New insights into Glasser-Manna integrals and the harmonic zeta function

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The Journal of Analysis, **33** (2025)

Abstract Using a new representation of the harmonic zeta function (also known as the Apostol-Vu zeta function), we establish an interesting connection between the special values of this function and a certain type of complex logarithmic integrals previously introduced by Glasser and Manna which proved to be useful in the study of the Laplace transform of the digamma function. To our knowledge, this unexpected connection has not been noticed before.

Keywords Harmonic numbers; Bernoulli numbers; Stieltjes constants; Digamma function; Harmonic zeta function; Glasser-Manna integrals.

1 Introduction

The harmonic zeta function ζ_H is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s} \,,$$

where, for all integers $n \geq 1$,

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

are the classical harmonic numbers. Forty years ago, Apostol and Vu [2], along with Matsuoka [17], building upon the foundational work of Euler [13] and Ramanujan [19], demonstrated that this function extends as a meromorphic function, featuring

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a double pole at $s = 1$, and simple poles at $s = 0$ and the odd negative integers. The special values of ζ_H at positive integers are given by Euler's formula [2, 13, 17]:

$$2\zeta_H(p) = (p+2)\zeta(p+1) - \sum_{k=1}^{p-2} \zeta(k+1)\zeta(p-k) \qquad (p \geq 2)\,,$$

whereas the special values at even negative integers are given by Matsuoka's formula [5, Equation (16)], [17]:

$$2\zeta_H(-2p) = (1-2p)\zeta(1-2p) = (2p-1)\frac{B_{2p}}{2p} \qquad (p \geq 1)\,,$$

where the B_{2p} are the Bernoulli numbers.

The Laurent expansion of the harmonic zeta function ζ_H around its double pole can be written as

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\,\gamma_{H,k}(s-1)^k \quad (0 < |s-1| < 1)\,.$$

The constant $\gamma = -\Gamma'(1)$ is known as the Euler-Mascheroni constant, whereas the coefficents $\gamma_{H,k}$ are referred to as harmonic Stieltjes constants, drawing an analogy to the classical Stieltjes constants. A common definition of the Stieltjes constants γ_k for arbitrary k is

$$\gamma_k = \lim_{s\to 1}\left\{(-1)^k\zeta^{(k)}(s) - \frac{k!}{(s-1)^{k+1}}\right\} \qquad (k \geq 0)\,,$$

where $\zeta^{(k)}(s)$ is the kth derivative of the Riemann zeta function [3, 8]. In particular, γ_0 is nothing else than Euler's constant γ. All these results are classic and well-known.

In the next section, we give a new representation of ζ_H which is a subtle improvement of an earlier formula due to Boyadzhiev, Gadiyar and Padma [5, Theorem 1] by a modification of the path of integration (see Theorem 1). We use this representation to derive a new expression of the harmonic Stieltjes constant $\gamma_{H,k}$ (see Proposition 1) to be compared with the cumbersome formula given by Kargin et al. [15, Equation (21)]. Moreover, we also derive new integral representations of $\zeta(n)$ and $\zeta'(n)$ (see Corollary 1).

The most remarkable consequence of this new representation of ζ_H given by Theorem 1 is that it allows us to establish a connection between the special values of this function at negative integers and a certain type of complex logarithmic integrals introduced earlier by Glasser and Manna [14] (see Definition 1) that proved to be useful in the study of the Laplace transform of the digamma function (see Remark 3). To our knowledge, this surprising connection between ζ_H and these integrals has not been recognized before, and we believe that it deserves to be clarified.

2 New representation of $\zeta_H(s)$

We start by giving a new expression of the harmonic zeta function which is a subtle improvement of [5, Theorem 1].

Theorem 1. *For all complex numbers s in $\mathbb{C} \setminus \mathbb{Z}$, we have*

$$\zeta_H(s) = \pi \cot(\pi s)\, \zeta(s) + \zeta(s+1) + \Gamma(1-s)\Phi(s) \tag{1}$$

with

$$\Phi(s) := \frac{1}{2\pi} \int_{-\pi}^{\pi} ix \left(\mathrm{Log}(1+e^{ix})\right)^{s-1} dx\,, \tag{2}$$

where Log *denotes the principal branch of the complex logarithm.*

Proof. Let $z = \mathrm{Log}(1+e^{ix}) = \ln\left(2\cos(\frac{x}{2})\right) + \frac{ix}{2}$, $-\pi < x < \pi$. When x varies from $-\pi$ to π, the variable z travels the path L defined by the parametric equations $\mathrm{Re}\, z = \ln(2\cos(x/2))$ and $\mathrm{Im}\, z = x/2$. This path extends from $-\infty$ below the line $\mathrm{Im}\, z = 0$, passes through the point $(\ln 2, 0)$, then extends back to $-\infty$ above the line $\mathrm{Im}\, z = 0$. The path L is homotopic to the Hankel contour used in [4] and [5]. Differentiation of the integral representation

$$\zeta(s) = \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1}e^z}{1-e^z}\, dz$$

leads to the following identity:

$$\psi(1-s)\zeta(s) + \zeta'(s) = \frac{\Gamma(1-s)}{2i\pi} \int_L \frac{z^{s-1}e^z \mathrm{Log}(z)}{1-e^z}\, dz \qquad (s \neq 1, 2, 3, \ldots),$$

where $\psi = \Gamma'/\Gamma$ is the digamma function [4, Equation (2.7)]. Writing

$$\frac{1}{2i\pi} \int_L \frac{z^{s-1}e^z \mathrm{Log}(z)}{1-e^z}\, dz = \phi(s) - \Phi(s)\,,$$

with

$$\phi(s) = \frac{1}{2i\pi} \int_L \frac{z^{s-1}e^z}{e^z-1} \mathrm{Log}\left(\frac{e^z-1}{z}\right) dz$$

and

$$\Phi(s) = \frac{1}{2i\pi} \int_L \frac{z^{s-1}e^z}{e^z-1} \mathrm{Log}(e^z-1)\, dz\,,$$

allows us to deduce the following identity:

$$\Gamma(1-s)\phi(s) - \psi(1-s)\zeta(s) - \zeta'(s) = \Gamma(1-s)\Phi(s)\,. \tag{3}$$

By means of the relation [5, Equation (25)]:

$$\zeta_H(s) = \pi \cot(\pi s)\, \zeta(s) + \zeta(s+1) + \Gamma(1-s)\phi(s) - \psi(1-s)\zeta(s) - \zeta'(s)$$

we then derive formula (1) from (3) with the expression of $\Phi(s)$ given by formula (2). □

3 Some remarkable consequences

3.1 New expression of $\gamma_{H,k}$

An explicit expression of the constant $\gamma_{H,0}$ is given by the following formula [6, Equation (6)]:

$$\gamma_{H,0} = \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) = \frac{1}{2}\,\Gamma^{(2)}(1) = 0.989055\ldots \tag{4}$$

This nice expression also derives from a special case of a general formula which applies to height one multiple zeta functions $\zeta(s, 1, \ldots, 1)$ of arbitrary depth [20, Equation (28)]. We can deduce from Theorem 1 an expression of $\gamma_{H,k}$ for each integer $k \geq 1$ in terms of integrals $\mathrm{L}_k = \Phi^{(k)}(1)$. More precisely, we have the following statement:

Proposition 1. For any positive integer k, let L_k, ξ_k, ω_k denote respectively

$$\begin{aligned}
\mathrm{L}_k &:= \Phi^{(k)}(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} ix\,\mathrm{Log}^k(\mathrm{Log}(1+e^{ix}))\,dx\,,\\
\xi_k &:= (-1)^k\,\Gamma^{(k)}(1) = (-1)^k\int_0^{+\infty} e^{-x}\ln^k(x)\,dx\,,\\
\omega_k &:= (-1)^k\zeta^{(k)}(2) = \sum_{n=1}^{\infty}\frac{\ln^k n}{n^2}\,.
\end{aligned}$$

Then we have

$$\mathrm{L}_1 = -\gamma_{H,0} - \gamma_1 - \zeta(2) = -\frac{3}{2}\zeta(2) - \frac{1}{2}\gamma^2 - \gamma_1 = -2.561174\ldots\,, \tag{5}$$

and, for each positive integer n, the following general relations:

$$\begin{aligned}
\gamma_{H,2n-1} = \omega_{2n-1} &+ \frac{1}{2n}\sum_{k=0}^{2n-1}\binom{2n}{k}\xi_k\,\mathrm{L}_{2n-k}\\
&- \frac{\gamma_{2n}}{2n} + \sum_{k=1}^{n}\frac{2(2n-1)!}{(2n-2k)!}\,\gamma_{2n-2k}\,\zeta(2k)\,,
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\gamma_{H,2n} = \omega_{2n} &- \frac{1}{2n+1}\sum_{k=0}^{2n}\binom{2n+1}{k}\xi_k\,\mathrm{L}_{2n+1-k}\\
&- \frac{\gamma_{2n+1}}{2n+1} + \sum_{k=1}^{n}\frac{2(2n)!}{(2n+1-2k)!}\,\gamma_{2n+1-2k}\,\zeta(2k)\\
&- 2(2n)!\,\zeta(2n+2)\,.
\end{aligned} \tag{7}$$

Example 1. Applying formulas (6) and (7) to the simplest case $n = 1$ and using (5), we then obtain

$$\gamma_{H,1} = \omega_1 + \frac{1}{2}\mathrm{L}_2 - \frac{1}{2}\gamma^3 - \frac{1}{2}\gamma_2 - \gamma\gamma_1 + \frac{1}{2}\gamma\zeta(2)\,,$$

and

$$\gamma_{H,2} = \omega_2 - \frac{1}{3}\mathrm{L}_3 - \gamma\mathrm{L}_2 + \frac{1}{2}\gamma^4 - \frac{1}{3}\gamma_3 + \gamma^2\gamma_1 + (2\gamma^2 + 5\gamma_1)\zeta(2) - \frac{1}{4}\zeta(4)\,.$$

Numerical evaluations of L_2 and L_3 are

$$\mathrm{L}_2 = -1.924491\ldots\,, \quad \text{and} \quad \mathrm{L}_3 = 7.158075\ldots$$

This leads to the following numerical evaluations:

$$\gamma_{H,1} = 0.400761\ldots\,, \quad \text{and} \quad \gamma_{H,2} = 0.971304\ldots$$

which match with the computations made by Kargin et al. [15, p. 3].

Proof of Proposition 1. The key formula to derive the relations (5)–(7) above is the splitting of $\zeta_H(s)$ given by formula (1). Fortunately, the Laurent series expansion of each component in (1) can be written explicitly.

a) The Laurent expansions of $\pi\cot(\pi s)$ and $\zeta(s)$ at $s = 1$ are respectively

$$\pi\cot(\pi s) = \frac{1}{s-1} - \sum_{k=1}^{\infty} 2\zeta(2k)(s-1)^{2k-1}\,,$$

and

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty}(-1)^k \frac{\gamma_k}{k!}(s-1)^k\,,$$

where the coefficents γ_k are the Stieltjes constants. The expansion of $\pi\cot(\pi s)\,\zeta(s)$ is then obtained by Cauchy product:

$$\begin{aligned}\pi\cot(\pi s)\,\zeta(s) = {} & \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} - \gamma_1 - 2\zeta(2) - \left(-\frac{1}{2}\gamma_2 + 2\gamma\zeta(2)\right)(s-1) \\ & + \frac{1}{2}\left(-\frac{1}{3}\gamma_3 + 4\zeta(2)\gamma_1 - 4\zeta(4)\right)(s-1)^2 \\ & - \frac{1}{6}\left(-\frac{1}{4}\gamma_4 + 6\zeta(2)\gamma_2 + 12\gamma\zeta(4)\right)(s-1)^3 + \cdots \end{aligned} \tag{8}$$

b) It follows from the definition of ξ_k that the Laurent expansion of $\Gamma(1-s)$ at $s = 1$ is given by

$$\Gamma(1-s) = -\frac{1}{s-1} - \gamma - \sum_{k=2}^{\infty}\frac{\xi_k}{k!}(s-1)^{k-1} \qquad (0 < |s-1| < 1)\,,$$

On the other hand, the function Φ defined by (2) is an entire function of s with $\Phi(1) = 0$, and the definition of L_k as $\Phi^{(k)}(1)$ implies that the Taylor expansion of $\Phi(s)$ at $s = 1$ is given by

$$\Phi(s) = \sum_{k=1}^{\infty} \frac{\mathrm{L}_k}{k!} (s-1)^k .$$

The Laurent expansion of $\Gamma(1-s)\Phi(s)$ then follows by Cauchy product:

$$\begin{aligned}\Gamma(1-s)\Phi(s) = -\mathrm{L}_1 - \left(\frac{1}{2}\mathrm{L}_2 + \gamma \mathrm{L}_1\right)(s-1) \\ + \frac{1}{2}\left(-\frac{1}{3}\mathrm{L}_3 - \gamma\mathrm{L}_2 - \xi_2 \mathrm{L}_1\right)(s-1)^2 \\ - \frac{1}{6}\left(\frac{1}{4}\mathrm{L}_4 + \gamma\mathrm{L}_3 + \frac{3}{2}\xi_2\mathrm{L}_2 + (2\zeta(3) + 3\gamma\zeta(2) + \gamma^3)\mathrm{L}_1\right)(s-1)^3 + \cdots \quad (9)\end{aligned}$$

c) The Taylor expansion of $\zeta(s+1)$ at $s = 1$ can be written as follows:

$$\zeta(s+1) = \zeta(2) + \sum_{k=1}^{\infty} (-1)^k \frac{\omega_k}{k!} (s-1)^k . \quad (10)$$

By assembling equations (8)–(10) above, we first obtain (by identifying the constant term) the relation $\gamma_{H,0} = -\gamma_1 \ - \zeta(2) - \mathrm{L}_1$ which, thanks to (4), gives (5). In the same way, the general formulae (6) and (7) are easily derived by identifying the coefficients of higher degree. □

Remark 1. For $2 \le k \le n$, the constants ξ_n involved in formulae (6)–(7) have a polynomial expression in terms of Euler's constant γ and $\zeta(k)$ [3, 7]. More precisely, let P_n be the polynomials defined, for arbitrary n, by the generating function

$$\sum_{n=0}^{\infty} P_n(x_1, x_2, \ldots, x_n) \frac{t^n}{n!} = \exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m}\right) = 1 + x_1 t + (x_1^2 + x_2)\frac{t^2}{2} + \cdots ,$$

then we have

$$\xi_n = P_n(\gamma, \zeta(2), \ldots, \zeta(n)) \qquad (n \ge 2) .$$

In particular, $\xi_1 = \gamma$, $\xi_2 = \gamma^2 + \zeta(2)$, and

$$\xi_3 = \gamma^3 + 3\gamma\zeta(2) + 2\zeta(3),\ \xi_4 = \gamma^4 + 6\gamma^2\zeta(2) + 8\gamma\zeta(3) + 3\left(\zeta(2)\right)^2 + 6\zeta(4), \text{ etc.}$$

It is worth noting that the polynomial P_n and the nth exponential complete Bell polynomial $\mathbf{Y}_n$ [9, Section 3.3] are linked by the following relation:

$$P_n(x_1, \ldots, x_n) = \mathbf{Y}_n(0!\, x_1, \ldots, (n-1)!\, x_n) .$$

Remark 2. With the notations used in [20], let $\gamma_k^{[2]}$ denote the kth Stieltjes constant of the height 1 zeta function of depth 2 defined for $\mathrm{Re}(s) > 1$ by the absolutely convergent series

$$\zeta(s,1) = \sum_{n>m>0} \frac{1}{n^s\, m}\,.$$

The constants $\gamma_{H,k}$ and $\gamma_k^{[2]}$ are linked by the simple relation [20, Equation (14)]:

$$\gamma_{H,k} = \gamma_k^{[2]} + \omega_k \qquad (k \geq 1)\,,$$

easily derived from the relation $\zeta_H(s) = \zeta(s,1) + \zeta(s+1)$. It follows that the above formulas (6) and (7) also apply to $\gamma_k^{[2]}$ with only a slight modification. Moreover, we can also deduce from a theorem of Ramanujan [8, Theorem 4] the following identity:

$$\omega_k = k! + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_{k+n} \qquad (k \geq 1)\,.$$

3.2 New integral representations of $\zeta(n)$ and $\zeta'(n)$

Applied in a neighborhood of $s = n$ (for any integer $n \geq 2$), formula (1) provides a relation between $\Phi(s)$ and $\zeta_H(s)$ which allows us to derive nice evaluations of $\Phi(n)$ and $\Phi'(n)$.

Proposition 2. For any integer $n \geq 2$, we have

$$\Phi(n) = (-1)^{n-1}(n-1)!\,\zeta(n)\,, \tag{11}$$

and

$$\Phi'(n) = (-1)^{n-1}(n-1)! \times \left\{ \zeta'(n) + \zeta(n)\,\psi(n) - \frac{1}{2}\,n\,\zeta(n+1) + \frac{1}{2}\sum_{k=1}^{n-2} \zeta(n-k)\,\zeta(k+1) \right\}. \tag{12}$$

Proof of Proposition 2. For $n \geq 2$, we can write

$$\pi\cot(\pi s)\,\zeta(s) = \frac{\zeta(n)}{s-n} + \zeta'(n) + \mathrm{O}(s-n)\,,$$

$$\Gamma(1-s) = \frac{(-1)^n}{(n-1)!}\left(\frac{1}{s-n} + (\gamma - H_{n-1})\right) + \mathrm{O}(s-n)\,,$$

and

$$\Gamma(1-s)\,\Phi(s) = \frac{(-1)^n}{(n-1)!}\frac{\Phi(n)}{s-n} + \frac{(-1)^n}{(n-1)!}\left(\Phi'(n) + \Phi(n)(\gamma - H_{n-1})\right) + \mathrm{O}(s-n)\,.$$

By applying formula (1) around $s = n$, we obtain the equation

$$\zeta_H(n) + \mathrm{O}(s-n) = \left(\zeta(n) + \frac{(-1)^n}{(n-1)!}\Phi(n)\right)\frac{1}{s-n}$$
$$+ \frac{(-1)^n}{(n-1)!}\left(\Phi'(n) + \Phi(n)(\gamma - H_{n-1})\right) + \zeta'(n) + \zeta(n+1) + \mathrm{O}(s-n)\,,$$

which allows us to derive the identities $\zeta(n) = \dfrac{(-1)^{n+1}}{(n-1)!}\,\Phi(n)$, and

$$\zeta_H(n) = \frac{(-1)^n}{(n-1)!}\left(\Phi'(n) - \Phi(n)\psi(n)\right) + \zeta\,'(n) + \zeta(n+1)\,.$$

Formulae (11) and (12) then follow using Euler's formula for $\zeta_H(n)$. □

We deduce from Proposition 2, the following corollary:

Corollary 1. For any integer $n \geq 2$, we have

$$\zeta(n) = \frac{(-1)^{n-1}}{(n-1)!} \times \frac{1}{2\pi}\int_{-\pi}^{\pi} ix\,\left(\mathrm{Log}(1+e^{ix})\right)^{n-1}\,dx\,. \tag{13}$$

and

$$\zeta\,'(n) = \frac{1}{2}\,n\,\zeta(n+1) - \zeta(n)\,\psi(n) - \frac{1}{2}\sum_{k=1}^{n-2}\zeta(n-k)\,\zeta(k+1)$$
$$+ \frac{(-1)^{n-1}}{(n-1)!} \times \frac{1}{2\pi}\int_{-\pi}^{\pi} ix\,\mathrm{Log}\left(\mathrm{Log}(1+e^{ix})\right)\left(\mathrm{Log}(1+e^{ix})\right)^{n-1}\,dx\,. \tag{14}$$

Example 2. Applying the previous formulas to the simplest cases $n = 2$ and $n = 3$, we get

$$\zeta(2) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} ix\,\mathrm{Log}(1+e^{ix})\,dx\,,$$
$$\zeta(3) = \frac{1}{4\pi}\int_{-\pi}^{\pi} ix\,\mathrm{Log}^2(1+e^{ix})\,dx\,,$$
$$-\zeta\,'(2) = \zeta(2)(1-\gamma) - \zeta(3) + \frac{1}{2\pi}\int_{-\pi}^{\pi} ix\,\mathrm{Log}\left(\mathrm{Log}(1+e^{ix})\right)\,\mathrm{Log}(1+e^{ix})\,dx\,.$$

3.3 Link with the Glasser-Manna integral

Definition 1. The Glasser-Manna function M [1, 12, 14] is defined by

$$M(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{ix}{\mathrm{Log}\left(e^{-z}(1+e^{ix})\right)}\,dx = \frac{4}{\pi}\int_{0}^{\pi/2} \frac{y^2}{y^2 + \ln^2(2e^{-z}\cos(y))}\,dy\,.$$

This function is analytic for z in the disc $\mathrm{D}(0, \ln 2)$ and verifies the relation

$$M^{(n)}(0) = n!\,\Phi(-n) \qquad (n \geq 0). \tag{15}$$

Remark 3. If $\mathcal{L}\psi(x+1)$ denotes the Laplace transform of the function $x \mapsto \psi(x+1)$, where ψ is the digamma function, then the following identity can be derived from [1, Corollary 2.1]:

$$M(z) = \frac{\gamma}{z} + \mathcal{L}\psi(x+1)(z) + \frac{1}{1-e^{-z}}\Big(z + \mathrm{Log}(1-e^{-z})\Big) \qquad (0 < |z| < \ln 2)\,.$$

Applied in a neighborhood of $s = -n$, formula (1) provides a relation between $\Phi(s)$ and $\zeta_H(s)$ which allows us to give a nice evaluation of $\Phi(-n)$ (and therefore of $M^{(n)}(0)$) for any non-negative integer n.

Proposition 3. We have

$$2\Phi(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{ix}{\mathrm{Log}(1+e^{ix})}\,dx = \ln(2\pi) - \gamma + 1\,, \tag{16}$$

and

$$\begin{aligned} 2\Phi'(0) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{ix\,\mathrm{Log}(\mathrm{Log}(1+e^{ix}))}{\mathrm{Log}(1+e^{ix})}\,dx \\ &= \gamma_1 + \frac{1}{2}\ln^2(2\pi) + \frac{1}{2}\gamma^2 - \gamma - \gamma\ln(2\pi) - \frac{7}{4}\zeta(2) + 2\beta\,, \end{aligned} \tag{17}$$

where β is the linear coefficient in the Laurent expansion of ζ_H at $s = 0$.

Proof of Proposition 3. Around 0, we have the decomposition given by (1):

$$\zeta_H(s) = \zeta(s+1) + \pi\cot(\pi s)\,\zeta(s) + \Gamma(1-s)\Phi(s)\,,$$

and the expansion [6, Equation (8)]

$$\zeta_H(s) = \frac{1}{2s} + \frac{1}{2}\gamma + \frac{1}{2} + \beta s + \mathrm{O}(s^2)\,.$$

On the other side, we have the expansions

$$\begin{aligned} \zeta(s+1) &= \frac{1}{s} + \gamma - \gamma_1\,s + \mathrm{O}(s^2)\,, \\ \pi\cot(\pi s)\,\zeta(s) &= -\frac{1}{2s} - \frac{1}{2}\ln(2\pi) + (\frac{1}{2}\gamma_1 - \frac{1}{4}\ln^2(2\pi) + \frac{1}{4}\gamma^2 + \frac{7}{8}\zeta(2))s + \mathrm{O}(s^2)\,, \\ \Gamma(1-s)\Phi(s) &= \Phi(0) + (\Phi'(0) + \gamma\Phi(0)) + \mathrm{O}(s^2)\,. \end{aligned}$$

By identifying the constant coefficient in the expansion of the right-hand side of (1), we deduce the equation

$$\frac{1}{2}\gamma + \frac{1}{2} = \gamma - \frac{1}{2}\ln(2\pi) + \Phi(0)\,,$$

which is equivalent to (16). In the same way, formula (17) is derived by identifying the linear coefficient in the expansion of the right-hand side of (1). □

Remark 4. The evaluation of $M(0)$ deduced from equation (16) was first proved by Oloa [18].

Remark 5. The constant β which occurs in equation (17) can be evaluated using [20, Corollary 4.2]; more precisely, we have

$$\beta = 1 + \frac{1}{2}\gamma - \frac{1}{4}\gamma^2 - \gamma_1 + \frac{1}{4}\zeta(2) - \sum_{n=2}^{\infty}\frac{|b_n|}{(n-1)^2} = 1.589935\ldots\,,$$

where the b_n are the Bernoulli numbers of the second kind defined by means of their generating function

$$\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} b_n\, x^n \qquad (|x|<1).$$

Proposition 4. We have

$$\Phi(-1) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{ix}{\operatorname{Log}^2(1+e^{ix})}\,dx = \ln(\mathcal{A}) - \frac{1}{12}\gamma + \frac{7}{24}\,, \tag{18}$$

where $\ln(\mathcal{A}) = \dfrac{1}{12} - \zeta'(-1)$ and, more generally, for all integers $n \geq 2$, we have

$$(2n-1)!\,\Phi(1-2n) = \ln(\mathcal{A}_n) - \frac{B_{2n}}{2n}\gamma + \sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{B_k\,B_{2n-k}}{(2n-k)^2}\,, \tag{19}$$

with $\ln(\mathcal{A}_n) := \dfrac{H_{2n-1}B_{2n}}{2n} - \zeta'(1-2n)$.

Remark 6. The constant $\mathcal{A}$ is referred to as the Glaisher-Kinkelin constant, denoted by A, in the literature, whereas the constants $\mathcal{A}_n$ are sometimes called *generalized Glaisher-Kinkelin constants* (see [10, 16] for more details on these constants).

Example 3. In particular,

$$\Phi(-3) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{ix}{\operatorname{Log}^4(1+e^{ix})}\,dx = \frac{1}{6}\ln(\mathcal{A}_2) + \frac{1}{720}\gamma + \frac{1}{320}\,.$$

Proof of Proposition 4. Around $s=-1$, we have the expansion [6, Equation (9)]

$$\zeta_H(s) = -\frac{1}{12(s+1)} - \frac{1}{12}\gamma - \frac{1}{8} + \mathrm{O}(s+1)\,.$$

On the other side, we have

$$\begin{aligned} \zeta(s+1) &= -\frac{1}{2} + \mathrm{O}(s+1)\,,\\ \pi\cot(\pi s)\,\zeta(s) &= \zeta'(-1) + \mathrm{O}(s+1)\,,\\ \Gamma(1-s)\Phi(s) &= \Phi(-1) + \mathrm{O}(s+1)\,. \end{aligned}$$

Formula (18) is then deduced by identifying the constant coefficient in the expansion of the right-hand member of (1). In the same way, from [6, Proposition 2], the Laurent expansion around $s = 1-2n$ for $n \geq 2$ is given by

$$\zeta_H(s) = \frac{\zeta(1-2n)}{s+2n-1} - \frac{B_{2n}}{2n}\gamma + \frac{H_{2n-1}B_{2n}}{2n} + \sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{B_k\,B_{2n-k}}{(2n-k)^2} + \mathrm{O}(s+2n-1)$$

which, by the same method, allows us to deduce formula (19). □

Proposition 5. For all positive integers n, we have

$$(2n)!\,\Phi(-2n) = -\zeta'(-2n) + (2n+1)\frac{B_{2n}}{4n}\,, \tag{20}$$

from which follows the reflection formula:

$$2(2n)!\,\Phi(-2n) = (-1)^{n+1}(2\pi)^{-2n}\,\Phi(2n+1) + (2n+1)\frac{B_{2n}}{2n}\,. \tag{21}$$

Example 4. In particular,

$$2\Phi(-2) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{ix}{\mathrm{Log}^3(1+e^{ix})}\,dx = -\zeta'(-2n) + \frac{1}{8} = \frac{1}{4\pi^2}\,\zeta(3) + \frac{1}{8}\,.$$

Proof of Proposition 5. Around $s=-2n$, we have by Matsuoka's formula

$$\zeta_H(s) = (2n-1)\frac{B_{2n}}{4n} + \mathrm{O}(s+2n)\,.$$

On the other side, we have the expansions

$$\begin{aligned} \zeta(s+1) &= -\frac{B_{2n}}{2n} + \mathrm{O}(s+2n)\,,\\ \pi\cot(\pi s)\,\zeta(s) &= \zeta'(-2n) + \mathrm{O}(s+2n)\,,\\ \Gamma(1-s)\Phi(s) &= (2n)!\Phi(-2n) + \mathrm{O}(s+2n)\,. \end{aligned}$$

By identifying the constant coefficient in the expansion of the right-hand member of (1), we obtain formula (20). Moreover, the well-known identity

$$-2\zeta'(-2n) = (-1)^{n+1}\,(2\pi)^{-2n}\,(2n)!\,\zeta(2n+1)$$

and the relation $(2n)!\,\zeta(2n+1) = \Phi(2n+1)$ given by (11) enables to deduce formula (21) from (20). □

4 Evaluation of Glasser-Manna integrals by means of shifted Mascheroni series

In this additional section, we reinterpret some of the previous formulas in terms of shifted Mascheroni series. More precisely, for any positive integer k, we consider the series

$$\sigma_k := \sum_{n=k+1}^{\infty} \frac{|b_n|}{n-k},$$

where the b_n are the Bernoulli numbers of the second kind defined by means of their generating function

$$\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} b_n\, x^n = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \cdots$$

These series σ_k, called *shifted Mascheroni series*, have been studied in detail in [11]. The nice identity

$$\sigma_1 = \frac{1}{2}\ln(2\pi) - \frac{1}{2}\gamma - \frac{1}{2}$$

is well-known [11, Proposition 2] and allows us to deduce from formula (16) an evaluation of $\Phi(0)$ in terms of σ_1:

$$M(0) = \Phi(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{ix}{\mathrm{Log}(1+e^{ix})}\, dx = \sigma_1 + 1 = 1.13033\ldots$$

This evaluation was first given by Glasser and Manna [14, Proposition 3.3] in a slightly different but equivalent form. More generally, the following formula:

$$\sum_{k=1}^{n} (-1)^{n+k} k!\, S(n,k)\, \sigma_{k+1} = -\zeta'(-n) - \frac{B_{n+1}}{n+1}\left(\gamma + \frac{1}{n+1}\right) \qquad (n \geq 1),$$

with $S(n,k)$ denoting the Stirling numbers of the second kind [11, Proposition 3], allows us to give reinterpretations of formulas (18)–(20) in terms of shifted Mascheroni series σ_k. In this way, we obtain

$$\begin{aligned}(2n-1)!\,\Phi(1-2n) &= M^{(2n-1)}(0) \\ &= \sum_{k=1}^{2n-1} (-1)^{k+1} k!\, S(2n-1,k)\, \sigma_{k+1} + \frac{B_{2n}\, H_{2n}}{2n} + \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{B_k\, B_{2n-k}}{(2n-k)^2}\end{aligned} \qquad (n \geq 1), \quad (22)$$

and

$$(2n)!\,\Phi(-2n) = M^{(2n)}(0) = \sum_{k=1}^{2n} (-1)^k k!\, S(2n,k)\, \sigma_{k+1} + (2n+1)\,\frac{B_{2n}}{4n} \qquad (n \geq 1). \quad (23)$$

In particular, we have

$$M'(0) = \Phi(-1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}^2(1+e^{ix})}\, dx = \sigma_2 + \frac{5}{12} = 0.49232\ldots$$

and

$$M''(0) = 2\Phi(-2) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{ix}{\operatorname{Log}^3(1+e^{ix})}\, dx = 2\sigma_3 - \sigma_2 + \frac{1}{8} = 0.15544\ldots$$

Acknowledgements The authors are grateful to the reviewers for their relevant comments and valuable suggestions.

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