has partial derivatives given by:

$$\frac{\partial f}{\partial \lambda_i} = \frac{(\sigma_1 - \lambda_i)\sigma_1 - \sigma_2}{\sigma_1^2} = 1 - \frac{\lambda_i}{\sigma_1} - \frac{f}{\sigma_1}.$$

Therefore, on the one hand we have:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i} = (n-1) - n\frac{f}{\sigma_1} \leq (n-1),$$

while, by McLaurin's inequality, $\sigma_1 \geq \frac{2n}{n-1} f$ or else $\frac{nf}{\sigma_1} \leq \frac{n-1}{2}$, hence on the other hand:

$$\frac{n-1}{2} \leq \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i},$$

and altogether condition (F) cannot hold. Actually, this is a general fact for elementary hessian quotients [12, p.311], leaving them a wide open problem (not to mention that of relaxing the curvature condition in theorems 2 and 3).

Appendix

For completeness[2], we prove here that formula (7) is valid for any symmetric function $f(\lambda)$.

Proposition 2 *Let Γ be a symmetric domain of $\mathbb{R}^n$ and f, a symmetric C^1 real function defined on Γ. Let $S_n(\Gamma)$ denote the open subset of real symmetric $n \times n$ matrices $A = (a_i^j)_{1 \leq i,j \leq n}$ whose n-tuple of eigenvalues $\lambda(A)$ lies in Γ. For $A \in S_n(\Gamma)$, set $\Phi(A) := f[\lambda(A)]$. Let $A_0 \in S_n(\Gamma)$ be diagonal, with diagonal entries $(\lambda_{01}, \ldots, \lambda_{0n}) =: \lambda_0$. Then the following identity holds:*

$$\frac{\partial \Phi}{\partial a_i^j}(A_0) = \delta_{ij} \frac{\partial f}{\partial \lambda_i}(\lambda_0).$$

Moreover, the symmetry of f can be dropped provided the eigenvalues of A_0 are distinct.

Proof. Fix i, j in $\{1, \ldots, n\}$ and, for t small in $\mathbb{R}$, set $A_t := A_0 + tS_{ij}$ and $\phi(t) := \Phi(A_t)$, where S_{ij} stands for the symmetric matrix whose entries labelled by (i,j) and (j,i) are equal to 1, all others vanishing. Indicating by a prime the derivative with respect to t, the chain rule yields:

$$\phi'(0) = \sum_{k=1}^{n} \frac{\partial f}{\partial \lambda_k}(\lambda_0) \frac{\partial \lambda_k}{\partial a_i^j}(A_0). \tag{11}$$

When $i = j$ the matrix A_t remains diagonal, with diagonal entries $\lambda_{0k} + t\delta_{ik}$, hence the desired formula readily holds.

When $i \neq j$, the characteristic polynomial reads:

$$\det(A_t - \lambda I) = [(\lambda_{0i} - \lambda)(\lambda_{0j} - \lambda) - t^2] \prod_{k \neq i,j} (\lambda_{0k} - \lambda).$$

[2]this appendix on a well-known fact will not appear in the published version

So the eigenvalues of A_t are, on the one hand, the λ_{0k}'s for $k \neq i, j$, on the other hand, the roots of $[(\lambda_{0i} - \lambda)(\lambda_{0j} - \lambda) - t^2] = 0$, namely:

$$\lambda^{\pm}(t) = \frac{1}{2}\left[\lambda_{0i} + \lambda_{0j} \pm \sqrt{(\lambda_{0i} - \lambda_{0j})^2 + 4t^2}\right].$$

Whenever $\lambda_{0i} \neq \lambda_{0j}$ we have $(\lambda^{\pm})'(0) = 0$ and (11) yields $\phi'(0) = 0$, proving in particular the last part of proposition 2. In case $\lambda_{0i} = \lambda_{0j}$, we write

$$[(\lambda_{0i} - \lambda)(\lambda_{0j} - \lambda) - t^2] = [(\lambda_{0i} + t) - \lambda][(\lambda_{0i} - t) - \lambda]$$

and get the roots $\lambda^{\pm}(t) = \lambda_{0i} \pm t$. The symmetry of f enters here in the proof; it yields the identities (written with *e.g.* $i < j$):

$$\begin{aligned}\phi(t) &\equiv f(\lambda_{0i} + t, \lambda_{0i} - t, \lambda_{01}, \ldots, \widehat{\lambda_{0i}}, \ldots, \widehat{\lambda_{0j}}, \ldots, \lambda_{0n}) \\ &\equiv f(\lambda_{0i} - t, \lambda_{0i} + t, \lambda_{01}, \ldots, \widehat{\lambda_{0i}}, \ldots, \widehat{\lambda_{0j}}, \ldots, \lambda_{0n}),\end{aligned}$$

where a hat indicates omission, as usual. Differentiating each identity at $t = 0$, we get:

$$\phi'(0) = \frac{\partial f}{\partial \lambda_1}(\lambda_0) - \frac{\partial f}{\partial \lambda_2}(\lambda_0) \equiv -\frac{\partial f}{\partial \lambda_1}(\lambda_0) + \frac{\partial f}{\partial \lambda_2}(\lambda_0)$$

which implies indeed $\phi'(0) = 0$. The proof of proposition 2 is complete.

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