On the $\frac{1}{H}$-variation of the divergence integral w.r.t. fBm with Hurst parameter $H < \frac{1}{2}$

EL HASSAN ESSAKY
joint work with : David Nualart

Cadi Ayyad University
Poly-disciplinary Faculty, Safi

Colloque Franco-Maghrébin d'Analyse Stochastique
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1. Properties of fBm

A Gaussian process $B = \{B_t, t \in [0, T]\}$ is called fractional Brownian motion(fBm) of Hurst parameter H if it has zero mean and the covariance function

$$R_H(t,s) := E(B_t B_s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),$$

for $s, t \in [0, T]$. The fractional Brownian motion has the following properties:

1. Self-similarity: for any constant $a > 0$, the processes $\{a^{-H} B_{at}, t \geq 0\}$ and $\{B_t, t \geq 0\}$ has the same probability distribution.
2. Stationary increments: $B_{t+s} - B_s \sim B_t$, $s, t \geq 0$.

From these properties, it follows that for every $\alpha > 0$

$$E\left(|B_t - B_s|^{\alpha}\right) = E\left(|B_1|^{\alpha}\right)|t-s|^{\alpha H}.$$

As a consequence of the Kolmogorov continuity theorem, we deduce that there exists a version of the fBm B which is a continuous process and whose paths are γ-Hölder continuous for every $\gamma < H$. The parameter H controls the regularity of the trajectories of fBm.

1. Stochastic integral with respect to fBm

The fBm with Hurst parameter $H \neq \frac{1}{2}$ is not a semimartingale and then the Itô approach to the construction of stochastic integrals with respect to fBm is not valid.
Two main approaches have been used in the literature to define stochastic integrals with respect to fBm with Hurst parameter H.

1. Pathwise Riemann-Stieltjes stochastic integrals can be defined using Young's integral [16] in the case $H > \frac{1}{2}$. When $H \in (\frac{1}{4}, \frac{1}{2})$, the rough path analysis introduced by Lyons [11] is a suitable method to construct pathwise stochastic integrals.
2. A second approach to develop a stochastic calculus with respect to the fBm is based on the techniques of Malliavin calculus. The divergence operator, which is the adjoint of the derivative operator, can be regarded as a stochastic integral, which coincides with the limit of Riemann sums constructed using the Wick product. This idea has been developed by Decreusefond and Üstünel [6], Carmona, Coutin and Montseny [4], Alòs, Mazet and Nualart [1, 2], Alòs and Nualart, among others. The integral constructed by this method has zero mean.

1. Malliavin calculus for fBm

Let $B = \{B_t, t \in [0, T]\}$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is generated by B. We denote by $\mathcal{H}$ the Hilbert space associated to B, defined as the closure of the linear space generated by the indicator functions $\{\mathbf{1}_{[0,t]}, t \in [0, T]\}$, with respect to the inner product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s), \qquad s, t \in [0, T].$$

The mapping $\mathbf{1}_{[0,t]} \to B_t$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian space generated by B. We denote by $B(\varphi) = \int_0^T \varphi_t dB_t$ the image of an element $\varphi \in \mathcal{H}$ by this isometry.

1. Malliavin calculus for fBm

For a smooth and cylindrical random variable $F = f(B(\varphi_1), \ldots, B(\varphi_n))$, with $\varphi_i \in \mathcal{H}$ and $f \in C_b^\infty(R^n)$ (f and all its partial derivatives are bounded), the derivative of F is the $\mathcal{H}$-valued random variable defined by

$$DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_j.$$

For any integer $k \geq 1$ and any real number $p \geq 1$ we denote by $\mathbb{D}^{k,p}$ the Sobolev space defined as the the closure of the space of smooth and cylindrical random variables with respect to the norm

$$\|F\|_{k,p}^p = E(|F|^p) + \sum_{j=1}^{k} E(\|D^j F\|_{\mathcal{H}^{\otimes j}}^p).$$

Similarly, for a given Hilbert space V we can define Sobolev spaces of V-valued random variables $\mathbb{D}^{k,p}(W)$.

1. Malliavin calculus for fBm

The divergence operator δ is introduced as the adjoint of the derivative operator. More precisely, an element $u \in L^2(\Omega; \mathcal{H})$ belongs to the domain of δ, denoted by $\mathrm{Dom}\,\delta$, if there exists a constant c_u depending on u such that

$$|E(\langle DF, u\rangle_{\mathcal{H}})| \leq c_u \|F\|_2,$$

for any smooth random variable $F \in \mathcal{S}$. For any $u \in \mathrm{Dom}\,\delta$, $\delta(u)$ is the element of $L^2(\Omega)$ given by the duality relationship

$$E(\delta(u)F) = E(\langle DF, u\rangle_{\mathcal{H}}),$$

for any $F \in \mathbb{D}^{1,2}$. We will make use of the property

$$F\delta(u) = \delta(Fu) + \langle DF, u\rangle_{\mathcal{H}}, \tag{1}$$

which holds if $F \in \mathbb{D}^{1,2}$, $u \in \mathrm{Dom}\,\delta$ and the right-hand side is square integrable.

1. Malliavin calculus for fBm

The covariance of the fractional Brownian motion can be written as

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du,$$

where $K_H(t,s)$ is a square integrable kernel, defined for $0 < s < t < T$. In what follows, we assume that $0 < H < \frac{1}{2}$. In this case, this kernel has the following expression

$$K_H(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{H-\frac{1}{2}} \int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}} du \right],$$

with $c_H = \left(\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}\right)^{\frac{1}{2}}$ and $\beta(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt$ for $x, y > 0$. Notice also that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H(H-\frac{1}{2})\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.$$

From these expressions it follows that the kernel K_H satisfies the following two estimates

$$\left|\frac{\partial K_H}{\partial t}(t,s)\right| \leq c_H (t-s)^{H-\frac{3}{2}}, \tag{2}$$

and

$$|K_H(t,s)| \leq d_H \left((t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}}\right), \tag{3}$$

1. Malliavin calculus for fBm

Let $\mathcal{E}$ be the linear span of the indicator functions on $[0, T]$. Consider the linear operator K_H^* from $\mathcal{E}$ to $L^2([0, T])$ defined by

$$K_H^*(\varphi)(s) = K_H(T, s)\varphi(s) + \int_s^T (\varphi(t) - \varphi(s))\frac{\partial K_H}{\partial t}(t, s)dt. \qquad (4)$$

Notice that

$$K_H^*(\mathbf{1}_{[0,t]})(s) = K_H(t, s)\mathbf{1}_{[0,t]}(s).$$

As a consequence $C^\gamma([0, T]) \subset \mathcal{H} \subset L^2([0, T])$.

1. Malliavin calculus for fBm

It should be noted that the operator K_H^* is an isometry between the Hilbert space $\mathcal{H}$ and $L^2([0,T])$. That is, for every $\varphi, \psi \in \mathcal{H}$,

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \langle K_H^* \varphi, K_H^* \psi \rangle_{L^2([0,T])}. \tag{5}$$

In this case the fBm is more irregular than the classical Brownian motion, and some Hölder continuity is required for a function to be integrable.

1. Malliavin calculus for fBm

Consider the following seminorm on the space $\mathcal{E}$

$$\|\varphi\|_K^2 = \int_0^T \varphi^2(s)[(T-s)^{2H-1} + s^{2H-1}]ds + \int_0^T \left(\int_s^T |\varphi(t) - \varphi(s)|(t-s)^{H-\frac{3}{2}} dt \right)^2 ds. \quad (6)$$

We denote by $\mathcal{H}_K$ the completion of $\mathcal{E}$ with respect to this seminorm. From the estimates (2) and (3), there exists a constant k_H such that for any $\varphi \in \mathcal{H}_K$,

$$\|\varphi\|_{\mathcal{H}}^2 = \|K_H^*(\varphi)\|_{L^2([0,T])}^2 \leq k_H \|\varphi\|_K^2. \quad (7)$$

As a consequence, the space $\mathcal{H}_K$ is continuously embedded in $\mathcal{H}$. This implies also that $\mathbb{D}^{1,2}(\mathcal{H}_K) \subset \mathbb{D}^{1,2}(\mathcal{H}) \subset \mathrm{Dom}\,\delta$.

1. Malliavin calculus for fBm

One can show also that $\mathcal{H} = I_{T-}^{\frac{1}{2}-H}(L^2([0,T])) \subset L^{\frac{1}{H}}([0,T])$ (see Decreusefond -Üstünel [6]), where $I_{T-}^{\frac{1}{2}-H}\varphi(s) = \frac{1}{\Gamma(\frac{1}{2}-H)}\int_0^t (s-t)^{H-\frac{3}{2}}\varphi(s)ds$ is the left-sided fractional operator.

For $H > \frac{1}{2}$, consider the Banach space $|\mathcal{H}|$ of measurable functions $\varphi : [0,T] \to R$

$$\|\varphi\|_{|\mathcal{H}|}^2 = C_H \int_0^T \int_0^T |r-u|^{2H-2}|\varphi_u||\varphi_r|dudr < \infty.$$

In this case, we have $L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}$.

1. q-variation of a stochastic process

Fix $q \geq 1$ and $T > 0$ and set $t_i^n := \frac{iT}{n}$, where n is a positive integer and $i = 0, 1, 2, \ldots, n$. We need the following definition.

Definition

Let X be a given stochastic process defined in the complete probability space $(\Omega, \mathcal{F}, P)$. Let $V_n^q(X)$ be the random variable defined by

$$V_n^q(X) := \sum_{i=0}^{n-1} |\Delta_i^n X|^q,$$

where $\Delta_i^n X := X_{t_{i+1}^n} - X_{t_i^n}$. We define the q-variation of X as the limit in $L^1(\Omega)$, as n goes to infinity, of $V_n^q(X)$ if this limit exists.

1. $\frac{1}{H}$-variation of fBm

Using the self-similarity of fBm and the Ergodic Theorem one can prove that the fBm has a finite $\frac{1}{H}$-variation on any interval $[0, T]$, equals to $T\, e_H$, where $e_H = E\left[|B_1|^{\frac{1}{H}}\right]$ (see, for instance, Rogers [15]). More precisely, we have, as n tends to infinity

$$\sum_{i=0}^{n-1} |B_{t_{i+1}^n} - B_{t_i^n}|^{\frac{1}{H}} \xrightarrow{L^1(\Omega)} T\, e_H. \tag{8}$$

In fact, the self-similarity property implies that the sequence

$$\sum_{i=0}^{n-1} |B_{t_{i+1}^n} - B_{t_i^n}|^{\frac{1}{H}}$$

has the same distribution as

$$\frac{T}{n}\sum_{i=0}^{n-1} |B_{i+1} - B_i|^{\frac{1}{H}},$$

and by the Ergodic Theorem this converges in L^1 and almost surely to $T\, e_H$.

1. Objective of the talk

This result has been generalized by Guerra and Nualart [9] to the case of divergence integrals with respect to the fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$. They have proved that

$$V_n^{\frac{1}{H}}(X) \stackrel{L^1(\Omega)}{\longrightarrow} e_H \int_0^T |u_s|^{\frac{1}{H}} ds,$$

as n tends to infinity, where $X_t = \int_0^t u_s \delta B_s$ and $e_H = \boldsymbol{E}\left[|B_1|^{\frac{1}{H}}\right]$.

The purpose of this talk is to study the $\frac{1}{H}$-variation of divergence processes $X = \{X_t, t \in [0, T]\}$ with respect to the fBm with Hurst parameter $H < \frac{1}{2}$.

1. L^p-estimates for the divergence integral with respect to fBm

Let V be a given Hilbert space. We introduce the following hypothesis for a V-valued stochastic process $u = \{u_t, t \in [0, T]\}$, for some $p \geq 2$.

Hypothesis $(\mathbf{A.1})_p$ *Let* $p \geq 2$. *Then,* $\sup_{0 \leq s \leq T} \|u_s\|_{L^p(\Omega;V)} < \infty$ *and there exist constants* $L > 0$, $0 \leq \alpha < \frac{1}{2}$ *and* $\gamma > \frac{1}{2} - H$ *such that,*

$$\|u_t - u_s\|_{L^p(\Omega;V)} \leq L s^{-\alpha} |t-s|^{\gamma},$$

for all $0 < s \leq t \leq T$.

For any $0 \leq a < b \leq T$, we will make use of the notation

$$\|u\|_{p,a,b} = \sup_{a \leq s \leq b} \|u_s\|_{L^p(\Omega;V)}.$$

The following lemma is a crucial ingredient to establish the L^p-estimates for the divergence integral with respect to fBm.

Lemma

Let $u = \{u_t, 0 \leq t \leq T\}$ *be a process with values in a Hilbert space* V, *satisfying assumption* $(\mathbf{A.1})_p$ *for some* $p \geq 2$. *Then, there exists a positive constant* C *depending on* H, γ *and* p *such that for every* $0 < a \leq b \leq T$

$$E\left(\|u\mathbf{1}_{[a,b]}\|^p_{\mathcal{H}\otimes V}\right) \leq C\left(\|u\|^p_{p,a,b}(b-a)^{pH} + L^p a^{-p\alpha}(b-a)^{p\gamma+pH}\right). \quad (9)$$

1. L^p-estimates for the divergence integral with respect to fBm

Idea of the proof. Suppose first that $a > 0$. By equalities (5) and (4) we obtain

$$E\left(\|u\mathbf{1}_{[a,b]}\|^p_{\mathcal{H}\otimes V}\right) = E\left(\|K_H^*(u\mathbf{1}_{[a,b]})\|^p_{L^2([0,T];V)}\right)$$
$$= E\left(\left\|K_H(T,s)u_s\mathbf{1}_{[a,b]}(s) + \int_s^T \left(u_t\mathbf{1}_{[a,b]}(t) - u_s\mathbf{1}_{[a,b]}(s)\right)\frac{\partial K_H}{\partial t}(t,s)dt\right\|^p_{L^2([0,T];V)}\right).$$

Consider the decomposition

$$\int_s^T \left(u_t\mathbf{1}_{[a,b]}(t) - u_s\mathbf{1}_{[a,b]}(s)\right)\frac{\partial K_H}{\partial t}(t,s)dt = \left[\int_s^b (u_t - u_s)\frac{\partial K_H}{\partial t}(t,s)dt\right]\mathbf{1}_{[a,b]}(s)$$
$$+\left[-\int_b^T u_s\frac{\partial K_H}{\partial t}(t,s)dt\right]\mathbf{1}_{[a,b]}(s) + \left[\int_a^b u_t\frac{\partial K_H}{\partial t}(t,s)dt\right]\mathbf{1}_{[0,a]}(s)$$
$$:= I_1 + I_2 + I_3.$$

Therefore

$$E\left(\|u\mathbf{1}_{[a,b]}\|^p_{\mathcal{H}\otimes V}\right) \leq C\sum_{i=0}^{3} A_i,$$

where $A_0 = E\left[\|K_H(T,\cdot)u\mathbf{1}_{[a,b]}\|^p_{L^2([0,T];V)}\right]$ and for $i = 1,2,3$, $A_i = E\left[\|I_i\|^p_{L^2([0,T];V)}\right]$.

1. L^p-estimates for the divergence integral with respect to fBm

Hypothesis $(\mathbf{A}.\mathbf{2})_p$ *Let $u \in \mathbb{D}^{1,2}(\mathcal{H})$ be a real-valued stochastic process, which satisfies Hypothesis $(\mathbf{A}.\mathbf{1})_p$ with constants L_u, α_1 and γ for a fixed $p \geq 2$. We also assume that the $\mathcal{H}$-valued process $\{Du_s, s \in [0,T]\}$ satisfies Hypothesis $(\mathbf{A}.\mathbf{1})_p$ with constants L_{Du}, α_2 and γ for the same value of p.*

Hypothesis $(\mathbf{A}.\mathbf{2})_p$ means that u_s and Du_s have bounded L^p norms in $[0,T]$ and satisfy

$$\|u_t - u_s\|_{L^p(\Omega)} \leq L_u s^{-\alpha_1}|t-s|^{\gamma} \quad (11)$$

$$\|Du_t - Du_s\|_{L^p(\Omega;\mathcal{H})} \leq L_{Du} s^{-\alpha_2}|t-s|^{\gamma}, \quad (12)$$

for all $0 < s \leq t \leq T$.

1. L^p-estimates for the divergence integral with respect to fBm

Theorem

Suppose that $u \in \mathbb{D}^{1,2}(\mathcal{H})$ is a stochastic process satisfying Hypothesis $(\mathbf{A.2})_p$ for some $p \geq 2$. Let $0 < a \leq b \leq T$. Then, there exists a positive constant C depending on H, γ and p such that

$$E\left(\left|\int_a^b u_s \delta B_s\right|^p\right) \leq C\left((\|u\|_{p,a,b}^p + \|Du\|_{p,a,b}^p)(b-a)^{pH} + (L_u^p a^{-p\alpha_1} + L_{Du}^p a^{-p\alpha_2})(b-a)^{p\gamma+pH}\right). \quad (13)$$

If $a = 0$, then

$$E\left(\left|\int_0^b u_s \delta B_s\right|^p\right) \leq C\left((\|u\|_{p,a,b}^p + \|Du\|_{p,a,b}^p)b^{pH} + (L_u^p b^{-p\alpha_1} + L_{Du}^p b^{-p\alpha_2})b^{p\gamma+pH}\right). \quad (14)$$

1. L^p-estimates for the divergence integral with respect to fBm

Proof. We have

$$E\left(\left|\int_a^b u_s \delta B_s\right|^p\right) \leq C_p \left(E(\|u\mathbf{1}_{[a,b]}\|_{\mathcal{H}}^p) + E(\|D_s(u_t\mathbf{1}_{[a,b]}(t))\|_{\mathcal{H}\otimes\mathcal{H}}^p\right).$$

The first and the second terms of the above inequality can be estimated applying Lemma 2 to the processes u and Du, with $V = \mathbb{R}$ and $V = \mathcal{H}$, respectively.

1. $\frac{1}{H}$-variation for fBm

Hypothesis (A.3) *Let $u \in \mathbb{D}^{1,2}(\mathcal{H})$ be a real-valued stochastic process which is bounded in $L^q(\Omega)$ for some $q > \frac{1}{H}$ and satisfies the Hölder continuity property (11) with $p = \frac{1}{H}$, that is*

$$\|u_t - u_s\|_{L^{\frac{1}{H}}(\Omega)} \leq L_u s^{-\alpha_1}|t-s|^{\gamma}. \tag{15}$$

Suppose also that the $\mathcal{H}$-valued process $\{Du_s, s \in [0,T]\}$ is bounded in $L^{\frac{1}{H}}(\Omega;\mathcal{H})$ and satisfies the Hölder continuity property (12) with $p = \frac{1}{H}$, that is

$$\|Du_t - Du_s\|_{L^{\frac{1}{H}}(\Omega;\mathcal{H})} \leq L_{Du} s^{-\alpha_2}|t-s|^{\gamma}. \tag{16}$$

Moreover, we assume that the derivative $\{D_t u_s, s,t \in [0,T]\}$ satisfies

$$\sup_{0\leq s\leq T} \|D_s u_t\|_{L^{\frac{1}{H}}(\Omega)} \leq K t^{-\alpha_3}, \tag{17}$$

for every $t \in (0,T]$ and for some constants $0 < \alpha_3 < 2H$ and $K > 0$.

1. $\frac{1}{H}$-variation for fBm

Consider the indefinite divergence integral of u with respect to the fBm B, given by

$$X_t = \int_0^t u_s \delta B_s := \delta(u\mathbf{1}_{[0,t]}). \tag{18}$$

Theorem

*Suppose that $u \in \mathbb{D}^{1,2}(\mathcal{H})$ is a stochastic process satisfying Hypothesis **(A.3)**, and consider the divergence integral process X given by (18). Then, we have*

$$V_n^{\frac{1}{H}}(X) \xrightarrow{L^1(\Omega)} e_H \int_0^T |u_s|^{\frac{1}{H}} ds,$$

as n tends to infinity, where $e_H = E\left[|B_1|^{\frac{1}{H}}\right]$.

1. $\frac{1}{H}$-variation for fBm

We need to show that the expression

$$F_n := E\left(\left|\sum_{i=0}^{n-1}\left|\int_{t_i^n}^{t_{i+1}^n} u_s \delta B_s\right|^{\frac{1}{H}} - e_H \int_0^T |u_s|^{\frac{1}{H}} ds\right|\right),$$

converges to zero as n tends to infinity. Using (1), we can write

$$\begin{aligned}\int_{t_i^n}^{t_{i+1}^n} u_s \delta B_s &= \int_{t_i^n}^{t_{i+1}^n} (u_s - u_{t_i^n})\delta B_s + \int_{t_i^n}^{t_{i+1}^n} u_{t_i^n} \delta B_s \\ &= \int_{t_i^n}^{t_{i+1}^n} (u_s - u_{t_i^n})\delta B_s - \langle D u_{t_i^n}, \mathbf{1}_{[t_i^n, t_{i+1}^n]}\rangle_{\mathcal{H}} + u_{t_i^n}(B_{t_{i+1}^n} - B_{t_i^n}). \\ &:= A_i^{1,n} - A_i^{2,n} + A_i^{3,n}.\end{aligned} \tag{19}$$

By the triangular inequality, we obtain

$$F_n \leq E\left(\sum_{i=0}^{n-1}\left||A_i^{1,n} - A_i^{2,n} + A_i^{3,n}|^{\frac{1}{H}} - |A_i^{3,n}|^{\frac{1}{H}}\right|\right) + D_n, \tag{20}$$

1. $\frac{1}{H}$-variation for fBm

where

$$D_n = E\left(\left|\sum_{i=0}^{n-1} |A_i^{3,n}|^{\frac{1}{H}} - e_H \int_0^T |u_s|^{\frac{1}{H}} ds\right|\right).$$

Using the mean value theorem and Hölder inequality, we can write

$$\begin{aligned}
& E\left(\sum_{i=0}^{n-1} \left| |A_i^{1,n} - A_i^{2,n} + A_i^{3,n}|^{\frac{1}{H}} - |A_i^{3,n}|^{\frac{1}{H}} \right|\right) \\
& \leq \frac{1}{H} E\left(\sum_{i=0}^{n-1} |A_i^{1,n} - A_i^{2,n}| \left[|A_i^{1,n} - A_i^{2,n} + A_i^{3,n}|^{\frac{1}{H}-1} + |A_i^{3,n}|^{\frac{1}{H}-1} \right]\right) \\
& \leq C \left[E\left(\sum_{i=0}^{n-1} |A_i^{1,n} - A_i^{2,n}|^{\frac{1}{H}}\right)\right]^H \\
& \qquad \times \left[E\left(\sum_{i=0}^{n-1} |A_i^{1,n} - A_i^{2,n} + A_i^{3,n}|^{\frac{1}{H}}\right) + E\left(\sum_{i=0}^{n-1} |A_i^{3,n}|^{\frac{1}{H}}\right)\right]^{1-H}. \qquad (21)
\end{aligned}$$

1. $\frac{1}{H}$-variation for fBm

Substituting (21) into (20) yields

$$F_n \leq C A_n^H (B_n + C_n)^{1-H} + D_n,$$

where

$$\begin{aligned}
A_n &= E\left(\sum_{i=0}^{n-1} |A_i^{1,n} - A_i^{2,n}|^{\frac{1}{H}}\right), \\
B_n &= E\left(\sum_{i=0}^{n-1} |A_i^{1,n} - A_i^{2,n} + A_i^{3,n}|^{\frac{1}{H}}\right), \\
C_n &= E\left(\sum_{i=0}^{n-1} |A_i^{3,n}|^{\frac{1}{H}}\right) \\
D_n &= E\left(\left|\sum_{i=0}^{n-1} |A_i^{3,n}|^{\frac{1}{H}} - e_H \int_0^T |u_s|^{\frac{1}{H}} ds\right|\right).
\end{aligned}$$

We first prove that B_n and C_n are bounded and second we show that A_n converges to zero.

1. $\frac{1}{H}$-variation for fBm

Remark that

$$\begin{aligned} B_n &= E\left(\left|\int_0^{\frac{T}{n}} u_s \delta B_s\right|^{\frac{1}{H}}\right) + E\left(\sum_{i=1}^{n-1}\left|\int_{t_i^n}^{t_{i+1}^n} u_s \delta B_s\right|^{\frac{1}{H}}\right) \\ &=: K_1^n + K_2^n. \end{aligned}$$

Using estimate (14) with $p = \frac{1}{H}$, it follows that

$$\begin{aligned} K_1^n &\leq C\left(\|u\|^{\frac{1}{H}}_{\frac{1}{H},0,\frac{T}{n}} + \|Du\|^{\frac{1}{H}}_{\frac{1}{H},0,\frac{T}{n}}\right) n^{-1} + \left(L_u^{\frac{1}{H}} n^{\frac{\alpha_1}{H}} + L_{Du}^{\frac{1}{H}} n^{\frac{\alpha_2}{H}}\right) n^{-\frac{\gamma}{H}-1} \\ &\leq C\left(n^{-1} + n^{\frac{\alpha_1}{H}-\frac{\gamma}{H}-1} + n^{\frac{\alpha_2}{H}-\frac{\gamma}{H}-1}\right). \end{aligned}$$

Therefore, K_1^n is bounded since $\alpha_1 < \gamma + H$ and $\alpha_2 < \gamma + H$.
Each term $A_i^{2,n}$ can be expressed as

$$A_i^{2,n} = \int_0^T D_s u_{t_i^n} \frac{\partial}{\partial s}\left(R(s, t_{i+1}^n) - (R(s, t_i^n)\right) ds.$$

1. $\frac{1}{H}$-variation for fBm

In order to show that the term D_n converges to zero as n tends to infinity, we replace n by the product nm and we let first m tend to infinity. That is, we consider the partition of interval $[0, T]$ given by $0 = t_0^{nm} < \cdots < t_{nm}^{nm} = T$ and we define

$$
\begin{aligned}
Z^{n,m} &:= \left| \sum_{i=0}^{nm-1} |u_{t_i^{nm}}|^{\frac{1}{H}} |\Delta_i^{nm} B|^{\frac{1}{H}} - e_H \sum_{j=0}^{n-1} |u_{t_j^n}|^{\frac{1}{H}} (t_{j+1}^n - t_j^n) \right| \\
&= \left| \sum_{j=0}^{n-1} \left[\sum_{i=jm}^{(j+1)m-1} \left(|u_{t_i^{nm}}|^{\frac{1}{H}} - |u_{t_j^n}|^{\frac{1}{H}} \right) |\Delta_i^{nm} B|^{\frac{1}{H}} \right.\right. \\
&\qquad \left.\left. + |u_{t_j^n}|^{\frac{1}{H}} \left(\sum_{i=jm}^{(j+1)m-1} |\Delta_i^{nm} B|^{\frac{1}{H}} - e_H (t_{j+1}^n - t_j^n) \right) \right] \right|. \\
&\leq \sum_{j=0}^{n-1} \sum_{i=jm}^{(j+1)m-1} \left| |u_{t_i^{nm}}|^{\frac{1}{H}} - |u_{t_j^n}|^{\frac{1}{H}} \right| |\Delta_i^{nm} B|^{\frac{1}{H}} \\
&\qquad + \sum_{j=0}^{n-1} |u_{t_j^n}|^{\frac{1}{H}} \left| \sum_{i=jm}^{(j+1)m-1} |\Delta_i^{nm} B|^{\frac{1}{H}} - e_H (t_{j+1}^n - t_j^n) \right| := Z_1^{n,m} + Z_2^{n,m}.
\end{aligned}
$$

1. $\frac{1}{H}$-variation for fBm

$$\lim_{n\to\infty} \sup_{m\geq 1} E(Z_1^{n,m}) = 0, \tag{22}$$

and for any $n \geq 1$,

$$\lim_{m\to\infty} E(Z_2^{n,m}) = 0. \tag{23}$$

Therefore, it follows from (22) and (23) that

$$\lim_{n\to\infty} \lim_{m\to\infty} E(Z^{n,m}) = 0. \tag{24}$$

1. $\frac{1}{H}$-variation for fBm

Example

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function, such that

$$|f(x)| + |f'(x)| + |f''(x)| \leq e^{\lambda x^2},$$

for some $\lambda < \frac{1}{2T^{2H}}$. Suppose that $H < \frac{1}{4}$. Then the process $u = \{f(B_t), t \in [0, T]\}$ satisfies Assumption **(**A.3) with constants $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\gamma = H$. As a consequence, Theorem 4.1 holds for the process $X_t = \int_0^t f(B_s)\delta B_s$. Indeed, let us just show condition (15). We can write, by the mean value theorem,

$$\|f(B_t) - f(B_s)\|_{L^{\frac{1}{H}}(\Omega)} \leq \left\| \exp\left(\lambda \max_{0 \leq t \leq T} B_t^2 \right) |B_t - B_s| \right\|_{L^{\frac{1}{H}}(\Omega)},$$

and we conclude using Hölder's inequality.

1. $\frac{1}{H}$-variation for d-fBm

Consider a d-dimensional fractional Brownian motion ($d \geq 2$)

$$B = \{B_t, t \in [0, T]\} = \{(B_t^{(1)}, B_t^{(2)}, \dots, B_t^{(d)}),\ t \in [0, T]\}$$

with Hurst parameter $H \in (0, 1)$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is generated by B. That is, the components $B^{(i)}$, $i = 1, \dots, d$, are independent fractional Brownian motions with Hurst parameter H. We can define the derivative and divergence operators, $D^{(i)}$ and $\delta^{(i)}$, with respect to each component $B^{(i)}$, as in Section 2. Denote by $\mathbb{D}_i^{1,p}(\mathcal{H})$ the associated Sobolev spaces. We assume that these spaces include functionals depending on of all the components of B and not only the ith component.

1. $\frac{1}{H}$-variation for d-fBm

Lemma

Let F be a bounded random variable with values in R^d. Then, we have

$$V_n^{\frac{1}{H}}(\langle F, B\rangle) \xrightarrow{L^1(\Omega)} \int_{R^d} \left[\int_0^T |\langle F, \xi\rangle|^{\frac{1}{H}} ds\right] \nu(d\xi),$$

as n tends to infinity, where ν is the normal distribution $N(0, I)$ on R^d.

Theorem

*Suppose that for each $i = 1, \ldots, d$, $u^{(i)} \in \mathbb{D}^{1,2}(\mathcal{H})$ is a stochastic process satisfying Hypothesis **(A.3)**. Set $u_t = (u_t^{(1)}, \ldots, u_t^{(d)})$ and consider the divergence integral process $X = \{X_t, t \in [0, T]\}$ defined by $X_t := \sum_{i=1}^d \int_0^t u_s^{(i)} \delta B_s^{(i)}$. Then, we have*

$$V_n^{\frac{1}{H}}(X) \xrightarrow{L^1(\Omega)} \int_{R^d} \left[\int_0^T |\langle u_s, \xi\rangle|^{\frac{1}{H}} ds\right] \nu(d\xi),$$

as n tends to infinity, where ν is the normal distribution $N(0, I)$ on R^d.

1. Extended Domain

The space $\mathcal{H}$ is too small for some purposes. For instance, it has been proved in Cheridito-Nualart [5], that the trajectories of the fBm B belongs to $\mathcal{H}$ if and only if $H > \frac{1}{4}$. This creates difficulties when defining the divergence $\delta(u)$ of a stochastic process whose trajectories do not belong to $\mathcal{H}$, for example, if $u_t = f(B_t)$ and $H < \frac{1}{4}$, because the domain of δ is included in $L^2(\Omega; \mathcal{H})$. To overcome this difficulty, we extend the domain of the divergence operator. The main ingredient in the definition of this extended domain is the extension of the inner produce $\langle \varphi, \psi \rangle_{\mathcal{H}}$ to the case where $\psi \in \mathcal{E}$ and $\varphi \in L^\beta([0,T])$ for some $\beta > \frac{1}{2H}$. More precisely, for $\varphi \in L^\beta([0,T])$ and $\psi = \sum_{j=1}^m b_j \mathbf{1}_{[0,t_j]} \in \mathcal{E}$ we set

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \sum_{j=1}^m b_j \int_0^T \varphi_s \frac{\partial R}{\partial s}(s, t_j) ds. \tag{25}$$

This expression coincides with the inner produce in $\mathcal{H}$ if $\varphi \in \mathcal{H}$, and it is well defined, because

$$|\langle \varphi, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}}| = \left| \int_0^T \varphi_s \frac{\partial R}{\partial s}(s,t) ds \right| \leq \|\varphi\|_{L^\beta([0,T])} \sup_{0 \leq t \leq T} \left(\int_0^T |\frac{\partial R}{\partial s}(s, t_j)|^\alpha ds \right)^{\frac{1}{\alpha}} < \infty.$$

1. Extended Domain

We need the following extension of the domain of the divergence operator to processes with trajectories in $L^{\beta}([0,T],R^d)$, where $\beta > \frac{1}{2H}$.

Definition

Fix $\beta > \frac{1}{2H}$. We say that a d-dimensional stochastic process $u = (u^{(1)},\dots,u^{(d)}) \in L^1(\Omega; L^{\beta}([0,T],R^d))$ belongs to the extended domain of the divergence $\mathrm{Dom}^*\delta$, if there exists $q > 1$ such that

$$|E\langle u, DF\rangle_{\mathcal{H}_d}| = \left|\sum_{i=1}^{d} E(\langle u^{(i)}, D^{(i)}F\rangle_{\mathcal{H}})\right| \leq c_u\|F\|_{L^q(\Omega)}, \tag{26}$$

for every smooth and cylindrical random variable $F \in \mathcal{S}_d$, where c_u is some constant depending on u. In this case $\delta(u) \in L^p(\Omega)$, where p is the conjugate of q, is defined by the duality relationship

$$E(\langle u, DF\rangle_{\mathcal{H}}) = E(\delta(u)F),$$

for every smooth and cylindrical random variable $F \in \mathcal{S}_d$.

1. Fractional Bessel process

Let B be a d-dimensional fractional Brownian motion ($d \geq 2$). The process $R = \{R_t, t \in [0, T]\}$, defined by $R_t = \|B_t\|$, is called the fractional Bessel process of dimension d and Hurst parameter H. It has been proved in [5] that, for $H > \frac{1}{2}$, the fractional Bessel process R has the following representation

$$R_t = \sum_{i=1}^{d} \int_0^t \frac{B_s^{(i)}}{R_s} \delta B_s^{(i)} + H(d-1) \int_0^t \frac{s^{2H-1}}{R_s} ds. \qquad (27)$$

This representation (27) is similar the one obtained for Bessel processes with respect to standard Brownian motion (see, for instance, Karatzas and Shreve [12]). Indeed, if W is a d-Brownian motion and $R_t = \|W_t\|$, then

$$R_t = \sum_{i=1}^{d} \int_0^t \frac{W_s^{(i)}}{R_s} dW_s^{(i)} + \frac{d-1}{2} \int_0^t \frac{ds}{R_s}.$$

1. Itô formula

Theorem

Let B a d-dimensional fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. Suppose that $F \in C^2(R^d)$ satisfies the growth condition

$$\max_{x \in \mathbf{R}^d} \left\{ |F(x)|, \left\| \frac{\partial F}{\partial x_i}(x) \right\|, \left\| \frac{\partial^2 F}{\partial x_i^2}(x) \right\|, i = 1, \ldots, d \right\} \leq c e^{\lambda x^2}, \tag{28}$$

where c and λ are positive constants such that $\lambda < \dfrac{T^{-2H}}{4d}$. Then, for each $i = 1, ..., d$ and $t \in [0, T]$, the process $\mathbf{1}_{[0,t]} \dfrac{\partial F}{\partial x_i}(B_t) \in \mathrm{Dom}^E \delta$, and the following formula holds

$$F(B_t) = F(0) + \sum_{i=1}^{d} \int_0^t \frac{\partial F}{\partial x_i}(B_s) \delta B_s^{(i)} + H \sum_{i=1}^{d} \int_0^t \frac{\partial^2 F}{\partial x_i^2}(B_s) s^{2H-1} ds, \tag{29}$$

where $\mathrm{Dom}^E \delta$ is the extended domain of the divergence operator in the sense of Definition 3.9 in Hu, Jolis, Tindel [7].

1. Fractional Bessel process

The next result is a change of variable formula for the fractional Bessel process in the case $H < \frac{1}{2}$.

Theorem

Let $H < \frac{1}{2}$, and let $R = \{R_t, \in [0, T]\}$ be the fractional Bessel process. Set $u_t^{(i)} = \frac{B_t^{(i)}}{R_t}$ and $u_t = (u_t^{(1)}, \dots, u_t^{(d)})$, for $t \in [0, T]$. Then, we have the following results:

(i) *For any $t \in (0, T]$, the process $\{u_s \mathbf{1}_{[0,t]}(s), s \in [0, T]\}$ belongs to the extended domain $\mathrm{Dom}^* \delta$ and the representation (27) holds true.*

(ii) *If $H > \frac{1}{4}$, for any $t \in [0, T]$, the process $u\mathbf{1}_{[0,t]}$ belongs to $L^2(\Omega; \mathcal{H}_d)$ and to the domain of δ in $L^p(\Omega)$ for any $p < d$.*

1. Fractional Bessel process

Idea of the proof. *i*) Let us first prove part (i). Since the function $\|x\|$ is not differentiable at the origin, the Itô formula (29) cannot be applied and we need to make a suitable approximation. For $\varepsilon > 0$, consider the function $F_\varepsilon(x) = (\|x\|^2 + \varepsilon^2)^{\frac{1}{2}}$, which is smooth and satisfies some growth condition. Applying Itô's formula we have

$$F_\varepsilon(B_t) = \varepsilon + \sum_{i=1}^{d} \int_0^t \frac{B_s^{(i)}}{(R_s^2 + \varepsilon^2)^{\frac{1}{2}}} \delta B_s^{(i)} + Hd \int_0^t \frac{s^{2H-1}}{(R_t^2 + \varepsilon^2)^{\frac{1}{2}}} ds - H \int_0^t \frac{s^{2H-1} R_s^2}{(R_s^2 + \varepsilon^2)^{\frac{3}{2}}} ds. \quad (30)$$

1. Fractional Bessel process

Taking into account that $\delta(u^{\varepsilon}\mathbf{1}_{[0,t]})$ converges to G_t in L^p, and that

$$\lim_{\varepsilon\to 0} E(\langle u^{\varepsilon}\mathbf{1}_{[0,t]}, DF\rangle_{\mathcal{H}_d}) = E(\langle u\mathbf{1}_{[0,t]}, DF\rangle_{\mathcal{H}_d}),$$

since the components of u are bounded by one, we deduce that

$$E(\langle u1_{[0,t]}, DF\rangle_{\mathcal{H}_d}) = E(G_t F).$$

This implies that $u\mathbf{1}_{[0,t]}$ belongs to the extended domain of the divergence and $\delta(u\mathbf{1}_{[0,t]}) = G_t$.
ii) Assume that $H > \frac{1}{4}$. We first show that for any $i = 1, \dots, d$, $u^{(i)} \in L^2(\Omega; \mathcal{H})$. We have

$$\begin{aligned} E(\|u_t^{(i)}\|_{\mathcal{H}}^2) \quad \leq \quad & k_H E\left(\int_0^T (u_s^{(i)})^2[(T-s)^{2H-1} + s^{2H-1}]ds\right) \\ & + k_H E\left(\int_0^T \left(\int_s^T |u_t^{(i)} - u_s^{(i)}|(t-s)^{H-\frac{3}{2}} dt\right)^2 ds\right). \end{aligned}$$

1. Fractional Bessel process

Theorem

*Suppose that $2dH^2 > 1$. Let $R = \{R_t, t \in [0, T]\}$ be the fractional Bessel process. Then, for $i = 1, 2, \ldots, d$, the process $u_t^{(i)} = \frac{B_t^{(i)}}{R_t}$ satisfies Hypothesis **(A.3)**.*

Idea of the proof : Fix $i = 1, \ldots, d$. The random variable $u_t^{(i)}$ is bounded and so, it is bounded in $L^q(\Omega)$ for all $q > \frac{1}{H}$. The Malliavin derivative $D^{(i)}u^{(i)}$ is given by

$$D_s^{(i)} u_t^{(i)} = \left(-R_t^{-3}(B_t^{(i)})^2 + R_t^{-1}\right) \mathbf{1}_{[0,t]}(s) := \phi_t \mathbf{1}_{[0,t]}(s).$$

1. Fractional Bessel process

We now discuss the properties of the process $\Theta = \{\Theta_t, t \in [0, T]\}$ defined by

$$\Theta_t := \sum_{i=1}^{d} \int_0^t \frac{B_s^{(i)}}{R_t} \delta B_s^{(i)}.$$

We have that for every $i = 1, \ldots, d$, $u_t^{(i)} = \dfrac{B_t^{(i)}}{R_t}$ satisfies Hypothesis **(A.3)** if $2dH^2 > 1$. Therefore, we have the following corollary.

Corollary

Suppose that $2dH^2 > 1$. Then we have the following

$$V_n^{\frac{1}{H}}(\Theta) \xrightarrow{L^1(\Omega)} \int_{R^d} \left[\int_0^T \left| \left\langle \frac{B_s}{R_s}, \xi \right\rangle \right|^{\frac{1}{H}} ds \right] \nu(d\xi),$$

as n tends to infinity, where ν is the normal distribution $N(0, I)$ on R^d.

1. Fractional Bessel process

Proposition

The process Θ is H-self-similar.

Let $a > 0$. By the representation (27) and the self-similarity of fBm, we have

$$\begin{aligned} \Theta_{at} &= R_{at} - H(d-1)\int_O^{at} \frac{s^{2H-1}}{R_s} ds \\ &\stackrel{d}{=} a^H R_t - H(d-1)a^H \int_0^t \frac{u^{2H-1}}{R_u} du = a^H \Theta_t, \end{aligned}$$

where the symbol $\stackrel{d}{=}$ means that the distributions of both processes are the same. This proves that Θ is H-self-similar.

1. Open problem : process of eigenvalues of a symmetric matrix-valued process

Open problem : Consider now a family of independent fractional Brownian motion with Hurst parameter $H \in)\frac{1}{4}, \frac{1}{2}($, $b = \{(b_{ij(t)}, t \geq 0), 1 \leq i \leq j \leq d\}$. We define the symmetric matrix fBm $B(t)$ with Hurst parameter H by: $B_{ij}(t) = b_{ij}$ if $i < j$ and $B_{ii}(t) = \sqrt{2} b_{ii}(t)$. We identify $B(t)$ as an element of $R^{\frac{d(d+1)}{2}}$.

Lemma

For every $i = 1, ..., d$, there exists a function $\Phi_i : R^{\frac{d(d+1)}{2}} \longrightarrow R$, which is C^∞ in an open subset G of $R^{\frac{d(d+1)}{2}}$, with $|G^c| = 0$, such that $\lambda_i(t) = \Phi_i(b(t))$. (Nualart-Abreu $H > \frac{1}{2}$ case).

To study :

$$u_t^i = \frac{\partial \Phi_i}{\partial x_{kh}}(b(t)), \quad \lambda_i(t) = \lambda_i(0) + Y_t^i + 2H \sum_{j \neq i} \int_0^t \frac{s^{2H-1}}{\lambda_i(s) - \lambda_j(s)} ds.$$

Y^i is the divergence integral of u^i.

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