Generalized Glaisher-Kinkelin constants and Ramanujan summation of series

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Abstract We study a sequence of constants known as the Bendersky-Adamchik constants which appear quite naturally in number theory and generalize the classical Glaisher-Kinkelin constant. Our main initial purpose is to elucidate the close relation between the logarithm of these constants and the Ramanujan summation of certain divergent series. In addition, we also present a remarkable, and previously unknown, expansion of the logarithm of these constants in convergent series involving the Bernoulli numbers of the second kind.

Keywords Bendersky-Adamchik constants; Bendersky's generalized gamma functions; Hurwitz zeta function; Bernoulli numbers of the second kind; Ramanujan summation of series.

1 Introduction: from Stirling to Bendersky and beyond

The story of the constants known today as the Bendersky-Adamchik constants begins with the famous Stirling formula for the factorial:

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \quad \text{as} \quad n \to \infty$$

which dates back from the middle of the 18th century. In fact, Stirling never explicitly stated this formula. The first appearance of this result occurs in a letter from Euler to Goldbach dated June 1744. The constant $\sqrt{2\pi}$ is commonly referred to as the Stirling constant.

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A similar but less well-known formula, due to Glaisher, also applies to the hyperfactorial:

$$\prod_{\nu=1}^{n} \nu^{\nu} = 1^1 2^2 \cdots n^n \sim A\, n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\, e^{-\frac{n^2}{4}} \quad \text{as} \quad n \to \infty\,,$$

where the constant $A = 1.282427\ldots$ is the Glaisher-Kinkelin constant [9, Prop. 11].

In an important article published in 1933, Bendersky [2] studied the product $\prod_{\nu=1}^{n} \nu^{\nu^k}$ for $k = 0, 1, 2, \ldots$ which reduces to the classical factorial when $k = 0$, and to the hyperfactorial when $k = 1$. For his purpose, Bendersky introduced a natural generalization Γ_k of the classical Γ-function, defined on the positive real axis, whose fundamental properties are $\Gamma_k(1) = 1$ and

$$\Gamma_k(x+1) = x^{x^k}\Gamma_k(x) \quad \text{for all } x > 0.$$

In particular, for each positive integer n,

$$\Gamma_k(n+1) = \prod_{\nu=1}^{n} \nu^{\nu^k} \quad \text{for} \quad k = 0, 1, 2, \ldots$$

Note that $\Gamma_0 = \Gamma$, and Γ_1 is the Kinkelin hyperfactorial K-function [11].

Bendersky showed, for any integer $k \geq 0$, the existence of a constant A_k and a couple of polynomials P_k and Q_k of degree $k+1$ such that

$$\Gamma_k(x+1) \sim A_k\, x^{P_k(x)} e^{-Q_k(x)} \quad \text{as} \quad x \to +\infty\,.$$

Using the Euler-Maclaurin summation formula, Bendersky managed to evaluate the constants A_k and the polynomials P_k and Q_k for $k \leq 4$.

Unaware of Bendersky's work and following an idea of Milnor, Kurokawa and Ochiai [10, Thm. 2] rediscovered much more later the function Γ_k and expressed appropriately this function in terms of the derivative of the Hurwitz zeta function $\zeta(s, x)$ at $s = -k$. More precisely, they established the following representation[1]

$$\ln \Gamma_k(x) = \zeta'(-k, x) - \zeta'(-k) \quad \text{for } x > 0 \text{ and } k \geq 0\,,$$

which extends a classical representation due to Lerch in the case $k = 0$ [6, p. 78].

Our initial interest in these constants comes from their interesting interpretation in terms of the Ramanujan summation of certain divergent series, as explained

1. According to Kellner [9, Rem. 27], this expression of $\ln \Gamma_k$ is due to Alexeiewsky in the special case where $x = n + 1$ is an integer.

in Section 4. Thanks to the properties of the function $\ln\Gamma_k$, we are in a position to completely elucidate the link between the logarithm of the constant A_k and the sum of the divergent series $\sum_{n\geq 1} n^k \ln n$ in the sense of Ramanujan's summation method, following the exposition in [4]. More precisely, using the notations of [4], we show (see Theorem 1) that for all integers $k \geq 0$,

$$\ln A_k = \sum_{n\geq 1}^{\mathcal{R}} n^k \ln n + \frac{H_k B_{k+1}}{k+1} + \frac{1}{(k+1)^2}\,,$$

where, in this expression, H_k and B_k denote respectively the kth harmonic number and the kth Bernoulli number.

2 Bendersky-Adamchik constants

Definition 1. The sequence of real numbers $\{A_k\}_{k\geq 0}$ can be defined as follows [13, Eq. (1.1)–(1.6)]: for all integers $k \geq 0$,

$$\ln A_k := \lim_{n\to\infty} \left\{ \sum_{\nu=1}^{n} \nu^k \ln \nu - P_k(n) \ln n + Q_k(n) \right\}\,, \tag{1}$$

where (P_k, Q_k) is a couple of polynomials of degree $k+1$ with rational coefficients such that

$$(P_0, Q_0) = (x + \frac{1}{2}, x)\,,$$
$$(P_1, Q_1) = (\frac{x^2}{2} + \frac{x}{2} + \frac{1}{12}, \frac{x^2}{4})\,,$$

and whose general expression is given for $k \geq 2$ by

$$P_k(x) = \frac{x^{k+1}}{k+1} + \frac{x^k}{2} + \sum_{r=1}^{[\frac{k+1}{2}]} \frac{B_{2r}}{(2r)!} \left(\prod_{j=1}^{2r-1} (k-j+1) \right) x^{k+1-2r}\,,$$

and

$$Q_k(x) = \frac{x^{k+1}}{(k+1)^2} - \sum_{r=1}^{[\frac{k+1}{2}]+\frac{(-1)^k-1}{2}} \frac{B_{2r}}{(2r)!} \left\{ \prod_{j=1}^{2r-1} (k-j+1) \sum_{j=1}^{2r-1} \frac{1}{k-j+1} \right\} x^{k+1-2r}\,,$$

where $\{B_{2r}\}_{r\geq 0}$ is the sequence of (even) Bernoulli numbers. The numbers A_k (for $k = 0, 1, 2, \ldots$) are called the *Bendersky-Adamchik constants* because Bendersky introduced these constants for the first time in 1933 [2], and Adamchik [1] rediscovered them more than 60 years later, giving a nice expression of $\ln A_k$ in terms of the derivatives of the Riemann zeta function [1, Prop. 4]. More precisely, this expression (called Adamchik's formula in the remainder of this article) is the following:

$$\ln A_k = \frac{H_k B_{k+1}}{k+1} - \zeta'(-k)\,, \tag{2}$$

where H_k and B_k denote respectively the kth harmonic number (with the usual convention $H_0 = 0$) and the kth Bernoulli number. In particular, A_0 is the Stirling constant, and $A_1 = A$ is the Glaisher-Kinkelin constant [1, 9, 12, 13].

3 Adamchik's formula revisited

The following identities:

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \qquad (k \geq 1)\,,$$

and

$$\zeta'(1-2k) = (-1)^{k+1} \frac{(2k)!}{k(2\pi)^{2k}} \zeta'(2k) + \frac{B_{2k}}{2k} \left(H_{2k-1} - \gamma - \ln 2\pi\right) \qquad (k \geq 1)\,,$$

where $\gamma = 0.577215\cdots$ is Euler's constant, are easily derived by differentiation of the functional equation of the zeta function [12, p. 384]. Furthermore, combined with the famous Euler formula [6, p. 17]

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k} \qquad (k \geq 1),$$

these identities allow us to rephrase formula (2) as follows:

$$\ln A_{2k} = \frac{B_{2k}}{4} \left(\frac{\zeta(2k+1)}{\zeta(2k)} \right) \qquad (k \geq 1)\,, \tag{3}$$

and

$$\ln A_{2k-1} = \frac{B_{2k}}{2k} \left(\gamma + \ln 2\pi - \frac{\zeta'(2k)}{\zeta(2k)} \right) \qquad (k \geq 1)\,. \tag{4}$$

Thus, Adamchik's formula means that $\ln A_k$ is closely related to $\zeta(k+1)/\zeta(k)$ when k is positive and even, and to $\zeta'(k+1)/\zeta(k+1)$ in the odd case. For example,

$$\ln A = \frac{1}{12} \left(\gamma + \ln 2\pi - \frac{\zeta'(2)}{\zeta(2)} \right) \quad \text{and} \quad \ln A_2 = \frac{1}{24} \left(\frac{\zeta(3)}{\zeta(2)} \right)\,.$$

4 Link with the Ramanujan summmation of series

Formula (1) strongly suggests that the constant $\ln A_k$ is closely related to the $\mathcal{R}$-sum of the divergent series $\sum_{n\geq 1} n^k \ln n$ (i.e. the sum of the series in the sense of Ramanujan's summation method, following the notations and exposition in [4]). More precisely, we have the following nice result:

Theorem 1. For all integers $k \geq 0$,

$$\ln A_k = \sum_{n\geq 1}^{\mathcal{R}} n^k \ln n + \frac{H_k B_{k+1}}{k+1} + \frac{1}{(k+1)^2}\,. \tag{5}$$

Proof. For any real number x with $x > -1$ and integer $k \geq 0$, let us consider the function

$$\varphi_k(x) := \ln \Gamma_k(x+1) = \zeta'(-k, x+1) - \zeta'(-k).$$

The function φ_k satisfies both $\varphi_k(0) = 0$ and the difference equation

$$\varphi_k(x) - \varphi_k(x-1) = x^k \ln x \qquad (x > 0)\,.$$

Thus, from [4, Eq. (1.30)], we can write the identity

$$\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n = \int_0^1 \varphi_k(x)\,dx = \int_0^1 \ln \Gamma_k(x+1)\,dx\,.$$

We then make use of the following identity [2, Eq. (V'$_k$), p. 280]:

$$\int_0^1 \ln \Gamma_k(x+1)\,dx = \ln A_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2}$$

which allows us to deduce formula (5). □

We also derive from (5) and Adamchik's formula (2) the following elegant corollary:

Corollary 1. For any $k \geq 0$,

$$\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n = -\zeta'(-k) - \frac{1}{(k+1)^2}\,. \tag{6}$$

Example 1.

$$\begin{aligned}
\sum_{n\geq 1}^{\mathcal{R}} \ln n &= \frac{1}{2}\ln 2\pi - 1 = -\zeta'(0) - 1\,,\\
\sum_{n\geq 1}^{\mathcal{R}} n\ln n &= \ln A - \frac{1}{3} = \frac{1}{12}\left(\gamma + \ln 2\pi - \frac{\zeta'(2)}{\zeta(2)} - 4\right)\,,\\
\sum_{n\geq 1}^{\mathcal{R}} n^2\ln n &= \ln A_2 - \frac{1}{9} = \frac{1}{24}\left(\frac{\zeta(3)}{\zeta(2)}\right) - \frac{1}{9}\,.
\end{aligned}$$

Remark 1. The following similar formulas can also be shown (cf. [4, p. 82]):

$$\begin{aligned}
\sum_{n\geqslant 1}^{\mathcal{R}} H_n &= \frac{3}{2}\gamma - \frac{1}{2}\ln 2\pi + \frac{1}{2}\,,\\
\sum_{n\geqslant 1}^{\mathcal{R}} nH_n &= \frac{5}{12}\gamma - \frac{1}{2}\ln 2\pi + \ln A + \frac{19}{24}\,,\\
\sum_{n\geqslant 1}^{\mathcal{R}} n^2H_n &= \frac{1}{3}\gamma - \frac{1}{2}\ln 2\pi + 2\ln A - \ln A_2 + \frac{13}{24}\,,
\end{aligned}$$

and more generally, for any integer $k \geq 2$,

$$\sum_{n\geqslant 1}^{\mathcal{R}} n^kH_n = \left(\frac{1-B_{k+1}}{k+1}\right)\gamma + \sum_{j=0}^{k}(-1)^{j+1}\binom{k}{j}\ln A_j + \frac{1}{k} + c_k\,,$$

with

$$c_k = \begin{cases} (k-1)\dfrac{B_k}{2k} & \text{if } k \text{ is even}\,,\\ \displaystyle\sum_{j=0}^{k}\binom{k}{j}\frac{B_j\,B_{k+1-j}}{(k+1-j)^2} & \text{if } k \text{ is odd.}\end{cases}$$

5 New expansion of $\ln A_k$ in convergent series

In his seminal work, Bendersky [2, pp. 295–299] presented two different (cumbersome) expansions of the logarithm $L_k = \ln A_k$ in convergent series. In this section, we give a new one, of a completely different kind, involving a convergent series with only rational terms.

For convenience, we first introduce the sequence of positive rational numbers $\{\lambda_n\}_{n\geq 1}$ (called non-alternating Cauchy numbers in [5]) defined by

$$\lambda_n := \left|\sum_{k=1}^{n}\frac{s(n,k)}{k+1}\right| \qquad (n \geq 1)\,,$$

where $s(n,k)$ denotes the (signed) Stirling numbers of the first kind. The first ones are the following:

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

The numbers λ_n are closely related to the Bernoulli numbers of the second kind b_n [8] defined by means of their generating function

$$\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} b_n\, x^n \qquad (|x|<1),$$

through the simple relation

$$\lambda_n = (-1)^{n-1} n!\, b_n = n!\, |b_n| \qquad (n \geq 1)\,.$$

The following identity is already known (see e.g. [8, Prop. 2]):

$$\ln A_0 = \frac{1}{2}\ln(2\pi) = \sum_{n=2}^{\infty} \frac{\lambda_n}{n!\,(n-1)} + \frac{1}{2}\gamma + \frac{1}{2}\,.$$

Combining Adamchik's formula with a result previously given in [8] enables us to considerabely expand the scope of this formula through the following theorem:

Theorem 2. For all integers k and r with $0 \leq r \leq k$, let $S(k,r)$ be the Stirling numbers of the second kind

$$S(k,r) = \frac{1}{r!}\sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} j^k\,,$$

and σ_r be the shifted Mascheroni series

$$\sigma_r := \sum_{n=r+1}^{\infty} \frac{\lambda_n}{n!\,(n-r)} \qquad (r \geq 0)\,.$$

Then, for all integers $k \geq 1$, we have

$$\ln A_k = (-1)^k \sum_{r=1}^{k} (-1)^r r!\, S(k,r)\,\sigma_{r+1} + \frac{B_{k+1}}{k+1}\left(H_{k+1} + \gamma\right)\,. \qquad (7)$$

in particular,

$$\ln A = \sigma_2 + \frac{1}{12}\gamma + \frac{1}{8}\,.$$

Proof. Formula (7) results from the decomposition of $\zeta'(-k)$ given by [8, Prop. 3] and from Adamchik's formula (2). $\square$

Corollary 2. For any positive integer k, we have

$$\ln A_{2k} = \sum_{n=2k+2}^{\infty} \frac{\lambda_n}{n!} \left\{ \sum_{r=1}^{2k} \frac{(-1)^r r!\, S(2k,r)}{n-1-r} \right\} + C_{2k}\,, \tag{8}$$

and

$$\ln A_{2k-1} = \sum_{n=2k+1}^{\infty} \frac{\lambda_n}{n!} \left\{ \sum_{r=1}^{2k-1} \frac{(-1)^{r-1} r!\, S(2k-1,r)}{n-1-r} \right\} + \frac{B_{2k}}{2k}\,(H_{2k}+\gamma) + C_{2k-1}\,, \tag{9}$$

where the constants C_k are given by $C_1 = 0$, and

$$C_k = (-1)^k \sum_{r=1}^{k-1} (-1)^r r!\, S(k,r) \sum_{j=r+2}^{k+1} \frac{\lambda_j}{j!\,(j-1-r)} \qquad (k \geq 2)\,.$$

Example 2.

$$\ln A_1 = \sum_{n=3}^{\infty} \frac{\lambda_n}{n!\,(n-2)} + \frac{1}{12}\gamma + \frac{1}{8}\,,$$

$$\ln A_2 = \sum_{n=4}^{\infty} \frac{\lambda_n\,(n-1)}{n!\,(n-2)(n-3)} - \frac{1}{24}\,,$$

$$\ln A_3 = \sum_{n=5}^{\infty} \frac{\lambda_n\, n(n-1)}{n!\,(n-2)(n-3)(n-4)} - \frac{1}{120}\gamma - \frac{29}{240}\,,$$

$$\ln A_4 = \sum_{n=6}^{\infty} \frac{\lambda_n\,(n-1)^2(n+4)}{n!\,(n-2)(n-3)(n-4)(n-5)} - \frac{113}{480}\,.$$

The relation between $\ln A_k$ and the $\mathcal{R}$-sum $\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n$ provided by Theorem 1 allows us to derive from Corollary 2 above the corresponding formulas:

Corollary 3. For all integers $k \geq 1$,

$$\sum_{n\geq 1}^{\mathcal{R}} n^{2k} \ln n = \sum_{n=2k+2}^{\infty} \frac{\lambda_n}{n!} \left\{ \sum_{r=1}^{2k} \frac{(-1)^r r!\, S(2k,r)}{n-1-r} \right\} - \frac{1}{(2k+1)^2} + C_{2k}\,, \tag{10}$$

and

$$\sum_{n\geq 1}^{\mathcal{R}} n^{2k-1} \ln n = \sum_{n=2k+1}^{\infty} \frac{\lambda_n}{n!} \left\{ \sum_{r=1}^{2k-1} \frac{(-1)^{r-1} r!\, S(2k-1,r)}{n-1-r} \right\} + \frac{B_{2k}}{2k}\gamma + D_{2k}\,, \tag{11}$$

with

$$D_{2k} = \frac{B_{2k}-1}{(2k)^2} + C_{2k-1} \qquad (k \geq 1)\,.$$

6 Bendersky-Adamchik constants and Blagouchine's integral

In this additional section, we complete an unpublished short note of Blagouchine [3] by establishing a link with the Bendersky-Adamchik constants through a binomial sum.

Theorem 3. For any integer $k \geq 0$, let $\mathcal{J}_k$ be the complex integral ("Blagouchine's integral")

$$\mathcal{J}_k := \int_{-\infty}^{+\infty} \frac{\zeta(\frac{1}{2} + ix)}{(2k+1+2ix)\cosh(\pi x)}\, dx\,.$$

Then we have $\mathcal{J}_0 = -1$, and for all $k \geq 1$,

$$\mathcal{J}_k = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln A_j - \frac{1}{k} - \frac{1}{(k+1)^2}\,. \tag{12}$$

Proof. Theorem 3 results directly from the following two lemmas. □

Lemma 1. For any integer $k \geq 0$, let ν_k be the infinite alternating series

$$\nu_k := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k}\,.$$

For all $k \geq 0$, we have the relation:

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{(k+1)^2} - \mathcal{J}_k\,. \tag{13}$$

In particular, since $\nu_0 = \gamma$ (a classical formula due to Euler), we have $\mathcal{J}_0 = -1$.

Proof. Formula (13) is a special case of [3, Thm. 1] in the case where $\omega = k$ is an integer. We give a new direct proof. For $k \geq 0$, let us consider the function

$$f_k(z) = \frac{\zeta(z)}{(k+z)\sin(\pi z)}\,.$$

This function f_k has poles at integers $n \in \mathbb{Z}$. For $n \geq 2$, the residue of f_k at $z = n$ is

$$\operatorname{Res}(f_k; n) = \frac{(-1)^n \zeta(n)}{(n+k)\pi}\,.$$

For $n = 1$, f_k has a double pole and

$$\operatorname{Res}(f_k; 1) = -\frac{1}{\pi}\left(\frac{\gamma}{k+1} - \frac{1}{(k+1)^2}\right).$$

If q is a positive odd integer with $1 < q$, then, by the residue theorem, we have

$$\int_{\mathrm{Re}(z)=1/2} f_k(z)\,dz - \int_{\mathrm{Re}(z)=q/2} f_k(z)\,dz = -2i\pi \sum_{\frac{1}{2}<n<\frac{q}{2}} \mathrm{Res}(f_k;n)\,. \qquad (*)$$

Moreover, there is a positive constant C such that

$$\left|\int_{\mathrm{Re}(z)=q/2} f_k(z)\,dz\right| \leq C \int_{-\infty}^{+\infty} \frac{1}{\left((k+\frac{q}{2})^2+t^2\right)^{\frac{1}{2}} (e^{\pi t}+e^{-\pi t})}\,dt\,,$$

and thus, by the dominated convergence theorem, we have

$$\left|\int_{\mathrm{Re}(z)=q/2} f_k(z)\,dz\right| \to 0 \quad \text{as} \quad q \to +\infty\,.$$

Therefore, taking the limit in $(*)$, we obtain

$$\int_{\mathrm{Re}(z)=1/2} f_k(z)\,dz = -2i\pi \sum_{n>\frac{1}{2}} \mathrm{Res}(f_k;n)\,.$$

This last identity allows us to deduce formula (13). □

Lemma 2. For all $k \geq 1$, we have the identity

$$\nu_k = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln A_j\,. \qquad (14)$$

Proof. From [7, Prop. 1], we know that

$$\nu_k = \frac{\gamma}{k+1} + \frac{1}{k} + \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \zeta'(-j) + \sum_{j=0}^{k-1} \binom{k}{j} \frac{H_j\,B_{j+1}}{j+1} \qquad (k \geq 1)\,.$$

Rephrasing this later expression thanks to Adamchik's formula (2) leads to the equivalent formula (14). □

Remark 2. Since $\mathcal{J}_0 = -1$, we can also rewrite (12) in a slightly different manner:

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln A_j = \mathcal{J}_k - \frac{k^2+3k+1}{k(k+1)^2}\,\mathcal{J}_0 \qquad (k \geq 1)\,. \qquad (15)$$

Example 3. For small positive values of k, we derive from (12) the following identities:

$$\mathcal{J}_1 = \frac{1}{2}\ln 2\pi - \frac{5}{4}\,,$$
$$\mathcal{J}_2 = \frac{1}{3}\ln 2\pi - \frac{1}{6}\gamma - \frac{11}{18} + \frac{1}{6}\frac{\zeta'(2)}{\zeta(2)}\,,$$
$$\mathcal{J}_3 = \frac{1}{4}\ln 2\pi - \frac{1}{4}\gamma - \frac{19}{48} + \frac{1}{4}\frac{\zeta'(2)}{\zeta(2)} + \frac{1}{8}\frac{\zeta(3)}{\zeta(2)}\,,$$

and, for $k \geq 3$, using (3) and (4), we get the general formula:

$$\mathcal{J}_k = \frac{1}{k+1}\ln 2\pi - \frac{(k-1)}{2(k+1)}\gamma - \frac{k^2+3k+1}{k(k+1)^2} + \sum_{j=1}^{[\frac{k}{2}]}\binom{k}{2j-1}\frac{B_{2j}\,\zeta'(2j)}{2j\,\zeta(2j)} + \frac{1}{4}\sum_{j=1}^{[\frac{k-1}{2}]}\binom{k}{2j}\frac{B_{2j}\,\zeta(2j+1)}{\zeta(2j)}\,. \quad (16)$$

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