \underbrace{\frac{|\nabla u||g|}{\lambda}}_{:=k_2} + \underbrace{\frac{|\nabla\lambda||g|}{\lambda^2}}_{:=k_3} + \underbrace{\frac{|g|^2}{\lambda^2}}_{:=k_4}.$$

By using the assumption $\lambda^{-1/2} \in W^{1,\sigma}(\Omega)$ we can see that $k_1 \in L^{\beta_1}(\Omega)$, with $\beta_1 = (2\sigma)/(2+\sigma) > 1$. In fact:

$$\|k_1\|_{L^{\beta_1}(\Omega)} = \left(\int_\Omega \lambda^{\beta_1/2}|\nabla u|^{\beta_1}\frac{|\nabla\lambda|^{\beta_1}}{\lambda^{\frac{3\beta_1}{2}}}\right)^{1/\beta_1} \le \left(\int_\Omega \lambda|\nabla u|^2\right)^{1/2}\left(\int_\Omega \frac{|\nabla\lambda|^\sigma}{\lambda^{\frac{3\sigma}{2}}}\right)^{1/\sigma}$$
$$\le \|u\|_\lambda \|\lambda^{-1/2}\|_{1,\sigma} < \infty.$$

We now use Eq. 29 to see that $k_2, k_3 \in L^{\beta_2}(\Omega)$ with $\beta_2 = (2p)/(2+p) > 1$:

$$\|k_2\|_{L^{\beta_2}(\Omega)} \le \|\lambda^{-1}\|^{3/2}_{L^\infty(\Omega)}\|u\|_\lambda\|g\|_{(L^p(\Omega))^n} < \infty,$$

$$\|k_3\|_{L^{\beta_2}(\Omega)} \le \|\lambda^{-1}\|^{1/2}_{L^\infty(\Omega)}\|\lambda^{-1/2}\|_{1,2}\|g\|_{(L^p(\Omega))^n} < \infty.$$

Finally we can show that $k_4 \in L^{\beta_3}(\Omega)$, with $\beta_3 = p/2 > 1$:

$$\|k_4\|_{L^{\beta_3}(\Omega)} \le \|\lambda^{-1}\|^2_{L^\infty(\Omega)}\|g\|^2_{(L^p(\Omega))^n} < \infty.$$

Hence $k \in L^{\beta}(\Omega)$, with $\beta = \min(\beta_1, \beta_2, \beta_3) > 1$. Let $\varepsilon_1 = 2(\beta - 1)/q > 0$. We have proved that there exists $\varepsilon \in (0, \varepsilon_1]$ such that

$$\left(\int_{\Omega} |\nabla u|^{2+\varepsilon'} \right)^{\frac{1}{2+\varepsilon'}} \leq C_{15}(C_{14}, \|u\|_{\lambda}, \|k\|_{L^{\beta}}) < \infty,$$

where $\varepsilon' := (2n\varepsilon)/(n+2) > 0$.

We have thus proved Theorem 1.

2.5 The Proof of Theorem 3 and Indications for Implementing the Remarks in Paragraph 1.2

Theorem 3 can be proved by using the same technique as for the last part of Theorem 1. We can carry over the arguments presented in paragraphs 2.2, 2.3 and 2.5 with only slight modifications. In the sequel, we will indicate the modifications needed.

Let $u \in H_0 \cap L^{\infty}(\Omega)$ be a weak solution of problem (P'), that is:

$$\int_{\Omega} \langle \mathcal{A}(x, u, \nabla u), \nabla \varphi(x) \rangle dx = \int_{\Omega} f(x, u, \nabla u)\varphi(x) dx, \quad \forall \varphi \in H_0.$$

Here $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Caratheodory functions that satisfy Eqs. 21–22. We recall that the parameters μ_i in Eqs. 21–22 are allowed to depend on t. We then have:

$$\mu_i = \mu_i\big(\|u\|_{L^{\infty}(\Omega)}\big), \ \mu_i < \infty, \ i = 1, 2, 3, \ \mu_2 > 0.$$

By using the same test functions as in paragraphs 2.2 and 2.3, (this is allowed because we have assumed here that $u \in L^{\infty}(\Omega)$) we again obtain the inequality (35), but with M depending also on μ_i now, and with k given by:

$$k := \frac{|\nabla u||\varphi_2|}{\lambda} + \frac{|\nabla u||\nabla\lambda|}{\lambda} + \frac{|\nabla\lambda||\varphi_1|}{\lambda^2} + \frac{|\varphi_1|^2}{\lambda^2} + |\nabla u|^{\alpha} + \frac{|\varphi_3|}{\lambda}.$$

Under the assumptions made on λ, φ_i and α we recover the fact that $k \in L^{\beta}(\Omega)$ for some $\beta > 1$. The proof can then be completed by following the reasoning presented in paragraph 2.4.

We now give some indications concerning the remarks at the end of paragraph 1.2.

Firstly, if we consider $\lambda \in \mathcal{K}'$ instead of $\lambda \in \mathcal{K}$ then the proofs presented previously work. It suffices to consider the test functions φ given by

$$\varphi = \frac{u - \bar{u}}{\rho}\psi^2,$$

instead of Eq. 36 and

$$\varphi = \frac{u}{\rho}\psi^2,$$

instead of Eq. 43.

If we remove the assumption (13), then, as explained in Section 1, we need not have $H_0 = W_0$. Nevertheless, the first part of Theorem 1 can be established for W-solutions by using Proposition 2.6 and Theorem 2.9 in [31]. For the last part, we can carry over the arguments presented in the paragraphs 2.2, 2.3 and 2.5. In fact, the test functions φ we have used were always in W_0 and they are thus admissible for the W-formulation.

3 Additional Remarks and Examples

In the first paragraph, we will present some examples in dimensions two and three of some weights satisfying our assumptions in $\mathcal{K}$ but which are not in $\mathcal{A}_2$. In dimension three, we present a critical example of a weight in our class $\mathcal{K}$. It is apparently close to satisfying the condition (UE) but we will prove that it does not satisfy (M). Moreover, this weight does not have bounded mean oscillations.

In a second paragraph, we will give some remarks concerning the one dimensional case and the case where $\mathcal{A}(x) = \lambda(x)Id$.

3.1 Example of Weights in the Class $\mathcal{K}$ but not in the Class $\mathcal{A}_2$

We present a first example in dimension two. Let Ω be the unit disc in $\mathbb{R}^2$. We denote by Ω^- the inferior half disc. We also consider the sectors Λ_2, Λ_0 and Λ_1 having a polar angle θ between the values 0 and $\pi/4$, $\pi/4$ and $(3\pi)/4$, $(3\pi)/4$ and π, respectively. We set

$$\begin{aligned} \lambda &= r^{-1/2} \quad \text{in } \Lambda_0, \\ \lambda &= 1 \qquad\;\; \text{in } \Omega^-, \end{aligned}$$

and otherwise we define λ by an affine interpolation with respect to θ, that is:

$$\begin{aligned} \lambda &= \frac{4}{\pi}\frac{1-\sqrt{r}}{\sqrt{r}}\theta + 1 && \text{in } \Lambda_2, \\ \lambda &= \frac{4}{\pi}\frac{\sqrt{r}-1}{\sqrt{r}}\theta + \frac{4-3\sqrt{r}}{\sqrt{r}} && \text{in } \Lambda_1. \end{aligned}$$

We can then verify the following:

$$\lambda \geq 1 \quad \text{everywhere on } \Omega, \tag{51}$$

$$\lambda \in W^{1,s}(\Omega) \quad \text{for each } s < 4/3, \tag{52}$$

$$\lambda^{-1/2} \in W^{1,7/3}(\Omega). \tag{53}$$

Consequently, the assumptions (16–19) are fulfilled. Moreover, by using the Corollary 4.4 in [40] we can see that λ is regular.

It follows that λ is in the class $\mathcal{K}$ for which Theorem 1 and Corollary 2 can be applied. Nevertheless $\lambda \notin \mathcal{A}_2$.

To see this, we consider the sequence of points $x_k = (0, -1/k)$. For k sufficiently large, the disc $B(x_k, 2/k)$ is included in Ω and we have:

$$\int_{B(x_k,\frac{2}{k})} \lambda dx \geq \int_{B(0,\frac{1}{k})\cap\Lambda_0} \lambda dx = \frac{1}{4}\int_{B(0,\frac{1}{k})} r^{-1/2} dx = \frac{\pi}{3k^{3/2}}.$$

On the other hand:

$$\int_{B(x_k,\frac{1}{k})} \lambda dx = \left| B\left(x_k, \frac{1}{k}\right)\right| = \frac{\pi}{k^2}.$$

Hence

$$\frac{\int_{B(x_k,\frac{2}{k})} \lambda dx}{\int_{B(x_k,\frac{1}{k})} \lambda dx} \geq \frac{\sqrt{k}}{3} \to \infty \quad \text{when } k \to \infty.$$

This implies that the measure λdx is not doubling, and thus $\lambda \notin \mathcal{A}_2$ (see [19]).

In this situation, Theorem 1 is not a consequence of Theorem 3.3.6 p. 135 in [35], and Corollary 2 cannot follow by the results in [11].

We consider now a critical example in dimension three.

Let $\Omega = B(0, e^{-4}) \subset \mathbb{R}^3$. We consider the partition $\Omega = \Omega^- \cup \Lambda_0 \cup \Lambda_3$, where:

$$\Omega^- = \{(x, y, z) \in \Omega \ : \ z < 0\},$$

$$\Lambda_0 = \{(x, y, z) \in \Omega \ : \varphi \in (0, \pi/4)\},$$

$$\Lambda_3 = \{(x, y, z) \in \Omega \ : \varphi \in (\pi/4, \pi/2)\}.$$

Here φ denotes the colatitude in spherical coordinates.
We set:

$$\lambda = \begin{cases} \ln(-\ln(r)) & \text{in } \Lambda_0, \\ 1 \quad \text{in } \Omega^-, \\ \dfrac{4}{\pi}(1 - \ln(-\ln(r))\varphi + 2\ln(-\ln r) - 1 & \text{in } \Lambda_3. \end{cases}$$

Notice that, on the sector Λ_3 we have defined λ by interpolating (with respect to φ) between the values on Λ_0 and on Ω^-.

We have:

$$\lambda \geq 1 \quad \text{in } \Omega, \tag{54}$$

$$\lambda \in W^{1,3}(\Omega). \tag{55}$$

Remark that Eq. 54 together with Eq. 55 implies that $\lambda^{-1/2} \in W^{1,3}(\Omega)$. We can also verify that $\sqrt{\lambda} \in H^1(\Omega)$, and by using Theorem 3.1 in [15], we can show that λ is regular. Hence λ is in the class $\mathcal{K}$ for which Theorem 1 and Theorem 3 work.

By applying the same method, as in the previous example, we can verify that $\lambda \notin \mathcal{A}_2$. Notice that here we have $\lambda^{-1} \in L^\infty(\Omega)$ and $\lambda \in W^{1,3}(\Omega) \subset \cap_{p\geq 1} L^p(\Omega)$. This is a limit case: apparently λ is nearly satisfying (UE), nevertheless $\lambda \notin \mathcal{A}_2$. Remark also that here Theorem 1 allows us to obtain a weighted higher integrability for the gradient of u. Namely, the weak solution of problem (P) satisfies:

$$\int_\Omega \lambda |\nabla u|^{2+\varepsilon} < \infty,$$

for some $\varepsilon > 0$.

Finally we can even verify that $\lambda \notin BMO$. In fact (see [34] p. 218), each BMO function ψ can be written in the form $\psi = c \ln \omega$, with $\omega \in \mathcal{A}_p$ and $p > 1$. Here $\mathcal{A}_p$ denotes the Muckenhoupt class of order p (see [19]). In particular the measure ωdx is doubling (see [19]). But, $e^{\lambda} dx$ is not doubling (use again the same arguments as for λdx). Consequently $\lambda \notin BMO$, and Theorem 1 cannot be deduced from [36].

3.2 Some Special Cases for Problem (P)

In some particular situations we can obtain the results contained in Theorem 1 and Corollary 2 more easily.

This is true for instance in the one dimensional situation. Let $I = (a, b)$ be a finite interval on the real line. The problem (P) takes the form:

$$\begin{aligned} (\lambda u')' &= f \quad \text{in } I, \\ u(a) &= u(b) = 0. \end{aligned}$$

If we consider $f \in L^1(I)$, then this problem can be solved without using the weighted Sobolev setting, and it suffices to make the assumptions (5–6) on λ. In fact, by a direct integration we explicitly obtain a distributional solution u. Namely:

$$\begin{aligned} u(x) &= \int_a^x \frac{1}{\lambda(t)} \left(c_1 + \int_a^t f(s)ds \right) dt, \\ c_1 &= \frac{-\displaystyle\int_a^b \int_a^t f(s)dsdt}{\displaystyle\int_a^b \frac{1}{\lambda(t)} dt}. \end{aligned} \tag{56}$$

We can see that the weak derivative of u is given by

$$u'(x) = \frac{1}{\lambda(x)} \left(\int_a^x f(t)dt + c_1 \right).$$

It follows that if $\lambda \in L^s(I)$ then $u \in W^{1,s}(I)$. Consequently $u \in \mathcal{C}^{0,(s-1)/s}(\bar{I})$ (see [5] Theorem VIII.2 p. 122).

Let now $n \geq 2$, and consider the particular case where $\mathcal{A}(x) = \lambda(x) Id$. Assume that the hypothesis of Theorem 1 is satisfied. By using the first part of Theorem 1 we know that there exists a unique weak solution $u \in H_0$ for problem (P). Moreover, in this case we have:

$$-\Delta u = \underbrace{-\frac{f}{\lambda}}_{:= f_1 \in W^{-1,p}(\Omega)} + \underbrace{\frac{\nabla u \nabla \lambda}{\lambda}}_{:= f_2}.$$

By using the assumption (16) we can see that

$$f_2 = \lambda^{1/2} \nabla u \frac{\nabla \lambda}{\lambda^{3/2}} \in L^{\beta}(\Omega),$$

with $\beta = (2\sigma)/(2+\sigma)$.

Let us denote by G the inverse of the Laplacian operator on Ω with homogeneous Dirichlet conditions on $\partial\Omega$, and consider the functions $u_1 = G(f_1)$, $u_2 = G(f_2)$. If we assume that Ω is of class $\mathcal{C}^2$ then by using classical regularity results (see for instance [33], Theorem 7.2 p. 123 and [5] Theorem IX.25 p. 181) we obtain:

$$u_1 \in W_0^{1,p}(\Omega), \quad u_2 \in W^{2,\beta}(\Omega).$$

By employing the Sobolev imbedding theorem we see that $u_2 \in W^{1,\beta^*}(\Omega)$, with $\beta^* = (n\beta)/(n-\beta)$. It follows that for $n = 2$ we obtain $\beta^* > 2$ and we recover the last part of Theorem 1. When $n > 2$ we have to assume that $\sigma > n$ in order to recover in this manner the last result of Theorem 1.

Acknowledgement This work is supported by the Swiss National Science Foundation under the Grant Number 200020-100051/1.

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