

# Reflected BSDEs when the obstacle is not right-continuous and optimal stopping

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# Plan

- Formulate a notion of RBSDE in the case where the obstacle is not necessarily right-continuous
- Investigate the question of existence and uniqueness of solutions
- Approximation and limit theorem
- The case of two barriers and Mokobodski hypothesis

# Notations

- $L^2(\mathcal{F}_T)$  is the set of random variables which are  $\mathcal{F}_T$ -measurable and square-integrable.
- $H^{2,T}$  is the set of real-valued predictable processes  $\phi$  such that  $\|\phi\|_{H^{2,T}}^2 := E \left[ \left( \int_0^T |\phi_t|^2 dt \right) \right] < \infty$ .
- $\mathcal{S}^{2,T}$  is the set of real-valued optional processes  $\phi$  such that  $\|\phi\|_{\mathcal{S}^{2,T}}^2 := E(\text{esssup}_{\tau \in \mathcal{T}_{0,T}} |\phi_\tau|^2) < \infty$ .

BSDE with two reflecting barriers (RBSDE for short) is an equation of the following type:

$$\left\{ \begin{array}{l} (i) Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s \\ (ii) \forall t \leq T; L_t \leq Y_t \leq U_t, \\ (iii) \int_0^T (Y_t - L_t) dK_t^+ = \int_0^T (U_t - Y_t) dK_t^- = 0. \end{array} \right.$$

- $L$  and  $U$  are two barriers.
- UNKNOWNNS ARE :  $Y \in R$ ,  $Z \in R^n$ ,  $K^+ \in R$  and  $K^- \in R$ .

RBSDE are considered in various situations:

- 1 Continuous barriers :  $L < U$  or Mokobodski condition (Karatzas-Cvitanic, El Karoui et al., among others).
- 2 Cadlag barriers  $L < U$  and  $L_- < U_-$  (Hamadene and Hassani, Essaky and Hassani, Ouknine,...).
- 3  $L^2$ -barriers (Xu and Peng, Essaky, Hassani and Ouknine,...).
- 4  $L$  and  $U$  optional,  $L < U$  up to evanescent set,  $Y$  is optional,  $Z$  is predictable and  $K$  is ladtog adapted process of bounded variation (Grigorova et al.).
- 5 Recent work : Bouchard et al.

It is well known (see Dellacherie, Lenglart, proposition 21), that any optional process  $\xi$  which is upper semicontinuous from the right can be approximated from above by a sequence  $(\xi^n)_{n \in \mathbf{N}}$  of cadlag processes satisfying

$$\begin{aligned}\xi^n &\geq \xi^{n+1}, \quad n \in \mathbf{N}, \\ \xi &= \lim_{n \rightarrow \infty} \xi^n.\end{aligned}\tag{1}$$

let  $f$  be a 2-integrable driver, which is Lipschitz in  $y$  and  $z$ .  
 Assume that  $\xi$  is upper semicontinuous from the right and satisfies

$$E(\text{ess sup}_{\tau \in \mathcal{T}_{0,T}} \xi_{\tau}^2) < \infty, \quad (2)$$

Assume further that a decreasing sequence of cadlag processes  $(\xi^n)_{n \in \mathbf{N}}$  is given satisfying (1). An inspection of the proof of proposition 21 in Dellacherie, Lenglart reveals that the sequence  $(\xi^n)_{n \in \mathbf{N}}$  may be chosen such that  $\xi^n \leq \xi + 1$  for all  $n \in \mathbf{N}$ .  
 Therefore we have

$$E(\text{ess sup}_{\tau \in \mathcal{T}_{0,T}} (\xi^n)_{\tau}^2) < \infty \quad (3)$$

as well.

Let  $\xi$  be an optional process and consider the sequence of RBSDE associated  $\xi^n$  for  $n \in \mathbf{N}$

$$Y_t^n = \xi_T + \int_t^T f(s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dW_s + K_T^n - K_t^n, \quad ,$$

$$Y_t^n \geq \xi_t^n, \quad \text{for all } t \in [0, T],$$

$K^n$  is cadlag, increasing with  $K_0^n = 0, E(K_T^n) < \infty$ , and satisfies

$$\int_0^T (Y_{t-}^n - \xi_{t-}^n) dK_t^n = 0.$$

## Lemma

- 1 The sequence  $(Z^n)_{n \in \mathbf{N}}$  converges in  $\mathbb{H}^2$  to a process  $Z$ .
- 2 We can deduce that for any  $t \in [0, T]$  the sequence  $(K_t^n)_{n \in \mathbf{N}}$  is decreasing, and we may define

$$K_t = \lim_{n \rightarrow \infty} K_t^n, \quad t \in [0, T]. \quad (4)$$

$K$  is increasing, satisfies  $K_0 = 0$  and  $E(K_T) < \infty$ .

## Definition

Let  $\xi$  be an optional process. Let

$$\mathcal{L} = \{X : X \text{ is a cadlag optional process, } X \geq \xi\},$$

$$\mathcal{L}_- = \{X_- : X \in \mathcal{L}\},$$

and

$$\bar{\xi} = \text{ess inf } \mathcal{L}, \quad \hat{\xi} = \text{ess inf } \mathcal{L}_-.$$

We call  $\bar{\xi}$  *upper cadlag envelope* of  $\xi$ ,  $\hat{\xi}$  *left upper cadlag envelope* of  $\xi$ .

## Lemma

Let  $\xi$  be an optional process which is upper semicontinuous from the right. Then

$$\bar{\xi}_t = \xi_t, \quad t \in [0, T],$$

i.e.  $\bar{\xi}$  is a version of  $\xi$ . Moreover, the sequence  $(X_n)_{n \in \mathbf{N}}$  may be chosen identical to the sequence  $(\xi^n)_{n \in \mathbf{N}}$  resulting from theorem 21 of Dellacherie, Lenglart. Finally, we have

$$\hat{\xi} = \lim_{n \rightarrow \infty} \xi_-^n.$$

## Theorem

*Let  $K$  be the laddag increasing process defined in (4). Then there exists an increasing cadlag predictable process  $A$  satisfying  $A_0 = 0, E(A_T) < \infty$ , as well as an increasing cadlag optional process  $C$  satisfying  $E(C_T) < \infty$ , and such that*

$$K = A + C_-. \quad (5)$$

## Theorem

Let  $\xi$  be an optional process which is upper semicontinuous from the right, and let  $(\xi^n)_{n \in \mathbf{N}}$  be given according to (1). Let  $Y = \lim_{n \rightarrow \infty} Y^n$  in  $\mathcal{S}^2$ ,  $Z = \lim_{n \rightarrow \infty} Z^n$  in  $\mathbb{H}^2$ , according to Lemma 1. Let  $K$  be given by (4), and  $A$  and  $C$  by Proposition 4. Then the quadruple  $(Y, Z, A, C)$  satisfies for any  $t \in [0, T]$

$$Y_t = \xi_T + \int_t^T f(\cdot, s, Y_s, Z_s) ds - \int_t^T Z_s dW_s + (A_T - A_t) + (C_{T-} - C_{t-}). \quad (6)$$

A version  $\tilde{Y}$  of  $Y$  satisfies for any stopping time  $\tau \in \mathcal{T}_{0,T}$

$$\tilde{Y}_\tau = \xi_T + \int_\tau^T f(\cdot, s, \tilde{Y}_s, Z_s) ds - \int_\tau^T Z_s dW_s + (A_T - A_\tau) + (C_{T-} - C_{\tau-}). \quad (7)$$

## Theorem

Moreover,

$$\tilde{Y} \geq \xi,$$

$A$  is increasing, cadlag, predictable, with  $A_0 = 0$ ,  $E(A_T) < \infty$ , and satisfies

$$\int_0^T (\tilde{Y}_t - \xi_t) dA_t^c = 0, \text{ and } (\tilde{Y}_{\tau-} - \hat{\xi}_{\tau-})(A_\tau^d - A_{\tau-}^d) = 0$$

for all  $\tau \in \mathbb{T}_{0,T}$ ,

$C$  is increasing, cadlag, optional, pure jump, with  $E(C_T) < \infty$ , and satisfies

$$(\tilde{Y}_\tau - \xi_\tau)(C_\tau - C_{\tau-}) = 0.$$

Here  $\hat{\xi}$  is the left upper cadlag envelope of  $\xi$ .

For the uniqueness we need the following theorem.

### Theorem (Gal'chouk-Lenglart)

Let  $n \in \mathbf{N}$ . Let  $X$  be an  $n$ -dimensional optional semimartingale, i.e.  $X = (X^1, \dots, X^n)$  is an  $n$ -dimensional optional process with decomposition  $X^k = X_0^k + M^k + A^k + B^k$ , for all  $k \in \{1, \dots, n\}$ , where  $M^k$  is a (cadlag) local martingale,  $A^k$  is a right-continuous process of finite variation such that  $A_0 = 0$ , and  $B^k$  is a left-continuous process of finite variation which is purely discontinuous and such that  $B_0 = 0$ . Let  $F$  be a twice continuously differentiable function on  $\mathbf{R}^n$ . Then, almost surely, for all  $t \geq 0$ ,

## Theorem

$$\begin{aligned}
F(X_t) &= F(X_0) + \sum_{k=1}^n \int_{]0,t]} D^k F(X_{s-}) d(A^k + M^k)_s \\
&+ \frac{1}{2} \sum_{k,l=1}^n \int_{]0,t]} D^k D^l F(X_{s-}) d \langle M^{kc}, M^{lc} \rangle_s \\
&+ \sum_{0 < s \leq t} \left[ F(X_s) - F(X_{s-}) - \sum_{k=1}^n D^k F(X_{s-}) \Delta X_s^k \right] \\
&+ \sum_{k=1}^n \int_{[0,t[} D^k F(X_s) d(B^k)_{s+} \\
&+ \sum_{0 \leq s < t} \left[ F(X_{s+}) - F(X_s) - \sum_{k=1}^n D^k F(X_s) \Delta_+ X_s^k \right],
\end{aligned}$$

where  $D^k$  denotes the differentiation operator with respect to the  $k$ -th coordinate, and  $M^{kc}$  denotes the continuous part of  $M^k$ .

By the same way we have proved the existence and uniqueness of solutions for BSDEs with two optional reflecting barriers  $L$  and  $U$  assuming a kind of strong Mokobodski condition, *i.e.*, it is the difference of two strong supemartingales between  $L$  and  $U$ .

# References

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