LOCAL INVERSION OF ELLIPTIC PROBLEMS ON COMPACT MANIFOLDS

- Dedicated to the memory of Mikio Ise -

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Abstract. Nonlinear elliptic problems on compact manifolds are often solved by means of the so-called *continuity method*. The "openess" part of that method is based on an inverse function theorem (IFT, for short). Dealing with finitely differentiable functions, the classical (Banach) IFT suffices. However, dealing with smooth (i.e., C^∞) functions, only IFT's of Nash-Moser type are available even though no "loss of derivatives" occurs on the infinitesimal level. The present note provides a suitable abstract "elliptic" IFT which fills the gap; applications are drawn which agree with the results of a previous article [20].

I. Introduction. "Ellipticity" means that no loss of derivatives occurs for infinitesimal inverses; somehow, it can be defined so [14]. This opens the way toward an abstract characterization of ellipticity.

Besides, it inclines to think that no inverse function theorem (hereafter abbreviated as I.F.T.) of the C^∞ smoothing type [19] should ever be used to deal with nonlinear elliptic problems on compact manifolds. Still though, global existence results in the smooth (i.e. C^∞) category are often obtained by means of the so-called *continuity method* which requires a local inversion argument. This gap was already sensed by Mikio Ise who suggested the paper [20]. The present article shares the same motivation; let us describe how it is organized.

In section II, an abstract "elliptic" category is introduced and an "elliptic" I.F.T. is established in its framework. It is based only on the classical (Banach) I.F.T. combined with an abstract form of the elliptic regularity.

Definitions and basic examples of elliptic manifolds and submanifolds are introduced in section III.

In section IV, we show how these notions can be applied to general constrained elliptic boundary systems on compact manifolds (at this stage we use "folk" results about elliptic regularity in little Hölder spaces proved, for completness, in appendix A). Applications to precise geometric problems have been drawn in [10] [9]. We conclude with several comments

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including one about the possibility of similar results on open manifolds.

Although the continuity method is the very basic one that yields existence for nonlinear elliptic problems, it has sometimes to be supplemented by fixed point methods. For the reader's convenience, a Fixed Point theorem taken from [18] and slightly modified to suit our context is stated in appendix B.

II. Elliptic inverse function theorem. We are going to prove an inverse function theorem in the framework of an auxiliary category, the elliptic category, which we now proceed to describe in the following set of definitions.

Definition 1. An elliptic object (or space) is a Fréchet space equipped with an increasing sequence of norms defining its (inductive limit) topology.

If E is such an elliptic space and n is an integer, we denote by E^n the completion of E for the norm $|\cdot|_n$, so for $n \geq m, E^n$ is viewed as a dense subspace of E^m.

Definition 2. Let E and F be two elliptic spaces. An elliptic morphism $u \in Mor(E,F)$, is a continuous linear map $u \in L(E,F)$ which extends continuously, for any integer n, as a map $u^n \in L(E^n, F^n)$ and which satisfies the following "elliptic regularity" property:

$$\forall n \in \mathbf{N}, \forall x_0 \in E^0, u^0(x_0) \in F^n \Rightarrow x_0 \in E^n.$$

Important remark: $Mor(E,F)$ is a Fréchet space whose topology is canonically defined by the sequence of norms of linear maps in $L(E^n, F^n)$ for all integers n.

Definition 3. $u \in Mor(E, F)$ is an elliptic isomorphism if, $\forall n \in \mathbf{N}$, $u^n \in \mathrm{Isom}(E^n, F^n)$. $\mathrm{Isom}(E,F)$ will denote the (obviously open) subset of elliptic isomorphisms in $Mor(E,F)$.

Given $u \in \mathrm{Isom}(E,F)$, the sequence of inverse $[(u^n)^{-1}]_{n \in N}$ defines, by the universal property of the inductive limit topology, an element of $L(F,E)$ which is the inverse of u in $L(E,F)$, thus denoted by u^{-1}. Since $u \in Mor(E,F)$ and since, by construction,

$$(u^{-1})^n \equiv (u^n)^{-1},$$

u^{-1} is easily seen to satisfy "elliptic regularity"; hence $u^{-1} \in Mor(F,E)$.

Definition 4. Let Ω^0 be an open subset of E^0. Set $\Omega = \Omega^0 \cap E$, and, $\forall n \in \mathbf{N}, \Omega^n = \Omega^0 \cap E^n$. An elliptic map $\mu : \Omega \to F$ is a continuous map which admits C^1 extensions $\mu^n : \Omega^n \to F^n, \forall n \in \mathbf{N}$, commuting together: $\forall n \geq m$,

$$\begin{array}{ccc} \Omega^n & \xrightarrow{\mu^n} & F^n \\ \downarrow & & \downarrow \\ \Omega^m & \xrightarrow[\mu^m]{} & F^m \end{array}$$

(vertical arrows will always denote canonical injections) and satisfying the following linear (or infinitesimal) and nonlinear "elliptic regularity" properties,

$$\text{(LER)} \quad \forall n \in \mathbf{N}, \forall x \in \Omega^n, \forall y \in E^0, D\mu^0(x)y \in F^n \Rightarrow y \in E^n$$

$$\text{(NLER)} \quad \forall n \in \mathbf{N}, \forall x \in \Omega^0, \mu^0(x) \in F^n \Rightarrow x \in E^n .$$

Let us record a few basic properties of elliptic maps. They are stable under composition. And given an elliptic map $\mu : \Omega \to F$ and $x \in \Omega$, by the universal property of the inductive limit topology, the sequence $[D\mu^n(x)]_{n \in N}$ induces a map $\mu'(x) \in L(E, F)$ easily seen to be elliptic: $\mu'(x) \in Mor(E, F)$. Moreover, the maps,

$$(x, u) \in \Omega \times E \to [\mu'(x)u] \in F,$$

and,

$$x \in \Omega \to \mu'(x) \in Mor(E, F),$$

are both continuous, the former coinciding with the Gateaux derivative of μ; hence μ is C^1 and we may write $D\mu$ instead of μ'.

Definition 5. An elliptic map $\mu : \Omega \to F$ is called an elliptic diffeomorphism if, $\forall n \in \mathbf{N}, \mu^n : \Omega^n \to F^n$ is a C^1-diffeomorphism; then, by the universal property of the inductive limit topology, so is μ.

We are now in a position to state the,

Theorem 1. *Let $\mu : \Omega \to F$ be an elliptic map. If, at $z \in \Omega, D\mu(z) \in Isom(E, F)$, then μ is an elliptic diffeomorphism in a neighborhood of z in Ω.*

Proof. Since $D\mu^0(z) \in Isom(E^0, F^0)$, the classical (Banach) inverse function theorem provides a neighborhood U^0 of z in E^0 on which μ^0 is a C^1-diffeomorphism. Moreover, since $Isom(E^0, F^0)$ is open in $L(E^0, F^0)$ and since the map $D\mu^0 : \Omega^0 \to L(E^0, F^0)$ is continuous, we may suppose (shrinking U^0 if necessary) that,

$$\forall x \in U^0, D\mu^0(x) \in Isom(E^0, F^0).$$

Set $U = U^0 \cap E$ and, $\forall n \in \mathbf{N}, U^n = U^0 \cap E^n$. For any fixed $n \in \mathbf{N}$, let $y \in U^n$. Recall that the following diagram commutes.

$$\begin{array}{ccc} E^n & \xrightarrow{D\mu^n(y)} & F^n \\ \downarrow & & \downarrow \\ E^0 & \xrightarrow[D\mu^0(y)]{} & F^0 \end{array}$$

Recall also that $D\mu^0(y) \in Isom(E^0, F^0)$. Infinitesimal elliptic regularity (LER) combined with the open mapping theorem thus yields,

$$D\mu^n(y) \in Isom(E^n, F^n).$$

Since y is arbitrary in U^n, the Banach I.F.T. implies that μ^n is a local C^1-diffeomorphism from U^n to its image V^n. Furthermore μ^n is injective on U^n, because μ^0 is, on U^0 and because the following diagram commutes,

$$\begin{array}{ccc} U^n & \xrightarrow{\mu^n} & F^n \\ \downarrow & & \downarrow \\ U^0 & \xrightarrow[\mu^0]{} & F^0 \end{array}$$

So μ^n is actually a global C^1-diffeomorphism on U^n. Last, the bijectivity of μ^0 between U^0 and its image V^0, combined with (NLER) yields,

$$V^n \equiv V^0 \cap F^n.$$

Since $n \in \mathbf{N}$ is arbitrary, μ is an elliptic diffeomorphism from U to $V = V^0 \cap F^{\cdot}$.

III. Elliptic Geometry.

III-1. Definitions. Among geometric elliptic objects, it will be enough for our purpose to define elliptic manifolds, submanifolds and maps. Anyhow, we would not be able to proceed much further, as in [13] (section I.4), since ellipticity is only a first order property, so, for instance, the tangent manifold to an elliptic manifold would not be an elliptic manifold anymore. In this respect, the elliptic category is just an auxiliary one.

Definition 6. An elliptic manifold is a Hausdorff topological space locally modelled on an elliptic space and whose transition maps are elliptic maps. The charts are called elliptic charts. An elliptic map between two elliptic manifolds is a continuous map which reads as an elliptic map in any couple of elliptic charts of the source and target manifolds.

Note that an elliptic manifold is a (particular type of) Fréchet manifold of class C^1.

Before defining elliptic submanifolds, we need to consider a splitting: $E = F \oplus G$, of an elliptic space E into two closed Fréchet subspaces F and G, and note that both F and G canonically inherit a structure of an elliptic space making the canonical embedding into E be an elliptic map. Incidentally, the projection of E onto F in the direction of G need not be an elliptic map; it is, however, if G is a Banach space, e.g. typically if G is finite-dimensional.

Definition 7. Let X be an elliptic manifold locally modelled on an elliptic space E. An elliptic submanifold Y of X is a closed subset of X for which there exists a splitting $E = F \oplus G$ (as above) such that, at any point of Y there exists an elliptic chart of X where the canonical injection $Y \to X$ reads locally as $F \to E$. As such, Y is itself an elliptic manifold locally modelled on F, the canonical injection $Y \to X$ being an elliptic map.

For finite-codimensional submanifolds, it is an easy exercise to check (using the remark just before Definition 7) that the following definition is equivalent.

Definition 7'. Y is a closed subset of X for which there exists a splitting $E = F \oplus G$, with $\dim(G)$ finite, such that in any elliptic chart of X, Y reads locally as the graph of a map from F to G, the corresponding embedding $F \to E$ being an elliptic map.

III-2. Basic Examples.

Elliptic manifolds. Let $f : X \to Y$ be a finite-dimensional smooth fibration (or fiber bundle) over a compact base. Recall that the set Σ of its smooth sections is a Fréchet manifold; let us describe the construction of its local charts.

Let $V := \ker(Tf)$ be the vertical sub-bundle of $TX \to X$ and let $s \in \Sigma$. Since by definition of a section, $f\ o\ s = Id_Y, s$ is an embedding and the following transversality condition holds,

$$s^*TX = Im(Ds) \oplus s^*V.$$

Here, s^* denotes the pull-back by s, and $Ds : TY \to s^*TX$ denotes the vector bundle morphism induced, over the identity Id_Y, by the tangent map Ts. So, there exists a smooth tubular neighborhood of $s(Y)$ in X in the direction of V; it provides a one-to-one map between a neighborhood of s in Σ and a neighborhood of the zero section in the Fréchet space $C^\infty(s^*V)$ of smooth sections of s^*V. This map is taken as a local chart of Σ at s.

Incidentally, one should note that $C^\infty(s^*V)$ is actually the tangent space $T_s\Sigma$ (it is clear by viewing Σ as the subset of smooth maps from Y to X that satisfy; $f\ o\ s = Id_Y$) and that whenever $f : X \to Y$ is a vector bundle, it is isomorphic to $s^*V \to Y, \forall s \in \Sigma$.

Now, given $\alpha \in (0,1)$, an integer b and $s \in \Sigma$, we may consider Σ as an elliptic manifold Σ_b locally modelled on $C^\infty(s^*V)$ viewed as the elliptic space E defined by,

$$\forall n \in \mathbf{N}, E^n := \lambda^{n+b,\alpha}(s^*V).$$

Here, $\forall n \in \mathbf{N}, \lambda^{n,\alpha}$ denotes a little Hölder space of sections, i.e. the closure of C^∞ in the usual (non-separable Banach) Hölder space $C^{n,\alpha}$: as such, $\lambda^{n,\alpha}$ is a closed separable (just as C^∞) strict subspace of $C^{n,\alpha}$. It is made of sections whose n^{th} derivatives satisfy α-Hölder conditions with a little o instead of a big O [22]. Note that the integer b plays here the same role as the "base" of tame inequalities in [13] (section II.1).

The transition maps of Σ_b are obviously elliptic since they just consist of composition with a change of smooth tubular neighborhoods. In the particular case where $f : X \to Y$ is a vector bundle, Σ_b is simply an elliptic space.

Elliptic submanifolds. Sticking to the preceding notations, let us state the

Proposition 1. *Let M be a locally closed finite codimensional C^2 Fréchet submanifold of Σ. Given any $s \in M$, there exist an integer b and a neighborhood Ω of s in Σ, such that $(M \cap \Omega)$ is an elliptic submanifold of the elliptic manifold Σ_b.*

The proof makes use of the following result (see e.g. [13] (section I.5)).

Theorem. *Let $f : U \subset E_1 \to E_2$, be a C^2 map between Fréchet spaces (U is an open subset of E_1). Then, given a sequence of semi-norms defining the topology of E_1 and a semi-norm $|\cdot|$ on $E_2, \forall u \in U$, there exist: a neighborhood V of u in U, three* *semi-norms* $|\cdot|_i, |\cdot|_j, |\cdot|_k$ *of the given sequence on* E_1 *and a constant* C, *such that:* $\forall v \in V, \forall (x,y) \in E_1 \times E_1$,

$$|Df(v)x| \le C|x|_i,$$
$$|D^2 f(v)(x,y)| \le C|x|_j|y|_k.$$

Proof of Proposition 1. Given $s \in M$, there exist a local chart of Σ at s valued in the Fréchet space,

$$E := T_s\Sigma \equiv C^\infty(s^*V)$$

and a splitting of E,

$E = F \oplus G$, with $\dim G =$ codim M finite, such that M reads locally at s as the graph of a C^2 map,

$$\Gamma : U \subset F \to G$$

U being some open subset of F.

Given any integer b, we may consider Σ_b as an elliptic manifold, E_b hence F_b as elliptic spaces (since all norms are equivalent on G, we need not specify any elliptic structure on it).

According to the preceding auxiliary theorem, applied to Γ and to the elliptic space F_0, there exist an integer b and a neighborhood V^0 of 0 in:

$$(F_0)^b \equiv (F_b)^0$$

(recall that the chart of Σ at s sends s to $0 \in E$), such that, $\forall v \in V := V^0 \cap F, D\Gamma(v)$ (resp. $D^2\Gamma(v)$) extends as a continuous linear (resp. bilinear) map from $(F_b)^0$ to G (take b as the maximum of i, j, k and the integers corresponding to the norms defining the neighborhood of the theorem). Applying the mean value theorem to $D\Gamma$ in V one easily infers that, $\forall n \in \mathbf{N}, \Gamma$ admits a unique C^1 extension,

$$\Gamma^n : V^n := V^0 \cap (F_b)^n \to G.$$

Now the associated embedding $V \to E$ given by: $v \to v + \Gamma(v)$, is readily an elliptic map from $V \cap F_b$ to E_b. According to Definition 7', this proves the proposition.$\therefore$

IV. Application to nonlinear elliptic problems on compact manifolds. This section illustrates how Theorem 1 applies to local inversion of general multi-degree nonlinear elliptic boundary systems on compact manifolds. Dealing with a local functional situation, we do not loose any generality in considering vector bundles and their Fréchet spaces of sections instead of (nonlinear) fibrations and their Fréchet manifolds of sections.

IV-1. Notations. Let us consider a finite-dimensional smooth compact manifold X with an orientable boundary δX, and three Whitney sums of (smooth) vector bundles,

$$U = U_1 \oplus \cdots \oplus U_m, V = V_1 \oplus \cdots \oplus V_p, \text{ over } X,$$

and,

$$W = W_1 \oplus \cdots \oplus W_q \text{ over } \delta X,$$

with $\dim(U) = \dim(V)$. Let us denote by E, F and G respectively their Fréchet spaces of (smooth) sections.

Let $\mu : \Omega \subset E \to F \oplus G$, be a smooth multi-degree nonlinear boundary system on X defined over some open subset Ω. It means that there exist three sequences of integers: $r = (r_1, \cdots, r_m), s = (s_1, \cdots, s_p), t = (t_1, \cdots, t_q)$, such that, if μ reads,

$$u = u_1 + \cdots + u_m \to \mu(u) = \mu_1(u) + \cdots + \mu_{p+q}(u)$$

then, $\forall (i, j, k) \in \{1, \cdots, m\} \times \{1, \cdots, p\} \times \{1, \cdots, q\}, \mu_j$ and μ_{p+k} are respectively of degree $(r_i - s_j)$ and $(r_i - t_k)$ with respect to u_i. In the sequel r, s and t are fixed.

IV-2. Ellipticity.. E and $(F \oplus G)$ may be viewed as elliptic spaces in the following way: given any integer b, define E_b and $(F \oplus G)_b$ by, $\forall n \in \mathbf{N}$,

$$(E_b)^n := \lambda^{b+n+r_1,\alpha}(U_1) \oplus \cdots \oplus \lambda^{b+n+r_m,\alpha}(U_m),$$

$$\begin{aligned}[(F \oplus G)_b]^n := (F_b)^n \oplus (G_b)^n : = & [\lambda^{b+n+s_1,\alpha}(V_1) \oplus \cdots \oplus \lambda^{b+n+s_p,\alpha}(V_p)] \\ & \oplus [\lambda^{b+n+t_1,\alpha}(W_1) \oplus \cdots \oplus \lambda^{b+n+t_q,\alpha}(W_q)].\end{aligned}$$

Here $\alpha \in (0,1)$ is fixed, and we consider direct sums of little Hölder spaces of sections of vector bundles U_i, V_j, W_k.

In the sequel, we assume that μ is elliptic in Ω, that is: $\forall s \in \Omega, D\mu(s)$ is a linear elliptic boundary system on X (see e.g. [15]).

Proposition 2. *Given $s \in \Omega$, there exists an integer b such that μ is an elliptic map from some open neighborhood of s in E_b to $(F \oplus G)_b$.*

Proof. By definition of the inductive limit topology of E, given any $s \in \Omega$ there exist an integer b and an open subset ω^0 of $(E_b)^0$ such that, $\omega := (\omega^0 \cap E)$ contains s and is contained in Ω. We still denote by μ the restriction of μ to ω. By construction, given any integer n, μ extends as a C^1 map,

$$\mu^n : \omega^n \to [(F \oplus G)_b]^n$$

where $\omega^n := \omega^0 \cap (E_b)^n$. Moreover, since μ is elliptic on Ω, a "folk" variant of [1] which is explained in Appendix A hereafter ensures of its (abstract) ellipticity in the sense of section II above.∴

IV-3. Local invertibility. Sticking to the preceding notations, let us state the

Theorem 2. *Let M and N be locally closed finite codimensional C^k Fréchet submanifolds respectively of E and $(F \oplus G), k \geq 2$, and let $\mu : \Omega \subset E \to (F \oplus G)$ be an elliptic smooth boundary system. Assume that the restriction μ_M of μ to M sends M into N and that its derivative,*

$$\begin{array}{ccc} TM & \xrightarrow{D\mu_M} & \mu^* TN \\ \downarrow & & \downarrow \\ M & \xrightarrow[Id]{} & M \end{array}$$

is an isomorphism at some point $s \in M$. Then μ_M is a C^k-diffeomorphism in a neighborhood of s.

Proof. According to Proposition 1 applied to M and N, we may choose an integer b such that M and N are elliptic submanifolds respectively of E_b and of $(F \oplus G)_b$ in neighborhoods of s and of $\mu(s)$. Now the restriction of μ_M going from the neighborhood of s to that of $\mu(s)$, is an elliptic map as a composition of such maps, provided b also meets the requirement of Proposition 2.

In order to apply Theorem 1 to this map, it remains only to check that its derivative at $s \in M$ is an elliptic isomorphism, possibly with a larger b. In other words, we must find an integer b such that,

$$\forall n \in \mathbf{N}, D[(\mu_M)^n](s) \in Isom\{(E'_b)^n, [(F \oplus G)'_b]^n\}, \tag{$*$}$$

where E' and $(F \oplus G)'$ are some finite codimensional closed Fréchet subspaces respectively of E and of $(F \oplus G)$, on which M (near s) and N (near $\mu(s)$) are respectively modelled.

But $D\mu_M(s) \in Isom[E', (F \oplus G)']$, so there exists an integer b such that $\forall n \geq b, [D\mu_M(s)]^{-1}$ extends as a continuous linear map from $[(F \oplus G)'_0]^n$ to $(E'_0)^0$. Since $D\mu_M(s)$ is elliptic i.e. satisfies (LER), $[D\mu_M(s)]^{-1}$ actually sends $[(F \oplus G)'_0]^n$ to $(E'_0)^n$; moreover, this correspondence is continuous according to the Schauder *a priori* estimates of [1] (II, section 9). Therefore, provided b meets that last requirement, (*) holds and Theorem 1 asserts that μ_M is a C^1 diffeomorphism near s.

Since M and N are $C^k, k \geq 2$, so is μ_M, hence so is its local inverse.·.

IV-4. Comments.

(a) Why constraints? It is a well-known fact that linear elliptic problems on compact manifolds are Fredholm (see e.g. [15]). Therefore an appropriate inverse function theorem for nonlinear elliptic problems should deal with possibly constrained problems, i.e. with their restrictions to suitable finite-codimensional functional submanifolds. This is for instance the case of the solution of the Calabi conjecture [21], [2], for which natural constraints dictated by the geometry make the problem locally invertible.

That very circumstance led us initially from [10] to the present article, because we would not be able to check all the hypotheses of R. Hamilton's I.F.T. [13] for constrained elliptic problems; in particular, the assumption at the top of page 158 of [13] would not seem to us so satisfactory.

Note also that Theorem 2 covers of course the case of unconstrained problems.

(b) About locality. Theorem 2 does not require μ to be a differential operator. What matters only is, first, that it extend as a C^1 map from $(E_b)^n$ to $[(F+G)_b]^n$, for some b and for all n, secondly, that it satisfy (LER) and (NLER).

This remark actually offers an alternative (less conceptual) way of applying Theorem 1 to the local inversion of concrete elliptic problems on compact manifolds. The device is to incorporate non-local terms (possibly representing suitable constraints) into the original elliptic operator, so that the new operator become locally invertible in the full functional space. It has been inaugurated in [6] where a real elliptic *Monge-Ampère* operator needed an adjunct average term to become well-posed; a similar device is used in various differential-geometric contexts [3], [7], [8], and in [4], [5].

(c) Sunada's earlier work. For several non-academic reasons, this research was completed a couple of years before the final preprint could be issued. Meanwhile, we found that T. Sunada [20] had already proved an "elliptic" implicit function theorem equivalent to our Theorem 2, under the suggestion of late Professor M. Ise (who was interested by the global analysis of *Monge-Ampère* operators and its geometrical applications [16] [17]). Sunada deals directly with elliptic operators on compact manifolds. He uses the Hodge-Kodaira decomposition theorem to explicit suitable local constraints, and sequences of special Hilbert-Sobolev norms depending on the linearization of the operator at a given section, as (what we called) elliptic structures.

(d) Open questions. We think that this study ought to be extended in two directions, namely, analyticity and open manifolds.

Indeed, dealing with elliptic problems it would be natural to seek an abstract framework designed to modelize operators with analytic coefficients acting on analytic sections of analytic bundles.

What can be said about possible extensions to elliptic systems on open manifolds?

Consider a smooth vector bundle over an open base. Given $\alpha \in (0,1)$ and $n \in \mathbf{N}$, denote by $C^\infty, C_b^\infty, C^{n,\alpha}, C_b^{n,\alpha}$, the vector spaces of its sections respectively, smooth, smooth with all derivatives bounded, n times continuously differentiable with n^{th} derivatives α-Hölderian, *idem* with bounded $C^{n,\alpha}$-norm. The latter is readily a Banach space, wheras $C^{n,\alpha}$ is certainly not. However, following [12], we may consider $C^{n,\alpha}$ equipped with the fine $C^{n,\alpha}$-topology, which we denote by $C_f^{n,\alpha}$. It is not a topological vector space, because the scalar multiplication is not continuous, but it may be canonically viewed as an affine Banach manifold locally modelled on $C_b^{n,\alpha}$. Hence, in particular, the canonical embeddings,

$$C_f^{n,\alpha} \to C_f^{k,\beta}, n+\alpha > k+\beta,$$

are compact. Similarly, we may consider C_f^∞: it is canonically an affine Fréchet manifold locally modelled on C_b^∞. Let $\lambda_f^{n,\alpha}$ be the closure of C^∞ in $C_f^{n,\alpha}$, and $\lambda_b^{n,\alpha}$ that of C_b^∞ in $C_b^{n,\alpha}$, then again, $\lambda_f^{n,\alpha}$ is canonically an affine Banach manifold locally modelled on $\lambda_b^{n,\alpha}$ and,

$$C_f^\infty = \text{inv.}\lim \lambda_f^{n,\alpha} \quad \text{as } n \to \infty.$$

So we still have at our disposal the two basic ingredients in order to do an elliptic theory, namely, the I.F.T. for C^1 maps between Banach manifolds and an elliptic regularity theory [11]. Note that a version of the Nash-Moser theorem has been proved on (possibly) open manifolds using fine topologies [12]. Unfortunately, real difficulties come from concrete applications since elliptic systems on open manifolds are not Fredholm and, what is worse, a *priori* estimates could hardly be carried out to infer global existence results from the continuity method.

So, given an elliptic problem on an open manifold, it is more interesting to seek a compactification of the manifold and an elliptic boundary value problem on it, thus Fredholm, which extend the original problem. Whenever that construction is possible, it amounts analytically to work with suitable *weighted* Banach-Hölder spaces of sections. Provided a Schauder type elliptic regularity theory is available for these spaces, our present study applies *in extenso.*

APPENDIX A

Elliptic regularity in little Hölder spaces. Elliptic regularity theory [1] (ERT, for short) has been worked out in the framework of projective sequences of Banach spaces, whereas smooth IFT's are proved in the framework of (what we called) elliptic spaces [19][13]. The duality, projective versus inductive, causes some difficulties when using Hölder norms, since $\lambda^{n,\alpha}$ is a *strict* subspace of $C^{n,\alpha}$ (cf. *supra*). As a matter of fact, the correct picture is the inductive one, namely,

$$C^\infty = \text{inv.}\lim \lambda^{n,\alpha} \quad \text{for } n \to \infty$$

because the canonical injections are dense.

In this appendix, we verify that ERT is valid in the framework of little Hölder spaces. This result seems so well-known that it is nowhere available in the literature: in a seminar talk, P. Grisvard sketched the argument to us. It goes as follows (we shall deal with an example instead of a general abstract proof [P. Delanoë, *unpublished*] that would be too lenghty here).

Let $U \to X$ and $V \to X$ be two vector bundles over a compact manifold, $C^\infty(U)$ and $C^\infty(V)$ be respectively the Fréchet spaces of their smooth sections. Let Ω be an open subset of $C^r(U)$ and $\mu : \Omega \cap C^\infty(U) \to C^\infty(V)$ be a nonlinear elliptic differential operator of order r. Given $\alpha \in (0,1)$, what we know from [1] is:

(LER) $\quad \forall n \in \mathbf{N}, \forall s \in \Omega \cap C^{n+r,\alpha}(U), \forall x \in C^{r,\alpha}(U),$

$$D\mu(s)(x) \in C^{n,\alpha}(V) \Rightarrow x \in C^{n+r,\alpha}(U)$$

(NLER) $\forall s \in \Omega \cap C^{r,\alpha}(U), \forall n \in \mathbf{N}, \mu(s) \in C^{n,\alpha}(V) \Rightarrow s \in C^{n+r,\alpha}(U)$

The proof of such statements with $\lambda^{n,\alpha}$ instead of $C^{n,\alpha}$ relies on the following easily established lemma [*Ibid.*].

Lemma. *Given $n \in \mathbf{N}$ and $s \in \Omega \cap \lambda^{n+r,\alpha}(U)$, there exist two splittings, $C^{n+r,\alpha}(U) = K^n \oplus D^n$ and $C^{n,\alpha}(V) = H^n \oplus \Delta^n$, satisfying the following properties: (i) K^n and H^n are finite-dimensional and spanned by smooth sections; (ii) $D\mu(s)$ induces an isomorphism $L^n : D^n \to \Delta^n$.*

Proof of (LER) *with little Hölder spaces.* Given n, s and x as in (LER) above, but with s and $D\mu(s)(x)$ in little Hölder spaces instead of usual ones, we want to show that $x \in \lambda^{n+r,\alpha}(U)$. To this end, consider the mapping,

$$M : [\Omega \cap \lambda^{n+r,\alpha}(U)] \oplus K^n \oplus D^n \oplus C^{n,\alpha}(V) \to \Delta^n$$

defined by,

$$M(s + y + z + t) := P^n[D\mu(s)(y + z) - t],$$

where P^n stands for the continuous projection of $C^{n,\alpha}(V)$ onto Δ^n with kernel H^n. By construction,

$$(\delta M/\delta z)(s + y + z + t) \equiv L^n.$$

Since L^n is an isomorphism, one may apply the (Banach) implicit function theorem to M at $[s + x + D\mu(s)(x)]$, where $M = 0$, with respect to the z variable. Approaching both s and $D\mu(s)(x)$ by sequences of smooth sections, one thus obtains a sequence in $C^\infty(U) \cap D^n$ approaching z in the $C^{n+r,\alpha}$ norm. Since, by property (i) of the lemma, the y component of $x = y + z$ is smooth the desired conclusion holds.$\therefore$

Proof of (NLER) *with little Hölder spaces.* Given $n \in \mathbf{N}$ and $s \in \Omega \cap \lambda^{r,\alpha}(U)$ such that $\mu(s) \in \lambda^{n,\alpha}(V)$, we want to show that $s \in \lambda^{n+r,\alpha}(U)$. To this end, consider the mapping,

$$F : (\Omega \cap K^n) \oplus (\Omega \cap D^n) \oplus C^{n,\alpha}(V) \to \Delta^n$$

defined by,

$$F(y + z + t) := P^n[\mu(y + z) - t].$$

By construction, $(\delta F/\delta z)(s + t) \equiv L^n$ is an isomorphism. Arguing as above yields the desired conclusion.$\therefore$

APPENDIX B

Fixed point theorem in Fréchet spaces. To start with, let us describe a typical situation where a Fixed Point Theorem has to supplement Theorem 2. Let again $U \to X$ and $V \to X$ be two vector bundles over a compact manifold, E and F be the Fréchet spaces of their smooth sections, $\mu : \Omega \subset E \to F$ and $\rho : \Omega \subset E \to F$ be two nonlinear differential operators on X, with μ elliptic on Ω and ρ of order strictly lower than μ. Assume that

μ is locally invertible in Ω and that, given $f \in F$, the continuity (or "surjectivity" [10]) method yields a unique solution $s \in \Omega$ of the equation: $\mu(s) = f$. Assume, without loss of generality, that $0 \in \Omega$ and $\mu(0) = 0$, but $\rho(0) \neq 0$.

One wishes to solve in Ω the new equation: $(\mu - \rho)(s) = 0$. Note that the operator $(\mu - \rho)$ need not be locally invertible any more. Nevertheless, local invertibility is now irrelevant; *a priori* estimates suffice. Indeed, introducing for $t \in [0,1]$ the family of equations,

$$(*) \quad (\mu - t\rho)(s) = 0$$

and viewing a solution of $(*)$ as a fixed point of the map $(s,t) \to T(s,t)$ well-defined by,

$$\mu[T(s,t)] = t\rho(s),$$

it is possible to consider the degree of the map, $[Id_E - T(\cdot,t)] : E \to E$, at the origin with respect to a sufficiently large but controlled pseudo-polysphere (i.e. the boundary of a pseudo-polydisc, cf. *infra*) S centered at 0, providing uniform *a priori* estimates on the solution $(*)$. By construction, $T(s,0) \equiv 0$ and $\text{degree}(Id_E, 0, S) = 1$; the homotopy invariance of the degree thus implies that T does have a fixed point for $t = 1$, i.e. that our original equation can be solved (probably non uniquely).

A degree in Fréchet spaces has been defined by M. Nagumo and applied to Fixed Point theorems [18]. Observe that they are incomparably easier than Fréchet I.F.T.'s. We now proceed to describe the result from [18] which best suits our context.

Let E be a Fréchet space equipped with a sequence of semi-norms defining its topology. Let us call a subset B of E, a pseudo-polydisc, if there exists a sequence of positive integers $(r_n)_{n \in \mathbf{N}}$ such that,

$$B = \{x \in E, \forall n \in \mathbf{N}, |x|_n < r_n).$$

Note that if B is a pseudo-polydisc, its boundary is closed in E. Of course, any bounded subset of E is contained in some pseudo-polydisc.

Recall that an operator $f : E \to E$ is called compact, if it is continuous and sends bounded subsets into relatively compact ones.

Theorem 3 (Nagumo). *Let $F : [0,1] \times E \to E$ be a continuous family of compact operators. Under the following two conditions the operator $F(1,\cdot)$ admits a fixed point:*

(i) *the fixed point set $\{x \in E, F(t,x) = x$ for some $t\}$ is bounded;*

(ii) $\forall x \in E, F(0,x) \equiv 0$.

Moreover, if $U \to X$ is a finite-dimensional vector bundle over a compact manifold and if E is the Fréchet space of its smooth sections, then one may relax the condition that, $\forall t \in [0,1], F(t,\cdot)$ be a compact operator.

Actually, Nagumo's formulation of the first part of the theorem [18] (p.504) concerns what he called "completely continuous mappings" i.e. continuous mappings defined on open subsets and valued in compact subsets (which is more restrictive than the usual definition, i.e. that of compact mappings). The proof here remains unchanged with compact mappings defined on pseudo-polydiscs of E instead of "completely continuous mappings" defined on open subsets.

The last part of Theorem 3, concerning concrete Fréchet spaces of sections on a manifold, simply relies on the fact that the closure of any pseudo-polydisc in E is now compact (as a well-known consequence of Ascoli Theorem). So the continuous image of any pseudo-polydisc is always contained in a compact subset of E. Besides, recall that the boundary of a polydisc is closed (cf. *supra*). These two facts just make Nagumo's argument work again (see [18]).

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