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Pushing Appell's variational heuristics to *second* order

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Abstract

Let M be a compact manifold and P a manifold diffeomorphic to M. Let c be a two point real function defined on an open subset $\Omega(c)$ of $M \times P$ that projects onto M and P. Given a probability volume measure μ on M and another such one ϖ on P, we assume the non-emptiness of the set $\mathrm{Diff}(\mu, \varpi, c)$ of diffeomorphisms from M to P that push μ to ϖ and have their graph lying $\Omega(c)$. Following Monge [1], let us consider the transportation (or allocation) cost functional:

$$\phi \in \mathrm{Diff}(\mu, \varpi, c) \to \mathcal{C}(\phi) = \int_M c\big(m, \varphi(m)\big)\, d\mu.$$

If a diffeomorphism $\varphi \in \mathrm{Diff}(\mu, \varpi, c)$ is *stationary* for the functional $\mathcal{C}$, we can find real functions $f(m)$ and $\tilde{f}(p)$ such that the two point function F given by $F(m,p) = c(m,p) + f(m) + \tilde{f}(p)$ is *stationary* at each point of the graph of φ. This variational heuristics is originally due to Appell [2]. If the *second* variation of the functional $\mathcal{C}$ at φ is *nonnegative*, we can further show that the Hessian of the two point function F is *nonnegative* at each point of graph(φ).

Geometric preliminary

A path $t \mapsto \phi_t \in \mathrm{Diff}(\mu, \varpi, c)$ near $t = 0$ may be written $\phi_t = \xi_t \circ \varphi$, where $\varphi = \phi_0$ and $t \mapsto \xi_t$ is a path of ϖ-preserving diffeomorphisms of the target manifold P. This way of writing ϕ_t indicates that $\mathrm{Diff}(\mu, \varpi, c)$ possesses a *manifold* structure locally modelled on that of the ϖ-preserving diffeomorphisms [3]. Accordingly, a tangent vector at the diffeomorphism $\varphi \in \mathrm{Diff}(\mu, \varpi, c)$ is written $(V \circ \varphi)$ with $V \in \ker \mathrm{div}_\varpi$. Similarly, setting $\tilde{c}(p,m) = c(m,p)$, we may consider the functional:

$$\psi \in \mathrm{Diff}(\varpi, \mu, \tilde{c}) \to \tilde{\mathcal{C}}(\psi) = \int_P c\big(\psi(p), p\big)\, d\varpi(p),$$

and note that, if $\psi_0 = \varphi^{-1}$, the equality $\tilde{\mathcal{C}}(\psi_0) \equiv \mathcal{C}(\varphi)$ holds. A path $t \mapsto \psi_t \in \mathrm{Diff}(\varpi, \mu, \tilde{c})$ near $t = 0$ is now written $\psi_t = \zeta_t \circ \varphi^{-1}$ with $\varphi^{-1} = \psi_0$ and $t \mapsto \zeta_t$ a path of μ-preserving diffeomorphisms of the source manifold M. A tangent vector to $\mathrm{Diff}(\varpi, \mu, \tilde{c})$ at φ^{-1} reads $(U \circ \varphi^{-1})$ for some vector field U on M lying in $\ker \mathrm{div}_\mu$.

Stationary condition

The first variation $\delta\mathcal{C}(\varphi)(V \circ \varphi)$ is equal to $\int_M d_P c\big(m, \varphi(m)\big)\big(V(\varphi(m)\big)\, d\mu(m) \equiv \int_P \alpha(V)\, d\varpi$ where $\alpha(p) = d_P c(m,p)|_{m=\varphi^{-1}(p)}(V(p))$ is a 1-form on P. If this variation vanishes for each vector field $V \in \ker \mathrm{div}_\varpi$, Helmholtz Lemma implies that the 1-form α is *exact*. So we can find a real function $\tilde{f}$ on P such that the equation:

$$d_P c(m,p) + d\tilde{f}(p) = 0 \qquad (1)$$

holds at each point (m,p) of the graph of the diffeomorphism φ. Dealing with the first variation $\delta\tilde{\mathcal{C}}(\varphi^{-1})(U \circ \varphi^{-1})$, with $U \in \ker \mathrm{div}_\mu$, Helmholtz lemma yields similarly a real function f on M such that the equation:

$$d_M c(m,p) + df(m) = 0 \qquad (2)$$

holds at each point (m,p) of graph(φ). The two point function $F(m,p) = c(m,p) + f(m) + \tilde{f}(p)$ which we obtain is *stationary* at each point (m,p) of graph(φ), as claimed.

Minimum condition

The integral of the two point function $(F - c)$ over the graph of the diffeomorphism $\phi_t \in \mathrm{Diff}(\mu, \varpi, c)$ is independent of t, equal to $\int_M f d\mu + \int_P \tilde{f} d\varpi$. Therefore, the minimum condition $\delta^2\mathcal{C}(\varphi) \geqslant 0$ may be written equivalently:

$$\frac{d^2}{dt^2}\left(\int_M F\big(m, \phi_t(m)\big)\, d\mu\right)_{t=0} \geqslant 0. \qquad (3)$$

The Hessian of the function $p \mapsto c(m,p) + \tilde{f}(p)$ at $p = \varphi(m)$ is *intrinsic* due to (1). We call it the c-Hessian of $\tilde{f}$ at the point p, denote it by $\mathrm{Hess}_c\tilde{f}(p)$. Moreover, from (1), the left-hand side of (3) reduces to $\int_P \mathrm{Hess}_c\tilde{f}(V,V)\, d\varpi$, for some vector field $V \in \ker \mathrm{div}_\varpi$. Altogether, the minimum condition $\delta^2\mathcal{C}(\varphi) \geqslant 0$ thus yields the inequality:

$$\int_P \mathrm{Hess}_c\tilde{f}(V,V)\, d\varpi \geqslant 0, \ \forall V \in \ker \mathrm{div}_\varpi. \qquad (4)$$

Similarly, we can define the c-Hessian of the function f at each point of M and the minimum condition $\delta^2\tilde{\mathcal{C}}(\varphi^{-1}) \geqslant 0$ implies:

$$\int_M \mathrm{Hess}_c f(U,U)\, d\mu \geqslant 0, \qquad (5)$$

for each vector field $U \in \ker \mathrm{div}_\mu$.

Pointwise conditions

In [4], we proved that the inequalities (4) and (5) imply *pointwise* quadratic form inequalities:

$$\mathrm{Hess}_c f \geqslant 0 \text{ and } \mathrm{Hess}_c\tilde{f} \geqslant 0. \qquad (6)$$

The proof goes by contradiction. If (6) does not hold for f (say), we can find a chart x of M pushing the measure $d\mu$ to the measure dx and such that the form $\mathrm{Hess}_c f(U,U)$ reads *smaller* than the form: $-2(u^1)^2 + \Sigma_{i=2}^n (u^i)^2$, in a small enough cube $\{\max_i |x^i| < \varepsilon\}$ of size ε. The local vector field $w = x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}$ satisfies $\mathrm{div} w = 0$ and its flow preserves any function of the form $h(x) = H\big(\sqrt{(x^1)^2 + (x^2)^2}, x^3, \ldots, x^n\big)$, hence $\mathrm{div}(hw) = 0$. Take $h(x) = \alpha\big(\sqrt{(x^1)^2 + (x^2)^2}\big) \prod_{i=3}^{n} \alpha(|x^i|)$, with α a cut-off function supported in the small interval $[0, \varepsilon)$. We get a *global* vector field $U = hw$ lying in $\ker \mathrm{div}_\mu$. Moreover, the integral $\int_M \mathrm{Hess}_c f(U,U)\, d\mu$ is less than $\int_{\mathbb{R}^n} h^2(x)\big((x^1)^2 - 2(x^2)^2\big) dx \leqslant -\pi \int_0^\varepsilon \alpha^2(r) r^3\, dr$, which is *negative* contradicting (5).

Finally, the Hessian of the two point function F is itself *intrinsic* at each point of graph(φ), since $dF = 0$ there. It vanishes along graph(φ) while, transversally, in the direction of M, it coincides with $\mathrm{Hess}_c f$. It is thus *nonnegative*, as claimed.

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