HISTORIA MATHEMATICA 3 (1976), 141-160

EULER'S 1760 PAPER ON DIVERGENT SERIES

BY E.J. BARBEAU AND P.J. LEAH,
UNIVERSITY OF TORONTO

SUMMARIES

That Euler was quite aware of the subtleties of assigning a sum to a divergent series is amply demonstrated in his paper De seriebus divergentibus *which appeared in* Novi commentarii academiae scientiarum Petropolitanae *5 (1754/55), 205-237 (=* Opera Omnia *(1) 14, 585-617) in the year 1760. The first half of this paper contains a detailed exposition of Euler's views which should be more readily accessible to the mathematical community.*

The authors present here a translation from Latin of the summary and first twelve sections of Euler's paper with some explanatory comments. The remainder of the paper, treating Wallis' hypergeometric series and other technical matter, is described briefly. Appended is a short bibliography of works concerning Euler which are available to the English-speaking reader.

Dans son oeuvre, De seriebus divergentibus, *publiée en 1760 dans* Novi commentarii academiae scientiarum *5 (1754/55), 205-237 (=* Opera omnia *(1) 14, 585-617), Euler fit montre de sa connaissance des subtilités de l'attribution d'une somme à une série divergente. Cet article commence par un exposé de l'optique d'Euler, dont les mathématiciens actuels devraient être plus au courant qu'ils ne le sont.*

Les auteurs présentent ci-dessous une traduction de latin en anglais du résumé et les douze premières sections de cet article et en plus ajoutent quelques notes explicatives. La dernière partie, qui traite de la série hypergéométrique de Wallis et d'autres questions techniques, est présentée en forme abrégée. Enfin, il y a une liste des publications en langue anglaise au sujet d'Euler.

I. INTRODUCTION

Apparently, it is a common view today that Leonhard Euler (1707-1783) was unaware of the deeper issues involved in summing series. This might be inferred, for example, from Knopp [1947, Sect. 59]. However, this opinion was not shared by other investigators, notably Bromwich [1931] and Hardy [1949], who explained Euler's ideas in some detail in their monographs on infinite series.

Euler frequently makes it clear that he is cognizant of the behaviour of infinite series and, in fact, distinguishes between convergent and divergent series along modern lines. While he recognizes that pure mathematics, like other branches of knowledge, can generate controversy, he believes that disputes can be resolved in a systematic and rational way, at least in mathematics. Thus his assignment of a sum to a divergent series is a matter of conscious decision, made on pragmatic grounds and defensible by the consistency of mathematical analysis. Bromwich [1931, 322] quotes a passage from [Euler, 1755, 82, Sect. 111] which captures this idea well:

Let us say, therefore, that the sum *of any infinite series is the finite expression, by the expansion of which the series is generated. In this sense, the sum of the infinite series* $1-x+x^2-x^3+\ldots$ *will be* $1/1+x$, *because the series arises from the expansion of the fraction, whatever number is put in place of* x. *If this is agreed, the new definition of the word* sum *coincides with the ordinary meaning when a series converges; and since divergent series have no sum, in the proper sense of the word, no inconvenience can arise from this new terminology. Finally, by means of this definition, we can preserve the utility of divergent series and defend their use from all objections.*

The paper presented below was published in the midst of Euler's working life, while he was at the Berlin Academy. In a short space, he unfolds his thinking both on the nature of mathematical certitude and on infinite series. For this reason it deserves the attention of a wider audience. The first part, consisting of explanatory passages, is translated by the authors from Latin in its entirety, while the latter part, with its long calculations and tables, is paraphrased. A number of footnotes on the translated portion is given at the end of the paper; these are indicated in the text by numbers in square brackets. Subsequent to the passage translated is a summary with comments on the remainder of the paper. Following a list of references pertinent to the paper is a short bibliography of works on Euler which will provide an introduction for the English-speaking mathematician.

The first author is grateful to Prof. R. E. Fantham for her assistance in making some rather formidable Latin sentences understandable during a preliminary study of the work.

II. TRANSLATION OF THE SUMMARY AND FIRST TWELVE SECTIONS OF 'DE SERIEBUS DIVERGENTIBUS'

The author here undertakes to clarify a concept causing up to now the greatest difficulties; he found himself at considerable odds with the widespread opinion that mathematical research is free from all controversy. Indeed, it cannot be denied that mathematics contains the sort of speculations which put eminent geometers in great disagreement. Not only applied mathematics, witness that notorious dispute about "live forces" (vires vivas)[1], but also pure and abstract mathematics itself, strange as it may be, has supplied remarkable sources of dissent, such as the conflict between Leibniz and John Bernoulli on the disturbing question of logarithms of negative numbers [2], or, also from geometry, the problem of the cusp of curves of the second genus, the so-called "birdbeaks" [3]. The author himself has examined these controversies elsewhere in such a way that the two parties if they were both still alive would accept his solution. The question of divergent series is quite similar. The author is seen to have dealt with the matter equally happily here, so that henceforth no further controversy is to be feared. Wherefore, even if analysis is not without occasions for dispute, nevertheless they are distinguished from other occasions in that when eventually all the evidence has been thoroughly weighed the matter can be completely settled.

And now, series are said to be convergent when their terms steadily become smaller and at length completely vanish [4], such as this one: $1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + etc.$, whose sum is in fact $= 2$, without any doubt. For as you add in more terms, you draw closer to 2; thus the sum of 100 terms falls short of 2 by a very small amount, indeed a fraction with numerator 1 and a denominator made up of 30 digits. Therefore, with such a series, there is no doubt that it indeed has a sum and that the sum which is assigned in analysis is correct.

On the other hand, series are called divergent, whose terms do not tend to zero but never decrease below a certain amount or even increase to infinity. Such are $1 + 1 + 1 + 1 + 1 + etc.$, and $1 + 2 + 3 + 4 + 5 + 6 + etc.$, for which the sum becomes larger as more terms are added. Consequently, such a series can be made larger than any given number and therefore is properly said to be infinite since all terms taken to infinity are regarded as being gathered into one sum.

But if the signs alternate, as in the series $1 - 1 + 1 - 1 + 1 - 1$ *etc.*, or $1 - 2 + 3 - 4 + 5 - 6 + 7 - etc.$, no one will consider designating its sum as infinite. Although in fact, if two consecutive terms are taken together, the second series is changed to $- 1 - 1 - 1 - 1 - etc.$, whose sum is $-\infty$, yet, when the first term is taken separately and the following terms taken in pairs, we get $1 + 1 + 1 + 1 + etc.$, whose sum is $+\infty$. Evidently, in the former case the total number of terms is even, in the latter, to be sure, odd. Therefore, since the number of terms of the series continued to infinity is neither even nor odd, the sum will be neither $-\infty$ nor $+\infty$, whence it can be reckoned as equal to some finite number.

Notable enough, however, are the controversies over the series $1 - 1 + 1 - 1 + 1 - etc.$, whose sum was given by Leibniz as *1/2* , although others disagree. No one has yet assigned another value to that sum, and so the controversy turns on the question whether series of this type have a certain sum. Understanding of the question is to be sought in the word "sum"; this idea, if thus conceived--namely the sum of a series is said to be that quantity to which it is brought closer as more terms of the series are taken--has relevance only for convergent series, and we should in general give up this idea of sum for divergent series. Wherefore, those who thus define a sum cannot be blamed if they claim they are unable to assign a sum to a series. On the other hand, as series in analysis arise from the expansion of fractions or irrational quantities or even of transcendentals, it will in turn be permissible in calculation to substitute in place of such a series that quantity out of whose development it is produced [5]. For this reason, if we employ this definition of sum, that is to say, the sum of a series is that quantity which generates the series, all doubts with respect to divergent series vanish and no further controversy remains on this score, inasmuch as this definition is applicable equally to convergent or divergent series. Accordingly Leibniz, without any hesitation, accepted for the series $1 - 1 + 1 - 1 +$ *etc.*, the sum $1/2$, which arises out of the expansion of the fraction $1/1+1$, and for the series $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8$ *etc.* the sum $1/4$, which arises out of the expansion of the formula $1/(1+1)^2$. In a similar way a decision for all divergent series will be reached, where always a closed formula from whose expansion the series arises should be investigated. However, it can happen very often that this formula itself is difficult to find, as here where the author treats an exceptional example, that divergent series par excellence $1 - 1 + 2 - 6 + 24 - 120 + 720 - 5040 +$ *etc.*, which is Wallis' hypergeometric series [6], set out with alternating signs; this series, in whatever formula it finds its origin and however much this formula is valid, is seen to be determinable by only the deepest study of higher Analysis. Finally, after various attempts, the author by a wholly singular method using continued fractions found that the sum of this series is about 0.596347362123, and in this decimal fraction the error does not affect even the last digit. Then he proceeds to other similar series of wider application and he explains how to assign them a sum in the same way, where the word "sum" has that meaning which he has here established and by which all controversies are cut off.

1. If convergent series are considered as those whose terms continually decrease and eventually, when the series is continued to infinity, completely vanish, it is readily accepted that those whose terms do not tend to zero at infinity but either remain finite or grow to infinity, are assigned, since they are not convergent, to the class of divergent series [4]. Insofar then as

the ultimate terms of the series which are arrived at by continuing the progression to infinity, are either of finite or infinite magnitude, we have two types of divergent series, which can be further subdivided into two classes in which either all terms possess the same sign or proceed with alternating + and - signs. Altogether, therefore, we will have four types of divergent series, of which, for the sake of greater clarity, I append several examples.

I $1 + 1 + 1 + 1 + 1 + 1 + etc.$
$1/2 + 2/3 + 3/4 + 4/5 + 5/6 + 6/7 + etc.$

II $1 - 1 + 1 - 1 + 1 - 1 + etc.$
$1/2 - 2/3 + 3/4 - 4/5 + 5/6 - 6/7 + etc.$

III $1 + 2 + 3 + 4 + 5 + 6 + etc.$
$1 + 2 + 4 + 8 + 16 + 32 + etc.$

IV $1 - 2 + 3 - 4 + 5 - 6 + etc.$
$1 - 2 + 4 - 8 + 16 - 32 + etc.$

2. There is much discord among mathematicians concerning such divergent series, as some deny and others affirm that they can have a well-defined sum. Now in the first place it is indeed clear that sums of series which belong to the first type are actually infinite, as by taking enough terms we can arrive at a sum exceeding any given number; whence nothing is in doubt, so that the sums of these series can be denoted by such expressions as $a/0$. Thus controversy among geometers is chiefly attached to the remaining types, and as well, arguments, which are submitted by both sides for their purported viewpoints, embody so much persuasive force that neither side thus far has felt itself compelled to admit any validity to the other.

3. Of the second type is this series, $1 - 1 + 1 - 1 + \ldots.$, first considered by Leibniz, whose sum he gave as equal to $1/2$, with the support of the following fairly sound reasoning: first, this series appears if the fraction $1/(1+a)$ is expanded in the usual way by continued division into the following series $1 - a + a^2 - a^3 + a^4 - a^5 + \ldots$, and the value of the letter a is taken equal to unity. Secondly, to confirm this the more and to persuade those who are not accustomed to the calculation, there is need of the following explanation: if the series is terminated somewhere and the number of terms is made even, then its sum is equal to 0, but if, on the other hand, the number of terms is odd, the sum of the series is equal to 1; now if, therefore, the series is taken to infinity and (consequently) the number of terms cannot be regarded as either even or odd, it cannot be concluded that the sum is either 0 or 1, but we ought to take a certain median value which differs equally from both, namely $1/2$.

4. Against this argument, there is usually put forward the following objection: "First, the fraction $1/(1+a)$ is not equal to the infinite series $1 - a + a^2 - a^3 + a^4 - a^5 + a^6 - etc.$ unless a is a fraction less than unity. If for instance the division

is broken off somewhere and the portion due to the remaining terms is added on, the source of the false reasoning will be revealed; there result $1/(1+a) = 1 - a + a^2 - a^3 + \ldots . \pm a^n \mp a^{n+1}/(1+a)$ and, even if the number n is taken to be infinite, yet the adjoined fraction $\mp a^{n+1}/(1+a)$ cannot be disregarded, unless it really vanishes, which occurs only if $a < 1$ and the series turns out to be convergent. In the other cases it is always necessary to take the remainder $\mp a^{n+1}/(1+a)$ into account, and, although it is prefixed by an ambiguous $\mp$ sign, according as n is even or odd, yet, if n is infinite, the remainder cannot be neglected just because an infinite number is neither even nor odd and thereby provides no criterion for choice of sign. For it is absurd to think that there is a whole number, even an infinite one, which is neither even nor odd." [7]

5. Those who attach sums to divergent series customarily voice the reproach that in this objection an infinite number is conceived as a determinate number and so is accounted as even or odd, when it is indeed indeterminate. For once a series is said to be continued to infinity, it is contrary to this idea if some term of the same series is thought of as last, even if it is infinitesimal. Therefore, the above-noted objection concerning the addition or subtraction of a remainder after the ultimate term disappears of its own accord. Since, therefore, we never reach the end of an infinite series, we never get besides to such a place where it is necessary to add that remainder; accordingly, this same remainder not only can be neglected, but also should be, because nowhere is a place for it found. And these arguments, which are put forward to accept or reject sums of divergent series, also apply to the fourth type, which is usually burdened with problems of its very own.

6. But those, who object to sums of divergent series, are judged to find their firmest support in the third type. For although the terms of these series continually increase and therefore it is possible for the terms to be gathered into a sum greater than an artitrarily given number, and this is the definition of infinite, yet the advocates for sums of such series are forced to admit that these sums are finite and indeed negative, that is less than zero. Namely, as the fraction $1/(1-a)$ yields upon division the series expansion $1 + a + a^2 + a^3 + a^4 + etc.$, we ought to have $-1 = 1 + 2 + 4 + 8 + 16 + etc.$, $-1/2 = 1 + 3 + 9 + 27 + 81 + etc.$, and this is seen by opponents, not undeservedly, to be most absurd, since it is never possible to arrive at a negative sum through the addition of positive numbers. For this reason, they insist all the more on the necessity of adding the remainder mentioned above, as with this inserted it is clear that

$-1 = 1 + 2 + 4 + 8 + \ldots . + 2^n + 2^{n+1}/(1-2)$ even if n is an infinite number.

7. Therefore, the defenders of sums for divergent series, to resolve this remarkable paradox, set up a distinction, subtle enough but scarcely accurate, between negative quantities when they argue that there are some less than zero and others greater than infinity, that is, more than infinite quantities. Namely, we must acknowledge one value for -1 when we think of it as coming from the subtraction of a larger number $a+1$ from a smaller a, and another value when -1 is found to be equal to that series $1 + 2 + 4 + 8 + 16 + etc.$ and comes from the division of $+1$ by -1; in the former case -1 is a number less than zero, in the latter greater than infinity. For greater corroboration, they adduce this instance [of a series] of fractions $1/4, 1/3, 1/2, 1/1, 1/0, 1/-1, 1/-2, 1/-3, etc.$ which, as, by the earlier terms, it is seen to increase, is reckoned to increase continually, whence they infer that $1/-1 > 1/0$ and $1/-2 > 1/-1$ and so on; and thus to the extent that $1/-1$ is expressed by -1 and $1/0$ by infinity ∞ , then $-1 > \infty$ and all the more so $-1/2 > \infty$; and by this convention they expel that apparent absurdity ingeniously enough [8].

8. Although this distinction seems cleverly devised, it little satisfies the adversaries and apparently does violence to the certitude of analysis. For if both those values of -1, insofar as it is either $= 1 - 2$ or $= 1/-1$, really do differ from one another, so that it is incorrect to confuse them, the certitude and the use of the rules we follow in calculations would be completely taken away, and this would certainly be more absurd than that for which the distinction was thought up. But if $1 - 2 = 1/-1$, as the laws of algebra require, the matter is by no means settled, since that very quantity -1 , which is set equal to the series $1 + 2 + 4 + 8 + etc.$, is less than zero; and so the difficulty remains the same. However, it seems in accord with the truth if we say that the same quantities which are less than zero can be considered to be greater than infinity. For not only from algebra but also from geometry, we learn that there are two jumps from positive quantities to negative ones, one through nought or zero, the other through infinity, and that quantities whether increasing from zero or decreasing come back on themselves and return to the same destination 0, so that quantities greater than infinity are thereby less than zero and quantities less than infinity coincide with quantities greater than zero.

9. Those who say the sums which are usually assigned to divergent series are incorrect, not only put forward no alternative but also are determined wholly to resist so much as the imagining of the sum of a divergent series. Indeed, for convergent series as this, for example, $1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + etc.$ only the sum 2 can be permitted, since the more we add terms of this particular series, the closer we approach to two; but for divergent series, the matter is different by far; namely, the more terms we add in, the further the sums which appear differ among

themselves and do not approach some fixed and determined value. Whence they conclude that not even the idea of a sum can be transferred to divergent series, and the work squandered in investigating sums of divergent series is clearly wasted and contrary to the true principles of analysis.

10. Yet however substantial this particular dispute seems to be, neither side can be convicted of any error by the other side, whenever the use of such series occurs in analysis, and this ought to be a strong argument that neither side is in error, but that all disagreement is solely verbal. For if in a calculation I arrive at this series *1 - 1 + 1 - 1 + 1 - 1 etc.* and if in its place I substitute *1/2*, no one will rightly impute to me an error, which however everyone would do had I put some other number in the place of this series. Whence no doubt can remain that in fact the series *1 - 1 + 1 - 1 + 1 - 1 + etc.* and the fraction *1/2* are equivalent quantities and that it is always permitted to substitute one for the other without error. Thus the whole question is seen to reduce to this, whether we call the fraction *1/2* the correct sum of *1 - 1 + 1 - 1 + etc.;* and it is strongly to be feared that those who insist on denying this and who at the same time do not dare to deny the equivalence have stumbled into a battle over words.

11. But I think all this wrangling can be easily ended if we should carefully attend to what follows. Whenever in analysis we arrive at a rational or transcendental expression, we customarily convert it into a suitable series on which the subsequent calculation can more easily be performed. Therefore infinite series find a place in analysis inasmuch as they arise from the expansion of some closed expression, and accordingly in a calculation it is valid to substitute in place of the infinite series that formula from which the series came. Just as with great profit rules are usually given for converting expressions closed but awkward in form into infinite series, so likewise the rules, by whose help the closed expression, from which a proposed infinite series arises, can be investigated, are to be thought highly useful. Since this expression can always be substituted without error for the infinite series, both must have the same value: it follows that there is no infinite series for which the closed expression equivalent to it cannot be conceived.

12. If therefore we change the accepted notion of sum to such a degree that we say the sum of any series is a closed expression out of whose development that series is formed, all difficulties which are stirred up by either side vanish of their own accord. For first that expression from whose expansion a convergent series arises displays the sum, this word being taken in its ordinary sense; and if the series is divergent, the search cannot be thought absurd if we hunt for that closed expression which expanded produces the series according to the rules of analysis. Since it is valid in a calculation to substitute that expression in place of the

series, we cannot doubt that it is equal to it. This established, we do not even depart from ordinary usage if we call that expression which is equal to some series its sum, provided that for divergent series we do not connect this notion with the idea of a sum for which, as more terms are added in, the series should approach nearer to the value of that sum.

III. THE REMAINDER OF EULER'S PAPER: SUMMARY AND COMMENTS

(i) *Sections 13 - 16:* In view of the opinions just expressed, Euler is confident that it is worthwhile to investigate the sum of the hypergeometric series

(1) $1 - 1 + 2 - 6 + 24 - 120 + 720 - 5040 + 40320 - \dots,$

especially since, even "in the geometric case, divergence does not hinder the series from being summable". Before finding the closed expression giving rise to this series, Euler approximates the sum by repeated application of what is now called "Euler's summability method".

If $s = a - b + c - d + e - f + \dots$ is a given series, one calculates the first differences of the terms, neglecting their signs: $b - a$, $c - b$, $d - c$, ... , the second differences $c - 2b + a$, $d - 2c + b$, $e - 2d + c$, and then the differences of higher order. If a, $\alpha = b - a$, $\beta = c - 2b + a$, γ, δ, ... denote the first terms of the respective sequences of differences, then

(2) $$s = a/2 - \alpha/4 + \beta/8 - \gamma/16 + \delta/32 \dots.$$

This method is justified by Euler elsewhere, for example in [Euler, 1755, 222]. He makes s depend on the variable x, thus $s = ax - bx^2 + cx^3 - dx^4 + ex^5 \dots$, and applies the substitution $(1+x)y=x$ or $y=x(1-y)$ to obtain

$$s = ay - (b-a)y^2 + (c - 2b + a)y^3 - (d - 3c + 3b - a)y^4 + \dots$$

Since $y = 1/2$ when $x = 1$, for $x = 1$, s is given by the formula (2). (A modern discussion of this technique is given by Knopp [1947, Sections 33, 35b, 59].)

After giving four examples, Euler takes A to represent the sum of (1), and finds

$$A/2 = 1 - 3 + 12 - 60 + 360 - 2520 + 20160 - 181440 + \dots$$

$$= 1/2 - 2/4 + 7/8 - 32/16 + 181/32 - \dots$$

whence

$$A = 7/4 - 32/8 + 181/16 - 1214/32 + 9403/64 - 82508/128 + \dots$$

so

$$A - 5/16 = 81/128 - 456/512 + 3123/2048 - 24894/8192 + \dots$$

$$= 81/256 - 132/2048 + 771/16384 - 4122/131072 + \dots$$

Thus A is approximately $5/16 + 516/2048 + 2046/131072 = 38015/65536$.

Since the terms of (1) increase in magnitude factorially, each application of Euler's summation technique results in a divergent series [Hardy, 1949, 28, 29, 196], but one which alternates and appears to converge initially and for which the cutoff error is dominated by the first omitted term. This is not the only place in Euler's work in which this type of asymptotic behaviour occurs. Euler specifically draws attention to it in his derivation of a formula for $\pi/4$ [Euler, 1750, 357] and he must have been aware of it in connection with the so-called Euler-Maclaurin Sum Formula. [Euler, 1738, 43; 1741a, 108; 1741b, 124]

(ii) *Sections 17 - 18:* Euler next determines A by extrapolating suitable series. Let P_n be the nth term of the sequence $\{1,2,5,16,65,326,1957,\ldots\}$ where $P_{n+1} = nP_n+1$. He observes that the kth order difference of increment 1, $\Delta^k P_1$, is the kth term in the series $\{1,2,6,24,120,720,\ldots\}$. By what amounts to an application of the formula $P_n = (1+\Delta)^{n-1}P_1$, Euler obtains

$$P_n = 1+(n-1)+(n-1)(n-2)+(n-1)(n-2)(n-3)+(n-1)(n-2)(n-3)(n-4)+\ldots$$

Thus A is equal to P_0.

In order to find P_0, Euler takes for granted that $1/P_0$ is the extrapolation to the zeroth term of the sequence

$$\{1/P_n\} = \{1,\ 1/2,\ 1/5,\ 1/16,\ 1/65,\ \ldots\}.$$

This yields $1/P_0 = 1.6517401$, whence $A = 0.6$. This is somewhat inaccurate because of the negativity of the higher order differences. An improvement is obtained by extrapolating the sequence $\{\log P_n\}$ to obtain $\log P_0 = 0.7779089$, whereupon $A = 0.59966$. Even so, Euler regards this method as being too inaccurate to be effective.

(In fact, Euler's assumption that if the extrapolation formula yields a_0 from the sequence $\{a_1,a_2,a_3,\ldots,a_n,\ldots\}$, it must yield $f(a_0)$ from the sequence $\{f(a_1),f(a_2),\ldots,f(a_n),\ldots\}$ is not well founded, even for an entire function f. For example, take $a_n=n$, $f(z) = (\sin\pi z)/(\pi z)$. This raises the question of the extent to which the formula

$$f(z) = (1+\Delta)^{z-1}f(1) = \sum_{k=0}^{\infty} \frac{f(1)}{k!}(z-1)^{(k)}$$

is valid for a given function f, where Δ is the difference operator of increment 1 and the factorial power $(z-1)^{(k)}$ is equal to 1 when $k=0$ and $(z-1)(z-2)\ldots(z-k+1)$ when k is a positive integer. (It is hoped to deal with this point elsewhere. Cf. [Milne-Thomson, 1933] and [Norlund, 1926]

(iii) *Sections 19 - 20:* Euler considers the function $s = x-x^2+2x^3-6x^4+24x^5-120x^6+\ldots$ which satisfies the differential equation $ds + (sdx)/x^2 = dx/x$. Integration of this yields

$$s = e^{1/x} \int \frac{e^{-1/x}}{x}\, dx$$

whereupon

(3) $$A = e \int_0^1 \frac{e^{-1/x}}{x}\, dx \quad .$$

Euler observes that the integrand is bounded and uses the Trapezoidal Rule to get $A = 0.59637255$.

The integral (3) is also calculated by Euler's use of the substitution

$$v = \exp[1-(1/x)]$$

so that

(4) $$A = \int_0^1 \frac{dv}{1-\log v}$$

By observing that

$$\int \frac{dv}{1-\log v} = \frac{v}{1-\log v} - \frac{1.v}{(1-\log v)^2} + \frac{1.2.v}{(1-\log v)^3} - \frac{1.2.3.v}{(1-\log v)^4} + \frac{1.2.3.4.v}{(1-\log v)^5} - \ldots$$

and setting $v = 1$, Euler verifies (4) and suggests that A can again be found reasonably accurately by an approximate integration.

(iv) *Sections 21 - 25:* Euler observes that the series $1-x+2x^2-6x^3+24x^4-120x^5+720x^6-5040x^7+\ldots$ admits a continued fraction expansion

$$\cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\ldots\ldots}}}}}}}$$

When $x = 1$, he notes that the successive convergents alternatively exceed and fall short of the sum of the hypergeometric series, and approach ever more closely to that sum. The approximation can be improved by taking the arithmetic means of consecutive convergents.

Euler writes

$$A = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{1 \quad \cdots\cdots \quad \cfrac{}{1 + \cfrac{20}{1 + \cfrac{20}{1 + r}}}}}}}$$

where

$$r = \cfrac{21}{1 + \cfrac{21}{1 + \cfrac{22}{1 + \cfrac{22}{1 + \cfrac{23}{1 + \cfrac{23}{1 + \cdots\cdots}}}}}}$$

He approximates r by replacing all integers exceeding 21 by 21 so that r nearly satisfies $r = 21/(1+r)$, i.e. $r \doteq \frac{\sqrt{85}-1}{2}$. This will be too small. In an attempt to do better, Euler defines

$$r = \cfrac{21}{1 + \cfrac{21}{1 + \cfrac{22}{1 + \cfrac{22}{1 + \cdots\cdots}}}} \qquad s = \cfrac{22}{1 + \cfrac{22}{1 + \cfrac{23}{1 + \cdots\cdots}}}$$

$$t = \cfrac{23}{1 + \cfrac{23}{1 + \cfrac{24}{1 + \cfrac{24}{1 + \cdots\cdots}}}}$$

Using these definitions, he expresses r and t in terms of s and, taking $r + t = 2s$ (presumably on intuitive grounds), derives the cubic equation $2s^3 + 2s^2 - 43s - 22 = 0$, which he solves by an approximation method.

This method is similar to the one expounded by Newton [1745]. Observing that the root of $2s^3 + 2s^2 - 43s - 22 = 0$ lies between 4 and 5, Euler takes $s = 4 + u$ and finds $34 = 69u + 26u^2 + 2u^3$. Straight neglect of higher powers of u (as employed by Newton) would lead to $u = 0.49$ which is evidently too high, so Euler scales his estimate of u to 0.4 and takes $u = 0.4 + v$. Putting this into the equation for u gives $2.112 = 90.76v + 28.4v^2 + 2v^3$, and neglecting higher powers of v gives $v = 0.023$, approximately. (To see that this iterative procedure is close to what we now call Newton's method, observe that if p is a polynomial and a is close to a root $a + c$, then $0 = p(a+c) = p(a) + c\,p'(a) + \ldots$ so that for small c, the approximation $c \doteqdot -p(a)/p'(a)$ can be used.)

Since $s \doteqdot 4.423$, r is about 4.31 and the continued fraction expression for A produces a value of 0.5963473621372. Transforming this value in the standard way into a continued fraction allows him to place A between the convergents $653/1095$ and $1600/2683$.

(iv) *Sections 26 - 29:* Euler remarks that he can apply (and elsewhere has applied) similar techniques to more general series and differential equations, in which numerous parameters appear. He finds particularly worthy of note the differential equation shown in facsimile in Figure 1. After giving the solution in series and as a continued fraction, he specializes to $p = m = 1$ and $q = 2$, takes $x = 1$, and finds that $1 - 1 + 1.3 - 1.3.5 + 1.3.5.7 - 1.3.5.7.9. + \ldots = 0.65568$.

NOTES ON THE TRANSLATED PASSAGE

1. *vires vivas:* Leibniz coined the term *vis viva* to denote the force associated with a moving object, as opposed to the force due to a body at rest, the "dead weight" of Galileo. Leibniz argued that such forces were proportional to the square of the velocity of the object, thus contradicting Descartes' view that the "quantity of motion" was proportional to the velocity. The controversy was joined in 1686, when Leibniz published a paper in the Acta Eruditorum on "the memorable error of Descartes". Eventually, Leibniz and John Bernoulli were ranged on one side against such mathematicians as Maclaurin and Stirling on the other. [Jammer, 1957] [Hankins, 1965]

234—235] DE SERIEBUS DIVERGENTIBUS 615

28. Quo hae expressiones fiant simplices neque tamen earum extensioni vis inferatur, ponatur

$$b=1,\quad f=1,\quad m+a=p,\quad m-n=q,$$

ut sit

$$a=p-m \quad \text{et} \quad n=m-q;$$

habebiturque haec aequatio differentialis

$$x^m dx = x^{q+1}dz + (p-m)x^q z dx + z dx,$$

cuius primo integrale est

$$z = e^{1:qx^q} x^{m-p} \int e^{-1:qx^q} x^{p-q-1} dx.$$

Idem porro valor quantitatis z per sequentem seriem infinitam exprimetur

$$z = x^m - px^{m+q} + p(p+q)x^{m+2q} - p(p+q)(p+2q)x^{m+3q} + \text{etc.}$$

Denique huic seriei aequivalebit ista fractio continua

$$z = \cfrac{x^m}{1+\cfrac{px^q}{1+\cfrac{qx^q}{1+\cfrac{(p+q)x^q}{1+\cfrac{2qx^q}{1+\cfrac{(p+2q)x^q}{1+\cfrac{3qx^q}{1+\cfrac{(p+3q)x^q}{1+\text{etc.}}}}}}}}}$$

quae expressio plane congruit cum ea, quam ante § 26 sumus adepti, et quoniam de modo, quo illam eruimus, adhuc dubitari posset, utrum numeratores secundum legem observatam in infinitum progrediantur necne, hoc dubium iam penitus erit sublatum. Suppeditat ergo haec consideratio methodum certam innumerabiles series divergentes summandi seu valores ipsis aequivalentes inveniendi; inter quas ea, quam tractavimus, est casus particularis.

FIGURE 1

2. *logarithms of negative numbers:* In an exchange of letters in 1712 and 1713, John Bernoulli held that $\log(-n) = \log n$ for every natural number n while Leibniz maintained that the logarithms of negative numbers cannot be real. Bernoulli's assertion was questioned also by Euler, who, in 1727, recalled that it was John Bernoulli himself who had found the area of a circular quadrant of radius a to be $a^2 \log\sqrt{-1}/2\sqrt{-1}$, a fact not consistent with $0 = (1/2)\log(-1) = \log\sqrt{-1}$. Shortly after the Bernoulli-Leibniz correspondence was published in 1745, Euler prepared two papers (one published during his lifetime) outlining the controversy and offering his solution [Euler, 1751a, 1862]. It was well into the nineteenth century when Euler's idea of considering the logarithm as a multivalued function was adopted. A detailed account of the whole affair is found in [Cajori, 1913].

3. *curves of the second type:* Euler handled this topic elsewhere [Euler, 1751b]. L'Hopital, in his book *Analyse des infiniment petits*, had observed that the evolute of a curve ABM (Figure 2) had a cusp at D corresponding to the inflection point B, and that, furthermore, both sides of the cusp were concave in the same way. Gua de Malves objected that this was clearly not possible if one took account of the analytic expression for the curve. Referred to its tangent as axis and D as origin, the equation of the evolute is of the form $y = ax^2 + \Sigma bx^m$ where the exponents m exceed 2. Near the point D, the evolute behaves like $y = ax^2$ which does not turn back on itself. Euler remarked that this analysis is acceptable as far as it goes, but that it breaks down if imaginaries are involved. For example, $y = \alpha x^2 + \beta x^2\sqrt{x}$ is not defined for $x < 0$ while it has two values corresponding to the two roots of x for $x > 0$. Thus it will give rise to a "birdbeak" graph. In this way, both L'Hopital and Gua de Malves can be vindicated.

4. Euler's definition of convergent series is unsatisfactory. Of course, he realizes that some series, whose terms approach zero, such as the harmonic series, diverge to infinity. In his study of this series [Euler, 1740, 88] he, in fact, invokes a principle which reads like a non-standard version of the Cauchy Criterion, to wit that a series will converge provided partial sums of terms at infinity are infinitesimal. Observe that convergent and divergent series are distinguished in quite a modern way; the question is one of assigning a sum to a divergent series, and this is seen to be desirable for practical reasons.

5. The use of analytic expressions in defining a sum is quite in keeping with Euler's opinion that variation of one quantity with respect to another can be given by some analytic expression.

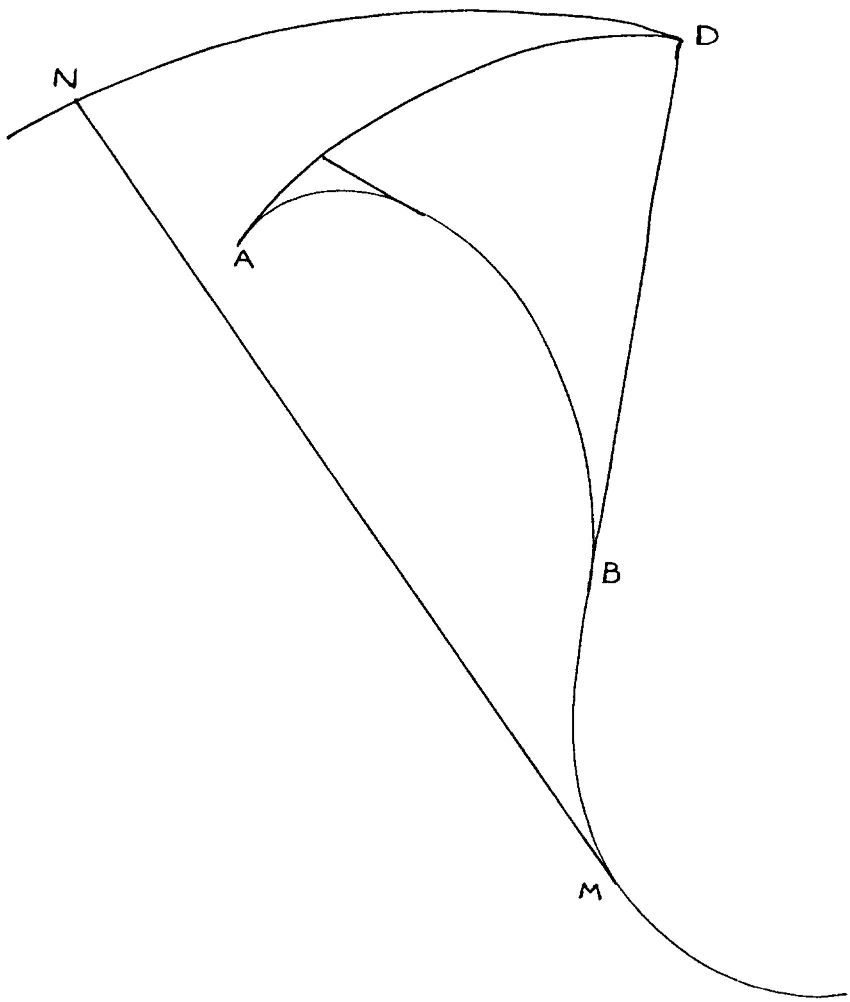

FIGURE 2

Thus he can regard the *n*th term of a series as defined even when n is non-integral or even infinite and he can differentiate the *n*th term with respect to n. There is none of this in the present work, although it permeates many of his other papers on series.

6. The hypergeometric series of Wallis is so called because each term is obtained from its predecessor through multiplication by a ratio which varies from term to term. Carl Boehm, an editor of Euler's complete works [Euler, 1741a, 114] places its appearance in Wallis' work in the Scholium to Proposition 190 of his *Arithmetica infinitorum* (1655).

7. Sections 4 and 5 reveal interesting insights on the perception of the infinite by Euler and his contemporaries. The infinite portion of the series is accessible in that its terms are subject to mathematical analysis, but at the same time endless, so that the remainder term is always beyond reach.

The ambiguity of the parity of the generic infinite integer, which throws the nature of the remainder into question, is not always as critical as here. In [Euler, 1748, 159, 166] he wishes to find the sum of the square reciprocals by factoring

$$\sinh x = \frac{1}{2}\left[\left(1+\frac{x}{n}\right)^n - \left(1-\frac{x}{n}\right)^n\right] \quad (n \text{ infinite}).$$

After observing that the factorization of $a^n - z^n$ contains or does not contain the term $a + z$ according as n is even or odd, he exploits the fact that, when $a = 1 + \frac{x}{n}$, $z = 1 - \frac{x}{n}$, $a + z$ can (if we care to include it) be absorbed into a constant factor and all uncertainty disappears.

8. The business of passing through infinity to get from the positive to the negative numbers is at least as old as Wallis [Scott, 1938, 44-45].

CITED LITERATURE

Bromwich, T J 1931 *An introduction to the theory of infinite series* London

Cajori, Florian 1913 History of the exponential and logarithm concepts *Amer. Math. Monthly* 20, 5-14, 35-47, 75-84, 107-117, 148-151, 173-182, 205-210 [esp. Ch. II-IV]

Euler, L 1738 Methodus generalis summandi progressiones *Commentarii academiae scientiarum Petropolitanae* 6 (1732/33), 68-97 = *Opera Omnia* (1)14 Leipzig and Berlin, 1925, 42-72

________ 1740 De progressionibus harmonicis observationes *Comm. acad. sci. Petrop.* 7 (1734/35), 150-161 = *Opera Omnia* (1)14 Leipzig and Berlin, 1925, 87-100

________ 1741a Inventio summae cuiusque seriei ex dato termino generali *Comm. acad. sci. Petrop.* 8 (1736), 9-22 =

Opera Omnia (1)14 Leipzig and Berlin, 1925, 108-123
________ 1741b Methodus universalis series summandi ulterius promota *Comm. acad. sci. Petrop.* 8 (1736), 147-158 = *Opera Omnia* 1(14) Leipzig and Berlin, 1925, 124-237
________ 1748 *Introductio in analysin infinitorum. Tomus primus* Lausanne = *Opera Omnia* 1(8) Leipzig and Berlin, 1922
________ 1750 Consideratio progressionis cuiusdam ad circuli quadraturam inveniendam idoneae *Comm. acad. sci. Petrop.* 11 (1739), 116-127 = *Opera Omnia* (1)14 Leipzig and Berlin, 1925, 350-363
________ 1751a De la controverse entre Mrs. Leibniz et Bernoulli sur les logarithmes des nombres negatifs et imaginaires *Mémoires de l'académie des sciences de Berlin* [5] (1749), 139-179 = *Opera Omnia* (1)17 Leipzig and Berlin, 1914, 195-232
________ 1751b Sur le point de rebroussement de la seconde espece de M. le Marquis de l'Hôpital *Mem. acad. sci. Berlin* [5] (1749), 203-221 = *Opera Omnia* (1)27 Lausanne, 1954, 236-252
________ 1755 *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum* Acad. Imp. Sci., St. Petersburg = *Opera Omnia* (1)10 Leipzig and Berlin, 1913
________ 1760 De seriebus divergentibus *Novi comm. acad. sci. Petrop.* 5 (1754/55), 19-23, 205-237 = *Opera Omnia* (1)14 Leipzig and Berlin, 1925, 585-617
________ 1862 Sur les logarithmes des nombres negatifs et imaginaires *Opera posthuma* 1 (Petrop.), 269-281 = *Opera omnia* (1)19 Leipzig and Berlin, 1932, 417-438
Gusev, A N 1965 The sources of L. Euler's investigations of divergent series (Russian) Kostrom. Gos. Ped. Inst. Učen. Zap. Vyp. 10, 10-23. MR 40 (1970), #2502
Hankins, Thomas L 1965 Eighteenth century attempts to resolve the *vis viva* controversy *Isis* 56, 281-297
Hardy, C H 1949 *Divergent series* (Oxford)
Jammer, Max 1957 *Concepts of force* (Harper)
Knopp, K 1947 *Theorie und Anwendung der unendlichen Reihen* (4th ed.), Berlin = *Theory and application of infinite series* (2nd ed.), London, Glasgow, New York
Milne-Thomson, L M 1933 *The calculus of finite differences* London
Newton, Isaac 1745 *Of analysis by equations of an infinite number of terms* London = *The mathematical works of Sir Isaac Newton, vol. 1* (ed. Whiteside) New York and London (Johnson Reprints), 1964, 3-25
Nörlund, N E 1926 *Leçons sur les séries d'interpolation* Paris (Gauthier-Villars)
Petrova, S S 1969 On the summation by Euler of $1-1!x + 2!x^2 - 3!x^3 + \dots$ (Russian) *Voprosy Istor. Estestvoznan i. Tehn. Vyp.* 1(26), 30-33 MR 40 (1970), #1245
Scott, J F 1938 *The mathematical work of John Wallis (1616-1703)* London

BIBLIOGRAPHY ON EULER

There does not appear to be a large corpus of English literature devoted to Euler. This short list will provide an introduction to his work; an asterisk indicates that translated portions of Euler's writing are included.

Archibald, R G 1935 Goldbach's theorem *Scripta math.* 3, 44-50

Ayoub, R 1974 Euler and the zeta function *Amer. Math. Monthly* 81, 1067-1086

Ball, W W R 1924 Euler's output, a historical note *Amer. Math. Monthly* 31, 83-84

Bos, H J M 1974 Differentials, higher-order differentials and the derivative in the Leibnizian calculus *Archive for History of Exact Sciences* 14, 1-90

Boyer, C B 1951 The foremost textbook of modern times *Amer. Math. Monthly* 58, 223-226 = *Selected papers on calculus* (M.A.A.), 32-35

Brown, B H 1942 The Euler-Diderot anecdote *Amer. Math. Monthly* 49, 302-303

Brykczynski, Alice 1937 The life and work of Euler *Amer. Math. Monthly* 44, 45-56

Cajori, Florian 1913 History of the exponential and logarithm concepts *Amer. Math. Monthly* 20, 5-14, 35-47, 75-84, 107-117, 148-151, 173-182, 205-210 [esp. Ch. II-IV]

Davis, P J 1959 Leonhard Euler's integral: a historical profile of the gamma function *Amer. Math. Monthly* 66, 849-869

Eberlein, W F 1975(?) On Euler's infinite product for the sine To appear

* Euler, L 1822 *Elements of algebra* Translated from French version by J Hewlett London

________ 1968 The Koenigsberg bridges *Mathematics in the modern world--readings from* Scientific American (ed. M Kline) San Francisco (Freeman), 141-144

Finkel, B 1897 Biography: Leonhard Euler *Amer. Math. Monthly* 4, 297-302

* Forbes, Eric G 1971 *The Euler-Mayer correspondence (1751-1755: a new perspective on eighteenth-century advances in lunar theory* London (Macmillan)

Gillings, R J 1954 The so-called Euler-Diderot incident *Amer. Math. Monthly* 61, 77-80

Grabiner, J V 1974 Is mathematical truth time dependent? *Amer. Math. Monthly* 81, 354-365

Langer, R E 1935 The life of Leonhard Euler *Scripta math.* 3, 61-66, 131-138

________ 1957 An excerpt from the works of Euler *Amer. Math. Monthly* 64, *H. E. Slaught Memorial Papers* 6, 37-44

Luxemburg, W A J 1973 What is nonstandard analysis? *Amer. Math. Monthly* 80, *H. E. Slaught Memorial Papers* 13, 38-67

Polya, G 1941 Heuristic reasoning and the theory of probability *Amer. Math. Monthly* 48, 450-465

* ________ 1954 *Mathematics and plausible reasoning: vol. 1* Princeton, 18, 90-100
* Smith, D E 1959 *A source book in mathematics* New York (Dover) 91, 95, 638
Struik, D J 1940 A story concerning Euler and Diderot *Isis* 31, 431-432
* ________ 1969 *A source book in mathematics, 1200-1800* (Harvard), 31, 36, 40, 99, 183, 345, 383, 399
Truesdell, C 1953 Generalization of a geometric theorem of Euler *Comment. Math. Helvet.* 27, 233-234
________ 1960 *The rational mechanics of flexible or elastic bodies, 1638-1738* Leonhardi Euleri Opera Omnia (2)11(2)
Youschkevitch, A P 1971 Euler *Dictionary of Scientific Biography* 4, 467-484 New York (Scribner)

****

HISTORY OF MATHEMATICS AT TEACHERS' CONFERENCES IN ENGLAND EASTER 1975

Association of Teachers of Mathematics, St. Martin's College, Lancaster. April 1-5. Some twenty participants attended three 90-minute seminars on (1) solution of cubic equations, (ii) perception, perspective and projective geometry, and (iii) calculating and computing devices. The availability of audio-visual and bibliographic materials was explored, and a "resourcefile" was circulated. About 120 teachers attended an evening session of two films, *Numbers Now and Then* and *$\sqrt{2}$, Geometry or Arithmetic,* used in the Open University history of mathematics course. [Information from Leo Rogers, Digby Stuart College, Roehampton Institute of Technology, London, S.W.15]

Mathematical Association, University of East Anglia, Norwich. April 2-5. At one session twelve participants discussed the history of mathematics as a dynamic study rather than a mere collection of facts. Most held that one should employ reasonable accuracy in teaching the subject but should not be obsessed with rigour. Two student-made films (directed by Derick Last, Ely Resource and Technology Centre, Back Hill, Ely, Cambridgeshire) were shown, *The Shadow of the Obelisk* and *Mathematics and the Mediterranean.* The historical contributions in the Association's journal *Mathematics in School* were noted, and some participants stressed the need for a report on the relevance of the history of mathematics to the teaching of mathematics. [Information from David Green, C.A.M.E.T, University of Technology, Loughborough, Leicester.] (See also *HM* 1, 325-326)