Three proofs of quadratic reciprocity and their impact on twentieth century mathematics

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Definition (Legendre symbol for odd prime p and coprime x)

$$\left(\frac{x}{p}\right) = \begin{cases} +1 & \text{if } x \text{ is a square in } \mathbb{F}_p^\times \\ -1 & \text{if } x \text{ is not a square in } \mathbb{F}_p^\times \end{cases}$$

Lemma (Euler's criterion)

$\left(\frac{x}{p}\right) = x^{\frac{p-1}{2}}$ in $\mathbb{F}_p$ so that $\left(\frac{x}{p}\right)\left(\frac{y}{p}\right) = \left(\frac{xy}{p}\right)$ in $\mathbb{F}_p$.

Proof.

$X^{p-1} - 1 = (X^{\frac{p-1}{2}} + 1)(X^{\frac{p-1}{2}} - 1)$ in $\mathbb{F}_p[X]$. □

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Theorem (Quadratic reciprocity law – Euler, Legendre, Gauss)

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- $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} +1 & \text{if } p \equiv \pm 1 \mod 8 \\ -1 & \text{if } p \equiv \pm 3 \mod 8 \end{cases}$

Example (Is 14 a square in $\mathbb{F}_{41}$?)

- $\left(\frac{2}{41}\right) = +1$
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Remark (Lemmermeyer 2000)

has referenced more than 300 proofs of the quadratic reciprocity law among which 8 by Gauss.
We outline three of them, a combinatorial, an algebraic, and a cyclotomic proof.

method	keyword1	keyword2	Gauss's proof	extended by
combinatorial	signature	Gauss's Lemma	3/5 (1808/18)	Eisenstein[a], Zolotarev[b], Frobenius[c]
algebraic	discriminant	Gauss reduction	2 (1801)	Stickelberger[d], Hensel[e,f], Artin[g]
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Remark (exécutaire testamentaire)

En 1866, la tuberculose emporte Riemann, et sa mort prématurée survient en pleine phase de grande créativité. Peu avant son décès, imitant Dirichlet, il avait fait de Dedekind le dépositaire de ses manuscrits, lequel s'attelle presque tout de suite à l'entreprise colossale que représente l'organisation des écrits de Riemann et la sélection des textes publiables ... Il est difficile d'évaluer le travail d'exécuteur testamentaire qu'accomplit Dedekind, d'autant qu'il avait vingt-huit ans quand Dirichlet est mort et quarante-cinq ans quand les Œuvres complètes (1876) de Riemann sont publiées, et que ces dix-sept ans correspondent à ses années les plus productives sur le plan mathématique. Dedekind a su faire passer les contributions de ses deux amis avant les siennes. (wikipedia)

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En 1866, la tuberculose emporte Riemann, et sa mort prématurée survient en pleine phase de grande créativité. Peu avant son décès, imitant Dirichlet, il avait fait de Dedekind le dépositaire de ses manuscrits, lequel s'attelle presque tout de suite à l'entreprise colossale que représente l'organisation des écrits de Riemann et la sélection des textes publiables ... Il est difficile d'évaluer le travail d'exécuteur testamentaire qu'accomplit Dedekind, d'autant qu'il avait vingt-huit ans quand Dirichlet est mort et quarante-cinq ans quand les Œuvres complètes (1876) de Riemann sont publiées, et que ces dix-sept ans correspondent à ses années les plus productives sur le plan mathématique. Dedekind a su faire passer les contributions de ses deux amis avant les siennes. (wikipedia)

Remark (heritage of Gauss (1777-1855))

- G. Dirichlet (1805-1859) from 1850 to 1859 in Göttingen: caractères de Dirichlet, fonctions L, progression arithmétique.
- B. Riemann (1826-1866) from 1849 to 1866 in Göttingen: géométrie Riemannienne, fonction ζ, hypothèse de Riemann.
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$\left(\frac{x}{p}\right) = \mathrm{sgn}(m_x)$ where $m_x : \mathbb{F}_p^\times \to \mathbb{F}_p^\times$ is multiplication by x.

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m_x has even/odd number of orbits iff x is/is not square in $\mathbb{F}_p$. □

Corollary (complementary laws of quadratic reciprocity)

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Definition (discriminant of a polynomial)

For $f(X) \in K[X]$ and $f(X) = (X - \alpha_1)\cdots(X - \alpha_n) \in L[X]$:

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Lemma ($p^* = (-1)^{\frac{p-1}{2}} p$)

$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$ if and only if $\left(\frac{p^*}{q}\right) = \left(\frac{q}{p}\right)$

Theorem (Dedekind 1877)

$\left(\frac{p^*}{q}\right) = \left(\frac{\Delta_{X^p-1}}{\mathbb{F}_q}\right) = \left(\frac{q}{p}\right)$

Proof.

$\Delta_f = (-1)^{\binom{n}{2}} \prod_i f'(\alpha_i)$ so $\Delta_{X^p-1} = (-1)^{\binom{p}{2}} \prod_{i=1}^p p\alpha_i^{p-1} = p^*$.
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Theorem (Stickelberger 1898, Hensel 1905, Swan 1962)

For $f \in \mathbb{F}_q[X]$ sth. $\Delta_f \neq 0$ one has $\left(\frac{\Delta_f}{\mathbb{F}_q}\right) = (-1)^{\deg(f) - \#\mathrm{irrfact}(f)}$

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$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$ if and only if $\left(\frac{p^*}{q}\right) = \left(\frac{q}{p}\right)$

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Definition (Dirichlet character)

$\chi : \mathbb{F}_p^\times \to \mathbb{C}^*$ sth. $\chi(xy) = \chi(x)\chi(y)$ and $\chi(1) = 1$. For pointwise multiplication, Dirichlet characters mod p form a cyclic group.

Remark

The Legendre symbol is the only Dirichlet character of order 2.

Definition (Gauss sums for $\zeta = e^{2\pi i/p}$)

- $\tau_\chi = \sum_{k=1}^{p-1} \chi(k)\zeta^k$
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Proposition (Discrete Fourier Transform)

$\hat{\chi} = \tau_\chi \bar{\chi}$ and $\hat{\hat{\chi}} = \tau_\chi \tau_{\bar{\chi}} \chi$ and $\tau_\chi \tau_{\bar{\chi}} = p\chi(-1)$.

Proof.

$\chi(s)\hat{\chi}(s) = \chi(s) \sum_{k=1}^{p-1} \chi(k)\zeta^{sk} = \sum_{k=1}^{p-1} \chi(sk)\zeta^{sk} = \tau_\chi$.
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Corollary (Eisenstein 1844)

$(\tau_p)^2 = p\left(\frac{-1}{p}\right) = p^*$ and $(\tau_p)^q \equiv \left(\frac{q}{p}\right)\tau_p \mod q\mathbb{Z}[\zeta]$

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$$\left(\frac{p^*}{q}\right) = (p^*)^{\frac{q-1}{2}} = (\tau_p)^{q-1} = \left(\frac{q}{p}\right)$$

Theorem (Gauss 1818, proof by Motose 2003)

$$\tau_p = \left(\frac{-2}{p}\right) i^{\frac{p-1}{2}} \sqrt{p} = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \mod 4 \\ i\sqrt{p} & \text{if } p \equiv 3 \mod 4 \end{cases}$$

Definition (cyclotomic Vandermonde matrix)

For $V = \begin{pmatrix} \zeta & \zeta^2 & \cdots & \zeta^{p-1} \\ \zeta^2 & \zeta^4 & \cdots & \zeta^{2(p-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \zeta^{p-1} & \zeta^{2(p-1)} & \cdots & \zeta^{(p-1)^2} \end{pmatrix}$ one has $V \begin{pmatrix} \chi(1) \\ \vdots \\ \vdots \\ \chi(p-1) \end{pmatrix} = \begin{pmatrix} \hat{\chi}(1) \\ \vdots \\ \vdots \\ \hat{\chi}(p-1) \end{pmatrix}$, i.e. $V\chi = \hat{\chi}$.

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$\left(\frac{-2}{p}\right) = (-1)^{\frac{(p-1)(p-3)}{8}} = (-1)^{1+2+\cdots+\frac{p-3}{2}}$ and $\prod_{1 \leq k \leq \frac{p-1}{2}} \zeta^k - \zeta^{-k} = i^{\frac{p-1}{2}} \sqrt{p}$.

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$$\det(V) = V(\zeta, \zeta^2, \ldots, \zeta^{p-1}) = \prod_{j=p-i, i<j} (\zeta^j - \zeta^i) \prod_{j \neq p-i, i<j} (\zeta^j - \zeta^i)$$

$$= ((-i)^{\frac{p-1}{2}} \sqrt{p}) \cdot (-1)^{\frac{1}{2}(\binom{p-1}{2} - \frac{p-1}{2})} p^{\frac{p-3}{2}} = (-1)^{\frac{p-1}{2}} i^{\frac{p-1}{2}} p^{\frac{p-3}{2}} \sqrt{p}$$

V is conjugate (with Dirichlet characters as basis) to a matrix decomposing into 2×2 block matrices. The characters of order 1 and 2 are real : χ_0 and $\chi_{\frac{p-1}{2}}$, the other's are complex : $\chi_k, \bar{\chi}_k$ for $k = 1, \ldots, \frac{p-3}{2}$.

Using the relation $\tau_{\chi_k} \tau_{\bar{\chi}_k} = p\chi_k(-1) = p(-1)^k$ we get :

$$\det(V) = \tau_{\chi_0} \tau_{\chi_{\frac{p-1}{2}}} \prod_1^{\frac{p-3}{2}} -\tau_{\chi_k} \tau_{\bar{\chi}_k} = -\tau_p (-p)^{\frac{p-3}{2}} (-1)^{1+2+\cdots+\frac{p-3}{2}} = (-1)^{\frac{p-1}{2} + \left(\frac{-2}{p}\right)} p^{\frac{p-3}{2}} \tau_p$$

so that $\tau_p = \left(\frac{-2}{p}\right) i^{\frac{p-1}{2}} \sqrt{p}$ as asserted. □

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$$\det(V) = V(\zeta, \zeta^2, \ldots, \zeta^{p-1}) = \prod_{j=p-i, i<j} (\zeta^j - \zeta^i) \prod_{j \neq p-i, i<j} (\zeta^j - \zeta^i)$$
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V is conjugate (with Dirichlet characters as basis) to a matrix decomposing into 2×2 block matrices. The characters of order 1 and 2 are real : χ_0 and $\chi_{\frac{p-1}{2}}$, the other's are complex : $\chi_k, \bar{\chi}_k$ for $k = 1, \ldots, \frac{p-3}{2}$.

Using the relation $\tau_{\chi_k} \tau_{\bar{\chi}_k} = p\chi_k(-1) = p(-1)^k$ we get :

$$\det(V) = \tau_{\chi_0} \tau_{\chi_{\frac{p-1}{2}}} \prod_{1}^{\frac{p-3}{2}} -\tau_{\chi_k} \tau_{\bar{\chi}_k} = -\tau_p (-p)^{\frac{p-3}{2}} (-1)^{1+2+\cdots+\frac{p-3}{2}} = (-1)^{\frac{p-1}{2} + \left(\frac{-2}{p}\right)} p^{\frac{p-3}{2}} \tau_p$$

so that $\tau_p = \left(\frac{-2}{p}\right) i^{\frac{p-1}{2}} \sqrt{p}$ as asserted. □

Proof.

$$\det(V) = V(\zeta, \zeta^2, \ldots, \zeta^{p-1}) = \prod_{j=p-i, i<j} (\zeta^j - \zeta^i) \prod_{j \neq p-i, i<j} (\zeta^j - \zeta^i)$$
$$= ((-i)^{\frac{p-1}{2}} \sqrt{p}) \cdot (-1)^{\frac{1}{2}(\binom{p-1}{2} - \frac{p-1}{2})} p^{\frac{p-3}{2}} = (-1)^{\frac{p-1}{2}} i^{\frac{p-1}{2}} p^{\frac{p-3}{2}} \sqrt{p}$$

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