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Numerical analysis of a non-singular boundary integral method: Part II: The general case

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SUMMARY

In order to numerically solve the interior and the exterior Dirichlet problems for the Laplacian operator, we have presented in a previous paper a method which consists in inverting, on a finite element space, a non-singular integral operator for circular domains. This operator was described as a geometrical perturbation of the Steklov operator, and we have precisely defined the relation between the geometrical perturbation and the dimension of the finite element space, in order to obtain a stable and convergent scheme in which there are non-singular integrals. We have also presented another point of view under which the method can be considered as a special quadrature formula method for the standard piecewise linear Galerkin approximation of the weakly singular single-layer potential.
In the present paper, we extend the results given in the previous paper to more general cases for which the Laplace problem is set on any $\mathscr{C}^\infty$ domains. We prove that the properties of stability and convergence remain valid. Copyright © 2002 John Wiley & Sons, Ltd.

1. INTRODUCTION

Let Ω be a simply connected, bounded, open domain in $\mathbb{R}^2$, the boundary Γ of which is assumed to be regular. For the sake of simplicity, we will suppose that Γ is $\mathscr{C}^\infty$. We denote by L the length of Γ and by x its parametrization by the curvilinear abscissa, i.e $x(\cdot)$ is the restriction to $[0,L]$ of a $\mathscr{C}^\infty$ L-periodic function, and

$$\Gamma=\{x(t),\ t\in[0,L],\ |x'(t)|=1\ \forall t\in[0,L]\} \tag{1}$$

If n_x is the outward unit normal vector at the point $x\in\Gamma$, and if n is a positive integer, we define the function $x^{(n)}(\cdot)$ by

$$x^{(n)}(t)=x(t)+\frac{\delta}{n}n_{x(t)},\quad t\in[0,L] \tag{2}$$

where δ is a positive number.

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With these functions, we can define a family of curves $(\Gamma^{(n)})_{n=1}^{\infty}$ surrounding Ω (if the parameter δ is chosen sufficiently small or n big) by

$$\Gamma^{(n)} = \{x^{(n)}(t),\ t \in [0, L]\} \tag{3}$$

Now, let $u_0 \in H^{1/2}(\Gamma)$ be a given function on Γ and let $u \in H^1(\Omega)$ be an harmonic function on Ω satisfying $u = u_0$ on the boundary Γ. Here $H^{1/2}(\Gamma)$ and $H^1(\Omega)$ denote the classical Sobolev spaces of functions on Γ or Ω. It is easy to show that if x does not belong to $\bar{\Omega} = \Omega \cup \Gamma$, then we have

$$\forall x \notin \bar{\Omega}: \quad \int_\Gamma G(x, y) \frac{\partial u}{\partial n_y}(y)\, \mathrm{d}s_y = \int_\Gamma \frac{\partial G}{\partial n_y}(x, y) u_0(y)\, \mathrm{d}s_y \tag{4}$$

where G denotes the Green kernel and $\partial G / \partial n_y$ its external normal derivative with respect to the variable y, i.e. if $x \in \mathbb{R}^2$, $y \in \Gamma$

$$G(x, y) = -\frac{1}{2\pi} \log |x - y| \tag{5}$$

$$\frac{\partial G}{\partial n_y}(x, y) = -\frac{(x - y, n_y)}{2\pi |x - y|^2} \tag{6}$$

where $(.,.)$ is the scalar product, and $|\cdot|$ the Euclidian norm in $\mathbb{R}^2$.

Let us assume that we are only interested in knowing the normal derivative $\partial u / \partial n_y$ on Γ instead of the whole u in Ω. If we call ζ the unknown $\partial u / \partial n_y$ on Γ, the integral of which is vanishing, we are in a position to give a variational formulation of (4) which is as follows:

Find $\zeta \in H_0^{-1/2}(\Gamma)$ satisfying

$$\forall \mu \in H_0^{-1/2}(\Gamma^{(n)}): \int_{\Gamma^{(n)}} \mathrm{d}s_x \int_\Gamma \mathrm{d}s_y G(x, y) \zeta(y) \mu(x) = \int_{\Gamma^{(n)}} \mathrm{d}s_x \int_\Gamma \mathrm{d}s_y \frac{\partial G}{\partial n_y}(x, y) u_0(y) \mu(x) \tag{7}$$

where $H^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$, the duality pairing being denoted by $\langle .,. \rangle_{-1/2, 1/2}$ and

$$H_0^{-1/2}(\Gamma) = \{\gamma \in H^{-1/2}(\Gamma):\ \langle \gamma, 1 \rangle_{-1/2, 1/2} = 0\}$$

Remark that in (7) the symbol 'integral' takes the meaning of duality $\langle .,. \rangle_{-1/2,1/2}$. As we said in the introduction of our previous paper [1], problem (7) is ill-posed because the continuous bilinear form $a_n : H_0^{-1/2}(\Gamma) \times H_0^{-1/2}(\Gamma^{(n)}) \to \mathbb{R}$ defined by

$$\zeta \in H_0^{-1/2}(\Gamma),\ \mu \in H_0^{-1/2}(\Gamma^{(n)}):\ a_n(\zeta, \mu) = \int_{\Gamma^{(n)}} \mathrm{d}s_x \int_\Gamma \mathrm{d}s_y G(x, y) \zeta(y) \mu(x) \tag{8}$$

does not satisfy the 'inf–sup condition' of Babuska–Ladyzenskaja (see Reference [2] for instance). It means the problem of finding $\zeta \in H_0^{-1/2}(\Gamma)$ satisfying

$$a_n(\zeta, \mu) = \langle \mu, g \rangle_{-1/2, 1/2}, \quad \forall \mu \in H_0^{-1/2}(\Gamma^{(n)})$$

where g is given in $H^{1/2}(\Gamma^{(n)})$, has in principle no solution, except if g is the trace of a harmonic function on $\Gamma^{(n)}$.

However, if $R_n : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma^{(n)})$ is the operator given by

$$\forall x \in \Gamma^{(n)}:\ (R_n u_0)(x) = \int_\Gamma \frac{\partial G}{\partial n_y}(x,y) u_0(y)\, \mathrm{d}s_y \tag{9}$$

then we know that the problem of finding $\zeta \in H_0^{-1/2}(\Gamma)$ such that

$$\forall \mu \in H_0^{-1/2}(\Gamma^{(n)}):\ a_n(\zeta, \mu) = \langle \mu, R_n u_0 \rangle_{-1/2, 1/2} \tag{10}$$

has at least one solution (it suffices to take $\zeta = \partial u / \partial n_y$ when u is the harmonic function in Ω satisfying $u = u_0$ on the boundary Γ). In this paper, we attack the numerical approximation of problem (10) by using a Galerkin method. By splitting up Γ and $\Gamma^{(n)}$ into $n+1$ simple arcs of curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_{n+1}$ and $\Gamma_1^{(n)}, \Gamma_2^{(n)}, \ldots, \Gamma_{n+1}^{(n)}$, respectively, and by defining X_n and Y_n as subspaces of piecewise constant functions on $(\Gamma_i)_{i=1}^{n+1}$ and $(\Gamma_i^{(n)})_{i=1}^{n+1}$ of $H_0^{-1/2}(\Gamma)$ and $H_0^{-1/2}(\Gamma^{(n)})$, respectively, we consider the problem of finding $\zeta_n \in X_n$ satisfying

$$\forall \mu_n \in Y_n:\ a_n(\zeta_n, \mu_n) = \langle \mu_n, R_n u_0 \rangle_{-1/2, 1/2} \tag{11}$$

We will prove that if we use a good choice for the splitting of Γ and $\Gamma^{(n)}$, then problem (11) has a unique solution ζ_n which converges to $\zeta = \partial u / \partial n_y$ when n tends to infinity. We will give error estimates between ζ_n and ζ.

In Reference [1] we have presented a complete history of the method: it was introduced in its first form by Kupradze [3] and studied by Christiansen (cf. References [4, 5]). But the scheme was unstable for the reasons we have explained. Later engineers have empirically found how to construct a stable method, which is sometimes used nowadays (cf. for instance References [6–8]). Our contribution in this work and in Reference [1] is to mathematically prove a stability property of the method and to establish results of convergence. Moreover, we give the rules for the choice of a quadrature formula when we want to numerically compute ζ_n. We have also explained in Reference [1] how the proposed scheme can be considered as a special quadrature formula method for the standard piecewise linear Galerkin approximation of the single-layer potential. From this point of view, the proposed method is related to a class that has been studied for many years, from the early papers [9, 10] up to recent papers (Reference [11] for instance). Nevertheless, the special quadrature obtained by introducing a neighbouring curve does not fall into a category which has been analysed in a previous work. The numerical analysis we present here shows that the efficiency of our scheme is asymptotically of the same order than the methods currently used, but its major feature resides in the simplicity of the ideas used for its construction. It can also be used (but without rigourous justification) in the case when the curve has corners (cf. Reference [7]), and the same simple ideas can be applied in the 3D case (cf. Reference [8]). It may be the main reason which has motivated engineers to employ it. We point out that similar but unstable methods seem to have a certain success with engineers (see for instance References [12, 13]). In addition to the papers [9–11] for a review of classical boundary element methods, we refer to References [14–17]. In practice, the boundary element methods are often used in combination with finite element methods (cf. Reference [18] for a reference article). We can see in Reference [19] or [20] how our method can be used with finite element methods in order to simulate a two-dimensional induction heating problem. See also Reference [21] for a reference book on the applications of integral methods in the acoustic and electromagnetic fields. Another interesting non-singular method has been presented in Reference [22].

2. DISCRETIZATION AND CONVERGENCE

In order to discretize problem (10) by using the Galerkin method (11), we set $h=L/(n+1)$, $t_j=jh,\ j=0,1,\dots,n+1,\ t_{j+1/2}=(j+\frac{1}{2})h,\ j=0,1,\dots,n$. To these points we associate the corresponding points on Γ and $\Gamma^{(n)}$ by using the parametrizations (1), (2), which are

$$x_j=x(t_j),\quad 0\leqslant j\leqslant n+1,\quad x_{j+1/2}=x(t_{j+1/2}),\quad 0\leqslant j\leqslant n \tag{12}$$

$$x_j^{(n)}=x^{(n)}(t_j),\quad 0\leqslant j\leqslant n+1,\quad x_{j+1/2}^{(n)}=x^{(n)}(t_{j+1/2}),\quad 0\leqslant j\leqslant n \tag{13}$$

Now we can define

$$\Gamma_j=\{x\in\mathbb{R}^2:\ x=x(t),\ t\in[t_{j-1/2},t_{j+1/2}]\},\quad 1\leqslant j\leqslant n \tag{14}$$

$$\Gamma_{n+1}=\{x\in\mathbb{R}^2:\ x=x(t),\ t\in[t_{n+1/2},L]\cup[0,t_{1/2}]\} \tag{15}$$

$$\Gamma_j^{(n)}=\{x\in\mathbb{R}^2:\ x=x^{(n)}(t),\ t\in[t_{j-1/2},t_{j+1/2}]\},\quad 1\leqslant j\leqslant n \tag{16}$$

$$\Gamma_{n+1}^{(n)}=\{x\in\mathbb{R}^2:\ x=x^{(n)}(t),\ t\in[t_{n+1/2},L]\cup[0,t_{1/2}]\} \tag{17}$$

and we recall that

$$X_n=\{\lambda\in H_0^{-1/2}(\Gamma):\ \lambda|_{\Gamma_i}\text{ is constant},\ i=1,2,\dots,n+1\} \tag{18}$$

$$Y_n=\{\mu\in H_0^{-1/2}(\Gamma^{(n)}):\ \mu|_{\Gamma_i^{(n)}}\text{ is constant},\ i=1,2,\dots,n+1\} \tag{19}$$

Our main result is the following:

Theorem 2.1. There exists $\delta_0>0$ such that for all $\delta\in]0,\delta_0[$, problem (11) has a unique solution $\zeta_n\in X_n$ for any integer n and we have $\lim_{n\to\infty}\|\partial u/\partial n-\zeta_n\|_{-1/2,\Gamma}=0$, where u is the harmonic function in Ω satisfying $u=u_0$ on the boundary Γ.

Moreover, if $\partial u/\partial n\in H^1(\Gamma)$, there exists a constant C independent of n such that

$$\left\|\frac{\partial u}{\partial n}-\zeta_n\right\|_{-1/2,\Gamma}\leqslant\frac{C}{n^{3/2}}$$

We can remark that in the circular case, i.e. when Γ is a circle, then Theorem 2.1 is exactly Theorem 2.1 of Reference [1]. The object here is to generalize the proof for any $\mathscr{C}^\infty$ closed curve. The main idea of this proof is related to a well-known property of the Steklov operator (see for instance Reference [15, p. 299]). It consists here in showing that the bilinear form a_n given in (8) can be written as the sum of the bilinear form corresponding to the circular case, with another bilinear form b_n with regular integrand. Moreover, when n tends to infinity, this form b_n tends (in the sense of the norm on the natural spaces) to a bilinear form which still has a regular integrand. In other words, the perturbation b_n is 'uniformly compact'. This property allows us to prove the stability and convergence of scheme (11).

We begin by introducing some notations in order to compare the general case with the circular case.

We denote by $\hat{\Gamma}$ the unit circle centred at the origin in $\mathbb{R}^2$, and by $\hat{\Gamma}^{(n)}$ its associated curve (which is also a circle centred at the origin). By using the complex notation, we have

$$\hat{\Gamma} = \{z(t) \in \mathbb{C}:\ z(t) = \mathrm{e}^{i(2\Pi/L)t},\ t \in [0, L]\} \tag{20}$$

$$\hat{\Gamma}^{(n)} = \left\{z^{(n)} \in \mathbb{C}:\ z^{(n)}(t) = \left(1 + \frac{\delta}{n}\right)\mathrm{e}^{i(2\Pi/L)t},\ t \in [0, L]\right\} \tag{21}$$

Moreover, we denote by $\hat{X}_n$ and $\hat{Y}_n$ the finite dimensional spaces corresponding to X_n and Y_n in the circular case, i.e.

$$\hat{X}_n = \{\lambda \in H_0^{-1/2}(\Gamma):\ \lambda|_{\hat{\Gamma}_i} \text{ is constant},\ i = 1, 2, \ldots, n+1\} \tag{22}$$

$$\hat{Y}_n = \{\mu \in H_0^{-1/2}(\Gamma^{(n)}):\ \mu|_{\hat{\Gamma}_i^{(n)}} \text{ is constant},\ i = 1, 2, \ldots, n+1\} \tag{23}$$

where $\hat{\Gamma}_i, \hat{\Gamma}_i^{(n)}$ are defined as in (14), (17) by replacing $x(t), x^{(n)}(t)$ by $z(t), z^{(n)}(t)$.

We consider the operators $\hat{\pi}: H^{-1/2}(\Gamma) \to H^{-1/2}(\hat{\Gamma})$, $r: H^{-1/2}(\Gamma^{(n)}) \to H^{-1/2}(\Gamma)$ and $\hat{r}: H^{-1/2}(\hat{\Gamma}^{(n)}) \to H^{-1/2}(\hat{\Gamma})$ defined by the following relations (see Figure 1):

$$(\hat{\pi}\zeta)(z(t)) \stackrel{\text{def}}{=} \hat{\zeta}(z(t)) = \zeta(x(t)), \quad \forall t \in [0, L] \tag{24}$$

$$(r\mu)(x(t)) = \mu(x^{(n)}(t)), \quad \forall t \in [0, L] \tag{25}$$

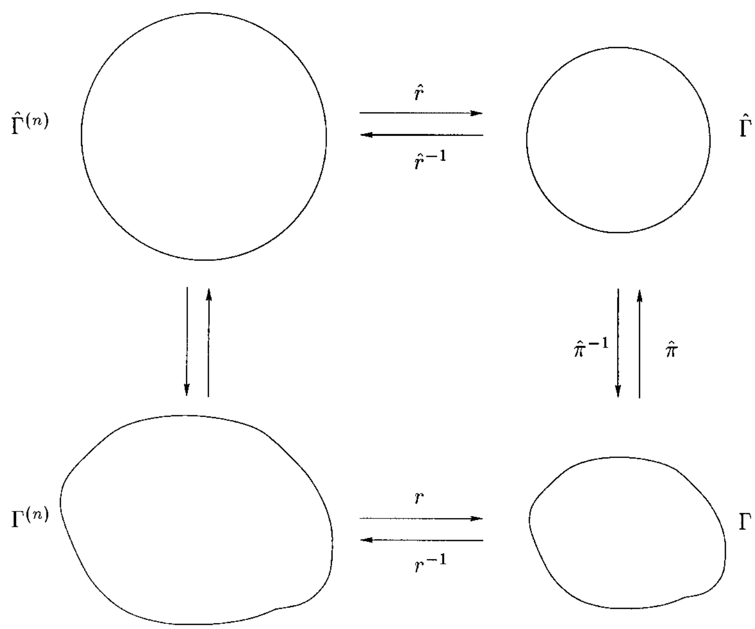

Figure 1. The operators $\hat{\pi}$, r and $\hat{r}$ for the change of variable.

$$(\hat{r}\mu)(z(t)) = \mu(z^{(n)}(t)), \quad \forall t \in [0, L] \tag{26}$$

Since we consider regular parametrizations, these operators are isomorphisms, i.e. they are linear bounded, invertible with a bounded inverse.

Let λ be a function defined on Γ. We can remark that if the mean value of λ on Γ is vanishing, then the mean value of $\hat{\lambda}$ of $\hat{\Gamma}$ is still vanishing because $|x'(t)| \equiv 1$. In consequence, the spaces X_n and $\hat{X}_n$ are uniformly isomorphic via the application $\hat{\pi}$. It is also easy to show that $\hat{r}(\hat{Y}_n) = \hat{X}_n$, but in general we do not have $r(Y_n) = X_n$ (we have only $r(Y_n) = X_n/\mathbb{R}$), because $|x^{(n)\prime}|$ is not necessarily constant. This fact is technically problematic, and it is convenient to correct this situation. Remark first that if $\gamma(t)$ is the curvature of Γ at point $x(t)$, then we have

$$|x'(t) - x^{(n)\prime}(t)| \leqslant \frac{\delta}{n} |\gamma(t)|, \quad \forall t \in [0, L] \tag{27}$$

and since $|x'(t)| \equiv 1$, then $|x^{(n)\prime}(t)|$ is close to 1 when δ is sufficiently small or n big. We next introduce the operators $M : H^{-1/2}(\Gamma^{(n)}) \to \mathbb{R}$ and $\tilde{r} : H^{-1/2}(\Gamma^{(n)}) \to H^{-1/2}(\Gamma)$ defined by

$$M\mu = \frac{1}{|\Gamma|} \int_\Gamma (r\mu)(x)\, \mathrm{d}s_x \tag{28}$$

$$\tilde{r}\mu = r\mu - M\mu \tag{29}$$

where $|\Gamma| = L$ is the measure of Γ.

We have the following result:

Lemma 2.1. There exists a positive constant C independant of n such that

$$|M\mu| \leqslant \frac{C\delta}{n} \|\mu\|_{-1/2, \Gamma^{(n)}}, \quad \forall \mu \in H_0^{-1/2}(\Gamma^{(n)}) \tag{30}$$

Moreover, there exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, the operator $\tilde{r}$ maps isomorphically and uniformly the spaces Y_n onto the spaces X_n, for all integer n.

Proof. Let $\mu \in H_0^{-1/2}(\Gamma^{(n)})$. Since $|x'(t)| \equiv 1$, we have

$$\begin{aligned}
|\Gamma||M\mu| &= \left| \int_0^L (r\mu)(x(t))\, \mathrm{d}t \right| = \left| \int_0^L \mu(x^{(n)}(t)\, \mathrm{d}t \right| \\
&= \left| \int_0^L \mu(x^{(n)}(t)|x^{(n)\prime}(t)|\, \mathrm{d}t + \int_0^1 (1 - |x^{(n)\prime}(t)|)\mu(x^{(n)}(t))\, \mathrm{d}t \right| \\
&\leqslant \left| \int_{\Gamma^{(n)}} \mu(x)\, \mathrm{d}s_x \right| + \frac{\delta}{n} \int_\Gamma |\gamma(x)||(r\mu)(x)|\, \mathrm{d}s_x \\
&\leqslant 0 + \frac{C\delta}{n} \|\mu\|_{-1/2, \Gamma^{(n)}}
\end{aligned}$$

with $C = \|\gamma\|_{1/2, \Gamma} \|r\|_{-1/2, 1/2, \Gamma}$. Consequently, we have obtained (30).

For $\mu \in H_0^{-1/2}(\Gamma^{(n)})$, we have

$$\|\tilde{r}\mu\|_{-1/2,\Gamma} \geqslant \|r\mu\|_{-1/2,\Gamma} - |M\mu| \geqslant \alpha \left(1 - \frac{\delta}{n}\right) \|\mu\|_{-1/2,\Gamma^{(n)}}$$

where α is a positive constant. This proves that for δ sufficiently small, the operator $\tilde{r}$ is an isomorphism (uniformly in n) between $H_0^{-1/2}(\Gamma^{(n)})$ and $H_0^{-1/2}(\Gamma)$. Moreover, it is easy to show that $\tilde{r}(Y_n) \subset X_n$, and we can conclude. □

We now define the bilinear forms

$$\tilde{a}_n : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma^{(n)}) \to \mathbb{R} \quad \text{and} \quad \tilde{a} : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{R} \quad \text{by}$$

$$\tilde{a}_n(\zeta, \mu) = -\frac{1}{2\pi} \int_0^L |x^{(n)\prime}(t)| \, \mathrm{d}t \int_0^L \log |z^{(n)}(t) - z(s)| \zeta(x(s)) \mu(x^{(n)}(t)) \, \mathrm{d}s \tag{31}$$

$$\tilde{a}(\zeta, \mu) = -\frac{1}{2\pi} \int_0^L \mathrm{d}t \int_0^L \log |z(t) - z(s)| \zeta(x(s)) \mu(x(t)) \, \mathrm{d}s \tag{32}$$

We have the following result:

Lemma 2.2. $\tilde{a}(\zeta, \zeta)^{1/2}$ is a norm equivalent to $\|\cdot\|_{-1/2,\Gamma}$ on $H_0^{-1/2}(\Gamma)$ and for all $\varepsilon > 0$, there exists δ_0 such that if $0 < \delta \leqslant \delta_0$, then

$$\sup_{\zeta \in X_n} \frac{|\tilde{a}_n(\zeta, r^{-1}\zeta) - \tilde{a}(\zeta, \zeta)|}{\tilde{a}(\zeta, \zeta)} \leqslant \varepsilon, \quad \text{for all } n \geqslant 1 \tag{33}$$

Proof. Clearly for all $\zeta, \mu \in H_0^{-1/2}(\Gamma)$, we have

$$\begin{aligned} \tilde{a}(\zeta, \mu) &= -\frac{1}{2\pi} \int_0^L \mathrm{d}t \int_0^L \log |z(t) - z(s)| \hat{\zeta}(z(s)) \hat{\mu}(z(t)) \, \mathrm{d}s \\ &= \frac{L^2}{4\pi^2} \int_{\hat{\Gamma}} \mathrm{d}s_x \int_{\hat{\Gamma}} G(x, y) \hat{\zeta}(y) \hat{\mu}(x) \, \mathrm{d}s_y \end{aligned}$$

where $\hat{\zeta}, \hat{\mu} \in H_0^{-1/2}(\hat{\Gamma})$ are defined by $\hat{\zeta} = \hat{\pi}\zeta$ and $\hat{\mu} = \hat{\pi}\mu$.

It is well known (see Reference [23] for instance) that this last bilinear form is coercive on $H_0^{-1/2}(\hat{\Gamma})$ and consequently $\tilde{a}(\zeta, \zeta)^{1/2}$ is a norm equivalent to $\|\cdot\|_{-1/2,\Gamma}$ on $H_0^{-1/2}(\Gamma)$.

Let now $\zeta \in X_n$. We have by using (27):

$$\begin{aligned} &2\pi |\tilde{a}_n(\zeta, r^{-1}\zeta) - \tilde{a}(\zeta, \zeta)| \\ &\quad = \left| -\int_0^L \mathrm{d}t \int_0^L \log |z^{(n)}(t) - z(s)| \zeta(x(s)) (r^{-1}\zeta)(x^{(n)}(t)) \, \mathrm{d}s \right. \\ &\qquad + \int_0^L \mathrm{d}t \int_0^L \log |z(t) - z(s)| \zeta(x(s)) \zeta(x(t)) \, \mathrm{d}s \end{aligned}$$

$$+\int_0^L (1-|x^{(n)\prime}(t)|)\,\mathrm{d}t\int_0^L \log|z^{(n)}(t)-z(s)|\zeta(x(s))(r^{-1}\zeta)(x^{(n)}(t))\,\mathrm{d}s\Bigg|$$

$$\leqslant \frac{L^2}{2\pi}\left|\frac{1}{(1+\delta/n)}\int_{\hat\Gamma_n}\mathrm{d}s_x\int_{\hat\Gamma}\mathrm{d}s_y G(x,y)\hat\zeta(y)(\hat r^{-1}\hat\zeta)(x)-\int_{\hat\Gamma}\mathrm{d}s_x\int_{\hat\Gamma}\mathrm{d}s_y G(x,y)\hat\zeta(y)\hat\zeta(x)\right|$$

$$+\frac{\delta}{n}\int_0^L|\gamma(t)|\,\mathrm{d}t\left|\int_0^L \log|z^{(n)}(t)-z(s)|\zeta(x(s))(r^{-1}\zeta)(x^{(n)}(t))\right|\mathrm{d}s$$

By using Lemma 2.2 of Reference [1], together with the property of equicontinuity in $\mathscr{L}(H^{-1/2}(0,L),H^{1/2}(0,L))$ of the operator $\zeta(x(.))\to\int_0^L\log|z^{(n)}(.)-z(s)|\zeta(x(s))\,\mathrm{d}s$ (cf. [19, p. 35]), we can conclude that (33) holds. □

Now we write the form $a_n(\zeta,\mu)$, $\zeta\in H_0^{-1/2}(\Gamma)$, $\mu\in H_0^{-1/2}(\Gamma^{(n)})$ given in (8) as follows:

$$\begin{aligned}a_n(\zeta,\mu)&=\int_{\Gamma^{(n)}}\mathrm{d}s_x\int_{\Gamma}\mathrm{d}s_y G(x,y)\zeta(y)\mu(x)\\&=-\frac{1}{2\pi}\int_0^L|x^{(n)\prime}(t)|\,\mathrm{d}t\int_0^L\log|x^{(n)}(t)-x(s)|\zeta(x(s))\mu(x^{(n)}(t))\,\mathrm{d}s\\&=\tilde a_n(\zeta,\mu)-\frac{1}{2\pi}\int_0^L|x^{(n)\prime}(t)|\,\mathrm{d}t\int_0^L\log\frac{|x^{(n)}(t)-x(s)|}{|z^{(n)}(t)-z(s)|}\zeta(x(s))\mu(x^{(n)}(t))\,\mathrm{d}s\end{aligned}$$

Let us define the bilinear forms
$b_n: H^{-1/2}(\Gamma)\times H^{-1/2}(\Gamma^{(n)})\to\mathbb{R}$ and $b: H^{-1/2}(\Gamma)\times H^{-1/2}(\Gamma)\to\mathbb{R}$ by

$$b_n(\zeta,\mu)=-\frac{1}{2\pi}\int_0^L|x^{(n)\prime}(t)|\,\mathrm{d}t\int_0^L\log\frac{|x^{(n)}(t)-x(s)|}{|z^{(n)}(t)-z(s)|}\zeta(x(s))\mu(x^{(n)}(t))\,\mathrm{d}s \tag{34}$$

$$b(\zeta,\mu)=-\frac{1}{2\pi}\int_0^L\mathrm{d}t\int_0^L\log\frac{|x(t)-x(s)|}{|z(t)-z(s)|}\zeta(x(s))\mu(x(t))\,\mathrm{d}s \tag{35}$$

Clearly with these definitions we have

$$a_n(\zeta,\mu)=\tilde a_n(\zeta,\mu)+b_n(\zeta,\mu),\quad \forall\zeta\in H^{-1/2}(\Gamma),\quad \forall\mu\in H^{-1/2}(\Gamma^{(n)}) \tag{36}$$

If $a: H^{-1/2}(\Gamma)\times H^{-1/2}(\Gamma)\to\mathbb{R}$ is defined by

$$a(\zeta,\mu)=\int_\Gamma\mathrm{d}s_x\int_\Gamma\mathrm{d}s_y G(x,y)\zeta(y)\mu(x)$$

we analogously obtain

$$a(\zeta,\mu) = \tilde{a}(\zeta,\mu) + b(\zeta,\mu) \tag{37}$$

Since we have evaluated $\tilde{a}(.,.) - \tilde{a}_n(.,.,)$, it remains to compare $b_n(.,.)$ to $b(.,.)$.

Lemma 2.3. We have

$$\lim_{n\to\infty} \sup_{\substack{\zeta,\mu\in H^{-1/2}(\Gamma)\\ \|\zeta\|_{-1/2,\Gamma}=\|\mu\|_{-1/2,\Gamma}=1}} |b_n(\zeta, r^{-1}\mu) - b(\zeta,\ \mu)| = 0 \tag{38}$$

Proof. By subtraction of (35) from (34) and by considering inequality (27), it suffices in the first step to set

$$\varphi_n(s,t) = \log \frac{|x^{(n)}(t) - x(s)|.|z(t) - z(s)|}{|x(t) - x(s)|.|z^{(n)}(t) - z(s)|}$$

and to prove that

$$\lim_{n\to\infty} \|\varphi_n\|_{H^1(0,L)^2} = 0 \tag{39}$$

The complete proof of (39) is given in Reference [19, Lemma 1.17]. However, we give here a sketch of the proof to make the reading easier. We can easily verify that when $s \neq t \bmod L$, we have

$$0 = \lim_{n\to\infty} \varphi_n(s,t) = \lim_{n\to\infty} \frac{\partial\varphi_n}{\partial s}(s,t) = \lim_{n\to\infty} \frac{\partial\varphi_n}{\partial t}(s,t) \tag{40}$$

where the first equality is still valid if $s = t \bmod L$. Since Γ is a regular curve, the function $\log|z(t) - z(s)|/|x(t) - x(s)|$ is in $\mathscr{C}^\infty(0,L)^2$. Moreover, by developing some calculations (see Reference [19, pp. 46–47]), it is not difficult to prove that the first partial derivative with respect to s and t of the functions $\log|x^{(n)}(t) - x(s)|/|z^{(n)}(t) - z(s)|$ are uniformly bounded with respect to n. Consequently, the convergence of φ_n to 0 is uniform in $[0,L]^2$. Moreover, by using Lebesgue's theorem, we can see that the convergence of $\partial\varphi_n/\partial s$ and $\partial\varphi_n/\partial t$ to zero hold in $L^2(0,L)^2$, and thus we obtain (39).

In the second step we use a Fourier analysis to show that

$$\left| \int_0^L \mathrm{d}s \int_0^L \mathrm{d}t\, \psi(s,t)\mu(s)\lambda(t) \right| \leqslant \|\psi\|_{H^1(0,L)^2} \|\zeta\|_{-1/2} \|\mu\|_{-1/2} \tag{41}$$

for all $\psi \in H^1(0,L)^2$, and all $\zeta,\mu \in H^{-1/2}(0,L)$. By using relationships (34) and (35), we write for $\zeta,\mu \in H^{-1/2}(\Gamma)$

$$\begin{aligned}
&\left|b_n(\zeta, r^{-1}\mu) - b(\zeta,\mu)\right| \\
&\quad\leqslant \frac{1}{2\pi} \left| \int_0^L \mathrm{d}t(|x^{(n)\prime}(t)| - 1) \int_0^L \log\frac{|x^{(n)}(t) - x(s)|}{|z^{(n)}(t) - z(s)|} \zeta(x(s))\mu(x(t))\,\mathrm{d}s \right| \\
&+ \frac{1}{2\pi} \left| \int_0^L \mathrm{d}t \int_0^L \varphi_n(s,t)\zeta(x(s))\mu(x(t))\,\mathrm{d}s \right|
\end{aligned}$$

By using (27) together with $|x'(t)| \equiv 1$, the fact that $\sup_{(s,t)\in(0,L)^2} |\log|x^{(n)}(t)-x(s)|/|z^{(n)}(t)-z(s)|$ is uniformly bounded with respect to n, and relationships (41) and (39), we obtain (38). □

Lemma 2.4. There exists δ_0 and $\alpha > 0$ independent of n such that the bilinear form $a_n(.,.)$ given by (8) satisfies

$$a_n(\zeta, r^{-1}\zeta) \geqslant \alpha \|\zeta\|^2_{-1/2,\Gamma}, \quad \forall \zeta \in X_n, \quad \forall n, \quad \forall \delta \leqslant \delta_0 \tag{42}$$

Proof. By using relationships (36) and (37), we write for all $\zeta \in X_n$:

$$\begin{aligned} a_n(\zeta, r^{-1}\zeta) &= \tilde{a}_n(\zeta, r^{-1}\zeta) + b_n(\zeta, r^{-1}\zeta) \\ &= a(\zeta,\zeta) + \tilde{a}_n(\zeta, r^{-1}\zeta) - \tilde{a}(\zeta,\zeta) + b_n(\zeta, r^{-1}\zeta) - b(\zeta,\zeta) \end{aligned}$$

It is well known that $a(.,.)$ is a coercive form on $H_0^{-1/2}(\Gamma)$ (see Reference [23] for instance). So, it suffices to use Lemmas 2.2 and 2.3 in order to conclude. □

Proof of Theorem 2.1. Lemma 2.4 together with Lemma 2.1 implies a uniform 'inf–sup' condition for $\delta \leqslant \delta_0$, i.e,

$$\inf_{\substack{\zeta \in X_n \\ \|\zeta\|_{-1/2,\Gamma}=1}} \sup_{\substack{\mu \in Y_n \\ \|\mu\|_{-1/2,\Gamma^{(n)}}=1}} a_n(\zeta,\mu) \geqslant \alpha, \quad n \geqslant 1 \tag{43}$$

where α is a positive constant independent of n.

In fact, let $\zeta \in X_n$ and $\mu \in Y_n$ satisfying $\|\zeta\|_{-1/2,\Gamma} = 1$ and $\|\mu\|_{-1/2,\Gamma^{(n)}} = 1$. We consider the function $\varphi \in X_n$ defined by $\varphi = \tilde{r}\mu$. By using Lemma 2.1 we have

$$\|\varphi\|_{-1/2,\Gamma} \geqslant C > 0$$

where C is a constant independent of μ and n. Moreover, since $\tilde{r}\mu = r\mu - M\mu$, then $\mu = r^{-1}\varphi + M\mu$ and consequently we have

$$\begin{aligned} \sup_{\substack{\mu \in Y_n \\ \|\mu\|_{-1/2,\Gamma^{(n)}}=1}} a_n(\zeta,\mu) &\geqslant \sup_{\substack{\varphi \in X_n \\ \|\varphi\|_{-1/2,\Gamma}=C}} a_n(\zeta, r^{-1}\varphi) - \sup_{\substack{\mu \in Y_n \\ \|\mu\|_{-1/2,\Gamma^{(n)}}=1}} |a_n(\zeta, M\mu)| \\ &\geqslant C a_n(\zeta, r^{-1}\zeta) - A\|\zeta\|_{-1/2,\Gamma}\|M\mu\|_{-1/2,\Gamma^{(n)}}, \end{aligned}$$

where A is a positive constant independent of n. At this point, by using Lemma 2.4 together with Lemma 2.1 we obtain (43), and then it is easy to conclude as in Reference [1]. □

3. NUMERICAL INTEGRATION

In order to solve the discrete problem (11), we have to build the matrix of coefficients $A_{ij} = a_n(\varphi_j, \psi_i)$ and the vector of coefficients $B_i = (\psi_i, R_n u_0)$, where $1 \leqslant i, \ j \leqslant n$, and $(\varphi_i)_{i=1}^n$, $(\psi_j)_{j=1}^n$ are the natural basis of X_n and Y_n, respectively.

In particular, we have to numerically evaluate expressions of the form

$$\int_{\Gamma_k^{(n)}} \mathrm{d}s_x \int_{\Gamma_l} \mathrm{d}s_y G(x, y) \tag{44}$$

$$\int_{\Gamma_k^{(n)}} \mathrm{d}s_x \int_{\Gamma_l} \mathrm{d}s_y\, u_0(y) \frac{\partial G}{\partial n_y}(x, y) \tag{45}$$

That is to say, we have to define a numerical quadrature in order to compute (44) and (45) and to replace the scheme (11) by the following:

find $\tilde{\zeta}_n \in X_n$ such that

$$\bar{a}_n(\tilde{\zeta}_n, \mu_n) = \bar{c}_n(\mu_n), \quad \forall \mu_n \in Y_n \tag{46}$$

where $\bar{a}_n(\tilde{\zeta}_n, \mu_n)$ and $\bar{c}_n(\mu_n)$ are the perturbations of $a_n(\tilde{\zeta}_n, \mu_n)$ and $c_n(\mu_n) \stackrel{\mathrm{def}}{=} \langle \mu_n, R_n u_0 \rangle_{-1/2,1/2}$ due to the use of the numerical quadrature.

In Reference [1], for the circular case, we have presented and analysed a quadrature formula such that the perturbated scheme was stable and as efficient as the scheme with exact integration (i.e. $\|\tilde{\zeta}_n - \zeta\|_{-1/2,\Gamma} = O(1/n^{3/2})$). It is easy to prove that the same results are still valid for the general case we have treated. However, we next present another quadrature formula that can also be employed. As in Reference [1], the formula we consider is based on the Gauss quadrature formula, but instead of keeping a constant number of integration points and using a subdivision of the arcs Γ_l or $\Gamma_k^{(n)}$, we use here a changeable number of integration points without any subdivision of the arcs. It is then possible to substantially reduce the computational cost of the corresponding scheme.

Let d be the distance between the arcs $\Gamma_k^{(n)}$ and Γ_l, we define the scaled distance $\tilde{d}$ by

$$\tilde{d} = \frac{2d}{h} \tag{47}$$

where $h = L/(n+1)$ is the length of the arcs Γ_l, $l = 1, \ldots, n+1$.

Lemma 3.1. Assume that Γ and u_0 are analytical data, and suppose that we use a standard $m \times m$ points Gauss quadrature formula with $m \geqslant \frac{3}{2} \log n / \log(1+\tilde{d})$ for the numerical evaluation of (44) and (45). Then the perturbated scheme (46) has the same properties of stability and convergence than scheme (11), and we obtain the same order of accuracy (it means, Theorem 2.1 still holds when we use this kind of numerical integration).

Proof. If, for $j = 1, \ldots, n$, we denote by φ_j the affine mapping from $[-1, 1]$ onto $[t_{j-1/2}, t_{j+1/2}]$, then we have

$$\int_{\Gamma_k^{(n)}} \mathrm{d}s_x \int_{\Gamma_l} \mathrm{d}s_y\, G(x, y) = \frac{h^2}{4} \int_{-1}^{1} \mathrm{d}s \int_{-1}^{1} G_1(s, t)\, \mathrm{d}t \tag{48}$$

$$\int_{\Gamma_k^{(n)}} \mathrm{d}s_x \int_{\Gamma_l} \mathrm{d}s_y\, u_0(y) \frac{\partial G}{\partial n_y}(x, y) = \frac{h^2}{4} \int_{-1}^{1} \mathrm{d}s \int_{-1}^{1} G_2(s, t)\, \mathrm{d}t \tag{49}$$

where

$$G_1(s,t) = |x^{(n)\prime}(s)| G(x^{(n)}(\varphi_k(s)), x(\varphi_l(t))),$$

$$G_2(s,t) = |x^{(n)\prime}(s)| \frac{\partial G}{\partial n_y}(x^{(n)}(\varphi_k(s)), x(\varphi_l(t))) u_0(x(\varphi_l(t)))$$

For $\alpha = 1,2$, we denote by $e^{\alpha}_{k,l}$ the errors due to the approximation of (48) and (49) by using a $m \times m$ points Gauss quadrature formula defined by the weights $\omega_1, \dots \omega_m$ and the integration points $\zeta_1, \dots, \zeta_m$. The important fact we use in the sequel is that the functions G_1 and G_2 admit an analytic extension for both variables s and t, and so we can estimate the errors $e^{\alpha}_{k,l}$ by using the theory of Davis–Rabinowitz (see Reference [24, Chapter 4.6]). In order to simplify the proof, we introduce the function $\tilde{G}$ defined by

$$\tilde{G}(s,t) = |x^{(n)}(\varphi_k(s)) - x(\varphi_l(t))|^{-1}$$

This function $\tilde{G}$ is more 'singular' than G_1 and G_2, in the sense that, for n sufficiently large, there exists a constant $C > 0$ such that

$$\max_{\beta_1+\beta_2=P} \max_{(s,t)\in[-1,1]^2} \left| \frac{\partial^P G_\alpha(s,t)}{\partial s^{\beta_1} \partial t^{\beta_2}}(s,t) \right| \leqslant C \max_{\beta_1+\beta_2=P} \max_{(s,t)\in[-1,1]^2} \left| \frac{\partial^P \tilde{G}(s,t)}{\partial s^{\beta_1} \partial t^{\beta_2}}(s,t) \right| \tag{50}$$

for $\alpha = 1,2$, and for all integer P.

Consequently, the errors $e^{\alpha}_{k,l}$ can be bounded by the error $\tilde{e}_{k,l}$ due to the approximation of the integral $h^2/4 \int_{-1}^{1} \mathrm{d}s \int_{-1}^{1} \tilde{G}(s,t)\, \mathrm{d}t$ by using a $m \times m$ points Gauss quadrature formula. Moreover, since for m fixed, this error $\tilde{e}_{k,l}$ is increasing when d is decreasing, we still have

$$e^{\alpha}_{k,l} \leqslant C\tilde{e}_{k,l} \leqslant Ch^2 \left| \int_{-1}^{1} \mathrm{d}s \int_{-1}^{1} f(s,t) - \sum_{i=1}^{m}\sum_{j=1}^{m} \omega_i \omega_j f(\zeta_i, \zeta_j) \right|, \quad \alpha = 1,2 \tag{51}$$

where $f(s,t) = (h^2/4)|(s-t)^2 + \tilde{d}^2|^{-1}$.

It is easy to show that for both variables s and t, this function admits an analytic extension in the complex domain $\{z \in \mathbb{C}\colon\ \mathrm{Im}(z) < \tilde{d}\}$. Moreover, this domain is included in the ellipse with focus at ± 1 and semi-axis sum equal $\tilde{d} + \sqrt{1+\tilde{d}^2}$. Then by using formula (4.6.1.11) of Reference [24], we can obtain the estimate

$$e^{\alpha}_{k,l} \leqslant \frac{C}{n(1+\tilde{d})^{2m}} \tag{52}$$

where C is a constant independent of k, l, n, m and $\tilde{d}$. Consequently, if we take $m \geqslant \frac{3}{2}\log n / \log(1+\tilde{d})$, then we have $e^{\alpha}_{k,l} \leqslant C/n^4$. In order to conclude, we use the results of Reference [1] Lemma 44 and its proof of Theorem 3.1. □

4. NUMERICAL RESULTS

Let Ω be the interior of an ellipse centred at the origin with semi-axis length $a=1$, $b=0.4$. We consider the harmonic function u defined on $\overline{\Omega}$ by $u(x_1,x_2) = \mathrm{e}^{x_1}\cos(x_2)$. Let $\zeta = \partial u/\partial n$ be

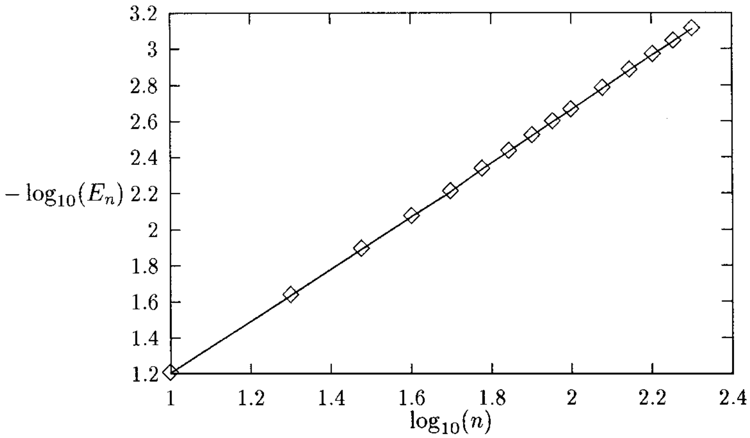

Figure 2. Relative L^2 error on the density.

the normal derivative of u on $\Gamma = \partial\Omega$, and u_0 the restriction of u on Γ. With the data u_0 and Γ, we obtain from numerical scheme (46), the function $\zeta_n \in X_n$ which is an approximation of ζ. For each integer n, we denote by $E_n = \|\zeta - \zeta_n\|_{L^2}/\|\zeta\|_{L^2}$, the relative L^2 error between the exact solution and its approximation by scheme (44). For the numerical test, we have chosen $\delta = 1$. The results are presented in Figure 2.

By using the well-known inverse inequality: $\|\zeta\|_{0,\Gamma} \leqslant C\sqrt{n}\|\zeta\|_{-1/2,\Gamma}$, $\forall \zeta \in X_n$, we can verify that this results are in agreement with Theorem 2.1.

Remark 4.1. If we want to solve a Laplace problem posed on the exterior of the domain Ω instead of its interior, we can also apply the method presented here. However, we have to choose δ negative in (2) in order to build a family of auxiliary curves which are contained in Ω. A numerical example can be found in Reference [19].

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