Cubical Feynman categories and derived modular envelopes

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[1] joint with Ralph Kaufmann (Purdue University)

Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$ moduli space of *hyperbolic metrics* on a surface $F_{g,n}$ of *genus* g with n *punctures* where $\chi(F_{g,n}) < 0$ and $n > 0$.
- $\mathcal{M}_G$ moduli space of *admissible metrics* on ribbon graph G.

Theorem (Mumford, Strebel, Penner, Kontsevich, ...)

$\mathcal{M}_{g,n} \simeq \bigcup_G \mathcal{M}_G$ where the metric ribbon graphs G are of type (g, n) and at least trivalent.

Proposition (Igusa)

$\bigcup_G \mathcal{M}_G \simeq |\mathrm{nerve}(\mathrm{rb}_{g,n})|$ where the *ribbon category* $\mathrm{rb}_{g,n}$ is generated by orientation preserving edge contractions between ribbon graphs of type (g, n).

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Definition (bordered case)

$\mathcal{M}^S_{g,n}$ moduli space of hyperbolic metrics on a surface $F^S_{g,n}$ of *genus* g with n *punctures* and S-marked boundary.

Theorem (Penner, Igusa, B-K)

$\mathcal{M}^S_{g,n} \simeq \bigcup_G \mathcal{M}_G \simeq |\mathrm{rb}^S_{g,n}|$ where the *flagged* ribbon graphs G are of type (g, n, S) and at least trivalent.

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Purpose of the talk

- define surface-modular operads (cf. Markl)
- show that the functor

$$J : (\text{planar-cyclic operads}) \longrightarrow (\text{surface-modular operads})$$

induces homotopy equivalences

$$\mathbb{L}J_!(\mathbb{1})(g, n, S) \simeq \mathcal{M}^S_{g,n}$$

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Proposition (May-Thomason, Elmendorf-Mandell, Hermida)

Each coloured operad $\mathcal{O}(i_1,\dots,i_k;i)$ induces a symmetric monoidal category $\mathfrak{F}_{\mathcal{O}}$ having as objects ordered sequences of colours and as morphisms ordered sequences of operations.

Remark (framed symmetric monoidal categories)

$\mathfrak{F}_{\mathcal{O}}$ contains the invertible unary operations of $\mathcal{O}$ as subgroupoid $\mathcal{V}_{\mathcal{O}}$ such that $(\mathcal{V}_{\mathcal{O}})^{\otimes} \simeq \mathrm{Iso}(\mathfrak{F}_{\mathcal{O}})$ (we call $\mathcal{V}_{\mathcal{O}}$ a *framing* of $\mathfrak{F}_{\mathcal{O}}$).

Proposition (B-K)

Coloured operads are *coreflective* inside framed sym. monoidal categories. The essential image consists of *Feynman categories*.

Definition (Kaufmann-Ward)

A Feynman category $\mathfrak{F}$ is a sym. mon. cat. with framing $\mathcal{V}^{\otimes} \simeq \mathrm{Iso}(\mathfrak{F})$ such that hereditary and size conditions are satisfied.

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Lemma ($\mathcal{O}$-algebra=$\mathfrak{F}_{\mathcal{O}}$-algebra)

Any $\mathcal{O}$-algebra extends to a strong sym. mon. functor $\mathfrak{F}_{\mathcal{O}} \to \mathrm{Sets}$.

Proposition (Kaufmann-Ward)

Any Feynman functor $j : \mathfrak{F} \to \mathfrak{F}'$ induces an adjunction

$$j_! : \mathfrak{F}\text{-alg} \longrightarrow \mathfrak{F}'\text{-alg} : j^*$$

such that the left adjoint is given by pointwise left Kan extension

$$(j_! P)(A') = \mathrm{colim}_{j(-)\downarrow A'} P(-).$$

Proposition (B-K)

Any Feynman functor $j : \mathfrak{F} \to \mathfrak{F}'$ factors essentially uniquely as a *connected* Feynman functor followed by a *covering* where j is connected (resp. a covering) iff $j_!(\mathbb{1}) = \mathbb{1}$ (resp. $\mathfrak{F} \cong \mathrm{el}_{\mathfrak{F}'}(j_!(\mathbb{1}))$).

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Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad $\mathcal{S}$ whose algebras are symmetric operads. Its associated Feynman category $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{sym}$ has

- as objects disjoint unions of *rooted* corollas
- as morphisms disjoint unions of *rooted* trees
- composition induced by *rooted* tree insertion

Lemma (Getzler-Kapranov)

The Feynman category $\mathfrak{F}_{cyc}$ for cyclic operads has

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Lemma (Borisov-Manin, Kaufmann-Ward)

There are Feynman functors $\mathfrak{F}_{sym} \to \mathfrak{F}_{cyc} \to \mathfrak{F}_{ctd}$ where $\mathfrak{F}_{ctd}$ has

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Proposition (Getzler-Kapranov)

The Feynman functor $h : \mathfrak{F}_{cyc} \to \mathfrak{F}_{ctd}$ factors as connected functor $j : \mathfrak{F}_{cyc} \to \mathfrak{F}_{mod}$ followed by a covering $k : \mathfrak{F}_{mod} \to \mathfrak{F}_{ctd}$ where $\mathfrak{F}_{mod}$ is the Feynman category for modular operads.

Corollary (B-K)

The $\mathfrak{F}_{ctd}$-algebra $h_!(1)$ counts the "number of loops".

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The $\mathfrak{F}_{ctd}$-algebra $h_!(1)$ counts the "number of loops".

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The $\mathfrak{F}_{ctd}$-algebra $h_!(1)$ counts the "number of loops".

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\begin{array}{ccccc}
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\big\downarrow{\scriptstyle p(\tau_{assoc})} & & \big\downarrow{\scriptstyle p(\tau_{planar})} & & \big\downarrow{\scriptstyle p(\tau_{surf})} \\
\mathfrak{F}_{sym} & \xrightarrow[\;i\;]{} & \mathfrak{F}_{cyc} & \xrightarrow[\;j\;]{} & \mathfrak{F}_{mod} \\
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where vertical arrows are coverings, and j, J are connected.

- τ_{assoc} is the $\mathfrak{F}_{sym}$-algebra for associative monoids
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- $i^*(\tau_{planar}) = \tau_{assoc}$ (τ_{planar} is the "cyclic" version of τ_{assoc})
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- equ. cl. of one-vertex ribbon graphs with γ loops and s flags
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Corollary (B-K)

The morphisms of the Feynman category $\mathfrak{F}_{surf}$ are γ/n-labeled polycyclic graphs.

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A Feynman category $\mathfrak{F}$ is *cubical* if there is a degree function $\deg : Mor(\mathfrak{F}) \to \mathbb{N}_0$ such that

- $\deg(\phi \otimes \psi) = \deg(\phi) + \deg(\psi)$
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- Degree 0 morphisms are invertible
- Each degree n morphism factors (up to iso) in $n!$ ways into degree 1 morphisms "compatibly with composition"

Remark

In the non-unital case without constants, the Feynman categories $\mathfrak{F}_{sym}, \mathfrak{F}_{cyc}, \mathfrak{F}_{mod}, \mathfrak{F}_{non\text{-}sym}, \mathfrak{F}_{plan\text{-}cyc}, \mathfrak{F}_{surf}$ are cubical. The degree of ϕ is the number of edges of the representing graph Γ_ϕ.

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Definition ($W_{\mathfrak{F}}$-construction)

Let P be an operad over a cubical Feynman category $\mathfrak{F}$. Put

$$(W_{\mathfrak{F}}P)(B) = \left(\coprod_{\phi \in \mathfrak{F}(A,B)} P(A) \times_{\mathrm{Aut}_{\mathfrak{F}\downarrow B}(\phi)} [0,1]^{\deg(\phi)} \right) / \sim$$

where identifications are on faces of $[0,1]^{\deg(\phi)}$ according to coarser factorisations of ϕ.

Proposition (Kaufmann-Ward, cf. Boardman-Vogt, B-Moerdijk)

For any cubical Feynman category $\mathfrak{F}$, the category of topological $\mathfrak{F}$-algebras admits a *transferred model structure*. If P has an underying cofibrant $\mathcal{V}$-collection then $W_{\mathfrak{F}}P$ is a *cofibrant $\mathfrak{F}$-algebra*.

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Example (cubically subdivided convex polytopes)

- $W_{sym}(\tau_{assoc})$(rooted corolla)=associahedron
- $W_{cyc}(\tau_{planar})$(corolla)=cyclohedron

Proposition (Igusa-flow)

Let $\phi : \mathfrak{F} \to \mathfrak{F}'$ be a functor of cubical Feynman categories.

- $(W_{\mathfrak{F}}\mathbb{1})(B) \simeq |\mathrm{nerve}(\mathfrak{F} \downarrow B)|$
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- $J_!(W_{plan\text{-}cyc}\mathbb{1}) = (\mathbb{L}J_!)(\mathbb{1})$?
  no for transferred *projective* model structure, but yes for transferred *equivariant* model structure, cf. Vogt.
- Since $p_!(\mathbb{1}_{plan\text{-}cyc}) = \tau_{planar}$, $j_!(W\tau_{planar})$ decomposes according to $p_!J_!(W_{plan\text{-}cyc}\mathbb{1})$. What about derived modular envelopes of other cyclic operads ?
- $\chi_{orbi}(\mathcal{M}_{g,1}) = \chi_{orbi}(\mathrm{rb}_{g,1}) = \zeta(1-2g) = \zeta(\chi(F_{g,1}))$ (Harer-Zagier).
  What about $\chi_{orbi}(\mathrm{rb}^S_{g,n})$?
  Relationship with multi-zeta functions ?
- Relationship with Kontsevich's graph homology or a flagged version of it ?

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