On some remarkable identities related to the harmonic zeta function

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Abstract Using Ramanujan's summation method applied to a certain type of divergent series involving harmonic numbers and powers of logarithms, we consider a variety of interesting constants and study their properties. In particular, we clarify their close connection with the classical Stieltjes constants and their analogues for the harmonic zeta function which allows us to deduce several remarkable identities.

Keywords Harmonic numbers; Bernoulli numbers; Stieltjes constants; Gamma function; Harmonic zeta function; Ramanujan summation of divergent series.

1 Introduction

The harmonic zeta function ζ_H is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s} \,,$$

where, for all integers $n \geq 1$,

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

are the classical harmonic numbers. Forty years ago, Apostol and Vu [1] and Matsuoka [12], building upon the foundational work of Euler [11], showed that this function extends as a meromorphic function, featuring a double pole at $s = 1$,

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and simple poles at $s = 0$ and the odd negative integers. The special values of ζ_H at even negative integers are given by Matsuoka's formula:

$$2\zeta_H(-2p) = (1-2p)\zeta(1-2p) = (2p-1)\frac{B_{2p}}{2p} \qquad (p \geq 1)\,,$$

where the B_{2p} are the classical Bernoulli numbers. The Laurent expansion of the harmonic zeta function ζ_H around its double pole can be written as

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\gamma_{H,k}(s-1)^k \quad (0 < |s-1| < 1)\,,$$

where $\gamma = -\Gamma'(1) = 0.577215\ldots$ is Euler's constant, and the coefficents $\gamma_{H,k}$ are referred to as *harmonic Stieltjes constants*, drawing an analogy to the classical Stieltjes constants. A common definition of the Stieltjes constants γ_k for arbitrary k is

$$\gamma_k = \lim_{s\to 1}\left\{(-1)^k\zeta^{(k)}(s) - \frac{k!}{(s-1)^{k+1}}\right\} \qquad (k \geq 0)\,,$$

where $\zeta^{(k)}(s)$ is the kth derivative of the Riemann zeta function [7]. In particular, γ_0 is Euler's constant γ. An explicit expression of $\gamma_{H,0}$ is given by the following formula [6, Equation (6)]:

$$\gamma_{H,0} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) = \frac{1}{2}\Gamma^{(2)}(1) = 0.989055\ldots\,. \tag{1}$$

As noted by Young, this nice expression derives from a special case of a general formula which applies to height one multiple zeta functions of arbitrary depth [13, Equation (28)].

Regarding the classical Stieltjes constants, we recall the well-known asymptotic representation of γ_k (see e.g. [7]):

$$\gamma_k = \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{(\ln n)^k}{n} - \frac{(\ln N)^{k+1}}{k+1}\right\} \qquad (k \geq 0)\,.$$

From the point of view adopted in this study, the constant γ_k can be interpreted as the $\mathcal{R}$-sum of the divergent series $\sum_{n\geqslant 1}\dfrac{(\ln n)^k}{n}$, i.e. the sum of the series in the sense of Ramanujan's summation method, following the exposition in [4] (see also [5, Chapter IX]). Thus, using these notations, we can write [4, p. 67], [5, p. 296]:

$$\sum_{n\geqslant 1}^{\mathcal{R}}\frac{(\ln n)^k}{n} = \gamma_k\,.$$

The existence of a similar asymptotic representation for $\gamma_{H,k}$:

$$\gamma_{H,k} = \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \frac{H_n(\ln n)^k}{n} - \frac{(\ln N)^{k+2}}{k+2} - \gamma \frac{(\ln N)^{k+1}}{k+1} \right\} \qquad (k \geq 0)\,,$$

which results from a generalization of a formula of Briggs and Buschman [2, 3], strongly suggests that the harmonic Stieltjes constant $\gamma_{H,k}$ and the $\mathcal{R}$-sum $\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n\,(\ln n)^k}{n}$ are closely linked. Although much less simple than in the previous case, such a connection exists: as we will see, each harmonic Stieltjes constant $\gamma_{H,k}$ differs from the corresponding $\mathcal{R}$-sum of $H_n(\ln n)^k/n$ by a quantity specified entirely in terms of the Riemann zeta function (see Proposition 1 below). In earlier studies [6, Equation (4)], [8, Equation (5.1)], we have already encountered and focus on the intriguing identity:

$$\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n} = \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2) + \gamma_1 + \tau_1 = \gamma_{H,0} + \tau_1 + \gamma_1 - \zeta(2) = 0.529052\ldots\,, \qquad (2)$$

where

$$\tau_1 := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \zeta(n+1) = 1.257746\ldots\,.$$

it should be noted that this constant τ_1 arises in several contexts in number theory (see e.g. [14, pp. 12–13] for a summary of these occurrences). The main purpose of this article is to determine to what extent this formula can be generalized and the consequences to be drawn. We start with the following definition:

Definition 1. For any positive integer k, the constant τ_k is defined by

$$\tau_k := \sum_{n=1}^{\infty} \frac{(-1)^{n+k}}{n^k} \zeta(n+1)\,.$$

Let us mention that the definition of the constant τ_k given above is, for $k > 1$, different than the definition used in [6] and [8]. We also give another expression of the constant τ_k which can be easily deduced from the previous definition:

$$\tau_k = (-1)^k \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Li}_k\left(-\frac{1}{n}\right) \qquad (k \geq 1)\,,$$

where $\operatorname{Li}_k(z)$ is the polylogarithm function. In particular, for $k = 1$, the known formula

$$\tau_1 = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n}\right)$$

is regained.

2 How the constants τ_k, γ_k and $\gamma_{H,k}$ are linked together

In this section, we clarify the relations between the constants defined above. The forthcoming Proposition 1 successfully generalizes formula (2).

Proposition 1. For any positive integer k, we have the following relations:

$$\begin{aligned}\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n\,(\ln n)^{2k-1}}{n} &= \gamma_{H,2k-1} + (2k-1)!\,\tau_{2k} \\ &\quad + \frac{\gamma_{2k}}{2k} + \sum_{j=1}^{k} \frac{(2k-1)!}{(2k-2j)!} \times \frac{(2^{2j-1}-1)}{2^{2j-2}}\,\gamma_{2k-2j}\,\zeta(2j)\,, \end{aligned}\tag{3}$$

and

$$\begin{aligned}\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n\,(\ln n)^{2k}}{n} &= \gamma_{H,2k} + (2k)!\,\tau_{2k+1} \\ &\quad + \frac{\gamma_{2k+1}}{2k+1} + \sum_{j=1}^{k} \frac{(2k)!}{(2k+1-2j)!} \times \frac{(2^{2j-1}-1)}{2^{2j-2}}\,\gamma_{2k+1-2j}\zeta(2j) \\ &\quad - (2k)!\frac{(2^{2k+1}-1)}{2^{2k}}\zeta(2k+2)\,. \end{aligned}\tag{4}$$

In particular, we have

$$\begin{aligned}\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n\,\ln n}{n} &= \gamma_{H,1} + \tau_2 + \frac{1}{2}\gamma_2 + \gamma\zeta(2)\,, \\ \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n\,(\ln n)^2}{n} &= \gamma_{H,2} + 2\tau_3 + \frac{1}{3}\gamma_3 + 2\gamma_1\zeta(2) - \frac{7}{2}\zeta(4)\,, \\ \sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n\,(\ln n)^3}{n} &= \gamma_{H,3} + 6\tau_4 + \frac{1}{4}\gamma_4 + 3\gamma_2\zeta(2) + \frac{21}{2}\gamma\zeta(4)\,. \end{aligned}$$

Proof of Proposition 1. The key formula to derive the general relations (3) and (4) is the splitting of $\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n^s}$ given by [6, Theorem 1] which is recalled below:

$$\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n^s} = \frac{\pi}{\sin(\pi s)}\,\zeta(s) + \int_0^1 \frac{\psi(x+1)+\gamma}{x^s}\,dx + \zeta_H(s)\,,\tag{5}$$

where $\psi = \Gamma'/\Gamma$ is the digamma function. This formula applies to all complex numbers s such that $\mathrm{Re}(s) < 2$ and $s \neq 1,\ 0,$ and $1-2k$ for each positive integer

k. Fortunately, the Laurent expansion of each component in (5) can be written explicitly.

a) The expansion of $\frac{\pi}{\sin(\pi s)}\zeta(s)$ at $s = 1$ can be obtained as follows: first, we write the successive equations:

$$\begin{aligned}\frac{-\pi}{\sin(\pi s)} = \frac{\pi}{\sin(\pi(s-1))} &= \frac{\exp(i\pi(s-1))}{s-1}\,\frac{2i\pi(s-1)}{\exp(2i\pi(s-1))-1}\\ &= \frac{1}{s-1}\left(\sum_{k\geq 0} i^k\pi^k\frac{1}{k!}(s-1)^k\right)\left(\sum_{k\geq 0} i^k(2\pi)^k\frac{B_k}{k!}(s-1)^k\right)\\ &= \frac{1}{s-1} + \sum_{k\geq 1}(-1)^k\left(\sum_{j=0}^{2k}\binom{2k}{j}2^jB_j\right)\frac{\pi^{2k}}{(2k)!}(s-1)^{2k-1}\,,\end{aligned}$$

where the B_j are the Bernoulli numbers. Euler's identity:

$$\zeta(2k) = (-1)^{k+1}2^{2k-1}B_{2k}\frac{\pi^{2k}}{(2k)!} \qquad (k \geq 1),$$

then allows us to rewrite this expansion as follows:

$$\frac{-\pi}{\sin(\pi s)} = \frac{1}{s-1} - \sum_{k\geq 1}\frac{2^{1-2k}}{B_{2k}}\sum_{j=0}^{2k}\binom{2k}{j}2^jB_j\,\zeta(2k)(s-1)^{2k-1}\,.$$

Morover, the latter expression can be simplified thanks to the identity

$$\sum_{j=0}^{k}\binom{k}{j}2^jB_j = 2^kB_k(\frac{1}{2}) = 2(1-2^{k-1})B_k \qquad (k \geq 2)\,.$$

Hence, the expansion of $\frac{\pi}{\sin(\pi s)}$ at $s = 1$ is given by

$$\frac{\pi}{\sin(\pi s)} = -\frac{1}{s-1} - \sum_{k=1}^{\infty}\frac{2^{2k-1}-1}{2^{2k-2}}\,\zeta(2k)(s-1)^{2k-1}\,.$$

On the other hand, the expansion of $\zeta(s)$ at $s = 1$ is

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\gamma_k(s-1)^k\,,$$

where γ_k are the Stieltjes constants. The expansion of $\dfrac{\pi}{\sin(\pi s)}\,\zeta(s)$ is then obtained by Cauchy product. We have

$$\begin{aligned}\frac{\pi}{\sin(\pi s)}\,\zeta(s) = &-\frac{1}{(s-1)^2} - \frac{\gamma}{(s-1)} + \gamma_1 - \zeta(2)\\ &- \Big(\frac{1}{2}\gamma_2 + \gamma\zeta(2)\Big)(s-1) + \Big(\frac{1}{6}\gamma_3 + \gamma_1\zeta(2) - \frac{7}{4}\zeta(4)\Big)(s-1)^2\\ &- \Big(\frac{1}{24}\gamma_4 + \frac{1}{2}\gamma_2\zeta(2) + \frac{7}{4}\gamma\zeta(4)\Big)(s-1)^3\\ &+ \Big(\frac{1}{120}\gamma_5 + \frac{1}{6}\gamma_3\zeta(2) + \frac{7}{4}\gamma_1\zeta(4) - \frac{31}{16}\zeta(6)\Big)(s-1)^4 - \cdots \qquad (6)\end{aligned}$$

In (6), the coefficient of $(s-1)^{2k}$ and $(s-1)^{2k-1}$ are respectively

$$\frac{\gamma_{2k+1}}{(2k+1)!} + \sum_{j=1}^{k} \frac{\gamma_{2k-2j+1}}{(2k-2j+1)!} \times \frac{(2^{2j-1}-1)}{2^{2j-2}}\,\zeta(2j) - \frac{(2^{2k+1}-1)}{2^{2k}}\zeta(2k+2)\,,$$

and

$$-\frac{\gamma_{2k}}{(2k)!} - \sum_{j=1}^{k} \frac{\gamma_{2k-2j}}{(2k-2j)!} \times \frac{(2^{2j-1}-1)}{2^{2j-2}}\,\zeta(2j)\,.$$

b) The well-known Taylor series expansion of the digamma function:

$$\psi(x+1) + \gamma = \sum_{n=1}^{\infty} (-1)^{n+1}\zeta(n+1)x^n \qquad (|x|<1),$$

allows us write

$$\int_0^1 \frac{\psi(x+1)+\gamma}{x^s}\,dx = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\zeta(n+1)}{n-(s-1)}.$$

This leads to the following expansion:

$$\begin{aligned}\int_0^1 \frac{\psi(x+1)+\gamma}{x^s}\,dx &= \sum_{k=0}^{\infty}\Big(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\zeta(n+1)}{n^{k+1}}\Big)(s-1)^k\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!} \times k!\,\tau_{k+1}\,(s-1)^k \qquad (|s-1|<1). \quad (7)\end{aligned}$$

c) To obtain the expansion of $\displaystyle\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n^s}$ at $s=1$, we proceed as follows: first we write the identities

$$\frac{H_n}{n^s} = \frac{H_n}{n}e^{-(s-1)\ln n} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!} \times \frac{H_n(\ln n)^k}{n}\,(s-1)^k\,,$$

so that

$$\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n^s} = \sum_{n\geqslant 1}^{\mathcal{R}} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \times \frac{H_n (\ln n)^k}{n} (s-1)^k \right) .$$

Moreover, [4, Theorem 9] allows us interchange $\sum_{n\geqslant 1}^{\mathcal{R}}$ and $\sum_{k=0}^{\infty}$. This leads to the expansion:

$$\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n^s} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n (\ln n)^k}{n} \right) (s-1)^k . \tag{8}$$

By combining the previous expansions (6), (7), and (8), we recover our formula (2) and we also obtain the desired formulas (3) and (4). □

3 Explicit expression of $\gamma_{H,k}$

The previously established Proposition 1 will enable us to derive an explicit expression of the constant $\gamma_{H,k}$ for each positive integer k.

If b_n denotes the Bernoulli numbers of the second kind defined by means of their generating function

$$\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} b_n\, x^n = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \cdots \qquad (|x| < 1),$$

then the following formula is well-known [4, Equation 4.29], [10, Equation 16]:

$$\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n} = \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} H_j = \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} = 0.529052\ldots$$

Hence formula (2) can be rewritten as follows:

$$\gamma_{H,0} = \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} - \gamma_1 - \tau_1 + \zeta(2) .$$

More generally, for each natural number k, we derive from [10, Equation (13)] the following identity:

$$\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n (\ln n)^k}{n} = \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} H_j\, (\ln j)^k \qquad (k \geq 0). \tag{9}$$

Using (9) and Proposition 1, we then obtain an expression of $\gamma_{H,k}$ to be compared with a similar (but simpler) formula for γ_k given long ago [9], namely:

$$\gamma_k = \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} (\ln j)^k \qquad (k \geq 0).$$

Corollary 1. For k odd, we have

$$\begin{aligned}\gamma_{H,k} &= \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} H_j \, (\ln j)^k \\ &\quad - \frac{\gamma_{k+1}}{k+1} - k! \, \tau_{k+1} - \sum_{j=1}^{k+1/2} \frac{(2j-1)! \, (2^{2j-1}-1)}{2^{2j-2}} \binom{k}{2j-1} \gamma_{k+1-2j} \zeta(2j) \,, \qquad (10)\end{aligned}$$

and, for k even, we have

$$\begin{aligned}\gamma_{H,k} &= \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} H_j \, (\ln j)^k \\ &\quad - \frac{\gamma_{k+1}}{k+1} - k! \, \tau_{k+1} - \sum_{j=1}^{k/2} \frac{(2j-1)! \, (2^{2j-1}-1)}{2^{2j-2}} \binom{k}{2j-1} \gamma_{k+1-2j} \zeta(2j) \\ &\quad + k! \, \frac{(2^{k+1}-1)}{2^k} \zeta(k+2) \,. \qquad (11)\end{aligned}$$

Example 1.

$$\gamma_{H,1} = \sum_{n=1}^{\infty} \frac{|b_n|}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} H_j \ln j - \frac{1}{2}\gamma_2 - \tau_2 - \gamma\zeta(2) = 0.400761\ldots$$

4 Another possible generalization

In this additional section, we examine the possibility of extending formula (2) by replacing the classical harmonic numbers H_n with the Roman harmonic numbers $H_{n,k}$ [10, 14]. This path was recently explored by Young [13]. More precisely, for an arbitrary $k \geq 1$, let

$$H_{n,k} := \sum_{n \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \, n_2 \cdots n_k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} j^{-k} \,,$$

and

$$\tau_1^{[k]} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \zeta(n+1, \underbrace{1, \ldots, 1}_{k-1}) = \sum_{n=1}^{\infty} \frac{|s(n,k)|}{n!} \ln\left(1 + \frac{1}{n}\right)$$

(where $s(n,k)$ are the Stirling numbers of the first kind), in such a way that $H_{n,1} = H_n$ and $\tau_1^{[1]} = \tau_1$. If $\gamma_1^{[k]}$ denotes the first Stieltjes constant of the height one zeta function of depth k, $\zeta(s, \underbrace{1, \ldots, 1}_{k-1})$, then the following beautiful relation is nothing else than a reformulation of [13, Theorem 6.1 d) and Corollary 5.2]:

$$\sum_{n \geq 1}^{\mathcal{R}} \frac{H_{n,k}}{n} = P_{k+1}(\gamma, -\zeta(2), \ldots, (-1)^k \zeta(k+1)) + \gamma_1^{[k]} + \tau_1^{[k]} = \sum_{n=1}^{\infty} \frac{|b_n|}{n^{k+1}} \,, \qquad (12)$$

where the P_n are the modified Bell polynomials defined by means of their generating function:

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(x_1,\dots,x_n)\,t^n &= \exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right)\\ &= 1 + x_1 t + \frac{1}{2}(x_1^2 + x_2)\,t^2 + \frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3)\,t^3 + \cdots\end{aligned}$$

In the simplest case $k=1$, formula (12) is nothing more than the original formula (2); for $k=2$, since $\gamma_1^{[2]} = \gamma_{H,1} + \zeta'(2)$ and $|s(n,2)| = (n-1)!\,H_{n-1}$, formula (12) translates into the following noteworthy identity:

$$\begin{aligned}\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} \sum_{k=1}^{n} \frac{H_k}{k} &= \gamma_{H,1} + \frac{1}{6}\gamma^3 - \frac{1}{2}\gamma\zeta(2) + \frac{1}{3}\zeta(3)\\ &\quad + \sum_{n=1}^{\infty} \frac{H_{n-1}}{n} \ln\left(1+\frac{1}{n}\right) + \zeta'(2) = 0.512673\dots\,.\end{aligned}$$

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References

[1] T. Apostol and T. H. Vu, Dirichlet series related to the Riemann zeta function, *J. Number Theory* **19** (1984), 85–102.

[2] U. Balakrishnan, Laurent Expansion of Dirichlet Series-II, *Monatshefte für Mathematik*, **103** (1987), 1–5.

[3] W. E. Briggs and R. G. Buschman, The power series coefficients of functions defined by Dirichlet series, *Illinois J. Math.* **5** (1961), 43–44.

[4] B. Candelpergher, *Ramanujan Summation of Divergent Series*, Lecture Notes in Math. 2185, Springer, 2017.

[5] B. Candelpergher, *Les séries divergentes, d'Euler à Ramanujan (Divergent series, from Euler to Ramanujan)*, Calvage & Mounet, Paris, 2023.

[6] B. Candelpergher and M-A. Coppo, Laurent expansion of harmonic zeta functions, *Journal of Mathematical Analysis and Applications* **491** (2020), 124309.

[7] B. K. Choudhury, The Riemann zeta-function and its derivatives, *Proceedings Royal Society London* **450** (1995), 477–499.

[8] M-A. Coppo and B. Candelpergher, On some formulae related to Euler sums, *Lithuanian Mathematical Journal*, 63 (2023), 31–43.

[9] M-A. Coppo, Nouvelles expressions des constantes de Stieltjes, *Expositiones mathematicae*, **17** (1999), 349-358.

[10] M-A. Coppo, Miscellaneous series identities with Cauchy and harmonic numbers, and their interpretation as Ramanujan summation, *Integers* **23** (2023), #A61.

[11] L. Euler, Meditationes circa singulare serierum genus, *Novi Comm. Acad. Sci. Petropol.* **20** (1776), 140–186. Reprinted in Opera Omnia ser. I, vol. 15. B. G. Teubner, Berlin, 1927, pp. 217–267.

[12] Y. Matsuoka, On the values of a certain Dirichlet series at rational integers, *Tokyo Journal of Mathematics* **5** (1982), 399–403.

[13] P. T. Young, Global series for height 1 multiple zeta functions, *European Journal of Mathematics* **9** (2023), 9:99.

[14] P. T. Young, Series of height one multiple zeta functions, *Integers* **24** (2024), #A43.