Perfect moment categories, cocartesian comonads and Joyal duality

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1. Introduction
2. Moment categories
3. Perfect moment categories
4. Strongly cocartesian comonads
5. Joyal duality

Purpose of the talk

(perfect moment categories) $\overset{calculus}{\rightsquigarrow}$ (strongly cocartesian comonads)

(strongly cocartesian comonads) $\overset{duality}{\rightsquigarrow}$ (strongly cartesian monads)

(strongly cartesian monads) $\overset{Barwick}{\rightsquigarrow}$ (perfect operator categories)

Example (Segal and Joyal duality)

- $\Gamma^{\mathrm{op}} \cong$ (category of finite sets and partial maps)
- $\Delta^{\mathrm{op}} \cong \Delta_{*,*}[-1] =$ (category of combinatorial 1-disks)
- $\Theta_n^{\mathrm{op}} \cong \mathbb{D}_n =$ (category of combinatorial n-disks)

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Definition (A moment category $\mathcal{C}$ is a category with an)

active/inert factorization system $f = f_{in}f_{act}$ such that

- each inert map admits a unique active retraction;
- if $fi = g$ for i inert and f, g active, then $gr = f$ where r is the unique active retraction of i.

$m_A = \{\phi : A \to A \mid \phi_{act}\phi_{in} = 1_A\}$ = the set of *moments* of A.

Proposition (pushing forward moments)

For $f : A \to B$ in $\mathcal{C}$ there is $f_* : m_A \to m_B$ such that:

1. $\phi_*(\psi) = \phi\psi \quad (\forall\, \phi, \psi \in m_A)$
2. $(gf)_* = g_* f_* \quad (\forall\, A \xrightarrow{f} B \xrightarrow{g} C)$
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Proof.

For $f : A \to B$ define $f_* : m_A \to m_B$ by

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Remark

1. (1), (3) imply m_A is a *left regular band*: $\psi\phi = \psi_*(\phi)\psi = \psi\phi\psi$.
2. (1), (2), (3) imply $f_*(\phi\psi) = f_*(\phi)f_*(\psi)$.
3. $f_*(1_A) = 1_B$ if and only if f is *active*.
4. If f is active then the outer square above is cocartesian in C_{act}.

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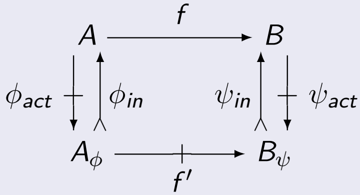

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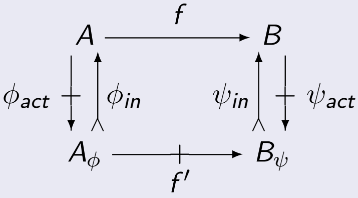

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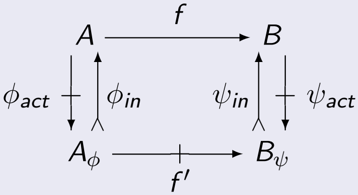

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Example (simplex category Δ and Segal's category Γ)

- $[m] \stackrel{\phi}{\to} [n]$ is active/inert iff ϕ endpoint/distance -preserving.
- $\underline{m} \xrightarrow{(\underline{n}_1,\dots,\underline{n}_m)} \underline{n}$ active/inert iff $\underline{n}_1 \sqcup \cdots \sqcup \underline{n}_m = \underline{n} / |\underline{n}_i| = 1 \, \forall i$.

Example (moment vs restriction categories – Cockett-Lack)

The dual of a restriction category with *splitting restriction idempotents* is a moment category with *commuting moments*. The dual of the category of finite sets and partial maps is Γ.

Definition (wreath product of moment categories, cf. Barr-Wells)

For a moment category $\mathcal{C}$ the wreath-product $\Delta \wr \mathcal{C}$ has as

- objects $([n], (C_1, \dots, C_n))$ with $[n] \in \Delta$ and $C_i \in \mathcal{C}$
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The dual of a restriction category with *splitting restriction idempotents* is a moment category with *commuting moments*. The dual of the category of finite sets and partial maps is Γ.

Definition (wreath product of moment categories, cf. Barr-Wells)

For a moment category $\mathcal{C}$ the wreath-product $\Delta \wr \mathcal{C}$ has as

- objects $([n], (C_1, \dots, C_n))$ with $[n] \in \Delta$ and $C_j \in \mathcal{C}$
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A moment category $\mathcal{C}$ is *perfect* if the inclusion $\mathcal{C}_{act} \hookrightarrow \mathcal{C}$ admits a left adjoint $\overline{(-)}$ such that the unit : $\eta_A : A \to \bar{A}$ is pointwise *inert*.

Examples

Γ, Δ, Θ_n are perfect moment categories.

- In Γ the unit of the reflection is given by the natural injections $\underline{n} \rightarrowtail \underline{n+1}$.
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The active retraction of $\eta_A : A \rightarrowtail \bar{A}$ induces a moment ω_A on $\bar{A}$ such that there is a bijection between general maps $f : A \to B$ and active maps $g : \bar{A} \to B$ with the property that $f_*(1_A) = g_*(\omega_A)$.

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Each perfect moment category $\mathcal{C}$ induces a moment classifier comonad (L, ν, ϵ) on $\mathcal{C}_{act}$ with

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Definition

A composable class of morphisms $\mathcal{R}$ is *stable* if arbitrary pushouts of members of $\mathcal{R}$ exist and still belong to $\mathcal{R}$. A comonad (L, ν, ϵ) is $\mathcal{R}$*-cocartesian* if the naturality squares of the counit are cocartesian, ϵ is pointwise in $\mathcal{R}$, and L preserves ϵ-pushouts.

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The moment classifier comonad on $\mathcal{C}_{act}$ is strongly $\mathcal{R}$-cocartesian, where $\mathcal{R}$ is the stable class of active retractions in $\mathcal{C}_{act}$.

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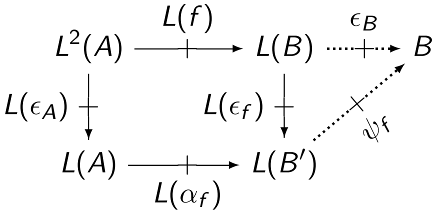

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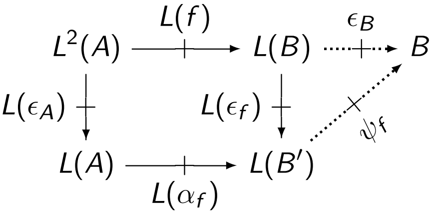

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