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A Cellular Nerve for Higher Categories

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We realise Joyal's cell category Θ as a dense subcategory of the category of ω-categories. The associated cellular nerve of an ω-category extends the well-known simplicial nerve of a small category. Cellular sets (like simplicial sets) carry a closed model structure in Quillen's sense with weak equivalences induced by a geometric realisation functor. More generally, there exists a dense subcategory Θ_A of the category of $\underline{A}$-algebras for each ω-operad A in Batanin's sense. Whenever A is contractible, the resulting homotopy category of $\underline{A}$-algebras (i.e. weak ω-categories) is equivalent to the homotopy category of compactly generated spaces.

Key Words: higher categories; globular operads; combinatorial homotopy.

The following text arose from the desire to establish a firm relationship between higher categories and topological spaces. Our approach combines the algebraic features of Batanin's *ω-operads* [2] with the geometric features of Joyal's *cellular sets* [25] and tries to mimick as far as possible the classical construction of the simplicial nerve of a small category.

Each ω-category has an underlying *ω-graph* (also called *globular set* [37]) and comes equipped with a family of *composition laws* governed by Godement's *interchange rules* [21, App.1.V]. The forgetful functor from ω-categories to ω-graphs is *monadic*. The left adjoint free functor may be deduced from Batanin's formalism of ω-operads; indeed, it turns out that ω-categories are the algebras for the *terminal ω-operad*. This leads to Batanin's definition of *weak ω-categories* as the *algebras* for a (fixed) *contractible ω-operad*, which may be compared with Boardman–Vogt–May's definition of E_∞-spaces [8, 29].

The main purpose here is to define a whole family of *nerve functors*, one for each ω-operad, and to study under which conditions these

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nerve functors define a well-behaved *homotopy theory* for the underlying algebras.

Nerve functors are induced by suitable *subcategories*. The simplicial nerve, for instance, is defined by embedding the simplex category Δ in the category of small categories. By analogy, we construct for each ω-operad A a *dense* subcategory Θ_A of the category of $\underline{A}$-algebras. The induced nerve $\mathcal{N}_A$ is then a *fully faithful* functor from $\underline{A}$-algebras to presheaves on Θ_A. Its image may be characterised by a certain *restricted sheaf condition*. Even in the case of ω-categories, the existence of such a fully faithful nerve functor is new, cf. [13, 36, 38]. We denote the corresponding dense subcategory by Θ and call presheaves on Θ *cellular sets*. This terminology has been suggested to us by the remarkable fact that the *operator category* Θ coincides with Joyal's *cell category* Θ although the latter has been defined quite differently. Indeed, Joyal's Θ plays the same role for ω-operads and weak ω-categories as Segal's Γ for symmetric operads and E_∞-spaces, cf. [34, App. B].

According to Joyal [25], cellular sets have a geometric realisation in which *simplex* and *ball* geometry are mixed through the combinatorics of *planar level trees*. It follows that ω-categories realise via their cellular nerve the same way as categories do via their simplicial nerve. *Weak ω-categories* also have a geometric realisation by means of the *left Segal extension* [34, App. A] of their A-cellular nerve along the canonical functor from Θ_A to Θ. This realisation induces a natural concept of *weak equivalence* between weak ω-categories.

Cellular sets carry a *closed model structure* in Quillen's sense [31]. Like for simplicial sets, the *fibrations* are defined by *horn filler* conditions. There is a whole *tower* of Quillen equivalent model categories beginning with simplicial sets and ending with cellular sets. Indeed, the cell category Θ is filtered by full subcategories $\Theta^{(n)}$ such that $\Theta^{(1)}$ equals the simplex category Δ and such that $\Theta^{(n)}$ is a Cauchy-complete extension of Simpson's [35] quotient $\Theta_n = \Delta^{\times n}/\sim$.

The *homotopy category* of cellular sets is equivalent to the homotopy category of compactly generated spaces. The cellular nerve, however, does not "create" a model structure for ω-categories, mainly because the left adjoint *ω-categorification* does not yield the correct homotopy type for all cellular sets. In order to solve this difficulty, we consider cellular sets as the *discrete objects* among *cellular spaces* and construct a convenient model structure for cellular spaces. Here, the ω-categorification yields a Quillen equivalence between cellular spaces and *simplicial ω-categories*. Both homotopy categories are determined by the discrete objects so that we end up with an equivalence between the homotopy categories of cellular sets and of ω-categories. More generally, for each *contractible* ω-operad A, there exists a model structure for A-cellular spaces such that the A-categorification induces a Quillen equivalence between A-cellular spaces and simplicial $\underline{A}$-algebras. Again, the discrete objects span the entire homotopy categories. Moreover, base change along $\Theta_A \to \Theta$ induces a Quillen equivalence between A-cellular spaces and cellular spaces.

Each topological space X defines a fundamental ω-graph ΠX whose n-cells are the continuous maps from the n-ball B^n to X. There is a contractible ω-operad acting on ΠX, inductively constructed by Batanin [2], so that via the above-mentioned Quillen equivalences, the homotopy type of X is entirely recoverable from this algebraic structure. In what sense the fundamental ω-graph is a *weak ω-groupoid* and to what extent weak ω-groupoids recover all homotopy types among weak ω-categories will be the theme of subsequent papers.

0. NOTATION AND TERMINOLOGY

We shall follow as closely as possible the expositions of Borceux [9], Gabriel-Zisman [20] and Quillen [31] concerning categorical, simplicial and model structures, respectively. Below, a summary of the most frequently used concepts.

A functor F is called (*co*)*continuous* if F preserves small (co)limits. A functor F *preserves* (resp. *detects*) a property P if, whenever the morphism f (resp. Ff) has property P, then also Ff (resp. f).

The *category of sets* (resp. *simplicial sets*) is denoted by $\mathscr{S}$ (resp. $s\mathscr{S}$).

0.1. Tensor Products

For functors $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{S}$ and $G : \mathscr{C} \to \mathscr{E}$, the *tensor product* $F \otimes_{\mathscr{C}} G$ is an object of $\mathscr{E}$ subject to the *adjunction* formula $\mathscr{E}(F \otimes_{\mathscr{C}} G, E) \cong \mathrm{Hom}_{\mathscr{C}}(F, \mathscr{E}(G, E))$, where $\mathrm{Hom}_{\mathscr{C}}(F, F')$ denotes the set of *natural transformations* $F \to F'$, and where $\mathscr{E}(G, E)$ denotes the presheaf defined by $\mathscr{E}(G, E)(-) = \mathscr{E}(G(-), E)$.

If the category $\mathscr{E}$ is cocomplete, the tensor product $F \otimes_{\mathscr{C}} G$ is the so-called *coend* of the bifunctor $(C', C) \mapsto F(C') \otimes G(C) := \coprod_{F(C')} G(C)$ and can thus be identified with the coequaliser

$$\coprod_{\phi : C \to C'} F(C') \otimes G(C) \rightrightarrows \coprod_{C} F(C) \otimes G(C) \twoheadrightarrow F \otimes_{\mathscr{C}} G.$$

For two functors $F : \mathscr{C} \to \mathscr{S}$ and $G : \mathscr{D} \to \mathscr{S}$ of the same variance, the tensor product $F \otimes G : \mathscr{C} \times \mathscr{D} \to \mathscr{S}$ is defined by $(F \otimes G)(-) = F(-) \times G(-)$.

0.2. Higher Graphs and Higher Categories

The globe category $\mathbb{G}$ has one object $\bar{n}$ for each integer $n \geqslant 0$. The reflexive globe category $\bar{\mathbb{G}}$ has same objects as $\mathbb{G}$.

The *globular operators* are generated by cosource/cotarget operators $s_n, t_n : \bar{n} \rightrightarrows \overline{n+1}$ and in the reflexive case also by coidentities $i_n : \overline{n+1} \to \bar{n}$ subject to the relations $s_{n+1}s_n = t_{n+1}s_n, s_{n+1}t_n = t_{n+1}t_n, i_ns_n = i_nt_n = \mathrm{id}_{\bar{n}}$, $n \geqslant 0$.

A presheaf on $\mathbb{G}$ (resp. $\bar{\mathbb{G}}$) is called an ω-*graph* (resp. reflexive ω-graph). Street [37] calls ω-graphs *globular sets*.

An ω-graph $X : \mathbb{G}^{\mathrm{op}} \to \mathcal{S}$ will often be denoted as an $\mathbb{N}$-graded family of sets $(X_n)_{n \geqslant 0}$ which comes equipped with source/target operations:

$$\cdots \rightrightarrows X_{n+1} \rightrightarrows X_n \rightrightarrows \cdots \rightrightarrows X_1 \rightrightarrows X_0.$$

An ω-graph which is empty in degrees strictly greater than n, is called an n-graph. The operations induced by s_n/t_n are called *source/target* maps. The operations induced by i_n are called *identity* maps. The representable functor $\mathbb{G}(-, \bar{n})$ is the *standard n-cell.*

A 2-*category* is a small Cat-enriched category, where Cat denotes the category of small categories. The objects of a 2-category $\mathscr{C}$ are the 0-cells, the objects (resp. morphisms) of the categorical hom-sets $\underline{\mathscr{C}}(-,-)$ are the 1-cells (resp. 2-cells) of $\mathscr{C}$. The *source/target* and *identity* maps define a *reflexive* 2-*graph* underlying $\mathscr{C}$. A 2-category comes equipped with three composition laws $\circ_i^j : \mathscr{C}_j \times_i \mathscr{C}_j \to \mathscr{C}_j$, $0 \leqslant i < j \leqslant 2$, subject to *Godement's interchange rules* [21].

An ω-*category* $\mathscr{C}$ [3, 36] is a reflexive ω-graph which comes equipped with composition laws $\circ_i^j : \mathscr{C}_j \times_i \mathscr{C}_j \to \mathscr{C}_j$, $i<j$, such that, for any triple of non-negative integers $i<j<k$, the family $(\mathscr{C}_i, \mathscr{C}_j, \mathscr{C}_k; \circ_i^j, \circ_i^k, \circ_j^k)$ has the structure of a 2-category with respect to the (iterated) source/target and identity maps of the underlying reflexive ω-graph.

The category of ω-categories is denoted by ω-Cat or $\mathrm{Alg}_{\underline{\omega}}$, cf. Theorem 1.12.

0.3. Monads and their Algebras

A *monad* on the category $\mathscr{E}$ is a monoid (T, η, μ) in the category of endofunctors of $\mathscr{E}$. A T-*algebra* is a pair (X, m_X) consisting of an object X of $\mathscr{E}$ and a T-*action* $m_X : TX \to X$ which is *unital* $(m_X\eta_X = \mathrm{id}_X)$ and *associative* $(m_X\mu_X = m_XTm_X)$. The *category of* T-*algebras* is denoted by Alg_T.

We shall use (slightly abusively) the same symbol to denote as well the monad T as well the *free functor* $T : \mathscr{E} \to \mathrm{Alg}_T$ since the free T-algebra on an object X of $\mathscr{E}$ is given by (TX, μ_X).

A pair of adjoint functors $F : \mathscr{E} \leftrightarrows \mathscr{E}' : G$ with left adjoint F induces a monad (GF, η, μ) on $\mathscr{E}$ where η is the unit of the adjunction and $\mu = G\varepsilon F$ is induced by the counit ε of the adjunction.

A functor $G : \mathscr{E}' \to \mathscr{E}$ is *monadic* if G has a left adjoint $F : \mathscr{E} \to \mathscr{E}'$ such that $Y \mapsto (GY, G\varepsilon_Y)$ induces an equivalence of categories $\mathscr{E}' \xrightarrow{\sim} \mathrm{Alg}_{GF}$.

0.4. Categories of Elements, Filtered Colimits and Finite Objects

For a set-valued presheaf F on $\mathscr{C}$, the *category of elements* $\mathrm{el}(F)$ has as objects the pairs (C, x) with $x \in F(C)$ and as morphisms $f : (C, x) \to (C', x')$ the $\mathscr{C}$-morphisms $f : C \to C'$ with $F(f)(x') = x$.

Every set-valued presheaf is the colimit of representable presheaves $\mathscr{C}(-, C)$ according to the formula: $\varinjlim_{(C,x)\in\mathrm{el}(F)} \mathscr{C}(-, C) \xrightarrow{\sim} F$ where the components $x : \mathscr{C}(-, C) \to F$ of the colimit cone are induced by the Yoneda-lemma.

A category $\mathscr{C}$ is *filtered* if the following three properties hold: $\mathscr{C}$ is non-empty; for any two objects A, B of $\mathscr{C}$ there is an object C of $\mathscr{C}$ such that the morphism-sets $\mathscr{C}(C, A)$ and $\mathscr{C}(C, B)$ are non-empty; for any parallel pair of $\mathscr{C}$-morphisms $f, g : A \rightrightarrows B$ there is a $\mathscr{C}$-morphism $h : C \to A$ such that $fh = gh$.

A *colimit* $\varinjlim_{\mathscr{C}} F$ is *filtered* if the *opposite* category $\mathscr{C}^{\mathrm{op}}$ of the indexing category is filtered.

An object A of a category $\mathscr{E}$ is called *finite* if the representable diagram $\mathscr{E}(A, -)$ preserves *filtered* colimits. Quillen [31] calls finite objects *small*, whence his *small object argument*. Finite objects are often called finitely presentable or ω-presentable where ω is the first infinite cardinal, cf. [9, II.5].

The finite objects of the category of sets are precisely the finite sets; the finite objects of a category of set-valued presheaves are precisely the quotients of finite coproducts of representable presheaves. The finite objects of an algebraic category are the objects of finite presentation, i.e., those having finitely many generators and finitely many relations, cf. [9, II.3.8.14].

0.5. Compactly Generated Spaces

The category Top of topological spaces has certain drawbacks among which the lack of a *cartesian closed* structure and the *non-finiteness* of compact (i.e. quasi-compact Hausdorff) spaces. These disadvantages disappear when we restrict to the full subcategory $\mathrm{Top}^{\mathrm{c}}$ of *compactly generated* spaces, which is the largest subcategory of Top with the property that the category $\mathscr{K}$ of compact spaces is *dense* in $\mathrm{Top}^{\mathrm{c}}$. According to Day [15], the category $\mathrm{Top}^{\mathrm{c}}$ embeds as a *reflective* subcategory in a category $\mathscr{D}$ of special presheaves on $\mathscr{K}$, which is cartesian closed and in which the compact spaces are finite; the reflector from $\mathscr{D}$ to $\mathrm{Top}^{\mathrm{c}}$ preserves finite products and finite objects, so that Top^c is cartesian closed and the compact spaces are finite in Top^c.

0.6. Orthogonal Morphisms

We call a morphism f *left orthogonal* to g (or g *right orthogonal* to f) if for each commutative diagram of unbroken arrows

$$\begin{tikzcd} A_1 \arrow[r, "\phi_1"] \arrow[d, "f"'] & B_1 \arrow[d, "g"] \\ A_2 \arrow[r, "\phi_2"] \arrow[ur, dashed, "\phi"] & B_2 \end{tikzcd}$$

there exists a *diagonal filler* $\phi : A_2 \to B_1$ such that $\phi f = \phi_1$ and $g\phi = \phi_2$.

An object A is left orthogonal to g if the unique morphism $\emptyset \to A$ is left orthogonal to g; an object B is right orthogonal to f if the unique morphism $B \to 1$ is right orthogonal to f, where $\emptyset$ (resp. 1) denotes an initial (resp. terminal) object of the category.

This terminology goes back to Max Kelly. In Quillen's terminology, the morphism f has the *left lifting property* with respect to g and g has the *right lifting property* with respect to f. In category theory, one often requires *uniqueness* of the diagonal filler. Uniqueness is automatic if either f is epic or g is monic. In the case of a Quillen closed model category there is no such uniqueness, but for a cofibration f and a fibration g, if either of the two is a weak equivalence, the diagonal filler exists by definition, and can be shown to be *unique up to homotopy*. This hopefully justifies our terminology.

1. GLOBULAR THEORIES AND CELLULAR NERVES

This section presents Batanin's ω-operads [2] from a point of view which resembles Boardman–Vogt's treatment [8] of symmetric operads [29] as topologically enriched algebraic theories [26] with special properties.

Indeed, an ω-operad A *generates* an operator category Θ_A which embodies the universal operations acting on an $\underline{A}$-algebra. We call such an operator category a *globular theory*. Since the underlying object of an $\underline{A}$-algebra is an ω-*graph*, domain and codomain of the universal operations are powers of a new kind, the so-called *tree-powers*. The tree-power X^S of an ω-graph X consists of those cell-configurations of X which have the "shape" of the *level tree* S.

A Θ_A-*model* is then an ω-graph X endowed with associative and unital operations $\Theta_A(S,T) \times X^T \to X^S$ for all (finite planar) level trees S, T.

Level trees are to ω-operads what natural numbers are to symmetric operads, cf. [37]. In particular, the globular theory Θ_A may be identified with the full subcategory of $\underline{A}$-algebras spanned by the *free* $\underline{A}$*-algebras* $\underline{A}(S_*)$ on the basic ω-graphs S_* of shape S. It is remarkable that for *each* ω-operad A, this defines a *dense* subcategory of the category of $\underline{A}$-algebras and hence a *fully faithful* nerve functor from $\underline{A}$-algebras to presheaves on Θ_A whose image is precisely the category of Θ_A-models. This full embedding of the category of $\underline{A}$-algebras into a presheaf category will be essential in the sequel.

The *terminal* ω*-operad* generates a globular theory denoted by Θ. This terminal ω-operadic theory coincides with Joyal's *cell category* Θ as will be shown in Section 2. Since the terminal ω-operad acts on ω-categories, the cell category Θ embeds densely in the category of ω-categories and the cellular nerve is a fully faithful functor from ω-categories to cellular sets. Neither Street's *simplicial* nerve [36] nor Brown's *cubical* nerve [13] nor Simpson–Tamsamani's *multi-simplicial* nerve [35, 38] have this property.

DEFINITION 1.1. A (finite planar) *level tree* is a finite graded set $T = (T(n))_{n\geqslant 0}$ endowed with a map $i_T : T_{>0} \to T$ which lowers degree by one, such that $T(0)$ is singleton, and such that all fibres $i_T^{-1}(x)$, $x \in T$, are linearly ordered.

A *subtree* (S, i_S) of (T, i_T) is a graded subset S of T such that $(i_T)|_S = i_S$. A subtree S of T is *plain* if for each vertex $x \in S$, the fibre $i_S^{-1}(x)$ is either empty or connected in $i_T^{-1}(x)$.

An element $x \in T(k)$ is called a *vertex of height* k, formally $k = \mathrm{ht}(x)$. If $n = \max_{x\in T} \mathrm{ht}(x)$, we say that T is an *n-level tree* and write $n = \mathrm{ht}(T)$. Any vertex with empty fibre is called an *input vertex* of T.

An *edge* of T is a pair of vertices (x, x') with $x = i_T(x')$. The set of edges will be written $e(T)$, the number of edges is the *dimension* $d(T)$ of T. A level tree T is *linear* if T has only one input vertex or, equivalently, if $d(T) = \mathrm{ht}(T)$.

For each vertex $x \in T$, the *set of incoming edges* $e_x(T)$ is linearly ordered:

$$e_x(T) = \{(x, x') \in e(T)\} \cong i_T^{-1}(x).$$

We adjoin "left and right horizontal" edges (x, x_-) and (x, x_+) to $e_x(T)$ which serve as the *new* minimum and maximum for $\bar{e}_x(T) = e_x(T) \cup \{(x, x_\pm)\}$.

A T*-sector* of height k is a triple $(x; x', x'')$ such that $x \in T(k)$ and such that $(x, x'), (x, x'')$ are *consecutive* edges in $\bar{e}_x(T)$. It is an *inner* T-sector if $(x, x'), (x, x'') \in e_x(T)$. Since input vertices of T have empty fibres, each input vertex supports a unique (outer) T-sector called *input sector*.

We write $\bar{0}$ or $[0]$ for the unique 0-level tree (consisting only of the root). If $n > 0$, we write $\bar{n}$ for the unique linear n-level tree, and $[n]$ for the unique 1-level tree with n input vertices.

LEMMA 1.2. *For each level tree T, the set T_* of T-sectors, graded by height, is an ω-graph. In particular, we get $\bar{n}_* \cong \mathbb{G}(-, \bar{n})$; more generally, T_* is an amalgamated sum of standard cells: one generating cell for each input sector of T, and one relation for each inner T-sector.*

Proof. The source (resp. target) of a T-sector $(y; y', y'')$ is given by $(i(y); x, y)$ (resp. $(i(y); y, z)$), where x, y, z are consecutive elements of the fibre $i^{-1}(i(y))$. If y is minimal (resp. maximal) then $x = i(y)_-$ (resp. $z = i(y)_+$). The globular identities hold, since the source and target maps only depend on the first component of a T-sector and moreover, the source and target of a T-sector have the same first component.

The linear n-level tree has a unique input sector of height n and two sectors at each level below; this corresponds precisely to the cells of the standard n-cell $\mathbb{G}(-, \bar{n})$. The source/target maps also coincide.

Finally, like any presheaf, the ω-graph T_* admits the following colimit decomposition: $\varinjlim_{\tau \in \mathrm{el}(T_*)} \overline{\mathrm{ht}(\tau)}_* \cong T_*$, where $\mathrm{el}(T_*)$ is the *category of elements* of T_*. An inspection of the category $\mathrm{el}(T_*)$ reveals that it is a partially ordered set with maximal elements given by the input sectors of T, and "intersections" given by the inner T-sectors. ∎

The star-construction is due to Batanin [2, p. 61]. The ω-graphs T_* are the cell-configurations we need for the underlying site of a globular theory. The following lemma explicitly describes the maps between these basic ω-graphs.

LEMMA 1.3. *Let S, T be level trees.*

(a) *Any map of ω-graphs $S_* \to T_*$ is* injective.

(b) *The inclusions $S_* \hookrightarrow T_*$ correspond bijectively to* cartesian subobjects[1] *of T_* or, equivalently, to* plain subtrees *S of T with a specific choice of T-sector for each input vertex of S.*

Proof. (a) Injectivity follows from the following three observations:

1. the S-sectors of a given height can be totally ordered "from left to right";

[1] We call a map of ω-graphs $f_* : X_* \to Y_*$ cartesian if for each $n > 0$, the square $(s_n, t_n) f_n = (f_{n-1} \times f_{n-1})(s_n, t_n)$ is cartesian.

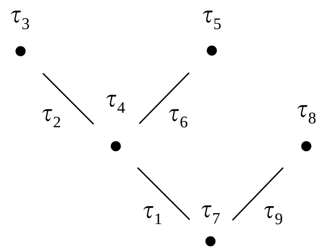
2. for any two *consecutive* S-sectors σ_1, σ_2 of a given height, there exists either an S-sector σ with $s(\sigma) = \sigma_1$ and $t(\sigma) = \sigma_2$, or an integer $k > 0$ with $t^k(\sigma_1) = s^k(\sigma_2)$;
3. there is no T-sector τ with $t^k(\tau) = s^k(\tau)$ for $k > 0$.

A map of ω-graphs $S_* \to T_*$ preserves the horizontal orders defined in (1), so that in view of (2), non-injectivity would produce a counter-example to (3).

(b) Cartesian subobjects X_* of T_* have the characteristic property that cells $y \in T_*$ for which source and target belong to X_* already belong to X_*. This implies that the colimit cone $\varinjlim_{\tau \in \mathrm{el}(T_*)} \overline{\mathrm{ht}(\tau)}_* \cong T_*$, pulled back to X_*, yields a colimit cone $\varinjlim_{\sigma \in \mathrm{el}(S_*)} \{\overline{\mathrm{ht}(\sigma)}\}_* \cong X_*$, where the level tree S is obtained from T by removing all vertices which do not support any T-sector in X_*. In particular, the subtree S of T is plain and $S_* \cong X_* \to T_*$. Different inclusions $S_* \to T_*$ correspond to different T-sectors for the input vertices of S. ∎

EXAMPLE 1.4.

$$\begin{array}{c} T(2) \\ \downarrow{\scriptstyle i_T} \\ T(1) \\ \downarrow{\scriptstyle i_T} \\ T(0) \end{array}$$



$$T_*: \quad \begin{array}{l} \tau_1 \xrightarrow{\tau_2} \tau_7 \\ \quad\ \ \| \\ \quad\ \ \tau_3 \\ \quad\ \ \Downarrow \\ \tau_1 \xrightarrow{\tau_4} \tau_7 \xrightarrow{\tau_8} \tau_9 \\ \quad\ \ \| \\ \quad\ \ \tau_5 \\ \quad\ \ \Downarrow \\ \tau_1 \xrightarrow{\tau_6} \tau_7 \end{array}$$

The 2-level tree T has three input sectors τ_3, τ_5, τ_8 and two inner sectors τ_4, τ_7. Its dimension is 4. The associated ω-graph T_* is an amalgamated sum of three standard cells: $T_* \cong (\bar{2}_* \cup_{\bar{1}_*} \bar{2}_*) \cup_{\bar{0}_*} \bar{1}_*$. Since T is binary, all its subtrees are plain. The truncation of T at level 1 embeds in three different ways as plain subtree of T according to the T-sectors τ_2, τ_4, τ_6.

The total order of the T-sectors (resp. the cells of T_*) given by the indices is the one obtained by running through T (resp. T_*) from left to right in the obvious manner. Street [37] defines a natural *partial order* for the cells of an arbitrary ω-graph and shows that the ω-graphs T_* can be characterised as those which are finite and for which this partial order is the above given total order. This leads to an alternative proof of Lemma 1.3 as pointed out by Mark Weber.

DEFINITION 1.5. The *globular site* Θ_0 has as objects the level trees and as morphisms the maps between the associated ω-graphs. Such a map is also called an *immersion*. The covering sieves are given by epimorphic families of immersions.

A *globular theory* is a category Θ_A having same object set as Θ_0 and containing Θ_0 as a subcategory in such a way that the representable presheaves $\Theta_A(-, T)$ are Θ_A-models.

A Θ_A*-model* is a presheaf on Θ_A which restricts to a sheaf on Θ_0.

For an ω-graph X and a level tree T, the T*th power* of X is defined by $X^T = \mathrm{Hom}_{\mathbb{G}}(T_*, X)$.

LEMMA 1.6. *The forgetful functor from sheaves on the globular site to ω-graphs is an equivalence of categories.*

Proof. The globe category $\mathbb{G}$ embeds in the globular site Θ_0 as the full subcategory spanned by the linear level trees, thus defining the forgetful functor. The minimal covering sieves of the globular site induce the standard colimit cones $\varinjlim_{\tau\in\mathrm{el}(T_*)}\overline{\mathrm{ht}(\tau)}_* \xrightarrow{\sim} T_*$. Therefore, a presheaf on Θ_0 is a sheaf if and only if for all level trees T,

$$X(T) \cong \varprojlim_{\tau\in\mathrm{el}(T_*)} X(\overline{\mathrm{ht}(\tau)}) = \mathrm{Hom}_{\mathbb{G}}\left(\varinjlim_{\tau\in\mathrm{el}(T_*)} \overline{\mathrm{ht}(\tau)}_*, X\right) = \mathrm{Hom}_{\mathbb{G}}(T_*, X).$$

The value $X(T)$ of a sheaf X at T can thus be identified with the Tth power X^T of the *underlying* ω-graph X. ∎

Remark 1.7. The formal analogy with Lawvere's *algebraic theories* [26] is instructive. The *algebraic site* $\mathscr{T}_0$ is the category of *finite sets* $\underline{n} = \{1, \dots, n\}$, $n \geqslant 0$, with epimorphic families as covering sieves. The *minimal* covering sieves induce the colimit cones $\underline{1} \sqcup \cdots \sqcup \underline{1} \xrightarrow{\sim} \underline{n}$, so that sheaves on the algebraic site are graded sets $X(\underline{n})$, $n \geqslant 0$, endowed with bijections $X(\underline{n}) \cong X(\underline{1})^n, n \geqslant 0$. An *algebraic theory* is then a "coproduct preserving" extension $\mathscr{T}_A$ of $\mathscr{T}_0$; a $\mathscr{T}_A$-model is a presheaf on $\mathscr{T}_A$ which restricts to a sheaf on $\mathscr{T}_0$.

Sheaves on the *algebraic site* are just *sets* with prescribed *powers*. Sheaves on the globular site are just *ω-graphs* with prescribed *tree-powers*. The operators of an algebraic theory induce thus operations of type $X^n \to X^m$, the operators of a globular theory induce operations of type $X^T \to X^S$.

The terminal operadic theory among algebraic theories is the theory of *commutative monoids*. The terminal operadic theory among globular theories is the theory of *ω-categories*. Spaces which are "up to homotopy" commutative monoids are so called *E_∞-spaces*.

ω-Graphs which are "up to homotopy" ω-categories are so-called *weak ω-categories*. Group-like E_∞-spaces are models for infinite loop spaces as shown by Boardman–Vogt [8], May [29] and Segal [34]. Weak ω-groupoids are models for general spaces as conjectured by Grothendieck [23] and Batanin [2]. The main motivation for this text was to give a proof of this conjecture; we succeed only partly, cf. Section 4.

Among algebraic theories only those "presentable" by operations of type $X^n \to X$ are operadic. Among globular theories only those presentable by operations of type $X^T \to X^{\bar{n}}$ for $\mathrm{ht}(T) \leqslant n$ are ω-operadic. The main point in the definition of ω-operads is the description of how these operations *compose*. This in turn relies on a thorough understanding of the free-forgetful adjunction between ω-graphs and ω-categories; geometrically, it involves *pasting of level trees*, which has to be opposed to the well-known *grafting of trees*.

DEFINITION 1.8. Let S, T be level trees.

- For an n-level tree T, the *truncation* $\partial_n T$ is the $(n-1)$-level tree obtained from T by removing all vertices of height n. For consistency, the truncation operator ∂_n is the identity on level trees of height less than n.
- The cosource (resp. cotarget) operator $(\partial_n T)_* \to T_*$ is specified by the left (resp. right) most T-sectors above the input vertices of $\partial_n T$, cf. Lemma 1.3(b). For an ω-graph X, these operators induce source/target maps $X^T \rightrightarrows X^{\partial_n T}$.
- The *monad* $(\underline{\omega}, \underline{\eta}, \underline{\mu})$ on the category of ω-graphs is defined by

$$\underline{\omega}(X)_n = \coprod_{\mathrm{ht}(T) \leqslant n} X^T$$

with unit $\underline{\eta} : \mathrm{id}_{\mathscr{S}^{\mathbb{G}^{\mathrm{op}}}} \to \underline{\omega}$ induced by Yoneda: $X_n \cong X^{\bar{n}}$, and multiplication $\underline{\mu} : \underline{\omega}^2 \to \underline{\omega}$ induced by pasting: each $\phi \in \underline{\omega}^2(X)_n = \coprod_{\mathrm{ht}(S) \leqslant n} \underline{\omega}(X)^S$ is considered as an *S-compatible family* $(\phi_\sigma \in \underline{\omega}(X)_{\mathrm{ht}(\sigma)})_{\sigma \in S_*}$. The pasting of this family is performed through the isomorphisms

$$\varprojlim_{\sigma \in \mathrm{el}(S_*)} X^{T_\sigma} \cong \mathrm{Hom}_{\mathbb{G}}\left(\varinjlim_{\sigma \in \mathrm{el}(S_*)} (T_\sigma)_*, X \right) \cong \mathrm{Hom}_{\mathbb{G}}(T_*, X) = X^T.$$

- The *cell category* Θ has as objects the level trees and as cellular operators $\Theta(S, T) = \mathrm{Alg}_{\underline{\omega}}(\underline{\omega}(S_*), \underline{\omega}(T_*)) = \mathrm{Hom}_{\mathbb{G}}(S_*, \underline{\omega}(T_*))$. A cellular operator $\phi : S \to T$ will be considered as an S-compatible family of immersions $(\phi_\sigma : (T_\sigma)_* \hookrightarrow T_*)_{\sigma \in S_*}$, cf. Lemma 1.3(a).

• A cellular operator $\phi \in \Theta(S,T)$ is a *cover* if the immersions ϕ_σ, $\sigma \in S_*$, form an epimorphic family. The subcategory of covers is denoted by Θ_{cov}. We identify the globular site Θ_0 with the $\underline{\omega}$-free $\underline{\omega}$-algebra maps in Θ.

• A cellular operator $\phi : S \to T$ is *level-preserving* if for all S-sectors $\sigma \in S_*$, we have $\mathrm{ht}(\sigma) = \mathrm{ht}(T_\sigma)$.

Remark 1.9. In [2], Batanin defines a *tree-diagram* to be a function which takes each vertex x of S to a plain subtree T_x of T such that $\mathrm{ht}(x) = \mathrm{ht}(T_x)$ and $T_{i_T(x)} = \partial_{\mathrm{ht}(x)} T$ and $T = \varinjlim_{x \in S} T_x$. Such *tree-diagrams correspond bijectively to level-preserving covers* $S \to T$, cf. Batanin–Street [3].

Indeed, a map of ω-graphs $\phi : S_* \to \underline{\omega}(T_*)$ is determined by the immersions $\phi_\sigma : (T_\sigma)_* \hookrightarrow T_*$ for input sectors $\sigma \in S_*$. These immersions are subject to relations determined by the *inner* S-sectors, cf. Lemma 1.2. In particular, the intersection $(T_\sigma)_* \cap (T_{\sigma'})_*$ inside T_* corresponds precisely to the embedded ω-graph $(T_{\sigma\cap\sigma'})_*$ where $\sigma \cap \sigma'$ is the inner S-sector which links the input sectors σ and σ'. Above this intersection, T_σ and $T_{\sigma'}$ have to be *distinct*. If ϕ is a cover, all input sectors of T belong to the image of ϕ_σ for some input sector $\sigma \in S_*$. This implies that a *cover* $\phi : S \to T$ is already determined by the *plain subtrees* T_σ of T for the input sectors $\sigma \in S_*$. For general cellular operators however, this is not true, cf. Lemma 1.3(b), Example 1.10 and Lemma 1.11.

EXAMPLE 1.10. We represent a cellular operator $\phi : S \to T$ by the corresponding family of immersions $\phi_\sigma : (T_\sigma)_* \hookrightarrow T_*$ (or simply their domains T_σ), where σ runs from left to right through the input sectors of S.

(a) For each level tree T and integer $n \geqslant \mathrm{ht}(T)$, there is a unique cover $\phi : \bar{n} \overset{(T)}{\to} T$. This cover is level-preserving if and only if $n = \mathrm{ht}(T)$.

(b) The full subcategory of Θ spanned by the linear trees $\bar{n}$ is isomorphic to the *reflexive* globe category $\bar{\mathbb{G}}$. The immersions correspond to cosource/cotarget operators, the covers to coidentity operators.

(c) For each level tree S, the identity $1_S : S \overset{(t_\sigma)}{\to} S$ is the only cellular operator that is at once an immersion and a cover. Here, t_σ is the linear subtree of S joining the input sector σ to the root.

(d) The full subcategory $\Theta^{(1)}$ of Θ spanned by the 0- and 1-level trees is isomorphic to the simplex category Δ.

The codegeneracy $s_i : [n+1] \to [n]$ is given by $s_i = (t_1, \ldots, r_{i\,i+1}, \ldots, t_n)$ where t_i denotes the ith edge of the 1-level tree $[n]$ and $r_{i\,i+1}$ denotes the root of $[n]$ embedded by the sector between the ith and $(i+1)$th edge.

The coface $\partial_i : [n-1] \to [n]$ is given by $\partial_i = (t_1, \ldots, v_{i\,i+1}, \ldots, t_n)$ where $v_{i\,i+1}$ denotes the subtree of $[n]$ consisting of the ith and $(i+1)$th edge. In both cases, the 0th and the $(n+1)$th edge of the 1-level tree $[n]$ are the *virtual* left and right horizontal edges at the root.

The codegeneracy operators are covers that are not level-preserving, the "outer" coface operators ∂_0, ∂_n are immersions, the remaining "inner" coface operators ∂_i, $0<i<n$, are level-preserving covers.

(e) Let V be the 2-level tree with two input vertices of height 2 and two vertices of height 1 having both one vertex above them. Let Y be the 2-level tree with two input vertices of height 2 and one vertex of height 1. In other words, for any 2-graph X, X^V consists of *horizontally* composable pairs of 2-cells and X^Y consists of *vertically* composable pairs of 2-cells of X.

Now, the unique cover $\bar{2} \to V$ factors in two ways as $\bar{2} \to Y \rightrightarrows V$ according to the level-preserving covers $w_{12}, w_{21} : Y \rightrightarrows V$ given by $w_{12} = (W_1, W_2)$ and $w_{21} = (W_2, W_1)$ where W_i is the subtree of V having same 1-level-truncation as V but only one vertex of height 2 at the ith place. This factorisation corresponds to the two ways a horizontal composite $X^V \to X^{\bar{2}}$ may be written as vertical composite $X^Y \to X^{\bar{2}}$ of whiskerings $X^V \to X^Y$. All Godement commutation rules of a 2-category can be deduced from this one.

LEMMA 1.11. *The cell category Θ is a globular theory. Each cellular operator factors uniquely into a cover followed by an immersion.*

Proof. The immersions of the globular site Θ_0 are identified with the $\underline{\omega}$-free $\underline{\omega}$-algebra maps in Θ. Each cellular operator $\phi : S \to T$ factors through the $\underline{\omega}$-free map $\underline{\omega}(\bigcup_{\sigma \in S_*}(T_\sigma)_* \hookrightarrow T_*)$. This is the *unique* factorisation of ϕ into a cover followed by an ω-free map for any two such factorisations have the same underlying immersion part and immersions are monic in the category of ω-graphs by Lemma 1.3(a).

Moreover, the representable presheaf $\Theta(-, T)$ restricted to the globular site is given by the functor $\mathrm{Hom}_{\mathbb{G}}(-, \underline{\omega}(T_*))$. This functor is a sheaf, since it transforms colimits into limits. ∎

THEOREM 1.12. *The category of ω-categories is isomorphic to the category of $\underline{\omega}$-algebras. The resulting embedding of the cell category Θ into* ω-Cat *is dense, i.e., the induced nerve $\mathcal{N}_\omega : \omega\text{-Cat} \to \mathcal{S}^{\Theta^{\mathrm{op}}}$ is a fully faithful functor from ω-categories to cellular sets. Its image consists precisely of the Θ-models.*

Proof. By definition, an $\underline{\omega}$-algebra is an ω-graph X endowed with a single operation $X^T \to X_n$ for each level tree T and each n with $\mathrm{ht}(T) \leqslant n$. Therefore, each level tree T induces a unique pasting operation $m_T : X^T \to X_{\mathrm{ht}(T)}$ together with identity cells in degrees $n > \mathrm{ht}(T)$. The level trees T with $d(T) = \mathrm{ht}(T) + 1$ define in this manner the *vertical compositions* and *whiskerings* of an ω-category. Since the cover $\bar{n} \to T$ of an n-level tree T with $d(T) = \mathrm{ht}(T) + 2$ factors in exactly two ways into non-identity level-preserving covers, the unicity of m_T yields the *associativity relations* and generating *Godement relations* of an ω-category. This proves that an $\underline{\omega}$-algebra is indeed an ω-category. Conversely, the underlying ω-graph of an ω-category has a uniquely determined $\underline{\omega}$-algebra structure by the well-known coherence theorem for ω-categories.

Therefore, the cell category Θ is canonically embedded in ω-Cat. Moreover, the cellular nerve of an ω-category X is given by $\mathcal{N}_\omega X = \omega\text{-Cat}(\underline{\omega}(-_*), X)$ which, when restricted to the globular site, yields the sheaf $\mathrm{Hom}_{\mathbb{G}}(-, X)$. It remains to show that the cellular nerve is an equivalence of categories onto the category Mod_Θ of Θ-models. For this it suffices to prove that

(a) the forgetful functor $j^* : \mathrm{Mod}_\Theta \to \mathrm{Shv}(\Theta_0)$ is *monadic*,
(b) the monad $j^*j_!$ may be identified under Lemma 1.6 with the monad $\underline{\omega}$.

To this end, consider the following commutative diagram whose left square is commutative since $X^T = \mathrm{Hom}_{\mathbb{G}}(T_*, X) = \mathrm{Hom}_{\underline{\omega}}(\underline{\omega}(T_*), X) = (\mathcal{N}_\omega X)(T)$:

$$\begin{tikzcd}
\omega\text{-Cat} \arrow[r, "\mathcal{N}_\omega"] \arrow[d, "k^*"] & \mathrm{Mod}_\Theta \arrow[r, hook] \arrow[d, "j^*"] & \mathcal{S}^{\Theta^{\mathrm{op}}} \arrow[d, "i^*"] \\
\mathcal{S}^{\mathbb{G}^{\mathrm{op}}} \arrow[r, "\sim"] & \mathrm{Shv}(\Theta_0) \arrow[r, hook] & \mathcal{S}^{\Theta_0^{\mathrm{op}}}
\end{tikzcd}$$

Since Θ-models are the models of a limit sketch on Θ^{op}, the category of Θ-models is complete and accessible, hence *cocomplete*, cf. [9, II.5.5.8]. In particular, the embedding $\mathrm{Mod}_\Theta \hookrightarrow \mathcal{S}^{\Theta^{\mathrm{op}}}$ admits a left adjoint *reflector* $a : \mathcal{S}^{\Theta^{\mathrm{op}}} \to \mathrm{Mod}_\Theta$. It follows that the forgetful functor j^* has a left adjoint functor $j_! = a \circ i_!$ where $i_!$ is the left adjoint of i^*.

The monadicity of j^* follows from the monadicity of i^* using *Beck's criterion*. Indeed, i^* detects isomorphisms, so j^* as well; moreover, for each parallel pair (f, g) in Mod_Θ for which $(j^*(f), j^*(g))$ has a split coequaliser, there is a lifted coequaliser in $\mathcal{S}^{\Theta^{\mathrm{op}}}$ whose reflection in Mod_Θ yields the required lift for (f, g).

Since $\mathrm{Shv}(\Theta_0)$ is reflective in $\mathcal{S}^{\Theta_0^{\mathrm{op}}}$ and since the monad $i^*i_!$ is cocontinuous, a sheaf X is sent to a *sheaf* $i^*i_!X$, whence $j^*j_!X = j^*ai_!X = i^*i_!X$. The factorisation property, Lemma 1.11, implies then that for any sheaf X on Θ_0:

$$(i_! X)(S) = \coprod_{\Theta_{\mathrm{cov}}(S,T)} X(T),$$

$$(j^* j_! X)(\bar{n}) = (i^* i_! X)(\bar{n}) = \coprod_{\Theta_{\mathrm{cov}}(\bar{n},T)} X^T = \coprod_{\mathrm{ht}(T) \leqslant n} X^T = \underline{\omega}(X)_n.$$

Therefore, the monad $j^* j_!$ corresponds under Lemma 1.6 to the monad $\underline{\omega} = k^* k_!$. ∎

Remark 1.13. The preceding theorem extends Grothendieck–Segal's [33] characterisation of nerves of small categories to nerves of ω-categories and implies moreover that the full subcategory $\Theta^{(1)}$ of Θ spanned by the 0- and 1-level trees is canonically isomorphic to the *simplex category* Δ, cf. Example 1.10(d).

Indeed, the globular site Θ_0 restricts to a one-dimensional site $\Theta_0^{(1)}$. Sheaves on this site are *graphs* $X_1 \rightrightarrows X_0$ with prescribed $[n]$-powers

$$X^{[n]} = \mathrm{Hom}_{\mathbb{G}^{(1)}}([n]_*, X) = X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1.$$

The factorisation property, Lemma 1.11, restricts to $\Theta^{(1)}$. By a simplified version of the above proof, the category of $\Theta^{(1)}$-models is equivalent (via a nerve functor) to the category of "graphical monoids", which is nothing but the category of *small categories*. In particular, $\Theta^{(1)}([m],[n]) \cong \mathrm{Cat}(\underline{\omega}([m]_*), \underline{\omega}([n]_*)) = \Delta([m],[n])$, since the free category generated by the graph $[n]_*$ is the finite ordinal $[n]$.

Immersions $[m] \to [n]$ correspond under this bijection to simplicial operators $\phi : [m] \to [n]$ such that $\phi(i+1) = \phi(i) + 1$ for all i. Covers $[m] \to [n]$ correspond to simplicial operators $\phi : [m] \to [n]$ such that $\phi(0) = 0$ and $\phi(m) = n$.

A simplicial set X is thus the nerve of a small category if and only if it is a Δ-model, i.e. $X([n]) \cong X^{[n]}$ for all n. A cellular set X is the nerve of an ω-category if and only if it is a Θ-model, i.e., $X(T) \cong X^T$ for all level trees T.

Let $\Theta^{(n)}$ be the full subcategory of Θ spanned by the level trees of height less than or equal to n. The filtration $\cdots \subset \Theta^{(n)} \subset \Theta^{(n+1)} \subset \cdots$ is compatible with the factorisation property, Lemma 1.11, and defines for each n an *n-graphical theory* extending the restricted site $\Theta_0^{(n)} = \Theta_0 \cap \Theta^{(n)}$. As above, this realises $\Theta^{(n)}$ as a dense subcategory of the category of n-categories. The nerve of an n-category is thus in a natural way a presheaf on $\Theta^{(n)}$ or as we shall say an *n-cellular set*.

There is another natural nerve for n-categories: the *n-simplicial nerve*, inductively defined using the fact that an n-category is a small category enriched in $(n-1)$-Cat. This results in a uniquely determined family of functors $m_n : \Delta^{\times n} \to \Theta^{(n)}$ such that m_n^* takes an n-cellular nerve to the corresponding n-simplicial nerve. The existence of these functors is central in Simpson and Tamsamani's approach to weak ω-category, cf. [35, 38].

The image of m_n contains precisely the level trees having *same edge-valency for all vertices of a given height*. The identifications induced by m_n are easily described using the fact that simplicial nerves are Δ-models. The induced quotient category $\Delta^{\times n}/\sim$ is Simpson's quotient Θ_n, cf. [35].

In view of Remark 2.5, the embedding $\Theta_n \hookrightarrow \Theta^{(n)}$ factors through the *Cauchy-completion* $\bar{\Theta}_n$ of Θ_n, cf. [9, I.6.5.9]. However, even this Cauchy-completed embedding is not a full embedding which makes the comparison between n-cellular and n-simplicial nerves rather difficult.

Remark 1.14. An ω-*operad* (n-*operad*) in Batanin's sense [2] is somehow the minimal data necessary to *generate* a globular (n-graphical) theory enjoying a factorisation property like Lemma 1.11. In order to describe this generation process, we recall here Batanin's definition of an ω-operad. There is a small difference between ours and Batanin's notation insofar as we do not allow level trees with *empty* levels: instead of $A(Z^kT)$, we shall thus write $A(T)_{\mathrm{ht}(T)+k}$.

An ω-*collection* is a doubly indexed family of sets $A(T)_n$, T being a level tree such that $\mathrm{ht}(T) \leqslant n$, endowed with (mutually compatible) ω-graph structures

$$\cdots \rightrightarrows A(T)_{n+1} \rightrightarrows A(T)_n \rightrightarrows A(\partial_n T)_{n-1} \rightrightarrows A(\partial_{n-1}\partial_n T)_{n-2} \rightrightarrows \cdots \rightrightarrows A(\bar{0})_0$$

for each n-level tree T. The truncation operator ∂_n has been defined in Definition 1.8. The *total* ω-*graph* associated to A is given by

$$(\mathrm{tot}\, A)_n = \coprod_{\mathrm{ht}(T) \leqslant n} A(T)_n.$$

The category of ω-collections carries a monoidal structure with respect to the following *circle product*: Given two ω-collections A, B, the ω-collection $A \circ B$ is defined by

$$(A \circ B)(T)_n = \coprod_{\phi \in \Theta_{\mathrm{cov}}(S,T)} A(S)_n \times B(\phi),$$

where $B(\phi) = \{f \in \mathrm{Hom}_{\mathbb{G}}(S_*, (\mathrm{tot}\, B)_*) \mid f(\sigma) \in B(T_\sigma)_{\mathrm{ht}(\sigma)} \text{ for all } \sigma \in S_*\}$.

The *associativity* of the circle product follows from the following parentheses independent description of the k-fold circle product, cf. [2, 6.1]:

$$(A_0 \circ \cdots \circ A_k)(T)_n = \coprod_{(\phi_1,\ldots,\phi_k)} A_0(S)_n \times A_1(\phi_1) \times \cdots \times A_k(\phi_k),$$

where the coproduct is indexed by cover-chains

$$S \xrightarrow{\phi_1} S_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{k-1}} S_{k-1} \xrightarrow{\phi_k} T.$$

The unit ω-collection $\emptyset$ is defined by requiring that $\emptyset(T)_n$ is singleton if $T = \bar{n}$ and empty otherwise.

An *ω-operad* is a $\circ$-monoid (A, η_A, μ_A) in the category of ω-collections. Each ω-operad A induces a monad $(\underline{A}, \underline{\eta}_A, \underline{\mu}_A)$ on the category of ω-graphs by

$$\underline{A}(X)_n = \coprod_{\mathrm{ht}(T) \leqslant n} A(T)_n \times X^T.$$

This follows either from Street's [37] isomorphism $\underline{A \circ B} \cong \underline{A} \circ \underline{B}$ or from the fact that ω-graphs are *enriched* in ω-collections in such a way that there is an adjunction between maps of ω-collections $A \to \underline{\mathrm{hom}}_{\mathbb{G}}(X, Y)$ and globular maps $\underline{A}(X) \to Y$. In particular, an $\underline{A}$-algebra can also be defined as an ω-graph X endowed with a map of *ω-operads* $A \to \underline{\mathrm{hom}}_{\mathbb{G}}(X, X)$, cf. [2, 7.2].

We invite the reader to check that the monad $(\underline{\omega}, \underline{\eta}, \underline{\mu})$ associated to the free-forgetful adjunction between ω-graphs and ω-categories, cf. Definition 1.8, is induced by the *terminal ω-operad* (ω, η, μ). Of course, $\omega(T)_n$ is singleton everywhere; this defines the unit $\eta : \emptyset \to \omega$. The multiplication $\mu : \omega \circ \omega \to \omega$ is the most elementary instance of the pasting of level trees.

Definition 1.15. A globular theory Θ_A is *homogeneous* if Θ_A contains a subcategory Θ^A_{cov} of A-covers such that any operator factors uniquely into an A-cover followed by an immersion.

A faithful monad $\underline{A}$ on ω-graphs defines a globular theory Θ_A with operators $\Theta_A(S, T) = \mathrm{Alg}_{\underline{A}}(\underline{A}(S_*), \underline{A}(T_*))$, where the immersions are identified with the $\underline{A}$-free $\underline{A}$-algebra maps. A faithful monad $\underline{A}$ is called homogeneous if the globular theory Θ_A is homogeneous and if moreover any $\underline{A}$-algebra map $\underline{A}(\bar{n}_*) \to \underline{A}(X)$ factors uniquely into an A-cover followed by an $\underline{A}$-free $\underline{A}$-algebra map.

Street defines *analytic* endofunctors $\underline{A}$ as those which come equipped with a *cartesian* transformation $\underline{A} \to \underline{\omega}$, where a natural transformation is cartesian if all naturality squares are cartesian. Analytic endofunctors (resp. monads) correspond bijectively to ω-collections (resp. ω-operads), see [37]. The following proposition shows that we have actually *three* equivalent concepts: ω-operads A, analytic monads $\underline{A}$ and homogeneous globular theories Θ_A.

PROPOSITION 1.16. *For a monad on ω-graphs, the following three properties are equivalent*:

1. *The monad is induced by an ω-operad.*
2. *The monad is analytic.*
3. *The monad is faithful and homogeneous.*

The generating ω-operad A of the homogeneous globular theory Θ_A is given by the set of A-covers with linear domain.

Proof. Condition (1) implies (2). Any ω-operad A comes equipped with a map of ω-operads $A \to \omega$ to the terminal ω-operad. Any ω-graph X comes equipped with a map of ω-graphs $X \to *_{\mathbb{G}}$ to the terminal ω-graph. We have thus for each $n \geqslant 0$ the following commutative squares:

$$\begin{array}{ccccccc} \coprod_{ht(T)\leq n} A(T)_n \times X^T & \longrightarrow & \coprod_{ht(T)\leq n} X^T & \qquad & \underline{A}(X)_n & \longrightarrow & \underline{\omega}(X)_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \coprod_{ht(T)\leq n} A(T)_n & \longrightarrow & \coprod_{ht(T)\leq n} \{*\} & & \underline{A}(*_{\mathbb{G}})_n & \longrightarrow & \underline{\omega}(*_{\mathbb{G}})_n \end{array}$$

The left square is visibly cartesian so that the isomorphic right square is cartesian too. This implies (by transitivity of cartesian squares) that the canonical transformation of monads $\underline{A} \to \underline{\omega}$ is cartesian, whence $\underline{A}$ is analytic.

Condition (2) implies (3). The unit $\underline{\eta}_A$ of an analytic monad $\underline{A}$ is monic, since the unit $\underline{\eta}$ of $\underline{\omega}$ is monic and $\underline{\eta}$ factors through $\underline{\eta}_A$. Therefore, $\underline{A}$ is faithful. By Theorem 1.12 and Lemma 1.11, an $\underline{\omega}$-algebra map $\underline{\omega}(S_*) \to \underline{\omega}(T_*)$ factors uniquely into a cover $\underline{\omega}(S_*) \to \underline{\omega}(T'_*)$ followed by a $\underline{\omega}$-free map $\underline{\omega}(T'_*) \to \underline{\omega}(T_*)$. The cartesianness of $\underline{A} \to \underline{\omega}$ implies then that an $\underline{A}$-algebra map $\underline{A}(S_*) \to \underline{A}(T_*)$ factors uniquely into an A-cover followed by an $\underline{A}$-free map; in other words, the A-covers are given by $\Theta^A_{\text{cov}} = p^{-1}\Theta_{\text{cov}}$, where $p : \Theta_A \to \Theta$ is the functor induced by the cartesian transformation $\underline{A} \to \underline{\omega}$. In particular, Θ^A_{cov} is a subcategory of Θ_A, whence the homogeneity of Θ_A. Since the monad $\underline{\omega}$ is homogeneous, any analytic monad $\underline{A}$ is homogeneous.

Condition (3) implies (1). The factorisation property of Θ_A implies that the family $A(T)_n = \Theta^A_{\text{cov}}(\bar{n}, T)$, $\text{ht}(T) \leqslant n$, is an ω-collection, endowed with maps

$$A(S)_n \times A(\phi) \to A(T)_n, \qquad \phi \in \Theta_{\mathrm{cov}}(S,T),$$

which are induced by composition $\Theta^A_{\mathrm{cov}}(\bar{n},S) \times \Theta^A_{\mathrm{cov}}(S,T) \to \Theta^A_{\mathrm{cov}}(\bar{n},T)$, since $\Theta^A_{\mathrm{cov}}(S,T) = \coprod_{\phi \in \Theta_{\mathrm{cov}}(S,T)} A(\phi)$. This yields a map $\mu_A : A \circ A \to A$ of ω-collections. The associativity of the composition of A-covers is equivalent to the associativity of μ_A, which shows that A is an ω-operad. The monad $\underline{A}$ induced by A coincides with the given monad $\underline{A}'$ since by homogeneity we have

$$\underline{A}'(X)_n = \coprod_{\Theta^A_{\mathrm{cov}}(\bar{n},T)} X^T = \coprod_{\mathrm{ht}(T) \leqslant n} A(T)_n \times X^T = \underline{A}(X)_n. \quad \blacksquare$$

THEOREM 1.17. *For each ω-operad A, the embedding $\Theta_A \hookrightarrow \mathrm{Alg}_{\underline{A}}$ is dense, i.e., the induced nerve $\mathcal{N}_A : \mathrm{Alg}_{\underline{A}} \to \mathcal{S}^{\Theta_A^{\mathrm{op}}}$ is a fully faithful functor from $\underline{A}$-algebras to A-cellular sets. Its image consists precisely of the Θ_A-models.*

Proof. We proceed exactly like in the proof of Theorem 1.12 using $i : \Theta_0 \hookrightarrow \Theta_A$:

$$\begin{tikzcd}
\mathrm{Alg}_{\underline{A}} \arrow[r, "\mathcal{N}_A"] \arrow[d, "k^*"] & \mathrm{Mod}_{\Theta_A} \arrow[r, hook] \arrow[d, "j^*"] & \mathcal{S}^{\Theta_A^{\mathrm{op}}} \arrow[d, "i^*"] \\
\mathcal{S}^{\mathbb{G}^{\mathrm{op}}} \arrow[r, "\sim"] & \mathrm{Shv}(\Theta_0) \arrow[r, hook] & \mathcal{S}^{\Theta_0^{\mathrm{op}}}
\end{tikzcd}$$

The factorisation property of Θ_A implies that for any presheaf X on Θ_0

$$(i_! X)(S) = \coprod_{\Theta^A_{\mathrm{cov}}(S,T)} X^T.$$

This yields properties (a) and (b) analogous to those in Theorem 1.12 so that $\mathcal{N}_A$ is an equivalence onto Mod_{Θ_A}, whence the density of Θ_A in $\mathrm{Alg}_{\underline{A}}$. $\blacksquare$

Remark 1.18. The outer rectangle above is the key diagram for most of the subsequent constructions. It is an example of a *lifted* adjunction, cf. [9, II.4.5]:

$$\begin{tikzcd}
\mathrm{Alg}_{\underline{A}} \arrow[r, "\mathcal{N}_A"] \arrow[d, "k^*"'] & \mathcal{S}^{\Theta_A^{\mathrm{op}}} \arrow[d, "i^*"] \\
\mathcal{S}^{\mathbb{G}^{\mathrm{op}}} \arrow[r, "\mathcal{N}_\emptyset"] & \mathcal{S}^{\Theta_0^{\mathrm{op}}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Alg}_{\underline{A}} & \mathcal{S}^{\Theta_A^{\mathrm{op}}} \arrow[l, "cat_A"'] \\
\mathcal{S}^{\mathbb{G}^{\mathrm{op}}} \arrow[u, "k_!"] & \mathcal{S}^{\Theta_0^{\mathrm{op}}} \arrow[l, "cat_\emptyset"'] \arrow[u, "i_!"']
\end{tikzcd}$$

Here, cat_A denotes the left adjoint of $\mathcal{N}_A$ which exists by Theorem 1.17. The two squares commute by construction, i.e. $i^*\mathcal{N}_A = \mathcal{N}_\emptyset k^*$ and $\mathrm{cat}_A i_! = k_!\mathrm{cat}_\emptyset$. The situation is even better, since we have an isomorphism $i_!\mathcal{N}_\emptyset \cong \mathcal{N}_A k_!$ expressing the fact that the monad $i^*i_!$ on $\mathcal{S}^{\Theta_0^{\mathrm{op}}}$ extends the monad $k^*k_!$ on $\mathcal{S}^{\mathbb{G}^{\mathrm{op}}}$. In particular, we get for each level tree T that the A-cellular nerve of the *free $\underline{A}$-algebra on T_** may be identified with the *representable model* $\Theta_A[T]$:

$$\mathcal{N}_A k_! T_* \cong i_!\mathcal{N}_\emptyset T_* = i_!\Theta_0[T] = \Theta_A[T].$$

There are also formulae for the "A-categorification" $\mathrm{cat}_A : \mathcal{S}^{\Theta_A^{\mathrm{op}}} \to \mathrm{Alg}_{\underline{A}}$. Indeed, for an A-cellular set X, the $\underline{A}$-algebra $\mathrm{cat}_A\, X$ is a canonical *quotient* of the *free $\underline{A}$-algebra $k_!Y$* on the ω-graph *Y underlying X*. More precisely, we have a cocartesian square of A-cellular sets:

$$\begin{tikzcd}
i_!i^*X \arrow[r] \arrow[d] & \mathcal{N}_A cat_A X i_!i^*X = \mathcal{N}_A k_! Y \arrow[d] \\
X \arrow[r] & \mathcal{N}_A cat_A X
\end{tikzcd}$$

In the truncated case of the simplex category $\Delta = \Theta^{(1)}$, this formula yields the well-known *categorification* of a simplicial set X, which is a canonical quotient of the *free category* on the *graph* underlying X, cf. [20].

Our adjoint square is also interesting from a topos-theoretic point of view. The category of $\underline{A}$-algebras may be considered as a monadic analogue of the usual comonadic factorisation of the *geometric morphism* $\mathcal{S}^{\mathbb{G}^{\mathrm{op}}} \rightarrowtail \mathcal{S}^{\Theta_0^{\mathrm{op}}} \twoheadrightarrow \mathcal{S}^{\Theta_A^{\mathrm{op}}}$, cf. [27]. The monadic and comonadic factorisations are in general distinct so that the category of $\underline{A}$-algebras is far away from being a topos. In the sequel, it will however be important that the free-forgetful monad $k^*k_!$ extends in this way to a *cocontinuous* monad $i^*i_!$ on the presheaf topos $\mathcal{S}^{\Theta_0^{\mathrm{op}}}$. This enables us to define a suitable model structure for simplicial $\underline{A}$-algebras, cf. Theorem 4.13, and yields

COROLLARY 1.19. *For each ω-operad A, the category of $\underline{A}$-algebras is locally finitely presentable. In particular, the forgetful functor from $\underline{A}$-algebras to ω-graphs preserves filtered colimits.*

Proof. The forgetful functor from $\underline{A}$-algebras to ω-graphs preserves filtered colimits as soon as the free-forgetful monad $\underline{A}$ does, cf. [9, II.4.3.2]. The monad $i^*i_!$ on $\mathcal{S}^{\Theta_0^{\mathrm{op}}}$ restricts to the monad $j^*j_!$ on $\mathrm{Shv}(\Theta_0)$ which may be identified with the monad $\underline{A} = k^*k_!$ on ω-graphs; the embedding

$\mathrm{Shv}(\Theta_0) \hookrightarrow \mathcal{S}^{\Theta_0^{\mathrm{op}}}$ detects filtered colimits, and the monad $i^*i_!$ preserves them, so that $j^*j_!$ (resp. $\underline{A}$) preserves them.

The category of ω-graphs has a dense set of finite generators, namely the set of standard n-cells, $n \geqslant 0$. Since the forgetful functor detects isomorphisms and preserves filtered colimits, the free $\underline{A}$-algebras generated by the standard n-cells form a strongly generating set of finite $\underline{A}$-algebras. Moreover, the category of $\underline{A}$-algebras is *cocomplete* as reflective subcategory of a presheaf category; therefore, it is locally finitely presentable, cf. [9, II.5]. ∎

DEFINITION 1.20. An ω-operad A is *contractible* if for any pair of elements x, y of $A(T)_n$ with same source and target, and for any level tree S with $\partial_{n+1} S = T$, there exists an element $z \in A(S)_{n+1}$ such that $s_n(z) = x$ and $t_n(z) = y$.

A weak *ω-category* is an $\underline{A}$-algebra X for a contractible ω-operad A.

There is a global description of this contractibility condition in terms of the associated analytic monad $\underline{A}$. Indeed, an ω-operad A is contractible if and only if for every ω-graph X and every integer $n \geqslant 0$, any commutative square

$$\begin{tikzcd} \partial\bar{n}_* \arrow[r] \arrow[d] & \underline{A}(X) \arrow[d] \\ \bar{n}_* \arrow[r] \arrow[ur, dashed] & \underline{\omega}(X) \end{tikzcd}$$

admits a diagonal filler as indicated. Here, $\partial\bar{n}_*$ denotes the *formal boundary* of $\bar{n}_*$, i.e., the union of all m-cells of $\bar{n}_*$ with $m < n$. For $X = *_{\mathbb{G}}$, this is just a reformulation of the contractibility condition; conversely, the special case implies the general case, since the monad $\underline{A}$ is analytic.

EXAMPLE 1.21. Since the terminal ω-operad is contractible, ω-categories are weak ω-categories in the above sense. Every n-operad A, which satisfies the contractibility condition at level trees of height less than n, embeds canonically in a *contractible* $(n+1)$-operad A by a kind of globular kernel construction.

Indeed, for a level tree T of height less than n, let $A(T)_{n+1} \rightrightarrows A(T)_n$ be the *equivalence relation* [9] that relates two elements precisely when they lie in the same fibre of $A(T)_n \rightrightarrows A(T)_{n-1}$; if $\mathrm{ht}(T) = n$, resp. $\mathrm{ht}(T) = n+1$, let $A(T)_{n+1}$ be the equivalence relation induced by the fibres of $A(T)_n \rightrightarrows A(\partial_n T)_{n-1}$, resp. of $A(\partial_{n+1} T)_n \rightrightarrows A(\partial_n \partial_{n+1} T)_{n-1}$, cf. the proof of [2, 8.1].

Bénabou's *bicategories* [6] are the algebras for a contractible 2-operad B: $B(\bar{0})_0$ is singleton and $B([n])_1$ consists of the set of complete bracketings of the input vertices of $[n]$, i.e., $B([2])_1 = \{(12)\}$, $B([3])_1 = \{((12)3), (1(23)), (123)\}$, and so on. For each cover $\phi \in \Theta^{(1)}_{\mathrm{cov}}([n],[m]) = \Delta_{\mathrm{cov}}([n],[m])$, there is an operad-multiplication $B([n])_1 \times B(\phi) \to B([m])_1$. For non-degenerate ϕ, this map is given by substituting bracketings and yields Stasheff's operad of *associahedra*. The so defined 1-operad is contractible at the root $\bar{0}$ and embeds thus in a *contractible* 2-*operad* B whose algebras are precisely the *bicategories* [2, App.]. The B-algebra maps are Bénabou's *homomorphisms* of bicategories.

Gray-categories [22] are the algebras for a contractible 3-operad G: The underlying 2-operad acts on *sesquicategories*, i.e., "2-categories without Godement's relations". $G(\bar{0})_0$ as well as $G([n])_k$ are singleton for $n \geqslant 1$ and $k \geqslant 1$. For each 2-level tree T, $G(T)_2$ is the *set of* T-*shuffles*, i.e., the set of those permutations of $T(2)$ which respect the linear orders of the fibres. For an n-fold bunch $\bar{2}^{\vee n}$ of linear 2-level trees, $G(\bar{2}^{\vee n})_2$ is the entire symmetric group $\mathscr{S}_n$ and the operad-multiplication $G(\bar{2}^{\vee n})_2 \times G(\phi) \to G(\bar{2}^{\vee m})_2$ yields Milgram's operad of *permutohedra*, cf. [7]. The so-defined 2-operad is contractible at level trees of height 0 and 1, and embeds thus in a *contractible* 3-*operad* G whose algebras are precisely the Gray-categories, cf. [2, p. 94].

2. CELLULAR SETS AND THEIR GEOMETRIC REALISATION

In this section, we recall Joyal's geometric realisation of cellular sets [25]. This geometric realisation is a cocontinuous and finite limit preserving functor which on representable cellular sets $\Theta[T] = \Theta(-, T)$ yields *convex cells* of dimension equal to the dimension $d(T)$ of the level tree T.

The tree combinatorics induce a clever mixing of ball geometry with simplicial geometry. Indeed, we get canonical homeomorphisms $|\Theta[\bar{n}]| \cong B^n$ and $|\Theta[[n]]| \cong \Delta_n$, extending to the usual realisations of the globe category and the simplex category. More generally, the "vertical dimension" of a level tree induces the globular structure, the "horizontal dimension" the simplicial structure.

Cellular sets share some of the "magic" features of simplicial sets. There is an epi-mono factorisation of cellular operators such that the cells of a cellular set are supported by uniquely determined *non-degenerate* cells. In particular, the geometric realisation of a cellular set X is a CW-*complex* with exactly one $d(T)$-dimensional cell for each non-degenerate T-cell of X. Moreover, there is a canonical "prismatic" decomposition of the cartesian product $\Theta[S] \times \Theta[T]$ into a union of standard cells $\Theta[U]$, where U runs through the set of all "shuffled bouquets" of S and T.

In order to apply this geometric realisation also to *weak* ω-*categories*, we have to borrow some techniques from the theory of *homotopy colimits*. Indeed, for a homogeneous globular theory Θ_A, the left Kan extension along the canonical functor $\Theta_A \to \Theta$ sends Θ_A-models to Θ-models, cf. Remark 1.18. It corresponds to the left adjoint of base change along $\underline{A} \to \underline{\omega}$, and sends the underlying $\underline{A}$-algebra to an $\underline{\omega}$-algebra, i.e., ω-category. This *strictification* is (alas !) *not* homotopy invariant. We therefore replace the left Kan extension by its "homotopy invariant version" baptised *left Segal extension*, cf. [34, App. A; 24].

DEFINITION 2.1 (Joyal [25]). An ω-*disk* (in sets) is a diagram $D : \overline{\mathbb{G}} \to \mathscr{S}$ such that for each n, each fibre $D(i_n)^{-1}(x)$ is linearly ordered with minimum $D(s_n)(x)$ and maximum $D(t_n)(x)$, and such that $D(\bar{0})$ is singleton.

The reflexive globe category has a realisation $B : \overline{\mathbb{G}} \to \mathrm{Top}^c$ by euclidean n-balls $B(\bar{n}) = B^n,\ n \geqslant 0$, with cosource and cotarget operators given by the lower and upper hemisphere inclusions and coidentity operators given by orthogonal projection $B(i_n) : B^{n+1} \to B^n$. The functor B is thus an ω-disk in spaces.

Each level tree T gives rise to an ω-*disk*

$$\cdots \longrightarrow T(n) \xrightarrow{T(i_{n-1})} T(n-1) \xrightarrow{T(i_{n-2})} \cdots \xrightarrow{T(i_0)} T(0)$$

since there is a universal way to adjoin cosources and cotargets analogous to the adjunction of "left and right horizontal edges", cf. Definition 1.1, [25].

We shall denote the so-defined ω-disk by $\bar{T} : \overline{\mathbb{G}} \to \mathscr{S}$. A map of ω-disks is a natural transformation of diagrams which is *order-preserving in each fibre*. The set of ω-disk maps from D to D' is denoted by $\mathrm{Disk}(D, D')$.

The following proposition shows that Definition 1.8 of Θ agrees with Joyal's original definition [25] and that cellular sets have the promised geometric realisation. An alternative proof has been given by Makkai and Zawadowski [28].

PROPOSITION 2.2. *For level trees* S, T, *there is a natural bijection*

$$\Theta(S, T) \cong \mathrm{Disk}(\bar{T}, \bar{S}).$$

The covariant functor $\mathrm{Disk}(-, B) : \Theta \to \mathrm{Top}^c$ *is* flat, *i.e.*, *tensoring is a finite limit preserving functor* $X \mapsto |X| = X \otimes_\Theta \mathrm{Disk}(-, B)$ *from cellular sets to compactly generated spaces.*

Proof. A disk-map $f \in \mathrm{Disk}(\bar{T}, \bar{S})$ labels the T-vertices of height n by elements of $\bar{S}(n)$ in such a way that consecutive T-vertices of the same fibre have increasing labels, and T-edges correspond to $\bar{S}$-edges. If instead of the vertices (except the root) we label the outgoing edges, we get an order- and level-preserving $\bar{e}(S)$-labelling of the T-edges. A disk-map $f \in \mathrm{Disk}(\bar{T}, \bar{S})$ is thus an *order- and level-preserving* $\bar{e}(S)$*-labelling of* $e(T)$, cf. Definition 1.1.

We claim that such an $\bar{e}(S)$-labelling of $e(T)$ determines a cellular operator $\phi : S \to T$. Indeed, each S-sector $\sigma = (x; x', x'') \in S_*$ induces three plain subtrees $T_x, T_{x'}, T_{x''}$ of T corresponding to the set of T-edges labelled by the S-edges below x, x', x''. One has $T_x \subseteq T_{x'}$ and $T_x \subseteq T_{x''}$, and each input vertex of T_x supports a *unique* T-sector lying between $T_{x'}$ and $T_{x''}$. This defines an immersion $(T_\sigma)_* \hookrightarrow T_*$, cf. Lemma 1.3(b). For $\sigma \in S_*$, these immersions assemble into a cellular operator $\phi : S \to T$. Conversely, any cellular operator arises this way from a unique $\bar{e}(S)$-labelling of $e(T)$.

For the flatness of the functor $T \mapsto \mathrm{Disk}(\bar{T}, B)$, it is sufficient to show that its category of elements is *filtered*, cf. [27, VII]. For simplicity, we abbreviate $|T| = \mathrm{Disk}(\bar{T}, B)$. Given two pairs $(S, s), (T, t)$ with $s \in |S|$, $t \in |T|$, we have to find (U, u) with $u \in |U|$ and cellular operators $\phi : U \to S$, $\psi : U \to T$ such that $|\phi|(u) = s$ and $|\psi|(u) = t$. Moreover, parallel maps of pairs should admit equalising maps.

Like above, s, t are given by edge-labels: $s = (s_\alpha)_{\alpha \in e(S)}$, $t = (t_\beta)_{\beta \in e(T)}$, where the labels are points of B which respect level and order. The level tree U is then defined as a "shuffled bouquet" of S and T; indeed, we have to shuffle the root-edges of S with the root-edges of T in order to obtain increasing edge-labels above the root; the remaining edges and edge-labels of (S, s) and (T, t) stay as they are. The cellular operator ϕ (resp. ψ) is the degeneracy (see below) which forgets about T (resp. S).

Assume a parallel pair $\phi_1, \phi_2 : (S, s) \rightrightarrows (T, t)$. The cellular *Eilenberg Lemma* 2.4 shows that there is a *unique* level-preserving cellular operator $\psi : S' \to S$ such that the pre-image $s' = |\psi|^{-1}(s)$ is an interior point of $|S'|$. Since $|\phi_1|$ and $|\phi_2|$ send s to the same point t, their composites $|\phi_1 \psi|$ and $|\phi_2 \psi|$ coincide on the cell-interior and hence everywhere, which implies $\phi_1 \psi = \phi_2 \psi$. ∎

DEFINITION 2.3. A cellular operator $\phi : S \to T$ is a *degeneracy* if the level tree T is a subtree of S such that ϕ associates to each S-sector σ the maximal linear subtree of T below σ, cf. Definition 1.8.

A cell $x \in X(S)$ of a cellular set X is *degenerate* if there exists a non-identity degeneracy $\phi : S \to T$ and a cell $y \in X(T)$ such that $X(\phi)(y) = x$.

LEMMA 2.4. *Let X be a cellular set.*

(a) *Each cellular operator can be written in a unique way as a degeneracy followed by a level-preserving cellular operator. Degeneracies lower dimension and level-preserving operators raise dimension.*

(b) *Level-preserving operators are sections, degeneracies are retractions. Each degeneracy is entirely determined by the set of its sections.*

(c) *For each cell* $x \in X(S)$, *there exists a unique non-degenerate cell* $y \in X(T)$ *and a unique degeneracy* $\phi : S \to T$ *such that* $X(\phi)(y) = x$.

(d) *For each point* $t \in |X|$, *there exists a unique non-degenerate cell* $x \in X(S)$ *and a unique interior point* $s \in |\Theta[S]|$, *such that* $|x|(s) = t$. *In particular*, *the points of* $|X|$ *admit a disjoint decomposition into cell-interiors indexed by the non-degenerate cells of* X.

Proof. A cellular operator factors uniquely into a cover followed by an immersion Lemma 1.11. A cover factors uniquely into a degeneracy followed by a level-preserving cover. Since immersions are level-preserving and the degenerate part of a cellular operator is a uniquely determined epimorphism. This proves (a).

Any immersion admits a retraction, and any degeneracy admits an immersion as section. The domain S of a non-degenerate cover $\phi : S \to T$ embeds as a subtree in the codomain T, whence a degeneracy $T \to S$ which serves as retraction of ϕ. Two distinct degeneracies are defined by distinct subtrees and have thus distinct sets of sections. This proves (b).

The cell category Θ is a *well-ordered DI-category with retractions* in Baues' sense by (a) and (b), which is sufficient to show (c), cf. [4, II]. Moreover, the realisation of level-preserving cellular operators is monic, since any level-preserving operator admits a retraction. This implies (d). ∎

Remark 2.5. Property (a) of Lemma 2.4 is the defining property of a *Reedy category* [17]. It follows that the representable Θ-model $\Theta[T]$ admits a *formal boundary* $\partial\Theta[T]$ defined as the cellular subset of $\Theta[T]$ consisting of all cellular operators $S \to T$ which factorise through a non-identity level-preserving cellular operator $S' \to T$. Restricted to the subcategory Δ, this yields the usual boundary $\partial\Delta[n]$.

Properties (a) and (b) of Lemma 2.4 imply that Θ is *Cauchy-complete*, i.e., its idempotents split, cf. [9, I.6.5]. Since the epi-mono factorisation of cellular operators is compatible with the *level-filtration* $\cdots \subset \Theta^{(n)} \subset \Theta^{(n+1)} \subset \cdots$ this shows that for each $n \geqslant 0$, the full subcategory $\Theta^{(n)}$ is also *Cauchy-complete*, which is well-known for $\Theta^{(1)} = \Delta$.

Proposition 2.6. *The geometric realisation of cellular sets has the following three properties*:

(a) *The realisation functor is cocontinuous and finite limit preserving.*

(b) *For each level tree* T, *the realisation of the standard* T*-cell* $\Theta[T]$ *is a convex closed subset of the cube* $[-1, 1]^{d(T)}$ *with non-empty interior. The simplex category is realised by simplices with the usual simplicial operators*;

the globe category is realised by balls with the usual globular operators. In general, we get

$$|\Theta[T]| = \{(t_\alpha) \in [-1,1]^{e(T)} | t_\alpha \leqslant t_\beta \textit{ if } \alpha \textit{ precedes } \beta \textit{ in } e_x(T),$$

$$(t_\alpha)_{(\alpha \textit{ below } x)} \in B^{\mathrm{ht}(x)}\}.$$

(c) *The geometric realisation of a cellular set X is a CW-complex with one $d(T)$-dimensional cell for each non-degenerate T-cell of X. In particular, the formal boundary $\partial\Theta[T]$ realises to the topological boundary of $|\Theta[T]|$.*

Proof. The geometric realisation $X \mapsto |X| = X \otimes_\Theta \mathrm{Disk}(-,B)$ has a right adjoint *cellular complex* $Y \mapsto \mathrm{Top}^{\mathrm{c}}(\mathrm{Disk}(-,B),Y)$ and is thus cocontinuous, whence (a) by Proposition 2.2.

A disk-map $\bar{T} \to B$ corresponds to a labelling of the edges $\alpha \in e(T)$ by real numbers $t_\alpha \in [-1,1]$ such that the labels of confluent edges are increasing from left to right, and such that the labels of a linear k-level subtree define a point of the k-ball B^k. This defines the indicated subset of $[-1,1]^{e(T)}$ which is convex and has interior points. In particular, for the 1-level tree $[n]$ with n input vertices we get the subset of n-tupels $(t_1,\dots,t_n) \in [-1,1]^n$ with $t_1 \leqslant \cdots \leqslant t_n$, which is homeomorphic to Δ_n, and for the linear n-level tree $\bar{n}$ we get the n-ball B^n. By Proposition 2.2, the cellular operators induce the usual simplicial, resp. globular, operators.

Since the Yoneda-embedding $\Theta \hookrightarrow \mathscr{S}^{\Theta^{\mathrm{op}}}$ is fully faithful, we have canonical homeomorphisms $|\mathrm{Disk}(\bar{T},B)| \cong |\Theta[T]|$. In particular, the topological boundary of $|\Theta[T]|$ is a sphere of dimension $d(T)-1$, which can be identified with the union of all images of $|\Theta[\phi]| : |\Theta[S]| \to |\Theta[T]|$ for level-preserving cellular operators $\phi : S \to T$ such that $d(S) = d(T)-1$. The latter are closed inclusions, so that the arguments of Lemma 2.4(c) and (d) imply (c). ∎

Example 2.7. Let T be the 4-dimensional 2-level tree of Example 1.4. The edges of T are ordered according to the total order of the sectors so that we get

$$|\Theta[T]| = \{(t_1,t_2,t_3,t_4) \in [-1,1]^4 | t_1 \leqslant t_4,\ t_2 \leqslant t_3,\ t_1^2 + t_2^2 \leqslant 1,\ t_1^2 + t_3^2 \leqslant 1\}.$$

The boundary of $|\Theta[T]|$ is made up by five codimension-one faces, three with domain W_1 and two with domain Y, see Example 1.10(e) for the notation. We have

$$|\Theta[W_1]| = \{(u_1,u_2,u_3) \in [-1,1]^3 | u_1 \leqslant u_3, u_1^2 + u_2^2 \leqslant 1\},$$

$$|\Theta[Y]| = \{(v_1,v_2,v_3) \in [-1,1]^3 | v_2 \leqslant v_3, v_1^2 + v_2^2 \leqslant 1, v_1^2 + v_3^2 \leqslant 1\}.$$

There is one immersion (resp. cover) $Y \to T$. Its geometric realisation is the map $(v_1, v_2, v_3) \mapsto (v_1, v_2, v_3, 1)$ (resp. the map $(v_1, v_2, v_3) \mapsto (v_1, v_2, v_3, v_1)$). There are two immersions $W_1 \to T$ whose geometric realisations are given by $(u_1, u_2, u_3) \mapsto (u_1, -\sqrt{1-u_1^2}, u_2, u_3)$, resp. $(u_1, u_2, +\sqrt{1-u_1^2}, u_3)$. Finally, there is a cover $W_1 \to T$ inducing $(u_1, u_2, u_3) \mapsto (u_1, u_2, u_2, u_3)$.

We just mention that for 1-level trees, the geometric realisation of cellular operators yields the simplicial formulae in their *homogeneous* form, which is somehow unusual. For instance, the three level-preserving cellular operators $W_1 \to T$ are shifted simplicial operators $[1] \to [2]$, cf. Example 1.10(d).

PROPOSITION 2.8. *For level trees S, T, the product of the standard S-cell with the standard T-cell is an amalgamated sum of standard U-cells according to*

$$\Theta[S] \times \Theta[T] = \bigcup_{U \in \mathrm{shuff}(S,T)} \Theta[U],$$

where $\mathrm{shuff}(S,T) = \{U \mid S \cup T = U, S \cap T = \bar{0}\}$ *denotes the set of all "shuffled bouquets" of S and T.*

Proof. Assume that the level tree U contains the level trees S and T as subtrees in such a way that $S \cup T = U$ and $S \cap T = \bar{0}$. The two degeneracies $U \to S$ and $U \to T$ which, respectively, forget about T and S then define an *injective* cellular map $\rho_U : \Theta[U] \to \Theta[S] \times \Theta[T]$ since different V-cells $\phi, \psi : V \to U$ differ at least on one of the two subtrees S, T of U. Moreover, in order to prove that any V-cell x of $\Theta[S] \times \Theta[T]$ lies in the image of some ρ_U, we can assume without loss of generality that x is non-degenerate, and that the projection of x onto $\Theta[S]$ (resp. $\Theta[T]$) is a degeneracy of the unique non-degenerate S-cell $1_S : S \to S$ (resp. the unique non-degenerate T-cell $1_T : T \to T$). It follows that $V = S \cup T$ for subtrees S, T such that $S \cap T = \bar{0}$, whence $x = \rho_V(1_V)$. ∎

Remark 2.9. The preceding proposition generalises the well-known decomposition of $\Delta[n] \times \Delta[m]$ into an amalgamated sum of $\frac{(n+m)!}{n!\,m!}$ copies of $\Delta[n+m]$. It is instructive to compare the above "tree-geometric" proof with the more common "path-geometric" proof, cf. [20, II.5.5]. Most importantly for us, this decomposition of the product of standard cells is an essential ingredient for the *closed model structure* on the category of cellular sets, cf. Theorem 3.9.

The simplest “non-simplicial” example of the cellular decomposition is the cylinder on the 2-disk: $\Theta[\bar{2}] \times \Theta[\bar{1}] = \Theta[W_1] \cup \Theta[W_2]$, cf. Example 1.10(e).

DEFINITION 2.10. A face of $\Theta[T]$ is inner (resp. outer) if it is induced by a (non-identity) level-preserving cover (resp. immersion) $\kappa : S \to T$. A κ-horn $\Lambda^{\kappa}[T] \hookrightarrow \Theta[T]$ is inner (resp. outer) if the missing face is inner (resp. outer).

The union of the outer faces of $\Theta[T]$ generates a cellular subset of $\Theta[T]$, called the *outer boundary*, and denoted by $\partial_{\text{out}}\Theta[T]$. Explicitly, the outer boundary consists of those cellular operators $S \to T$ which factorise through a non-identity *immersion* $S' \to T$. This concept makes sense for each ω-operad A so that we can speak of the *outer boundary* $\partial_{\text{out}}\Theta_A[T]$ of a representable Θ_A-model $\Theta_A[T]$. The globular site Θ_0 is a Reedy category without degeneracies and with immersions as the only level-preserving operators, i.e., $\partial\Theta_0[T] = \partial_{\text{out}}\Theta_0[T]$. Moreover, the left Kan extension along the embedding $i : \Theta_0 \hookrightarrow \Theta_A$ sends the boundary $\partial\Theta_0[T]$ to the *outer* boundary $\partial_{\text{out}}\Theta_A[T]$.

Restricted to the subcategory $\Theta^{(1)} = \Delta$, we get as *inner faces* of $\Delta[n]$ those induced by $\partial_i : [n-1] \to [n]$, $0 < i < n$, and as *outer faces* those induced by $\partial_0, \partial_n : [n-1] \to [n]$, whence the terminology, cf. Example 1.10(d) and Joyal [25].

We also extend the simplicial k-horns to cellular sets. Indeed, we define a κ-horn $\Lambda^{\kappa}[T] \hookrightarrow \Theta[T]$ to be the union of all proper faces of $\Theta[T]$ except the one induced by the level-preserving operator $\kappa : S \to T$ with $d(T) = d(S) + 1 > 0$.

LEMMA 2.11. *The outer boundary of $\Theta[T]$ equals the whole boundary if T is linear, and is weakly contractible if T is non-linear.*

Proof. All faces of $\Theta[\bar{n}]$ are outer. In general, the outer faces of $\Theta[T]$ form a poset which is isomorphic to the poset of proper cartesian subobjects of T_*, cf. Lemma 1.13. If T is *non-linear*, two proper cartesian subobjects of T_* without any common cartesian subobject are contained in a *proper* cartesian subobject of T_*. This shows that the poset of outer faces is weakly contractible; therefore, since all faces are contractible, cf. Theorem 3.9, the outer boundary is weakly contractible. ∎

PROPOSITION 2.12. *For a cellular set X, the following three properties are equivalent*:

(1) *X is the cellular nerve of an ω-category*;

(2) *for each non-linear level tree* T, *the inclusion of the outer boundary induces a bijection* $X^{\Theta[T]} \xrightarrow{\sim} X^{\partial_{\mathrm{out}}\Theta[T]}$;

(3) *each inner horn* $\Lambda^\kappa[T] \hookrightarrow \Theta[T]$ *induces a bijection* $X^{\Theta[T]} \xrightarrow{\sim} X^{\Lambda^\kappa[T]}$.

Proof. A cellular set X is the nerve of an ω-category if and only if for each level tree T, there is a bijection $(1)_T : X(T) \cong \varprojlim_{\tau \in \mathrm{el}(T_*)} X(\overline{\mathrm{ht}(\tau)})$, cf. Theorem 1.12. In particular, if some cellular subset B of $\Theta[T]$ contains $\mathrm{im}(\Theta[\bar{n}] \hookrightarrow \Theta[T])$ for all immersions of maximal linear subtrees of T, then the canonical map $X^{\Theta[T]} \to X^B$ is a bijection. This is the case if B is the *outer* boundary of $\Theta[T]$ for a *non-linear* level tree T, thus $(1)_T$ implies the bijection $(2)_T : X^{\Theta[T]} \xrightarrow{\sim} X^{\partial_{\mathrm{out}}\Theta[T]}$.

Assume given an inner κ-horn $x : \Lambda^\kappa[T] \to X$ in the cellular nerve X. Then T must be non-linear and the κ-horn $\Lambda^\kappa[T]$ contains the outer boundary of $\Theta[T]$. Therefore, $(1)_T$ implies also the bijection $(3)_T : X^{\Theta[T]} \xrightarrow{\sim} X^{\Lambda^\kappa[T]}$.

For linear level trees, $(1)-(3)$ are trivially equivalent so that we can assume by induction that $(1)_S, (2)_S, (3)_S$ are equivalent for level trees S of dimension $d(S) < d(T)$. We shall show that $(3)_T$ implies $(2)_T$ implies $(1)_T$.

Indeed, since $\Lambda^\kappa[T] \supset \partial_{\mathrm{out}}\Theta[T]$, the filler for an inner κ-horn induces at least one filler for the outer boundary. If we had two fillers x, y which coincide on $\partial_{\mathrm{out}}\Theta[T]$, then there would be an *inner* face $\Theta[S]$ of $\Theta[T]$ on which x and y differ. The outer boundary of $\Theta[S]$ is contained in the outer boundary of $\Theta[T]$, since the composition of a proper immersion with a cover cannot be a cover. Therefore $x|_{\partial_{\mathrm{out}}\Theta[S]} = y|_{\partial_{\mathrm{out}}\Theta[S]}$, but $x|_{\Theta[S]} \neq y|_{\Theta[S]}$ for a level tree S with $d(S) < d(T)$ contradicting $(2)_S$, whence $(3)_T$ implies $(2)_T$.

Finally, each element of $\varprojlim_{\tau \in \mathrm{el}(T_*)} X(\overline{\mathrm{ht}(\tau)})$ defines a compatible family of "globular" cells in $\Theta[T]$. These *outer* faces *span* the whole outer boundary of $\Theta[T]$ by successive (uniquely determined) fillings of *inner* κ-horns $\Lambda^\kappa[S] \hookrightarrow \Theta[S]$ for *outer* faces $\Theta[S]$ of $\Theta[T]$. Therefore, $(2)_T$ implies $(1)_T$. ∎

Remark 2.13. The preceding proposition generalises a well-known property of simplicial sets: a simplicial set is the nerve of a category (i.e., a Δ-model) if and only if every *inner* horn has a *unique* filler.

The *weak Kan complexes* of Boardman–Vogt [8, III.4.8] are simplicial sets with (non-necessarily unique) fillers for inner horns. Weak Kan complexes are thus a *compromise* between simplicial nerves (with unique fillers for inner horns) and Kan complexes (with fillers for all horns). Joyal [25] defines in analogy a θ-*category* to be a *cellular set* with fillers for inner horns.

Remark 2.14. We shall now indicate how to extend the geometric realisation of cellular sets to A-*cellular sets* $X : \Theta_A^{\mathrm{op}} \to \mathcal{S}$. The idea is to define the realisation of X as the realisation of its *left Segal extension* $\tilde{p}_! X$

along the canonical functor $p : \Theta_A \to \Theta$. Strictly speaking, the left Segal extension is a cellular object in simplicial sets or, as we shall say, a *cellular space* $\tilde{p}_! X : \Theta^{\mathrm{op}} \to s\mathscr{S}$.

"Homotopy" Kan extensions exist in the literature since the early 1970s, cf. [8, 11, 29, 34, 40]. In guise of simplicity, we split the left Segal extension $\tilde{p}_! : \mathscr{S}^{\Theta_A^{\mathrm{op}}} \to s\mathscr{S}^{\Theta^{\mathrm{op}}}$ into a simplicial resolution $w_A : \mathscr{S}^{\Theta_A^{\mathrm{op}}} \to s\mathscr{S}^{\Theta_A^{\mathrm{op}}}$ followed by the left Kan extension $p_! : s\mathscr{S}^{\Theta_A^{\mathrm{op}}} \to s\mathscr{S}^{\Theta^{\mathrm{op}}}$. It remains to define a convenient simplicial resolution functor w_A. Our method is to define w_A as the *cofibrant replacement functor* with respect to a *closed model structure* for *A-cellular spaces*, cf. Section 4. The left Segal extension $\tilde{p}_!$ coincides then by definition with Quillen's *left derived functor* $Lp_!$ of $p_!$, cf. [31]. This is the most satisfactory way to explain its *homotopy invariance*.

Of course, such a definition depends on the choice of the model structure. It can be shown that a convenient choice subsumes all existing approaches to homotopy Kan extensions. They actually belong to two *distinct* families: Bousfield–Kan [11], Segal [34] and Vogt [40] construct the cofibrant replacement functor using the free-forgetful adjunction between "reflexive graphs" and "small categories", see [18, 19, 24] for an explicit description of the resulting replacement functor.

Boardman–Vogt [8] and May [29] construct the cofibrant replacement functor using the free-forgetful adjunction between spaces and algebras for a given topological operad. The closed model structure for A-cellular spaces constructed in Section 4 is closely related to this operadic approach, cf. Definition 4.10.

3. A CLOSED MODEL STRUCTURE FOR CELLULAR SETS

In this section, we endow the category of cellular sets with a *closed model structure* in Quillen's sense [31]. Its *homotopy category* is equivalent to the homotopy category of compactly generated spaces; in particular, cellular sets have a well-behaved *homotopy theory*. There is, however, no direct way to define a closed model structure on the category of ω-categories. This is related to the fact that *ω-categorification* modifies the homotopy type of certain cellular sets. We shall repair this defect in Section 4 by considering cellular sets as the discrete objects among cellular spaces. The latter carry a closed model structure which behaves well under ω-categorification.

Closed model categories are a powerful tool, insofar as they capture homotopy structures in contexts which may be far away from topology. Fundamental is the closed model category of *simplicial sets*, in which the *Kan fibrations* play an eminent role. This model structure extends in a fairly natural way to cellular sets. It actually suffices to find the cellular analog of the *simplicial k-horns*. Our proof avoids the use of *minimal fibrations* and focuses rather on the combinatorial properties of the simplicial (resp. cellular) *cylinder*. We also make frequent use of Dwyer–Hirschhorn–Kan's concept of a *generating set of* (*trivial*) *cofibrations* [17] which axiomatises Gabriel–Zisman's *anodyne extensions* [20].

DEFINITION 3.1 (Quillen). A *closed model category* $\mathscr{E}$ is a category with finite limits and colimits and three distinguished classes of morphisms $\mathscr{E}_{\mathrm{cof}}, \mathscr{E}_{\mathrm{we}}, \mathscr{E}_{\mathrm{fib}}$ called, respectively, *cofibrations, weak equivalences, fibrations* such that:

(M1) If two out of f, g and gf are weak equivalences, then so is the third.

(M2) Cofibrations, weak equivalences and fibrations compose, contain all isomorphisms and are stable under retract.

(M3) Trivial cofibrations are left orthogonal to fibrations; cofibrations are left orthogonal to trivial fibrations.

(M4) Any morphism can be factored into a trivial cofibration followed by a fibration as well as into a cofibration followed by a trivial fibration.

A *trivial* cofibration (resp. fibration) is by definition a cofibration (resp. fibration) which is also a weak equivalence.

For any closed model category $\mathscr{E}$, the localisation $\mathbf{Ho}(\mathscr{E}) = \mathscr{E}[\mathscr{E}_{\mathrm{we}}^{-1}]$ *exists* and is called the *homotopy category* of $\mathscr{E}$. Moreover, $\mathbf{Ho}(\mathscr{E})(A, B) \cong [cA, fB]$, where $cA \overset{\sim}{\twoheadrightarrow} A$ is a *cofibrant replacement* of A, and $B \overset{\sim}{\rightarrowtail} fB$ is a *fibrant replacement* of B, and $[X, Y]$ denotes the set of *homotopy classes* of morphisms $X \to Y$, well defined as soon as X is cofibrant and Y is fibrant.

An object of $\mathscr{E}$ is called *cofibrant* (resp. *fibrant*) if the unique map $\emptyset \to A$ (resp. $B \to 1$) is a cofibration (resp. fibration). The replacements exist due to (M4). As usual, cofibrations are depicted by arrows of the form $\rightarrowtail$, weak equivalences by $\overset{\sim}{\to}$, and fibrations by $\twoheadrightarrow$.

A *Quillen pair* between two closed model categories is a pair of adjoint functors such that the *left* adjoint preserves *cofibrations* and the *right* adjoint preserves *fibrations*. The left (resp. right) adjoint preserves cofibrations if and only if the right (resp. left) adjoint preserves trivial fibrations (resp. trivial cofibrations). Brown's lemma [12] implies then that the left (resp. right) adjoint of a Quillen pair preserves weak equivalences between cofibrant (resp. fibrant) objects. In particular, each Quillen pair (F, G) induces a *derived adjoint pair* (LF, RG) between the homotopy categories.

A Quillen pair $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$ is a *Quillen equivalence* if its derived adjoint pair $LF : \mathbf{Ho}(\mathscr{C}) \leftrightarrows \mathbf{Ho}(\mathscr{D}) : RG$ is an equivalence of categories. The precise criterion for a Quillen equivalence is that for each cofibrant object X of $\mathscr{C}$ and each fibrant object Y of $\mathscr{D}$, a $\mathscr{D}$-morphism $FX \to Y$ is a weak equivalence if and only if the adjoint $\mathscr{C}$-morphism $X \to GY$ is so. If F (resp. G) preserves *and* detects weak equivalences, then (F, G) is a Quillen equivalence if and only if the counit (resp. unit) is a weak equivalence at fibrant (resp. cofibrant) objects.

For a class J of maps, the *right orthogonal* $J^{\perp}$ is the class of maps ϕ such that ϕ is right orthogonal to all j in J. The *left orthogonal* is defined analogously. The *closure* of J is the left orthogonal of the right orthogonal, i.e., $\bar{J} = {}^{\perp}(J^{\perp})$.

DEFINITION 3.2. A closed model category $\mathscr{E}$ is *finitely generated*[2] if $\mathscr{E}$ is cocomplete and if there is a set I of cofibrations with finite domains and a set J of trivial cofibrations with finite domains, such that $\bar{I} = \mathscr{E}_{\mathrm{cof}}$ and $\bar{J} = \mathscr{E}_{\mathrm{cof}} \cap \mathscr{E}_{\mathrm{we}}$.

The elements of I (resp. J) are referred to as the *generating* cofibrations (resp. trivial cofibrations) of $\mathscr{E}$.

Quillen's *small object argument* [31, II.3.4] shows that the sets I and J induce functorial factorisations: any morphism factors into a relative I-complex followed by a morphism that is right orthogonal to I (resp. a relative J-complex followed by a morphism that is right orthogonal to J).

A *relative I-complex* is a sequential colimit $X \to \varinjlim_{n \geqslant -1} X^{(n)}$ of morphisms *under* X, which are induced by cobase changes of the form

$$\begin{array}{ccc} \coprod_i A_i & \longrightarrow & X^{(n-1)} \\ {\scriptstyle \coprod_i \phi_i}\downarrow & & \downarrow \\ \coprod_i B_i & \longrightarrow & X^{(n)} \end{array}$$

where $\phi_i \in I$ and $X \overset{=}{\to} X^{(-1)}$. Whitehead's relative CW-complexes are relative I-complexes for $I = \{S^{n-1} \hookrightarrow B^n,\ n \geqslant 0\}$, cf. Example 3.12(a); any monomorphism of simplicial sets is a relative I-complex for $I = \{\partial\Delta[n] \hookrightarrow \Delta[n],\ n \geqslant 0\}$.

In a finitely generated closed model category $\mathscr{E}$, the trivial fibrations are precisely the morphisms that are right orthogonal to I, and the fibrations are

[2] This is stronger than being finitely cofibrantly generated in Dwyer–Hirschhorn–Kan's sense [17], where I-maps (resp. J-maps) have only $\bar{I}$-finite (resp. $\bar{J}$-finite) domains.

precisely the morphisms that are right orthogonal to J. Moreover, the cofibrations are precisely codomain-retracts of relative I-complexes, and the trivial cofibrations are precisely codomain-retracts of relative J-complexes.

The proof that cellular sets carry a closed model structure follows closely the analogous proof for simplicial sets. In order to give a common framework, we introduce the concept of *a closed cylinder category* and determine which complementary property ensures that such a category is a closed model category.

DEFINITION 3.3 (cf. Baues [5]). A *closed cofibration category* is a finitely cocomplete category with cofibrations, weak equivalences and a distinguished subclass J of trivial cofibrations such that (M1) and (M2) hold and, moreover,

(C3) any morphism factors into a cofibration followed by a weak equivalence;

(C4) relative J-complexes are trivial cofibrations, and any morphism factors into a trivial cofibration followed by a morphism right orthogonal to J.

If the closed cofibration category is *cocomplete*, and J is a *set* of trivial cofibrations with *finite* domains, then the existence of (C4)-factorisations is automatic by Quillen's small object argument.

The morphisms (resp. weak equivalences) of a closed cofibration category that are right orthogonal to J will henceforth simply be called *fibrations* (resp. *trivial fibrations*). This is slightly abusive, since it is in general *neither* true that the fibrations of a closed cofibration category are right orthogonal to *all* trivial cofibrations *nor* that trivial fibrations are right orthogonal to cofibrations. However, the latter statement implies the former:

LEMMA 3.4. *A closed cofibration category with finite limits is a closed model category if and only if trivial fibrations are right orthogonal to cofibrations.*

Proof. In a closed model category $\mathscr{E}$, cofibrations are left orthogonal to trivial fibrations, so that the condition is necessary. For the converse, observe that $J^\perp$ is closed under retract, so that (M2) holds also for fibrations. Condition (C4) yields the first part of (M4). According to (C3) and (C4), any morphism ϕ factors as $\phi = \phi_{\mathrm{we}}\phi_{\mathrm{cof}} = \phi_{\mathrm{fib}}\bar{\phi}\phi_{\mathrm{cof}}$ with $\phi_{\mathrm{fib}} \in J^\perp$ and $\bar{\phi}$ a trivial cofibration. By (M1), ϕ_{fib} is a trivial fibration, whence the factorisation into a cofibration $\bar{\phi}\phi_{\mathrm{cof}}$ followed by a trivial fibration ϕ_{fib}, which is the second part of (M4).

For (M3), it remains to show that any trival cofibration ϕ belongs to $\bar{J}$. According to (C4), ϕ decomposes as $\phi_{\mathrm{fib}}\bar{\phi}$. By (M1), ϕ_{fib} is a *trivial* fibration right orthogonal to the *cofibration* ϕ. This yields a section of ϕ_{fib} realising ϕ as a *retract* of $\bar{\phi}$. Since $\bar{J}$ is closed under retract, ϕ itself belongs to $\bar{J}$. ∎

According to Quillen, a *cylinder object* for X is a (C3)-factorisation of its codiagonal: $X \sqcup X \rightarrowtail \mathrm{Cyl}(X) \xrightarrow{\sim} X$. We shall write $(i_0, i_1) : X \sqcup X \rightarrowtail \mathrm{Cyl}(X)$ and $r\colon \mathrm{Cyl}(X) \xrightarrow{\sim} X$. Any morphism $\phi : X \to Y$ factors then through its *mapping cylinder* $\mathrm{Cyl}(X) \bigcup_\phi Y$.

Two morphisms $f, g : X \to Y$ are *homotopic* under $u : U \rightarrowtail X$ (formally $f \overset{u}{\sim} g$) if there is a homotopy $h\colon \mathrm{Cyl}(X) \to Y$ and a cylinder map $\bar{u}$: $\mathrm{Cyl}(U) \to \mathrm{Cyl}(X)$ such that $h(i_0, i_1) = (f, g)$ and $h\bar{u} = fur = gur$.

DEFINITION 3.5. A *cylinder* for a closed cofibration category is a *functorial cylinder object* such that:

(I0) The mapping cylinder factorisation is a (C3)-factorisation.

(I1) For any cofibration $\phi : X \rightarrowtail Y$, the induced map $\mathrm{Cyl}(X) \bigcup\limits_\phi Y \to \mathrm{Cyl}(Y)$ is a relative J-complex.

(I2) For any cofibrant object X, there are homotopies: $i_0 r \overset{i_0}{\sim} \mathrm{id}_{\mathrm{Cyl}(X)} \overset{i_1}{\sim} i_1 r$.

A *closed cylinder category* is a closed cofibration category with cylinder.

A retraction r of a cofibration $i : X \rightarrowtail Y$ is a *deformation retraction* if the composite map $ir : Y \to Y$ is homotopic to id_Y under i.

An object X is *contractible* if there is a *cofibration* $* \rightarrowtail X$ with deformation retraction for a cofibrant terminal object $*$.

LEMMA 3.6. *In a closed cylinder category, the following properties hold:*

(a) *Fibrations have the homotopy lifting property rel. to cofibrant objects.*

(b) *Cofibrations have the homotopy extension property rel. to fibrations.*

(c) *Base change of a fibration along homotopic maps with cofibrant domain yields fibre homotopy equivalent fibrations.*

Proof. (a) For a cofibrant object X, the inclusions $i_{0,1} : X \rightrightarrows \mathrm{Cyl}(X)$ are relative J-complexes by (I1).

(b) Consider the following diagrams of unbroken arrows:

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail, "\phi"'] & E \arrow[d, two heads, "p"] \\ Y \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \qquad \begin{tikzcd} Cyl\,(X) \cup_\phi Y \arrow[r, "h"] \arrow[d, tail, "\Phi"'] & E \arrow[d, two heads, "p"] \\ Cyl\,(Y) \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

where the upper triangle of the left square commutes up to homotopy *over* B. This *defines* the right square. Since Φ is a relative J-complex and p a fibration, (I1) induces a homotopy over B extending the given homotopy h.

(c) Any cofibrant object B' is a deformation retract of its cylinder $\mathrm{Cyl}(B')$ with respect to both inclusions i_0, i_1 by (I2). Base change of $p: E \twoheadrightarrow B$ along $h: f \sim g$ yields a fibration $h^*E \twoheadrightarrow \mathrm{Cyl}(B')$ admitting (due to (a)) a deformation retraction to f^*E and a deformation retraction to g^*E. Composition of these two homotopy equivalences yields a *fibre* homotopy equivalence over B'. ∎

PROPOSITION 3.7. *A finitely complete closed cylinder category with a generating set of cofibrations having contractible codomains is a closed model category if and only if the fibres of any trivial fibration are right orthogonal to cofibrations.*

Proof. The condition is necessary, since in a closed model category, trivial fibrations are stable under base change and right orthogonal to cofibrations. For its sufficiency, it remains, by Lemma 3.4, to show that the right orthogonality of the fibres implies the right orthogonality of the morphism. It suffices to establish this right orthogonality with respect to cofibrations $i: S \rightarrowtail D$ with contractible codomain D. For a trivial fibration $p: E \overset{\sim}{\twoheadrightarrow} B$ consider the following diagram:

$$\begin{tikzcd} S \arrow[r, "j"] \arrow[dr, tail, "i"'] & f^*E \arrow[r] \arrow[d, two heads] & E \arrow[d, two heads, "p", "\sim"'] \\ & D \arrow[r, "f"] \arrow[u, dashed, bend right, "s"'] & B \end{tikzcd}$$

We have to construct a section s of f^*p such that $j = si$. Since D is contractible, Lemma 3.6(c) implies that the fibration f^*p is fibre homotopy equivalent to a projection $D \times F \to D$ where F is a fibre of p. Since F is right orthogonal to cofibrations, there is a section s' of $D \times F \to D$ such that $j' = s'i$, inducing thus a section s'' of f^*p such that j is *fibre homotopic* to $s''i$

over D. The homotopy extension property, Lemma 3.6(b), of i rel. to f^*p yields the required section s. ∎

EXAMPLE 3.8. Kan–Quillen's closed model structure for simplicial sets.

Simplicial sets form a *closed cylinder category* with *monomorphisms* as cofibrations, *realisation weak equivalences* as weak equivalences, and the set of *k-horns* $\Lambda^k[n] \hookrightarrow \Delta[n]$, $0 \leqslant k \leqslant n$ $(n \neq 0)$, as distinguished subset J of trivial cofibrations. The k-horn $\Lambda^k[n]$ is defined to be the union of all proper faces of $\Delta[n]$ except the one induced by the coface $\partial_k : [n-1] \to [n]$.

Indeed, conditions (M1) and (M2) are immediate. The simplicial interval $\Delta[1]$ defines a functorial cylinder object $X \sqcup X \rightarrowtail X \times \Delta[1] \xrightarrow{\sim} X$, whose mapping cylinder factorisation yields (C3), since geometric realisation is a cocontinuous and finite limit preserving functor. Condition (C4) holds by Quillen's small object argument, since the geometric realisation of a relative J-complex is a (weak) homotopy equivalence. Condition (I1) holds, i.e., for any cofibration $X \rightarrowtail Y$, the mapping-cylinder embedding $X \times \Delta[1] \cup Y \times \Delta[0] \hookrightarrow Y \times \Delta[1]$ is a relative J-complex, because the prism $\Delta[n] \times \Delta[1]$ is an amalgamated sum of $(n+1)$ copies of $\Delta[n+1]$ and any cofibration is a relative I-complex for $I = \{\partial\Delta[n] \hookrightarrow \Delta[n],\ n \geqslant 0\}$. Finally, (I2) holds, since the simplicial interval $\Delta[1] = \mathcal{N}_\Delta([1])$ carries two multiplications (image under $\mathcal{N}_\Delta$ of $\min, \max : [1] \times [1] \to [1]$) inducing the required deformation retractions.

Simplicial fibrations have been introduced by Daniel Kan and are usually called *Kan fibrations*; the fibrant simplicial sets are often called *Kan complexes*. Moreover, retracts of relative J-complexes have been introduced by Gabriel–Zisman [20] under the name *anodyne extensions*.

Since the standard n-simplex $\Delta[n]$ is contractible, the *closed model structure* relies by Proposition 3.7 on the property that the fibres of trivial Kan fibrations are right orthogonal to cofibrations. Lemma 3.6(c) shows that the geometric realisation of a Kan fibration is locally fibre homotopy equivalent to a projection, and thus at least a *quasi-fibration*. It remains therefore to show that *weakly contractible* Kan complexes F are right orthogonal to cofibrations. This follows from an inductive argument using the existence of a simplicial path fibration $\Omega F \to PF \twoheadrightarrow F$ with *contractible* total space, cf. the proof of Theorem 3.9.

The closed model structure is *finitely generated* by I and J.

We now extend the preceding proof to the category of cellular sets. We denote by $\Theta[T]$ the representable functor $\Theta(-, T)$, and by $\partial\Theta[T]$ the union of all its codimension-one faces. The geometric realisation of $\Theta[T]$ is a convex $d(T)$-dimensional cell with boundary $|\partial\Theta[T]|$, cf. Proposition 2.6. A *κ-horn* $\Lambda^\kappa[T] \hookrightarrow \Theta[T]$ is the union of all proper faces of $\Theta[T]$ except the one induced by the level-preserving operator $\kappa : S \to T$ with $d(T) = d(S) + 1 > 0$.

THEOREM 3.9. *The category of cellular sets is a finitely generated closed model category with* monomorphisms *as cofibrations and* realisation weak equivalences *as weak equivalences. The set of* κ-horns *is a generating set of trivial cofibrations. Geometric realisation is the left adjoint part of a Quillen equivalence between cellular sets and compactly generated spaces.*

Proof. We show first that cellular sets form a *closed cylinder category*. Again, (M1) and (M2) are immediate and (C3) follows from the existence of a functorial cylinder object $X \sqcup X \rightarrowtail X \times \Theta[\bar{1}] \xrightarrow{\sim} X$ using that geometric realisation is finite limit preserving and cocontinuous by Proposition 2.6(a). This also implies that the realisation of a relative J-complex is a weak homotopy equivalence, since κ-horns realise to deformation retracts, and since in the category of CW-complexes, weak homotopy equivalences are closed under coproducts and countable direct compositions. Quillen's small object argument yields then (C4).

For condition (I1), we can restrict ourselves to the generating set I of cofibrations of the form $\partial\Theta[T] \rightarrowtail \Theta[T]$, since any cofibration is a relative I-cell complex, cf. Lemma 2.4(c). In general, the product $\Theta[S] \times \Theta[T]$ is an amalgamated sum of standard U-cells, where U runs through all "shuffled bouquets" of S and T, cf. Proposition 2.8. In particular, we have

$$\Theta[T] \times \Theta[\bar{1}] = \bigcup_{\sigma \in T_*, \mathrm{ht}(\sigma)=0} \Theta[T_\sigma^+],$$

where T_σ^+ is the level tree obtained from T by adjoining an edge inside the T-sector σ of height 0. Moreover, for two consecutive T-sectors σ, σ', the intersection $\Theta[T_\sigma^+] \cap \Theta[T_{\sigma'}^+]$ is a standard T-cell, so that the mapping-cylinder embedding $\partial\Theta[T] \times \Theta[\bar{1}] \cup \Theta[T] \times \Theta[\bar{0}] \hookrightarrow \Theta[T] \times \Theta[\bar{1}]$ is isomorphic to the composition of successive attachments of T_σ^+-cells along κ-horns $\Lambda^\kappa[T_\sigma^+] \to \Theta[T_\sigma^+]$ with missing T-face. This shows that mapping-cylinder embeddings are relative J-complexes. Condition (I2) follows like in the simplicial case from two multiplications on the cellular interval $\Theta[\bar{1}] \cong \mathcal{N}_\omega(\underline{\omega}([1]_*))$. By Proposition 3.7 it suffices now to show that the standard cells are contractible, and that the fibres of a trivial fibration are right orthogonal to cofibrations.

For the first point, we associate to each T_σ^+-cell of the prism $\Theta[T] \times \Theta[\bar{1}]$ the degeneracy operator $T \to T'$ defined by the subtree $T' \subseteq T$ lying on the right of the T-sector σ. These degeneracy operators are compatible on intersections $\Theta[T_\sigma^+] \cap \Theta[T_{\sigma'}^+]$ and define thus a cylinder contraction of the standard cell $\Theta[T]$.

For the second point, observe first that by Lemma 3.6(c) the geometric realisation of a fibration $p : E \to B$ is *locally fibre homotopy equivalent* to a projection $U \times |F| \to U$ and hence a *quasi-fibration*. The fibres of a trivial fibration are therefore weakly contractible and it remains to show that fibrant and weakly contractible cellular sets F are right orthogonal to cofibrations. Since the category of cellular sets is cartesian closed, it follows from Lemma 3.6(a) that any fibrant cellular set F defines a *cocylinder* $\underline{\mathrm{Hom}}_{\Theta}(\Theta[\bar{1}], F) \twoheadrightarrow F \times F$, and hence (by restriction) a *path fibration* $\Omega F \to PF \twoheadrightarrow F$ having *contractible* total space.

A fibrant cellular set F is said to be *n-connected* if every cellular map $x : (\Theta[T], \partial\Theta[T]) \to (F, *)$ with $d(T) \leqslant n$ is homotopic to the constant map $*$. Weakly contractible F are 0-connected. Assume then by induction, that weakly contractible F are $(n-1)$-connected. For any cellular map x like above with $d(T) = n$, there is a lift $\bar{x} : \Theta[T] \to PF$ mapping some κ-horn $\Lambda^{\kappa}[T]$ to $*$. The missing face defines a cellular map $y : (\Theta[S], \partial\Theta[S]) \to (\Omega F, *)$, where $\kappa : S \to T$ has dimension $d(S) = d(T) - 1$. Since by induction hypothesis, y is homotopic to a constant map, $\bar{x}$ is homotopic to a map which is constant on the boundary $\partial\Theta[T]$. Since PF is contractible, this implies that x itself is homotopic to the constant map, and hence F is n-connected. Finally, the homotopy extension property, Lemma 3.6(b), shows that ∞-connected fibrant cellular sets are right orthogonal to cofibrations. By Proposition 3.7, this implies that cellular sets form a closed model category, finitely generated by the sets I and J.

The geometric realisation preserves cofibrations by Proposition 2.6(c) and hence also trivial cofibrations so that it is the left adjoint part of a Quillen pair $(|-|, \mathrm{Cell}(-))$, where the *total cellular complex* is given by $\mathrm{Cell}(Y)(T) = \mathrm{Top}^{c}(|\Theta[T]|, Y)$.

The counits $\varepsilon_Y : |\mathrm{Cell}(Y)| \to Y$ are weak homotopy equivalences by the usual inductive argument, which begins with componentwise contractible spaces and applies the five lemma to the long exact homotopy sequence of the path fibration. Since geometric realisation preserves *and* detects weak equivalences, this shows that $(|-|, \mathrm{Cell}(-))$ is a Quillen equivalence. ∎

The following *transfer theorem* is a quite direct consequence of Quillen's small object argument. The first explicit "transfers" are due to Quillen [31] and Thomason [39], the first general statement is due to Crans [14, 3.3], cf. also [17].

THEOREM 3.10. *Let $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$ be an adjoint pair such that the left adjoint F preserves* finiteness. *Assume that $\mathscr{C}$ is a closed model category finitely generated by I and J, that $\mathscr{D}$ is cocomplete and finitely complete, and that G takes any relative FJ-complex to a weak equivalence.*

Then (F, G) is a Quillen pair for a closed model structure on $\mathscr{D}$, which is finitely generated by FI and FJ. The weak equivalences (resp. fibrations) of the transferred model structure are the $\mathscr{D}$-morphisms sent to weak equivalences (resp. fibrations) under G.

Proof. The functor F preserves finiteness, so that Quillen's small object argument applies to FI and FJ and yields factorisations of type $(\overline{FI}, (FI)^{\perp})$ and $(\overline{FJ}, (FJ)^{\perp})$. If the weak equivalences and fibrations of $\mathscr{D}$ are defined like indicated, we get $(FI)^{\perp} = \mathscr{D}_{\mathrm{we}} \cap \mathscr{D}_{\mathrm{fib}}$ and $(FJ)^{\perp} = \mathscr{D}_{\mathrm{fib}}$ by adjunction.

Thus if the *cofibrations* of $\mathscr{D}$ are those maps which are left orthogonal to trivial fibrations, then all axioms of a closed model structure on $\mathscr{D}$ are satisfied, provided any relative FJ-complex is a trivial cofibration. For logical reasons, the closure of FJ is contained in the closure of FI, so that relative FJ-complexes are cofibrations. By assumption they are also weak equivalences. The resulting closed model structure on $\mathscr{D}$ is finitely generated, since $\overline{FI} = \mathscr{D}_{\mathrm{cof}}$ by definition, and $\overline{FJ} = ^{\perp} \mathscr{D}_{\mathrm{fib}} = \mathscr{D}_{\mathrm{cof}} \cap \mathscr{D}_{\mathrm{we}}$. ∎

Remark 3.11. If in a finitely generated closed model category, not only the domains but also the codomains of the generating (trivial) cofibrations are finite, then weak equivalences are closed under coproducts and countable compositions. Under this hypothesis, it is sufficient for the existence of a transfer, Theorem 3.10, to require that G sends FJ-attachments (i.e., cobase changes of elements of FJ) to weak equivalences. This is in particular the case if $\mathscr{D}$ is a reflective subcategory of $\mathscr{C}$ for a reflector which takes trivial cofibrations with $\mathscr{D}$-domain to weak equivalences. Indeed, it suffices to observe that a cobase change in $\mathscr{D}$ equals the reflection of the corresponding cobase change in $\mathscr{C}$.

Example 3.12. (a) *Quillen–Serre's closed model structure for compactly generated spaces.* The adjoint pair $|-| : s\mathscr{S} \leftrightarrows \mathrm{Top}^{\mathrm{c}} : Sing$ satisfies the hypothesis of the transfer theorem (cf. Section 0.5) and induces Quillen's model structure on $\mathrm{Top}^{\mathrm{c}}$:

$$\mathrm{Top}^{\mathrm{c}}_{\mathrm{cof}} = \{\text{retracts of relative } I\text{-complexes}\};$$

$$\mathrm{Top}^{\mathrm{c}}_{\mathrm{we}} = \{\text{weak homotopy equivalences}\};$$

$$\mathrm{Top}^{\mathrm{c}}_{\mathrm{fib}} = \{\text{Serre fibrations}\}.$$

The closed model structure is finitely generated by $I = \{S^{n-1} \hookrightarrow B^n \mid n \geqslant 0\}$ and $J = \{B^{n-1} \hookrightarrow B^n \mid n > 0\}$. The Quillen pair $(|-|, Sing)$ induces an equivalence of homotopy categories: $\mathbf{Ho}(s\mathscr{S}) \sim \mathbf{Ho}(\mathrm{Top}^{\mathrm{c}})$.

(b) *Thomason's closed model structure for small categories*. Thomason [39] showed that the adjoint pair

$$\mathrm{cat} \circ Sd^2 : s\mathcal{S} \leftrightarrows \mathrm{Cat} : Ex^2 \circ \mathcal{N}_\Delta$$

satisfies the hypothesis of the transfer theorem and induces an equivalence of homotopy categories $\mathbf{Ho}(s\mathcal{S}) \sim \mathbf{Ho}(\mathrm{Cat})$. The doubly subdivided categorification yields thus a satisfactory homotopy theory for small categories. Thomason's transfer may be performed in two steps: a transfer along (Sd^2, Ex^2) followed by a transfer along $(\mathrm{cat}, \mathcal{N}_\Delta)$. The first step may be interpreted as a *strengthening of the cofibration structure* for simplicial sets.

(c) *Moerdijk's closed model structure for bisimplicial sets*. The diagonal $d : \Delta \to \Delta \times \Delta$ induces an adjunction $d_! : s\mathcal{S} \leftrightarrows ss\mathcal{S} : d^*$ between simplicial and bisimplicial sets. According to Moerdijk [30], this adjunction satisfies the hypothesis of the transfer theorem. It is instructive to see how the existence of a *right* adjoint functor d_* for d^* enters in the argument. The latter implies that d^* is cocontinuous so that $d_!$ preserves finiteness. The condition on $d_!J$-attachments follows then from the fact that $d^*d_!$ is cocontinuous and takes k-horns $\Lambda^k[n] \rightarrowtail \Delta[n]$ to trivial cofibrations. The Quillen pair $(d_!, d^*)$ is a Quillen equivalence, since d^* preserves and detects weak equivalences, and since the unit $\eta_X : X \to d^*d_!X$ is a weak equivalence.

Remark 3.13. The model structure for cellular sets does not transfer to ω-categories since the reflector from cellular sets to Θ-models does not take *outer horns* $\Lambda^\kappa[T] \hookrightarrow \Theta[T]$ to weak equivalences. It can be shown that the reflector from cellular sets to ω-*groupoids* sends *certain* outer horns to weak equivalences, namely those associated to immersions $\kappa : S \to T$ with $\mathrm{ht}(S) = \mathrm{ht}(T)$. However, outer horns associated to immersions $\kappa : S \to T$ with $\mathrm{ht}(S) < \mathrm{ht}(T)$ behave badly. For instance, the existence of fillers for the horn $\Lambda^{[2]}[W_1] \hookrightarrow \Theta[W_1]$, cf. Example 1.10(e), implies that any 2-cell must be an identity cell. This has been observed by James Dolan. In general, the cellular nerve of an ω-category is fibrant if and only if the ω-category is a 1-groupoid. This shows that the filler conditions are *too strong* when they are applied to *cellular nerves*. Our method to circumvent this difficulty is to embed ω-categories into the category of simplicial ω-categories and to define a suitable model structure for the latter.

There exists no Quillen equivalence between cellular sets and ω-groupoids, since the latter are known *not* to model all homotopy types. In [7], 3-groupoids are shown to be models for homotopy 3-types with vanishing second Postnikov invariant. Brown–Higgins [13] have shown that the category of ω-groupoids is equivalent to the category of *crossed complexes*. This strongly suggests that *all* higher Postnikov invariants of the cellular nerve of an ω-groupoid vanish. Godement's interchange rules for horizontal composition are too restrictive. The correct definition of a *weak* ω*-groupoid* relies to a large extent on a *coherent weakening* of Godement's interchange rules.

4. A HOMOTOPY STRUCTURE FOR WEAK ω-CATEGORIES

In this section, we construct for each contractible ω-operad A a closed model structure for *simplicial* $\underline{A}$*-algebras*. Its homotopy category is equivalent to the homotopy category of compactly generated spaces. This is a first step towards a positive answer to conjectures of Grothendieck [23] and Batanin [2], since the $\underline{A}$-algebras recover all homotopy types among simplicial $\underline{A}$-algebras.

The model structure for simplicial $\underline{A}$-algebras is obtained by transfer from the model structure for A*-cellular spaces*, i.e., simplicial presheaves on Θ_A. With the correct notion of weak equivalence, A*-categorification* is a Quillen equivalence between A-cellular spaces and simplicial $\underline{A}$-algebras. For contractible ω-operads A, this yields equivalences $\mathbf{Ho}(\mathrm{Alg}_{\underline{A}}) \sim \mathbf{Ho}(\mathscr{S}^{\Theta_A^{\mathrm{op}}}) \sim \mathbf{Ho}(\mathrm{Top}^{\mathrm{c}})$.

In more detail, the construction involves the following three steps:

- A model structure for A-cellular spaces obtained by *transfer* from Reedy's model structure for simplicial presheaves on the globular site. The weak equivalences are the *pointwise* weak equivalences.
- A Bousfield localisation [10] with respect to *realisation* weak equivalences. The *localised* model structure transfers to simplicial $\underline{A}$-algebras in such a way that A-categorification is the left adjoint part of a Quillen equivalence between A-cellular spaces and simplicial $\underline{A}$-algebras.
- For *contractible* ω-operads A, base change along $\underline{A} \to \underline{\omega}$, resp. $\Theta_A \to \Theta$, is a Quillen equivalence. Moreover, the homotopy categories are (up to isomorphism) spanned by the simplicially discrete objects.

Recall that any ω-operad A generates a homogeneous globular theory Θ_A and that the representable Θ_A-models are denoted by $\Theta_A[T]$. The *initial* globular theory is the *globular site* Θ_0. Its objects (the level trees) are graded by dimension and any non-identity immersion $S \to T$ *raises* dimension, i.e., $d(S) < d(T)$. The globular site is thus a *directed category* in the sense of Dwyer–Hirschhorn–Kan [17], which is a particular instance of a *Reedy category*. Therefore, the simplicial presheaves on Θ_0 carry a Reedy model structure with *monomorphisms* as cofibrations, *pointwise weak equivalences* as weak equivalences and *Reedy fibrations* as fibrations. We want to *transfer* this model structure to *A-cellular spaces* by means of the adjunction

$i_! : s\mathcal{S}^{\Theta_0^{\mathrm{op}}} \leftrightarrows s\mathcal{S}^{\Theta_A^{\mathrm{op}}} : i^*$, where $i : \Theta_0 \to \Theta_A$ is the canonical embedding of the globular site.

A *Reedy fibration* of simplicial presheaves on Θ_0 is a map $f : X \to Y$ such that the canonical map $X^{\Theta_0[T]} \to Y^{\Theta_0[T]} \times_{Y^{\partial\Theta_0[T]}} X^{\partial\Theta_0[T]}$ is a Kan fibration for each level tree T. Here, we denote by $\partial\Theta_0[T]$ the formal boundary $\Theta_0[T] - \{1_T\}$, cf. Definition 2.10. An i^*-Reedy fibration of A-cellular spaces is a morphism whose image under i^* is a Reedy fibration.

PROPOSITION 4.1. *For each ω-operad A, the category of A-cellular spaces carries a finitely generated right proper model structure with pointwise weak equivalences as weak equivalences, and with i^*-Reedy fibrations as fibrations.*

The cofibrations (resp. trivial cofibrations) are generated by

$$I_A = \{\partial_{\mathrm{out}}\Theta_A[T] \otimes \Delta[m] \cup \Theta_A[T] \otimes \partial\Delta[m] \hookrightarrow \Theta_A[T] \otimes \Delta[m] \mid T, m\};$$

$$J_A = \{\partial_{\mathrm{out}}\Theta_A[T] \otimes \Delta[m] \cup \Theta_A[T] \otimes \Lambda^k[m] \hookrightarrow \Theta_A[T] \otimes \Delta[m] \mid T, m, k\}.$$

Proof. The generating sets $I_\emptyset$ and $J_\emptyset$ for the *initial* ω-operad $\emptyset$ are the generating cofibrations and trivial cofibrations of the above-mentioned Reedy model structure for simplicial presheaves on $\Theta_0 = \Theta_\emptyset$, cf. [17]. Moreover, the left Kan extension $i_!$ sends $I_\emptyset$ (resp. $J_\emptyset$) to I_A (resp. J_A), cf. Definition 2.10. Transfer Theorem 3.10 applies: indeed, the forgetful functor $i^* : s\mathcal{S}^{\Theta_A^{\mathrm{op}}} \to s\mathcal{S}^{\Theta_0^{\mathrm{op}}}$ is monadic (for the embedding $i : \Theta_0 \to \Theta_A$ is bijective on objects, cf. [27]), the associated monad $i^*i_!$ preserves colimits, monomorphisms and pointwise weak equivalences, since $(i^*i_!X)(S) = \coprod_{\Theta^A_{\mathrm{cov}}(S,T)} X(T)$, cf. Theorem 1.17, so that the hypotheses of Theorem 3.10 are easily verified.

The model category of simplicial sets is right proper (i.e., weak equivalences are stable under base change along fibrations) because of Kan's fibrant replacement functor Ex^∞. Therefore, the Reedy model structure for simplicial presheaves on the globular site is right proper. Moreover, right properness is preserved under transfer. ∎

Remark 4.2. There are other model structures with pointwise weak equivalences for A-cellular spaces. For instance, for the terminal ω-operad $A = \omega$, there exists a Reedy model structure on cellular spaces, since Θ is a Reedy category by Lemma 2.4(a). The choice of model structure, Proposition 4.1, was guided by Proposition 4.12 and Theorem 4.13 which we have been unable to establish otherwise.

The *fibrant* objects are those A-cellular spaces X for which the *outer* boundary inclusions $\partial_{\mathrm{out}}\Theta_A[T] \hookrightarrow \Theta_A[T]$ induce Kan fibrations $X^{\Theta_A[T]} \twoheadrightarrow X^{\partial_{\mathrm{out}}\Theta_A[T]}$.

The *cofibrant* objects are retracts of I_A-complexes. An important class of cofibrant A-cellular spaces consists of the *degeneracy-free* A-cellular spaces.

An A-cellular space is degeneracy-free, if there is a *degeneracy diagram* $X_0 : \Theta_0^{\mathrm{op}} \to \mathscr{S}^{\Delta^{\mathrm{op}}_{\mathrm{surj}}}$ such that left Kan extension along $i : \Theta_0 \hookrightarrow \Theta_A$ yields an isomorphism $i_! X_0 \cong X|_{\Delta^{\mathrm{op}}_{\mathrm{surj}}}$. Degeneracy-freeness implies cofibrancy by an induction on the *simplicial* degree.

The *Godement resolution* (also called *bar resolution* or *cotriple resolution*) induced by the adjunction $i_! : \mathscr{S}^{\Theta_0^{\mathrm{op}}} \leftrightarrows \mathscr{S}^{\Theta_A^{\mathrm{op}}} : i^*$ defines a cocontinuous functor $w_A : \mathscr{S}^{\Theta_A^{\mathrm{op}}} \to s\mathscr{S}^{\Theta_A^{\mathrm{op}}}$ such that $\pi_0(w_A X) \cong X$, cf. [21, App. 3; 29, 9.6, 24]. By means of the diagonal $d : \Delta \to \Delta \times \Delta$, the Godement resolution may be extended to a cocontinuous endofunctor $w_A : s\mathscr{S}^{\Theta_A^{\mathrm{op}}} \to s\mathscr{S}^{\Theta_A^{\mathrm{op}}}$ such that there is a weak equivalence $w_A X \xrightarrow{\sim} X$ natural in A-cellular spaces. The explicit formulae for the Godement resolution readily imply that $w_A X$ is degeneracy-free, hence cofibrant. In other words, the model category of A-cellular spaces admits a *cocontinuous cofibrant replacement functor* w_A. This functor plays a similar (dual) role for A-cellular spaces as Kan's fibrant replacement functor Ex^∞ for simplicial sets. For instance, it follows from Brown's lemma and the existence of w_A that the model category of A-cellular spaces is *left proper*, i.e., weak equivalences are stable under cobase change along cofibrations.

The homotopy category of cellular spaces is not equivalent to the homotopy category of cellular sets, since a pointwise weak equivalence between *discrete* cellular spaces is an *isomorphism*. To remedy this, we embed the class of pointwise weak equivalences into the larger class of *realisation weak equivalences* and restrict the class of fibrations to the smaller class of *full fibrations* in such a way that the intersection of the two classes coincides with the original class of trivial fibrations. The cofibrations of the new model structure are thus the same as the original cofibrations. This process is known as *Bousfield localisation* [10].

Even in our case of a finitely generated model structure on a locally finitely presentable category, the existence of a Bousfield localisation with respect to a given class of "local equivalences" is not obvious. The main difficulty consists in finding a generating set of "locally trivial" cofibrations. Rather than recurring to general existence results, we shall use the following lemma.

LEMMA 4.3. *Let $\mathscr{E}$ be a closed model category finitely generated by I and J. Assume that there is a class $\mathscr{E}_{\mathrm{lwe}}$ of local equivalences fulfilling* (M1/2) *as well as a set J_{loc} of locally trivial cofibrations with finite domains such that*:

(1) *weak equivalences are local equivalences*;

(2) *relative* J_{loc}*-complexes are local equivalences*;
(3) *local equivalences right orthogonal to* J_{loc} *are trivial fibrations.*

Then the Bousfield localisation with respect to local equivalences exists and yields a closed model structure finitely generated by I *and* J_{loc}.

Proof. Conditions (1) and (2) imply that $(\mathscr{E}, \mathscr{E}_{\mathrm{cof}}, \mathscr{E}_{\mathrm{lwe}}, J_{\mathrm{loc}})$ is a closed cofibration category in the sense of Definition 3.3. It follows from Lemma 3.4 that the Bousfield localisation exists if and only if condition (3) holds. ∎

Corollary 4.4. *Let* $d^* : \mathscr{E} \leftrightarrows \mathscr{E}' : d_*$ *be a Quillen pair between finitely generated closed model categories such that*:

(1) d^* *has a cofibration preserving left adjoint* $d_!$;
(2) *the unit* $\mathrm{id}_{\mathscr{E}'} \to d^*d_!$ *is a weak equivalence at each cofibrant object*;
(3) *the counit* $d_!d^* \to \mathrm{id}_{\mathscr{E}}$ *admits a section at each generating cofibration.*

Then, $\mathscr{E}$ *admits a finitely generated Bousfield localisation with respect to* d^**-weak equivalences such that* $(d_!, d^*)$ *becomes a Quillen equivalence between the given model structure of* $\mathscr{E}'$ *and the localised model structure of* $\mathscr{E}$. *The full fibrations are those* $\mathscr{E}$*-fibrations whose image under* d^* *is an* $\mathscr{E}'$*-fibration.*

Proof. We apply Lemma 4.3 with d^*-weak equivalences as the local equivalences. Since d^* preserves trivial cofibrations *and* trivial fibrations (1), it preserves weak equivalences, whence Lemma 4.3(1).

The left adjoint $d_!$ preserves finiteness, since d^* is cocontinuous as left adjoint of d_*. Therefore, $J_{\mathrm{loc}} = J_{\mathscr{E}} \cup d_!J_{\mathscr{E}'}$ is a set of *cofibrations* with *finite* domains. By (2), the elements of $d_!J_{\mathscr{E}'}$ are d^*-weak equivalences. Moreover, $d_!J_{\mathscr{E}'}$-attachments are d^*-weak equivalences by cocontinuity of d^*, whence Lemma 4.3(2).

In order to establish Lemma 4.3(3), we have to show that an $\mathscr{E}$-fibration f is trivial as soon as its image under d^* is a trivial $\mathscr{E}'$-fibration. Consider, for a generating cofibration $A \rightarrowtail B$, the following commutative diagrams:

$$
\begin{tikzcd}
d_!d^*A \arrow[r] \arrow[d, tail] & A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads, "f"] \\
d_!d^*B \arrow[r] & B \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
d^*A \arrow[r] \arrow[d, tail] & d^*X \arrow[d, two heads, "d^*(f)"] \\
d^*B \arrow[r] \arrow[ur, dashed] & d^*Y
\end{tikzcd}
$$

The outer left rectangle has a diagonal filler since the adjoint right square does. Here, we use that d^* preserves cofibrations and that $d^*(f)$ is a trivial $\mathscr{E}'$-fibration. Since the left square of the rectangle admits a horizontal section, the right square also has a diagonal filler. Therefore, f is right orthogonal to all generating cofibrations and thus a trivial $\mathscr{E}$-fibration as required.

Since for the localised model structure, d^* preserves *and* detects weak equivalences, (2) implies that $(d_!, d^*)$ is a Quillen equivalence between the given model structure of $\mathscr{E}'$ and the localised model structure of $\mathscr{E}$. The characterisation of the full fibrations follows immediately from the definition of the generating locally trivial cofibrations. ∎

We will localise the model category of cellular spaces with respect to *realisation weak equivalences*, i.e., with respect to those maps $f : X \to Y$ that induce a homotopy equivalence upon *geometric* realisation $|f| = f \otimes_{\Theta\times\Delta} |\Theta[-]| \times |\Delta[-]|$. Since the simplex category $\Delta = \Theta^{(1)}$ embeds in the cell category Θ, geometric realisation factors through cellular sets via

$$\mathrm{real} : s\mathscr{S}^{\Theta^{\mathrm{op}}} \to \mathscr{S}^{\Theta^{\mathrm{op}}} : X \mapsto X \otimes_\Delta \Theta[[-]].$$

This *combinatorial* realisation may be identified with a *diagonal* d^* like in the case of bisimplicial sets. In particular, a map of cellular spaces is a *realisation weak equivalence* if and only if it is a *diagonal weak equivalence*. This point of view goes back to Anderson's paper on geometric realisation [1] and is also the cornerstone of Moerdijk's diagonal structure for bisimplicial sets, cf. Example 3.12(c).

LEMMA 4.5. *The combinatorial realisation of cellular spaces is isomorphic to the inverse image d^* with respect to the following truncated diagonal*:

$$d = (\mathrm{id}_\Theta, v) : \Theta \to \Theta \times \Delta : T \mapsto T \times \partial_2 \cdots \partial_{\mathrm{ht}(T)} T.$$

Proof. Observe first that the truncation of level trees at level 1 extends to a functor $v : \Theta \to \Theta^{(1)} = \Delta$. Indeed, any cellular operator $S \to T$ is given by an S-compatible family of immersions $((T_\sigma)_* \hookrightarrow T_*)_{\sigma \in S_*}$ whose restriction to the S-sectors of height $\leqslant 1$ defines a cellular operator $\partial_2 \cdots \partial_{\mathrm{ht}(S)} S \to \partial_2 \cdots \partial_{\mathrm{ht}(T)} T$, since $\mathrm{ht}(T_\sigma) \leqslant \mathrm{ht}(\sigma)$ for all $\sigma \in S_*$.

Since the functors *real* and d^* are cocontinuous, it suffices to show that they coincide on the representable presheaves $\Theta[T] \otimes \Delta[n]$. It follows from the definitions that $\mathrm{real}(\Theta[T] \otimes \Delta[n]) = \Theta[T] \times \Theta[[n]]$. The diagonal $d^*(\Theta[T] \otimes \Delta[n])$ yields the same cellular set, since for each level tree S and each integer $n \geqslant 0$, any cellular operator $S \to [n]$ factors uniquely through $\partial_2 \cdots \partial_{\mathrm{ht}(S)} S$. ∎

PROPOSITION 4.6. *The Reedy model category of cellular spaces admits a finitely generated Bousfield localisation with respect to realisation weak equivalences.*

Proof. We apply Corollary 4.4 to the adjunction $d^* : s\mathscr{S}^{\Theta^{\mathrm{op}}} \leftrightarrows \mathscr{S}^{\Theta^{\mathrm{op}}} : d_*$. The Reedy model structure on cellular spaces exists, since by Lemma 2.4(a) the cell category Θ is a Reedy category. Its cofibrations are the monomorphisms, its weak equivalences are the pointwise weak equivalences and its fibrations are the Reedy fibrations, i.e., the morphisms right orthogonal to

$$\{\partial\Theta[T] \otimes \Delta[n] \cup \Theta[T] \otimes \Lambda^k[n] \hookrightarrow \Theta[T] \otimes \Delta[n] \mid T \in Ob\Theta, \quad 0 \leqslant k \leqslant n \neq 0\}.$$

The diagonal d^* preserves monomorphisms and takes the generating trivial cofibrations of cellular spaces to weak equivalences of cellular sets. Moreover, the left adjoint $d_!$ takes the generating cofibrations of cellular sets to monomorphisms of cellular spaces, whence Corollary 4.4(1). For Corollary 4.4(2) it suffices (by Reedy's patching lemma) to observe that $d^*d_!$ is cocontinuous and that the unit $\Theta[T] \to d^*d_!\Theta[T] = \Theta[T] \times \Theta[v(T)]$ is a weak equivalence.

For Corollary 4.4(3), we need a section of the counit $d_!d^*(\Theta[T] \otimes \Delta[n]) \to \Theta[T] \otimes \Delta[n]$ which restricts to the boundary $\partial\Theta[T] \otimes \Delta[T] \cup \Theta[T] \otimes \partial\Delta[n]$. This will follow if we construct a section which commutes with face-inclusions in both variables.

Since $d^*(\Theta[T] \otimes \Delta[n]) = \Theta[T] \times \Theta[[n]]$ and since a product of standard cells decomposes into standard cells, we get (see Definition 2.8 and Lemma 4.5 for the notation)

$$d_!d^*(\Theta[T] \otimes \Delta[n]) = \bigcup_{U \in \mathrm{shuff}(T,[n])} \Theta[U] \otimes \Delta[v(U)],$$

where the counit is induced by the two projections of $\rho_U : \Theta[U] \hookrightarrow \Theta[T] \times \Theta[[n]]$. The second projection factorises through truncation $\Theta[U] \to \Theta[v(U)] \to \Theta[[n]]$. Let U be the bouquet $T \vee [n]$ with T to the left and $[n]$ to the right. This singles out the summand $\Theta[U] \otimes \Delta[v(U)]$ of $d_!d^*(\Theta[T] \otimes \Delta[n])$. There is an obvious immersion $T \to U$ (resp. $[n] \to U$) inducing a cellular map $\Theta[T] \to \Theta[U]$ (resp. simplicial map $\Delta[n] \to \Delta[v(U)]$). The tensor product of the two yields a map of cellular spaces $\Theta[T] \otimes \Delta[n] \to \Theta[U] \otimes \Delta[v(U)] \hookrightarrow d_!d^*(\Theta[T] \otimes \Delta[n])$ which is a section of the counit commuting with face-inclusions in both variables. ∎

Proposition 4.7. *A Reedy fibration f of cellular spaces is full* (*i.e.*, *right orthogonal to realisation trivial monomorphisms*) *if and only if one of the following three equivalent conditions is satisfied*:

(1) $d^*(f)$ *is a fibration of cellular sets*;

(2) f *is right orthogonal to*

$$\{\Theta[S]\otimes\Delta[n]\cup\Theta[T]\otimes\partial\Delta[n]\hookrightarrow\Theta[T]\otimes\Delta[n]\mid\Theta[\phi]:\Theta[S]\hookrightarrow\Theta[T],\quad n\geqslant 0\};$$

(3) f *is right orthogonal to*

$$\{\Lambda^{\kappa}[T]\otimes\Delta[n]\cup\Theta[T]\otimes\partial\Delta[n]\hookrightarrow\Theta[T]\otimes\Delta[n]\mid\Lambda^{\kappa}[T]\hookrightarrow\Theta[T],\quad n\geqslant 0\}.$$

Proof. That condition (1) is equivalent to fullness follows from the localisation of the Reedy model structure using Corollary 4.4.

Condition (1) implies (2). By (1), f is right orthogonal to realisation trivial monomorphisms, whence (2), since the diagonals of the indicated monomorphisms are weak equivalences.

Condition (2) implies (3). The right orthogonality properties Proposition 4.7(2), resp. Proposition 4.7(3) are equivalent to the homotopy cartesianness of all squares obtained from mapping $\Theta[S]\hookrightarrow\Theta[T]$, resp. $\Lambda^{\kappa}[T]\hookrightarrow\Theta[T]$, against f. The homotopy cartesianness of the first family implies the homotopy cartesianness of the second family, since any κ-horn may be obtained by gluing of lower dimensional faces along face-inclusions, whilst the lowest dimensional κ-horns are face-inclusions.

Condition (3) implies (1). By adjunction, it suffices to show that f is right orthogonal to the image under $d_!$ of each κ-horn $\Lambda^{\kappa}[T]\hookrightarrow\Theta[T]$. This image is a colimit of face-inclusions $\Theta[S]\otimes\Delta[v(S)]\hookrightarrow\Theta[T]\otimes\Delta[v(T)]$. As above, it suffices to show that f is right orthogonal to these face-inclusions. We have the factorisation

$$\Theta[S]\otimes\Delta[v(S)]\hookrightarrow\Theta[S]\otimes\Delta[v(T)]\hookrightarrow\Theta[T]\otimes\Delta[v(T)].$$

Since f is a Reedy fibration, f is right orthogonal to the first inclusion. Since f satisfies (3), f is a "dual Reedy fibration" (for the model category of simplicial objects in cellular sets) and thus right orthogonal to the second inclusion. ∎

Remark 4.8. The localisation of the Reedy model structure for *simplicial* spaces (i.e., bisimplicial sets) has been studied by Dugger [16] and Rezk–Schwede–Shipley [32]. The second reference takes Proposition 4.7(2) as the *definition* of the full Reedy fibrations, called there *equifibred* Reedy fibrations. Both references show that the localised Reedy model category is a simplicial model category. Our proof of the existence of a localisation for the Reedy model category of cellular (resp. simplicial) spaces is more combinatorial than the one given in [32, 4.3–4] and yields also new characterisations of the class of full Reedy fibrations.

We now turn back to the model structure, Proposition 4.1, for cellular spaces. Since any cofibration is a monomorphism, we get by orthogonality that any Reedy fibration is a fibration. The converse implications, however, are wrong. In particular, we *cannot* apply Corollary 4.4 in order to localise model structure, Proposition 4.1, since the left adjoint $d_!$ does not take cofibrations of cellular sets to cofibrations of cellular spaces. We shall instead use the localisation of the Reedy model structure together with the right properness of model structure, Proposition 4.1.

PROPOSITION 4.9. *The model structure, Proposition* 4.1, *for cellular spaces admits a finitely generated Bousfield localisation with respect to realisation weak equivalences.*

A fibration of cellular spaces is full precisely when it is right orthogonal to

$$\{\Theta[S] \otimes \Delta[n] \cup \Theta[T] \otimes \partial\Delta[n] \hookrightarrow \Theta[T] \otimes \Delta[n] \mid \phi : S \to T \textit{ immersion}, n \geqslant 0\}.$$

Proof. We choose as generating realisation trivial cofibrations the generating trivial cofibrations of model structure, Proposition 4.1, plus the above indicated set. We have to check Lemma 4.3(1–3). Observe that the new elements are realisation trivial cofibrations. They are images of monomorphisms under the left Kan extension along $i : \Theta_0 \hookrightarrow \Theta$, and hence cofibrations. They are realisation weak equivalences since their diagonal equals $\Theta[S] \times \Theta[[n]] \cup \Theta[T] \times \partial\Theta[[n]] \hookrightarrow \Theta[T] \times \Theta[[n]]$.

Lemma 4.3(1) and (2) follow from the localisation, Proposition 4.6, of Reedy's model structure and the fact that any cofibration is monic.

Condition (3) of Lemma 4.3 requires any realisation trivial full fibration $f : X \twoheadrightarrow Y$ to be a trivial fibration. According to Reedy's model structure, f embeds in the following commutative square:

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, two heads, "f"'] & \tilde{X} \arrow[d, two heads, "\tilde{f}"] \\ Y \arrow[r, "\sim"] & \tilde{Y} \end{tikzcd}$$

where the horizontal morphisms are pointwise weak equivalences with Reedy fibrant codomains and where $\tilde{f}$ is a Reedy fibration. Each immersion $\phi : S \to T$ induces two commutative *squares*, front and back of a commutative *cube*:

$$\begin{tikzcd} X(T) \arrow[r] \arrow[d, two heads] & X(S) \arrow[d, two heads] & & \tilde{X}(T) \arrow[r] \arrow[d, two heads] & \tilde{X}(S) \arrow[d, two heads] \\ Y(T) \arrow[r] & Y(S) & & \tilde{Y}(T) \arrow[r] & \tilde{Y}(S) \end{tikzcd}$$

Since f is a full fibration, the left square has a *trivial* Kan fibration as comparison map and is therefore *homotopy cartesian*. Since the horizontal maps of the cube are weak equivalences, the right square is homotopy cartesian as well. Even better, since cellular operators $\bar{0} \to T$ are necessarily *immersions*, the factorisation $(\bar{0} \to S \to T) = (\bar{0} \to T)$ for level-preserving operators $\phi : S \to T$ implies that the above squares are homotopy cartesian for *all* level-preserving operators $\phi : S \to T$. According to Proposition 4.7(2), this implies that $\tilde{f}$ is a *full Reedy fibration*. Therefore, if f (and thus $\tilde{f}$) is realisation trivial, the localisation of the Reedy model structure yields that $\tilde{f}$ is a trivial Reedy fibration, right orthogonal to all monomorphisms. Since cofibrations are monic and the model structure Proposition 4.1 is right proper, this implies that f is right orthogonal to all cofibrations and hence a trivial fibration. ∎

Recall that a *cofibrant replacement* of a morphism $f : X \to Y$ consists of a cofibrant replacement $cX \xrightarrow{\sim} X$ of the domain X together with a factorisation $cX \overset{c(f)}{\rightarrowtail} cY \xrightarrow{\sim} Y$ of the composite morphism $cX \xrightarrow{\sim} X \xrightarrow{f} Y$.

DEFINITION 4.10. A map of A-cellular spaces is a realisation weak equivalence if the left Kan extension along $p : \Theta_A \to \Theta$ of some cofibrant replacement is a realisation weak equivalence of cellular spaces.

A map of simplicial $\underline{A}$-algebras is a weak equivalence if its A-cellular nerve is a realisation weak equivalence of A-cellular spaces.

PROPOSITION 4.11. *For each ω-operad A, the model structure, Proposition* 4.1, *for A-cellular spaces admits a finitely generated Bousfield localisation with respect to realisation weak equivalences.*

A fibration of A-cellular spaces is full precisely when it is right orthogonal to

$$\{\Theta_A[S] \otimes \Delta[n] \cup \Theta_A[T] \otimes \partial\Delta[n] \hookrightarrow \Theta_A[T] \otimes \Delta[n] | \phi{:}S \to T \ \textit{immersion}, n \geqslant 0\}.$$

Proof. We have a canonical factorisation $(\Theta_0 \xrightarrow{i_A} \Theta_A \xrightarrow{p_A} \Theta) = (\Theta_0 \xrightarrow{i} \Theta)$. Since left Kan extensions compose, it follows from the definition of a realisation weak equivalence that it is sufficient to localise the Reedy model structure for simplicial presheaves on Θ_0, since for an arbitrary ω-operad A, the localisation of model structure, Proposition 4.1, may then be obtained by transfer, Theorem 3.10. The left Kan extension along $i : \Theta_0 \hookrightarrow \Theta$ sends $I_\emptyset$ (resp. $J_\emptyset$) to I_ω (resp. J_ω). We choose as generating set $J_\emptyset^{\mathrm{loc}}$ of realisation trivial cofibrations the above indicated set together with $J_\emptyset$ and apply Lemma 4.3. Conditions (1) and (2) of Lemma 4.3 are satisfied since $i_! J_\emptyset^{\mathrm{loc}} = J_\omega^{\mathrm{loc}}$ and all simplicial presheaves on Θ_0 are cofibrant. Condition (3) of Lemma 4.3 requires any realisation trivial full fibration to be a trivial fibration:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "j"] & X \arrow[r] \arrow[d, two heads, "f"] & i^*i_!X \arrow[d, two heads, "i^*i_!(f)"] \\ B \arrow[r] & Y \arrow[r] & i^*i_!Y \end{tikzcd}$$

By adjunction, the morphisms j and $i^*i_!(f)$ are orthogonal if and only if $i_!(j)$ and $i_!(f)$ are orthogonal. Therefore, since $i_! J_\emptyset^{\mathrm{loc}} = J_\omega^{\mathrm{loc}}$ and $i_! I_\emptyset = I_\omega$, the left Kan extension $i_!$ preserves full fibrations and trivial fibrations. The latter implies that $i_!$ preserves also realisation weak equivalences. The image $i_!(f)$ of a realisation trivial full fibration f is thus a trivial fibration of cellular spaces so that $i^*i_!(f)$ is a trivial fibration. The explicit formulae for $i_!$, cf. Theorem 1.12, imply that the right square above is cartesian. Therefore, f is right orthogonal to cofibrations and hence a trivial fibration.

Proposition 4.12. *The unit of the adjunction* $\mathrm{cat}_A : s\mathscr{S}^{\Theta_A^{\mathrm{op}}} \leftrightarrows s\mathrm{Alg}_A : \mathscr{N}_A$ *is a realisation weak equivalence at each object in the image of the Godement resolution functor* w_A, cf. *Remark* 4.2.

Proof. We first show that the unit is a realisation weak equivalence at A-cellular spaces of the form $i_!X'$ for a simplicial presheaf X' on the globular site. In this case we get $i_!X' \to \mathscr{N}_A\, \mathrm{cat}_A i_! X' = \mathscr{N}_A k_!\, \mathrm{cat}_\emptyset X' \cong i_! \mathscr{N}_\emptyset\, \mathrm{cat}_\emptyset X'$, cf. Remark 1.18. It therefore suffices to show that $X' \to \mathscr{N}_\emptyset\, \mathrm{cat}_\emptyset X'$ is a realisation weak equivalence.

Since all simplicial presheaves on the globular site are cofibrant, the functor $i_!$ preserves realisation weak equivalences and we can (without loss of generality) replace X' by a *fully fibrant* simplicial presheaf Y. Observe that $(\mathscr{N}_\emptyset\, \mathrm{cat}_\emptyset Y)(T) = Y^T$ in the notation of Lemma 1.6. Since for non-linear level trees, the inclusion of the outer boundary $\partial_{\mathrm{out}}\Theta[T] \hookrightarrow \Theta[T]$ is a weak equivalence, Lemma 2.11, we get a *trivial* Kan fibration $Y^{\Theta[T]} \to Y^{\partial_{\mathrm{out}}\Theta[T]}$, cf. Remark 4.2. For linear level trees, the Yoneda-lemma yields an isomorphism $Y(\bar{n}) \cong Y^{\bar{n}}$. By induction on $d(T)$, we get thus

$$\begin{tikzcd} Y^{\Theta[T]} = Y(T) \arrow[r] \arrow[d, two heads, "\sim"'] & Y^T = (\mathcal{N}_\emptyset cat_\emptyset Y)(T) \arrow[d, two heads, "\cong"'] \\ Y^{\partial_{out}\Theta[T]} \arrow[r, two heads, "\sim"] & \varprojlim_{S_* \subsetneq T_*} Y^S \end{tikzcd}$$

The lower horizontal map is a trivial Kan fibration, since it is a limit of trivial Kan fibrations induced by level trees S of dimension $d(S) < d(T)$. The right vertical map is an isomorphism, since T is non-linear and Y^- are the components of a Θ_0-model. Therefore, the upper horizontal map is a trivial Kan fibration.

Let us now show that the unit $w_A X \to \mathcal{N}_A \,\mathrm{cat}_A\, w_A X$ is a weak equivalence, where $w_A X$ denotes the Godement resolution of X, cf. Remark 4.2. The latter is the simplicial diagonal of a bisimplicial A-cellular set $w_A^{\bullet;\bullet} X$. By construction, this bisimplicial object in A-cellular sets is a simplicial object in $i^* i_!$-*free* A-cellular spaces. According to the above, the monad $\mathcal{N}_A \,\mathrm{cat}_A$ takes $w_A^{\bullet;\bullet} X$ to a simplicial object in A-cellular spaces which is degreewise realisation weakly equivalent to $w_A^{\bullet;\bullet} X$. Taking the simplicial diagonal yields thus a realisation weak equivalence $w_A X \to \mathcal{N}_A \,\mathrm{cat}_A\, w_A X$. ∎

THEOREM 4.13. *For each ω-operad A, the localised model structure for A-cellular spaces transfers to simplicial $\underline{A}$-algebras in such a way that the adjunction* $\mathrm{cat}_A : s\mathcal{S}^{\Theta_A^{\mathrm{op}}} \leftrightarrows s\mathrm{Alg}_{\underline{A}} : \mathcal{N}_A$ *is a Quillen equivalence.*

Proof. The elements of $\mathrm{cat}_A(I_A)$ and $\mathrm{cat}_A(J_A^{\mathrm{loc}})$ have finite domains and codomains. Moreover, since $\mathcal{N}_A$ may be considered as the embedding of a reflective subcategory, it suffices to show that $\mathcal{N}_A \,\mathrm{cat}_A$ takes realisation trivial cofibrations $f : X \rightarrowtail Y$ with domain $X \in \mathrm{im}(\mathcal{N}_A)$ to realisation weak equivalences by Remark 3.11. We may assume that f is obtained by attachment of an element in J_A^{loc}, and hence, cf. Propositions 4.1 and 4.11, of a morphism in the image of $\mathcal{N}_A$. Remark 1.18 implies then that for the A-cellular space Y, the Godement resolution commutes with A-categorification, i.e., $w_A \mathcal{N}_A \,\mathrm{cat}_A\, Y \cong \mathcal{N}_A \,\mathrm{cat}_A\, w_A Y$. Therefore, we get

$$\begin{tikzcd} \omega_A Y \arrow[r, "\sim", "(4.12)"'] \arrow[d, "\sim"'] & \mathcal{N}_A cat_A \omega_A Y \cong \omega_A \mathcal{N}_A cat_A Y \arrow[d, "\sim"'] \\ Y \arrow[r] & \mathcal{N}_A cat_A Y \end{tikzcd}$$

whence $\mathcal{N}_A\,\mathrm{cat}_A(f) = (X \rightarrowtail Y \to \mathcal{N}_A\,\mathrm{cat}_A\,Y)$ is a realisation weak equivalence.

Since the simplicial $\underline{A}$-algebras carry a transferred model structure, the composite functor $\mathcal{N}_A\,\mathrm{cat}_A$ takes realisation trivial cofibrations to realisation weak equivalences. By Brown's lemma, Propositions 4.2 and 4.12, this implies that the unit of the adjunction is a weak equivalence at each cofibrant A-cellular space. Therefore, since the A-cellular nerve preserves *and* detects realisation weak equivalences, the Quillen pair $(\mathrm{cat}_A, \mathcal{N}_A)$ is a Quillen equivalence. ∎

THEOREM 4.14. *For a contractible ω-operad A, base change along $\Theta_A \to \Theta$ (resp. $\underline{A} \to \underline{\omega}$) is a Quillen equivalence between the localised model structures.*

Proof. The adjunctions $p_! : s\mathcal{S}^{\Theta_A^{\mathrm{op}}} \leftrightarrows s\mathcal{S}^{\Theta^{\mathrm{op}}} : p^*$ and $q_! : s\mathrm{Alg}_{\underline{A}} \leftrightarrows s\mathrm{Alg}_{\underline{\omega}} : q^*$ are Quillen pairs, since for any ω-operad A, the forgetful functor $i^* : s\mathcal{S}^{\Theta_A^{\mathrm{op}}} \to s\mathcal{S}^{\Theta_0^{\mathrm{op}}}$ preserves *and* detects realisation weak equivalences and full fibrations, cf. the proof of Proposition 4.11. According to Theorem 4.13, both Quillen pairs are thus simultaneously Quillen equivalences. We shall prove the statement for the Quillen pair $(q_!, q^*)$.

The right adjoint q^* preserves and detects weak equivalences. It remains to show that the unit $X \to q^*q_!X$ is a weak equivalence for each *cofibrant* simplicial $\underline{A}$-algebra X. The free-forgetful adjunction $(k_!, k^*)$ induces a Godement resolution v_A which fulfills by Remark 1.18 the relation $\mathcal{N}_A v_A = w_A \mathcal{N}_A$. Therefore, v_A is a cofibrant replacement functor for simplicial $\underline{A}$-algebras. A diagonal trick allows us to restrict ourselves to the case of a simplicial $\underline{A}$-algebra of the form $X = v_A Y$ where Y is an $\underline{A}$-algebra. Since the natural transformation $\underline{A} \to \underline{\omega}$ is cartesian and the structural maps of the terminal analytic monad $\underline{\omega}$ are also cartesian, cf. [37; 2, 6.4], the unit $v_A Y \to q^*q_!v_A Y$ has the special property that it is a cartesian transformation with respect to globular operators and with respect to simplicial operators. An inspection of the generating trivial cofibrations $\mathrm{cat}_A(J_A^{\mathrm{loc}})$ reveals that such a morphism is a *fibration* so that its A-cellular nerve is a full fibration of A-cellular spaces.

The contractibility condition, Definition 1.20, implies that $v_A Y \to q^*q_!v_A Y$ is right orthogonal to the set $\{k_!(\partial\bar{n}_*) \hookrightarrow k_!(\bar{n}_*), n \geqslant 0\}$. Since $\mathcal{N}_A$ is fully faithful, and since $\mathcal{N}_A k_! \cong i_! \mathcal{N}_\emptyset$, Remark 1.18, this is equivalent to $\mathcal{N}_A v_A Y \to \mathcal{N}_A q^*q_!v_A Y$ being right orthogonal to $\{\partial\Theta_A[\bar{n}] \otimes \Delta[0] \hookrightarrow \Theta_A[\bar{n}] \otimes \Delta[0], n \geqslant 0\}$.

Since $p_!$ detects realisation weak equivalences and preserves full fibrations, it remains to show that a full fibration of cellular spaces is trivial as soon as it is right orthogonal to the set $\{\partial\Theta[\bar{n}] \otimes \Delta[0] \hookrightarrow \Theta[\bar{n}] \otimes \Delta[0], n \geqslant 0\}$. This follows from the fact that the latter set represents a complete set of sphere-inclusions and that the localised model category of cellular spaces is Quillen equivalent to the model category of compactly generated spaces, cf. the proof of Proposition 4.17. ∎

Remark 4.15. The preceding theorem may be compared with Dwyer–Kan's theorem [19] that a functor $p : \mathscr{C} \to \mathscr{D}$ induces an equivalence of homotopy categories $\mathbf{Ho}(s\mathscr{S}^{\mathscr{C}}) \sim \mathbf{Ho}(s\mathscr{S}^{\mathscr{D}})$ with respect to *pointwise weak equivalences* whenever $p : \mathscr{C} \to \mathscr{D}$ satisfies the hypothesis of Quillen's Theorem A (i.e., the "categorical homotopy fibers" are weakly contractible). It seems, however, unlikely that for a contractible ω-operad A, the canonical functor $p : \Theta_A \to \Theta$ satisfies this hypothesis, whence the necessity of a localisation of the pointwise structures.

We shall now show that for each ω-operad A, the localised homotopy category of A-cellular spaces is already determined by the *discrete* objects, i.e., by the A-cellular *sets*. Unfortunately, we have been unable to define an explicit model structure for A-cellular sets. The main difficulty is the correct definition of a cofibration of A-cellular sets since there are *too few* discrete cofibrations in the model category of A-cellular spaces. There is, nevertheless, a quite tractable concept of *discrete model structure* for the simplicial objects of a category $\mathscr{E}$.

The embedding $i : \mathscr{E} \hookrightarrow s\mathscr{E}$ identifies the objects of $\mathscr{E}$ with the *simplicially discrete* objects. This embedding has a right adjoint *coreflector* $r : s\mathscr{E} \to \mathscr{E}$ given by $rX = X([0])$ as well as a left adjoint *reflector* $\pi_0 : s\mathscr{E} \to \mathscr{E}$ given by the "simplicial path components". The above constructed model structures, however, do *not* transfer along (π_0, i) essentially because the boundary of the simplicial interval $\Delta[1]$ has *two* path components.

Definition 4.16. A closed model structure on the category $s\mathscr{E}$ of simplicial objects of $\mathscr{E}$ is *discrete* if the counit $irX \to X$ is a weak equivalence for each fibrant object X of $s\mathscr{E}$.

It follows that the homotopy category $\mathbf{Ho}(s\mathscr{E})$ is equivalent to the homotopy category $\mathbf{Ho}(\mathscr{E})$ spanned by the discrete objects, explicitly defined as follows: $\mathbf{Ho}(\mathscr{E})(X, Y) = \mathbf{Ho}(s\mathscr{E})(iX, iY)$ with the obvious composition law.

The *canonical model structure* on $s\mathscr{E}$ constructed by Rezk–Schwede–Shipley [32] for certain model categories $\mathscr{E}$ is an example of a discrete model structure.

Proposition 4.17. *For each ω-operad A, the localised model structure on the category of A-cellular spaces is discrete.*

Proof. We first treat the terminal ω-operad $A = \omega$, i.e., the case of cellular spaces. We have two composable Quillen pairs (d^*, d_*) and

$(|-|, \mathrm{Cell})$:

$$s\mathcal{S}^{\Theta^{\mathrm{op}}} \leftrightarrows \mathcal{S}^{\Theta^{\mathrm{op}}} \leftrightarrows \mathrm{Top}^{\mathrm{c}}.$$

The right Quillen pair is a Quillen equivalence by Theorem 3.9, the composite Quillen pair is a Quillen equivalence by essentially the same reason, cf. Proposition 4.9. It follows that the left Quillen pair is also a Quillen equivalence. Therefore, since d^* preserves and detects realisation weak equivalences and since all cellular sets are cofibrant, the unit $X \to d_*d^*X$ is a realisation weak equivalence for all cellular spaces X. Since $rd_* = \mathrm{id}_{\mathcal{S}^{\Theta^{\mathrm{op}}}}$, we get the following commutative square:

$$\begin{tikzcd} X \arrow[r, "\sim"] & d_*d^*X \\ irX \arrow[u] \arrow[r, "\sim"] & id^*X \arrow[u, "\sim"'] \end{tikzcd}$$

The right vertical map is a realisation weak equivalence since its diagonal may be identified with $d^*(X \xrightarrow{\sim} d_*d^*X)$. The lower horizontal map is a realisation weak equivalence for *fully fibrant* cellular spaces, since it corresponds to the embedding of $X([0])$ into the sequential colimit $X \otimes_\Delta \Theta[[-]]$, cf. Lemma 4.5. The successive inclusions of this colimit are weak equivalences by a well-known gluing argument, since for a fully fibrant cellular space X, the simplicial face-inclusions $\Delta[n-1] \hookrightarrow \Delta[n]$ induce trivial fibrations $\partial_i^X : X^{\Delta[n]} \to X^{\Delta[n-1]}$. Indeed, ∂_i^X is right orthogonal to horn-inclusions $\Lambda^\kappa[T] \hookrightarrow \Theta[T]$, as X is fibrant, Proposition 4.1, and to outer boundary-inclusions $\partial_{\mathrm{out}}\Theta[T] \hookrightarrow \Theta[T]$, as X is fully fibrant, Proposition 4.9; in particular, ∂_i^X is a fibration right orthogonal to the complete set of sphere-inclusions $\{\partial\Theta[\bar{n}] \hookrightarrow \Theta[\bar{n}] \mid n \geqslant 0\}$ and hence a trivial fibration, since the model category of cellular sets is Quillen equivalent to the model category of compactly generated spaces. It follows that $irX \to X$ is a realisation weak equivalence for each fully fibrant cellular space X so that the localised model structure for cellular spaces is discrete.

For an arbitrary ω-operad A, the localised model structure on the category of A-cellular spaces is discrete: the left Kan extension along the canonical functor $p : \Theta_A \to \Theta$ commutes with the coreflections ir and moreover preserves fully fibrant objects and detects realisation weak equivalences, cf. Proposition 4.11. ∎

THEOREM 4.18. *For each ω-operad A, the closed model structure for simplicial $\underline{A}$-algebras is discrete. If A is contractible, the resulting homotopy* *category of $\underline{A}$-algebras is equivalent to the homotopy category of compactly generated spaces.*

Proof. For each ω-operad A, the model structure for simplicial $\underline{A}$-algebras is obtained by transfer from the localised model structure for A-cellular spaces, cf. Theorem 4.13, which readily implies that it is a *discrete* model structure by Proposition 4.17. The second assertion follows from the following chain of equivalences:

$$\begin{tikzcd}
\mathbf{Ho}\,(\mathrm{Alg}_{\underline{A}}) \arrow[r, "\sim", "(4.18)"'] & \mathbf{Ho}\,(s\mathrm{Alg}_{\underline{A}}) \arrow[d, "\sim"', "(4.14)"] & \mathbf{Ho}(s\mathscr{S}^{\Theta_A^{\mathrm{op}}}) \arrow[l, "\sim"', "(4.13)"] \arrow[d, "\sim"', "(4.14)"] & \\
& \mathbf{Ho}\,(s\mathrm{Alg}_{\underline{\omega}}) & \mathbf{Ho}(s\mathscr{S}^{\Theta^{\mathrm{op}}}) \arrow[l, "\sim"', "(4.13)"] \arrow[r, "\sim", "(4.9)"'] & \mathbf{Ho}(\mathrm{Top}^c)
\end{tikzcd}$$

where all but the first equivalence are induced by Quillen equivalences. The direction indicates the left adjoint part of the Quillen equivalence. ∎

Remark 4.19. There are n-truncated versions of the above theorem, giving rise to model structures which are not anymore discrete (the proofs of Theorem 4.14, Proposition 4.17 need the existence of the complete set of standard n-cells $\Theta[\bar{n}]$), but only discrete up to the nth Postnikov section. For instance, if we consider the terminal n-operad and the associated full subcategory $\Theta^{(n)}$ of Θ, we get a model structure for *simplicial n-categories*, which is discrete up to the nth Postnikov section. Even the case $n = 1$ is interesting (observe that $\Theta^{(1)} = \Delta$), since we get a model structure on *bisimplicial sets* which is Quillen equivalent under categorification to the transferred model structure on *simplicial categories*. It is perhaps worthwhile noting that a simplicial category is fibrant for our model structure, Theorem 4.13 precisely when the simplicial set of objects is fibrant and the simplicial set of morphisms is a *path-object* in Quillen's sense via the source/target maps.

Remark 4.20. There are many questions left open, here is a small outline:

1. Is there a model structure for A-cellular sets (resp. $\underline{A}$-algebras) inducing the above constructed homotopy category $\mathbf{Ho}(\mathscr{S}^{\Theta_A^{\mathrm{op}}})$ (resp. $\mathbf{Ho}(\mathrm{Alg}_{\underline{A}})$)? Is the localised model category of A-cellular spaces, resp. of simplicial $\underline{A}$-algebras, a *simplicial* model category in Quillen's sense [31]?

2. For a given contractible ω-operad A, what are the *weak ω-groupoids*, i.e., what are the $\underline{A}$-algebras for which all cells of positive dimension are weakly invertible in a suitable sense? See [2, 9.2] for a possible answer. There

is another drastic method to define weak ω-groupoids, namely to require *strict* vertical invertibility so that the weakness only concerns the composition laws. In order to make this meaningful, we should have available uniquely determined identity cells and vertical compositions. We suggest calling an ω-operad A *pointed* if the sets $A(T)_n$ are pointed for all level trees T such that $d(T) \leqslant ht(T) + 1$. This includes an explicit choice for *identity cells* as well as for all *vertical compositions* and *whiskerings* inside a given $\underline{A}$-algebra, see [2, 8.4] for a slightly different option. An *$\underline{A}$-groupoid* is then an $\underline{A}$-algebra for which all cells of positive dimension are *vertically* invertible in such a way that vertical inversion commutes with whiskering.

3. It follows readily from the above definition that $\underline{A}$-groupoids define a *reflective* subcategory of the category of $\underline{A}$-algebras. In view of Theorem 4.18, Grothendieck's [23] and Batanin's [2, 9.1] conjectures may be reformulated as follows: For *certain* pointed contractible ω-operads A, *the $\underline{A}$-groupoids recover all homotopy types among $\underline{A}$-algebras*. Is there a criterion for A equivalent to (or at least sufficient for) the latter property to hold?It is known that for the *terminal ω-operad $A = \omega$*, this property fails. More precisely, 1-groupoids yield all homotopy 1-types, 2-groupoids yield all homotopy 2-types, but 3-groupoids yield only the homotopy 3-types with vanishing second Postnikov invariant. Replacing the terminal 3-operad by Gray's operad G, cf. Example 1.21, repairs this defect: G-groupoids yield all homotopy 3-types, cf. [7]. How to generalise this result to higher orders?

4. Each topological space X admits a *fundamental ω-graph ΠX* defined by $(\Pi X)_n = \mathrm{Top}^c(B^n, X)$. There is a tautological action on ΠX by the ω-operad $E(T)_n = \mathrm{Top}^c_{\mathbb{G}}(B^n, B^T)$, where B^T denotes a T-amalgamated sum of balls according to the ω-disk structure of B, cf. Definition 2.1, [2, 9.2]. This action is completely analogous to the tautological action of the symmetric operad $S(n) = \varinjlim_k \mathrm{Top}^c_*(S^k, (S^k)^{\vee n})$ on infinite loop spaces. Boardman–Vogt's little cubes operad [8, 29], sits inside S and acts thus on infinite loop spaces. Batanin's universal contractible ω-operad K sits inside E so that the fundamental ω-graph ΠX is a $\underline{K}$-algebra. Is ΠX a $\underline{K}$-groupoid and do the $\underline{K}$-groupoids recover all homotopy types among $\underline{K}$-algebras?

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Last but not least, my initial source of inspiration were infinite loop spaces. The reader acquainted with this beautiful, almost 30-year old theory will observe without difficulty that I did not so much more than consider weak ω-categories as if they were E_∞-spaces or Γ-spaces.

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