On some alternating series involving zeta and multiple zeta values

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Journal of Mathematical Analysis and Applications, **475** (2019)

Abstract In this article, we study a class of conditionally convergent alternating series including, as a special case, the famous series $\sum_{n\geqslant 2}(-1)^n\frac{\zeta(n)}{n}$ which links Euler's constant γ to special values of the Riemann zeta function at positive integers. We give several new relations of the same kind. Among other things, we show the existence of a similar relation for the Apostol-Vu harmonic zeta function which have never been noticed before. We also highlight a deep connection with the Ramanujan summation of certain divergent series which originally motivated this work.

Keywords Riemann zeta function; harmonic zeta function; Stirling numbers of the first kind; Stirling numbers of the second kind; Bernoulli numbers; Bernoulli numbers of the second kind; harmonic numbers; multiple zeta values; Ramanujan summation of divergent series.

Mathematics Subject Classification (2010) 11B73; 11B75; 11M06; 11M32; 40G99; 41A58.

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Introduction

The first part of this article is devoted to the study of the conditionally convergent alternating series ν_k defined by

$$\nu_k := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k}\,,$$

where $\zeta(s)$ is the Riemann zeta function and k denotes an integral parameter. By a classical result (cf. [9, p. 66], [16, p. 62]), it is well known that ν_0 is Euler's constant

$$\gamma = \lim_{n\to\infty} \left\{ \sum_{j=1}^{n} \frac{1}{j} - \ln n \right\} = 0.5772156649\ldots\,.$$

This remarkable connection between γ and the special values at positive integers of the Riemann zeta function goes back to Euler's early works on harmonic series [14]. Less famous but yet fairly well-known (cf. [9, p. 93], [17, Eq. (5.1)], [18, Eq. (1.5)]) is the relation

$$\nu_1 = \frac{\gamma}{2} - \frac{1}{2}\ln 2\pi + 1$$

sometimes called Suryanarayana formula. Recently, Blagouchine [7, p. 413] gave a general expression of these series ν_k in the case where k is a positive integer:

$$\begin{aligned}\nu_k = {} & \frac{\gamma}{2} - \frac{\ln 2\pi}{k+1} + \frac{1}{k} \\ & + \sum_{r=1}^{[\frac{k}{2}]} (-1)^r \binom{k}{2r-1} \frac{(2r)!}{r(2\pi)^{2r}} \zeta'(2r) + \sum_{r=1}^{[\frac{k-1}{2}]} (-1)^r \binom{k}{2r} \frac{(2r)!}{2(2\pi)^{2r}} \zeta(2r+1)\,. \quad (1)\end{aligned}$$

This formula seems quite cumbersome but can be much simplified using the functional equation of ζ. After some elementary transformations, we show that equation (1) can be reduced to the following equivalent (but much more pleasant) expression:

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k\,, \qquad (2)$$

where C_k is a rational number (see Proposition 1). Moreover, this expression allows to highlight a deep connection between ν_{2k} and the sum (in the sense of the Ramanujan summation of divergent series) of the series $\sum_{n\geqslant 1} n^{2k} H_n$, where H_n is the nth harmonic number (see Remark 3).

Next, in a second part, we introduce a generalization of these series series ν_k replacing the zeta values by certain multiple zeta values. A natural extension may be defined as follows: for all integers $k \geq -1$ and $p \geq 0$, we consider the class of series $(\nu_{k,p})$ with

$$\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k} \zeta(n, \underbrace{1, \ldots, 1}_{p}),$$

where

$$\zeta(s_1, s_2, \cdots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}},$$

so that the previous series ν_k become $\nu_{k,0}$. Then we establish (see Proposition 2) the following identity which is the main result of this work: we have

$$\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|g_n^{(k+1)}|}{n^{p+1}}, \tag{3}$$

where the rational numbers $g_n^{(k)}$ are defined by

$$g_n^{(k)} := \frac{1}{n!} \sum_{j=1}^{n} \frac{s(n,j)}{j+k} \qquad (k \geq 0, n \geq 1), \tag{4}$$

where $s(n,j)$ denotes the Stirling numbers of the first kind (comprehensive informations on the Stirling numbers of the first and the second kind may be found in [3, 7, 13, 16, 19]). In particular, for $k = 1$, $g_n^{(1)}$ are nothing else than the *Bernoulli numbers of the second kind* [3, 13, 16]. One can prove easily (see Lemma 4) that $g_n^{(k)} = (-1)^{n+1}|g_n^{(k)}|$, so that the rationals numbers $g_n^{(k)}$ alternate in sign. As a special case of equation (3), we derive the following result:

$$\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|g_n^{(k)}|}{n} \qquad (k \geq 0). \tag{5}$$

In the case $k = 1$, we recover the classical Mascheroni's series for γ (cf. [7, p. 406], [16, p. 280]):

$$\gamma = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362\,880} + \cdots.$$

Another notable consequence of formula (3) is the deduction of this nice formula (see Corollary 1):

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12},$$

where $\zeta_H(s)$ denotes the Apostol-Vu harmonic zeta function [4, 5, 6] and γ_1 is the first Stieltjes constant [8, 9].

Finally, in the last section, we highlight a relation between the series ν_k, the Stirling numbers of the second kind $S(n,k)$, and the shifted Mascheroni series σ_r whose study was the main subject of [13] (see Proposition 3 and Example 4).

1 The case of a positive integer

In this section, we focus on the case of a positive integer k and give two independant proofs of our formula (2). More precisely, we prove the following proposition:

Proposition 1. For any positive integer k, we have

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1}(-1)^j \binom{k}{j}\zeta'(-j) + C_k$$

with

$$C_k = \frac{1}{k} + \sum_{j=0}^{k-1}\binom{k}{j}\frac{H_j\,B_{j+1}}{j+1}\,, \tag{6}$$

where H_n are the harmonic numbers,

$$H_0 = 0\,, \quad H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (n \geq 1)\,,$$

and B_n are the Bernoulli numbers defined by means of the exponential generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (|x| < 2\pi)\,.$$

In particular, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_{2r+1} = 0$ for $r \geq 1$.

Proof. We can quite easily derive (2) from (1). Differentiation of the functional equation [5, Eq. (25.4.2)]

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2}$$

leads to the two relations

$$(-1)^r \frac{(2r)!}{2(2\pi)^{2r}}\zeta(2r+1) = \zeta'(-2r) \qquad (r \geq 1),$$

and

$$(-1)^r \frac{(2r)!}{r(2\pi)^{2r}}\zeta'(2r) = -\zeta'(1-2r) + \frac{B_{2r}}{2r}\left(H_{2r-1} - \gamma - \ln 2\pi\right) \qquad (r \geq 1)\,.$$

Substituting these relations into (1) and grouping together the terms under the two symbols Σ, leads to the expression

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k-1}(-1)^j \binom{k}{j}\zeta'(-j) + C_k\,,$$

where the rational constant C_k is given by equation (6). □

Remark 1. Formula (2) can also be rewritten as follows:

$$\nu_k = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} L_j \,,$$

where

$$L_j := \frac{H_j \, B_{j+1}}{j+1} - \zeta'(-j) \qquad (j \geq 0).$$

Furthermore, the numbers $A_j := \exp(L_j)$ are known as the *Bendersky-Adamchik constants* [1, 2].

Example 1. For the first values of k, we have the following relations:

$$\begin{aligned}
\nu_1 &= \frac{\gamma}{2} - \frac{1}{2}\ln 2\pi + 1\,, \\
\nu_2 &= \frac{\gamma}{3} - \frac{1}{2}\ln 2\pi - 2\zeta'(-1) + \frac{2}{3}\,, \\
\nu_3 &= \frac{\gamma}{4} - \frac{1}{2}\ln 2\pi - 3\zeta'(-1) + 3\zeta'(-2) + \frac{7}{12}\,, \\
\nu_4 &= \frac{\gamma}{5} - \frac{1}{2}\ln 2\pi - 4\zeta'(-1) + 6\zeta'(-2) - 4\zeta'(-3) + \frac{47}{90}\,, \\
\nu_5 &= \frac{\gamma}{6} - \frac{1}{2}\ln 2\pi - 5\zeta'(-1) + 10\zeta'(-2) - 10\zeta'(-3) + 5\zeta'(-4) + \frac{167}{360}\,.
\end{aligned}$$

Remark 2. Starting from the Maclaurin series expansion [5, Eq. (25.8.5)]

$$\psi(x+1) + \gamma = \sum_{n=2}^{\infty} (-1)^n \zeta(n) x^{n-1} \qquad (|x| < 1)$$

where $\psi(x)$ denotes the digamma function (i.e. the logarithmic derivative of the Γ-function), and multiplying each side by x^k (with $k \geq 1$), then an integration between 0 and 1 gives

$$\nu_k = \frac{\gamma}{k+1} + \int_0^1 x^k \psi(x+1)\, dx.$$

Thus, it follows from formula (2) that

$$\int_0^1 x^k \psi(x+1)\, dx = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + C_k \qquad (k \geq 1).$$

Remark 3 (Link with the Ramanujan summation: part I)**.** Candelpergher et al. [11, Corollary 1] (see also [9, p. 82]) have established that

$$\sum_{n\geqslant 1}^{\mathcal{R}} H_n = \frac{3}{2}\gamma - \frac{1}{2}\ln 2\pi + \frac{1}{2}\,,$$

and for any positive integer k,

$$\sum_{n\geqslant 1}^{\mathcal{R}} n^k H_n = \left(\frac{1-B_{k+1}}{k+1}\right)\gamma - \frac{1}{2}\ln 2\pi + \sum_{j=1}^{k}(-1)^j\binom{k}{j}\zeta'(-j) + R_k \quad \text{with } R_k \in \mathbb{Q},$$

where the symbol $\sum^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series, i.e. the sum in the sense of Ramanujan's summation method [8, 9, 10, 11]. For an even integer k, we have $B_{k+1} = 0$ and $R_k = C_k + \frac{B_k}{2} - \frac{B_k}{2k}$, then, in view of formula (2), these relations may be translated into the following identities:

$$\sum_{n\geqslant 1}^{\mathcal{R}} H_n = \nu_1 + \gamma - \frac{1}{2},$$

and for $k \geq 1$,

$$\sum_{n\geqslant 1}^{\mathcal{R}} n^{2k} H_n = \nu_{2k} + \zeta'(-2k) + \frac{1-2k}{2}\zeta(1-2k) = \nu_{2k} + \zeta'(-2k) + (2k-1)\frac{B_{2k}}{4k}. \quad (7)$$

In particular, we have

$$\sum_{n\geqslant 1}^{\mathcal{R}} n^2 H_n = \nu_2 + \zeta'(-2) + \frac{B_2}{4} = \nu_2 - \frac{\zeta(3)}{4\pi^2} + \frac{1}{24}.$$

2 The case $k = -1$

The case $k = -1$ behaves differently from the previous case and must be studied separately. Denoting by τ_1 the constant ν_{-1}, we have the following identities [12, p. 142]

$$\tau_1 = \int_0^1 \frac{\psi(x+1)+\gamma}{x}\,dx = \sum_{k=1}^{\infty}\frac{\ln(k+1)}{k(k+1)} = -\sum_{n=2}^{\infty}\zeta'(n) = 1.2577468869\ldots$$

Another interesting representation (communicated by I. V. Blagouchine) is

$$\tau_1 = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\zeta(3/2+ix)}{(1/2+ix)\cosh(\pi x)}\,dx.$$

Furthermore, we can write yet another relation that will be useful in the next section: for any integer $p \geq 0$, let κ_p be the constant defined by

$$\kappa_p := \sum_{n=1}^{\infty}\frac{|b_n|}{n^{p+1}},$$

where b_n are the *Bernoulli numbers of the second kind* defined by way of their generating function

$$\frac{x}{\log(x+1)} = \sum_{n=0}^{\infty} b_n\, x^n = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \frac{3x^5}{160} - \frac{863x^6}{60480} + \cdots$$

The relation

$$g_n^{(1)} = \frac{1}{n!}\sum_{j=1}^{n} \frac{s(n,j)}{j+1} = b_n$$

is well-known [16, p. 267]. We have in particular,

$$\kappa_0 = \sum_{n=1}^{\infty} \frac{|b_n|}{n} = \gamma \quad \text{and} \quad \kappa_1 = \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} = 0.5290529699\ldots$$

The constants κ_1 and τ_1 are linked by the relation [9, Eq. (3.23) p. 105]

$$\kappa_1 = \tau_1 + \gamma_1 + \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2)\,, \tag{8}$$

where γ_1 denotes the first Stieljes constant [5, 8, 9] .

$$\gamma_1 = \lim_{n\to\infty} \left\{ \sum_{j=1}^{n} \frac{\ln j}{j} - \frac{1}{2}\ln^2 n \right\} = -0.07281584548\ldots .$$

In terms of Ramanujan summation, γ_1 is $\sum_{n\geqslant 1}^{\mathcal{R}} \frac{\ln n}{n}$ (cf. [9, p. 67]), whereas κ_1 is $\sum_{n\geqslant 1}^{\mathcal{R}} \frac{H_n}{n}$ (cf. [9, Eq. (4.29) p. 133]).

3 Alternating series involving multiple zeta values

In this section, we consider a more general class of series of the previous type replacing zeta values with certain multiple zeta values. We prove our formula (3) and deduce some interesting consequences.

Proposition 2. For all integers $p \geq 0$ and $k \geq -1$, let

$$\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k}\zeta(n, \underbrace{1,\ldots,1}_{p})\,;$$

then

$$\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|g_n^{(k+1)}|}{n^{p+1}},$$

where the rational numbers $g_n^{(k)}$ are defined by equation (4). In particular, for $p = 0$, we have $\nu_{k,0} = \nu_k$ and thus the identity

$$\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|g_n^{(k)}|}{n} \qquad (k \geq 0),$$

whereas for $k = 0$, we have

$$\nu_{0,p} = \sum_{n=1}^{\infty} \frac{|g_n^{(1)}|}{n^{p+1}} = \sum_{n=1}^{\infty} \frac{|b_n|}{n^{p+1}} = \kappa_p \qquad (p \geq 0).$$

In order to prove Proposition 2, we begin by stating the following lemmas:

Lemma 1. For all integers $j \geq 1$ and $p \geq 0$, we have

$$\int_0^1 \frac{\ln^j(1-x)\,\ln^p(x)}{x}\,dx = (-1)^{j+p}\,j!\,p!\,\zeta(j+1,\underbrace{1,\ldots,1}_{p}). \tag{9}$$

Proof. This follows directly from [19, Eqs. (2.27), (2.28)]. □

Lemma 2. The Stirling numbers of the first kind $s(n, j)$ with fixed $j \geq 1$ admit the (vertical) exponential generating function [3, Eq. (2.8)]

$$\frac{\ln^j(1+x)}{j!} = \sum_{n=j}^{\infty} s(n,j)\frac{x^n}{n!} \qquad (|x| < 1). \tag{10}$$

Lemma 3. For all integers $n \geq 1$ and $p \geq 0$, we have

$$(-1)^p \int_0^1 x^{n-1} \ln^p(x)\,dx = \frac{p!}{n^{p+1}} \tag{11}$$

Proof. This is nothing else than [8, Eq. (41)] in the case where p is an integer. □

Lemma 4. For all integers $n \geq 1$ and $k \geq 0$, we have

$$g_n^{(k)} = \frac{(-1)^{n+1}}{n!} \int_0^1 x^k (1-x)_{n-1}\,dx,$$

where $(z)_n = z(z+1)(z+2)\cdots(z+n-1)$ is the Pochhammer symbol. In particular, this implies that

$$g_n^{(k)} = (-1)^{n+1} |g_n^{(k)}|. \tag{12}$$

Proof. Integration between 0 and 1 of the expansion

$$x^{k-1}x(x-1)\cdots(x-n+1)=\sum_{j=1}^{n}s(n,j)x^{j+k-1}$$

gives the required result. □

Proof of Proposition 2. Using successively formulas (9)–(12) above, we can write the following equalities:

$$\begin{aligned}
\nu_{k,p} &= \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j+k+1}\zeta(j+1,\underbrace{1,\ldots,1}_{p}) \\
&= \frac{(-1)^{p+1}}{p!}\sum_{j=1}^{\infty}\frac{1}{j+k+1}\int_0^1\frac{\ln^j(1-x)}{j!}\frac{\ln^p(x)}{x}\,dx \\
&= \frac{(-1)^{p+1}}{p!}\sum_{j=1}^{\infty}\frac{1}{j+k+1}\int_0^1\left(\sum_{n=j}^{\infty}(-1)^n s(n,j)\frac{x^n}{n!}\right)\frac{\ln^p(x)}{x}\,dx \\
&= \frac{(-1)^{p+1}}{p!}\sum_{j=1}^{\infty}\frac{1}{j+k+1}\sum_{n=j}^{\infty}(-1)^n\frac{s(n,j)}{n!}\int_0^1 x^{n-1}\ln^p(x)\,dx \\
&= -\sum_{j=1}^{\infty}\frac{1}{j+k+1}\sum_{n=j}^{\infty}(-1)^n\frac{s(n,j)}{n!\,n^{p+1}} \\
&= -\sum_{n=1}^{\infty}\frac{(-1)^n}{n!\,n^{p+1}}\sum_{j=1}^{n}\frac{s(n,j)}{j+k+1} \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n!}\sum_{j=1}^{n}\frac{s(n,j)}{j+k+1}\right)\frac{1}{n^{p+1}} \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{g_n^{(k+1)}}{n^{p+1}} = \sum_{n=1}^{\infty}\frac{|g_n^{(k+1)}|}{n^{p+1}}\,.
\end{aligned}$$

This completes the proof. □

Example 2. For the first values of $k \geq -1$, we have the following expansions:

$$\tau_1 = \sum_{n=1}^{\infty} \frac{|g_n^{(0)}|}{n} = 1 + \frac{1}{8} + \frac{5}{108} + \frac{3}{128} + \frac{251}{18000} + \frac{95}{10368} + \cdots,$$

$$\nu_0 = \sum_{n=1}^{\infty} \frac{|g_n^{(1)}|}{n} = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362\,880} + \cdots,$$

$$\nu_1 = \sum_{n=1}^{\infty} \frac{|g_n^{(2)}|}{n} = \frac{1}{3} + \frac{1}{48} + \frac{7}{1080} + \frac{17}{5760} + \frac{41}{25\,200} + \frac{731}{725\,760} + \cdots,$$

$$\nu_2 = \sum_{n=1}^{\infty} \frac{|g_n^{(3)}|}{n} = \frac{1}{4} + \frac{1}{80} + \frac{1}{270} + \frac{11}{6720} + \frac{89}{100\,800} + \frac{5849}{10\,886\,400} + \cdots.$$

Corollary 1. Let ζ_H be the Apostol-Vu harmonic zeta function [4, 5, 6, 11] defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s}.$$

We have the following elegant evaluation:

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{\pi^2}{12} = 0.916240149\ldots. \tag{13}$$

Another expression of this constant is $\zeta''(0) + \frac{1}{2}\ln^2(2\pi) + \frac{\pi^2}{8}$ [5, Eq. (25.6.12)].

Proof. Using Proposition 2 , we can write

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta_H(n)}{n} = \nu_{0,1} - \tau_1 + \zeta(2) = \kappa_1 - \tau_1 + \zeta(2),$$

and thus, formula (13) is derived from equation (8). □

Remark 4 (Link with the Ramanujan summation: part II)**.** For $s \in \mathbb{C}$, let $\zeta_H^{\mathcal{R}}$ be the function $s \mapsto \sum_{n\geqslant 1}^{\mathcal{R}} H_n \, n^{-s}$ where $\sum^{\mathcal{R}}$ stands for the Ramanujan summation. The function $\zeta_H^{\mathcal{R}}$ is an entire function linked to the harmonic zeta function ζ_H by the relation [11, Eq. (84)]

$$\zeta_H^{\mathcal{R}}(s) = \zeta_H(s) - \int_1^{\infty} x^{-s}\left(\psi(x+1) + \gamma\right) dx \quad \text{for } \mathrm{Re}(s) > 1.$$

We have the identities

$$\zeta_H^{\mathcal{R}}(1) = \nu_{0,1} = \kappa_1, \quad \zeta_H^{\mathcal{R}}(0) = \nu_1 + \gamma - \frac{1}{2},$$

and formula (7) may be nicely rewritten

$$\zeta_H^{\mathcal{R}}(-2k) = \zeta_H(-2k) + \zeta'(-2k) + \nu_{2k}.$$

4 Link with the shifted Mascheroni series

Let us consider now the forward shifted Mascheroni series which are defined by

$$\sigma_r := \sum_{n=1}^{\infty} \frac{|b_{n+r}|}{n}, \qquad \text{for } r = 0, 1, 2, \cdots .$$

We have in particular $\sigma_0 = \nu_0 = \gamma$. The study of these series σ_r was the main subject of [13]. Among other things, we have established the following decomposition of $\zeta'(-j)$ on the "basis" of σ_r [13, Proposition 3]:

$$\zeta'(-j) = \sum_{r=2}^{j+1} (-1)^{j-r}(r-1)! S(j, r-1)\sigma_r - \frac{B_{j+1}}{j+1}\gamma - \frac{B_{j+1}}{(j+1)^2}, \qquad \text{for } j = 1, 2, 3, \cdots,$$

where $S(j, r)$ are Stirling numbers of the second kind; moreover, for $j = 0$, we also have a similar relation:

$$-\zeta'(0) = \frac{1}{2}\ln 2\pi = \sigma_1 + \frac{\gamma}{2} + \frac{1}{2}.$$

Then, substituting these relations into (2) enables us to write each series ν_k with $k \geq 1$ as an integral linear combination of γ, $\sigma_1, \sigma_2, \cdots, \sigma_k$ plus a rational number D_k which is closely linked to C_k. In this combination, the coefficient of γ is zero since it is equal to $\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j$ which vanishes by a well-known property of the Bernoulli numbers. Finally, equation (2) may be nicely rewritten in terms of σ_r as follows:

Proposition 3. For all integers $k \geq 1$, we have the relation

$$\nu_k = D_k - \sigma_1 + \sum_{r=2}^{k} (-1)^r (r-1)! \left(\sum_{j=r-1}^{k-1} \binom{k}{j} S(j, r-1) \right) \sigma_r \tag{14}$$

with

$$D_k = \frac{1}{k} + \sum_{j=1}^{k} \binom{k}{j} \frac{B_j H_j}{k+1-j}.$$

Example 3. For the first values of k, we have the following relations:

$$\begin{aligned}
\nu_1 &= \frac{1}{2} - \sigma_1, \\
\nu_2 &= \frac{1}{4} - \sigma_1 + 2\sigma_2, \\
\nu_3 &= \frac{5}{24} - \sigma_1 + 6\sigma_2 - 6\sigma_3, \\
\nu_4 &= \frac{13}{72} - \sigma_1 + 14\sigma_2 - 36\sigma_3 + 24\sigma_4, \\
\nu_5 &= \frac{109}{720} - \sigma_1 + 30\sigma_2 - 150\sigma_3 + 240\sigma_4 - 120\sigma_5.
\end{aligned}$$

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