

# Cubical Feynman categories and derived modular envelopes

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<sup>1</sup>joint with Ralph Kaufmann (Purdue University)

- 1 Moduli spaces of bordered surfaces
- 2 Feynman categories
- 3 Symmetric, cyclic and modular operads
- 4 Non-symmetric, planar-cyclic and surface-modular operads
- 5 Cubical Feynman categories and  $W$ -construction
- 6 Perspectives and open problems

## Definition (moduli space for oriented surfaces/ribbon graphs)

- $\mathcal{M}_{g,n}$  moduli space of *hyperbolic metrics* on a surface  $F_{g,n}$  of genus  $g$  with  $n$  punctures where  $\chi(F_{g,n}) < 0$  and  $n > 0$ .
- $\mathcal{M}_G$  moduli space of *admissible metrics* on ribbon graph  $G$ .

## Theorem (Mumford, Strebel, Penner, Kontsevich, ...)

$\mathcal{M}_{g,n} \simeq \bigcup_G \mathcal{M}_G$  where the metric ribbon graphs  $G$  are of type  $(g, n)$  and at least trivalent.

## Proposition (Igusa)

$\bigcup_G \mathcal{M}_G \simeq |\text{nerve}(\text{rb}_{g,n})|$  where the *ribbon category*  $\text{rb}_{g,n}$  is generated by orientation preserving edge contractions between ribbon graphs of type  $(g, n)$ .

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$\mathcal{M}_{g,n}^S$  moduli space of hyperbolic metrics on a surface  $F_{g,n}^S$  of genus  $g$  with  $n$  punctures and  $S$ -marked boundary.

### Theorem (Penner, Igusa, B-K)

$\mathcal{M}_{g,n}^S \simeq \bigcup_G \mathcal{M}_G \simeq |\text{rb}_{g,n}^S|$  where the *flagged* ribbon graphs  $G$  are of type  $(g, n, S)$  and at least trivalent.

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## Purpose of the talk

- define surface-modular operads (cf. Markl)
- show that the functor

$$J : (\text{planar-cyclic operads}) \longrightarrow (\text{surface-modular operads})$$

induces homotopy equivalences

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Proposition (May-Thomason, Elmendorf-Mandell, Hermida)

Each coloured operad  $\mathcal{O}(i_1, \dots, i_k; i)$  induces a symmetric monoidal category  $\mathfrak{F}_{\mathcal{O}}$  having as objects ordered sequences of colours and as morphisms ordered sequences of operations.

Remark (framed symmetric monoidal categories)

$\mathfrak{F}_{\mathcal{O}}$  contains the invertible unary operations of  $\mathcal{O}$  as subgroupoid  $\mathcal{V}_{\mathcal{O}}$  such that  $(\mathcal{V}_{\mathcal{O}})^{\otimes} \simeq \text{Iso}(\mathfrak{F}_{\mathcal{O}})$  (we call  $\mathcal{V}_{\mathcal{O}}$  a *framing* of  $\mathfrak{F}_{\mathcal{O}}$ ).

Proposition (B-K)

Coloured operads are *coreflective* inside framed sym. monoidal categories. The essential image consists of *Feynman categories*.

Definition (Kaufmann-Ward)

A Feynman category  $\mathfrak{F}$  is a sym. mon. cat. with framing  $\mathcal{V}^{\otimes} \simeq \text{Iso}(\mathfrak{F})$  such that hereditary and size conditions are satisfied.

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Lemma ( $\mathcal{O}$ -algebra =  $\mathfrak{F}_{\mathcal{O}}$ -algebra)

Any  $\mathcal{O}$ -algebra extends to a strong sym. mon. functor  $\mathfrak{F}_{\mathcal{O}} \rightarrow \text{Sets}$ .

## Proposition (Kaufmann-Ward)

Any Feynman functor  $j : \mathfrak{F} \rightarrow \mathfrak{F}'$  induces an adjunction

$$j_! : \mathfrak{F}\text{-alg} \longrightarrow \mathfrak{F}'\text{-alg} : j^*$$

such that the left adjoint is given by pointwise left Kan extension

$$(j_! P)(A') = \text{colim}_{j(-) \downarrow A'} P(-).$$

## Proposition (B-K)

Any Feynman functor  $j : \mathfrak{F} \rightarrow \mathfrak{F}'$  factors essentially uniquely as a *connected* Feynman functor followed by a *covering* where  $j$  is connected (resp. a covering) iff  $j_!(1) = 1$  (resp.  $\mathfrak{F} \cong \text{el}_{\mathfrak{F}'}(j_!(1))$ ).

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### Lemma (Ginzburg-Kapranov, B-Moerdijk, Kontsevich-Soibelman)

There is a coloured operad  $\mathcal{S}$  whose algebras are symmetric operads. Its associated Feynman category  $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{sym}$  has

- as objects disjoint unions of rooted corollas
- as morphisms disjoint unions of rooted trees
- composition induced by rooted tree insertion

### Lemma (Getzler-Kapranov)

The Feynman category  $\mathfrak{F}_{cyc}$  for cyclic operads has

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## Lemma (Borisov-Manin, Kaufmann-Ward)

There are Feynman functors  $\mathfrak{F}_{sym} \rightarrow \mathfrak{F}_{cyc} \rightarrow \mathfrak{F}_{ctd}$  where  $\mathfrak{F}_{ctd}$  has

- objects: connected graphs
- morphisms: disjoint unions of connected graphs
- composition: induced by graph insertion

## Proposition (Getzler-Kapranov)

The Feynman functor  $h : \mathfrak{F}_{cyc} \rightarrow \mathfrak{F}_{ctd}$  factors as connected functor  $j : \mathfrak{F}_{cyc} \rightarrow \mathfrak{F}_{mod}$  followed by a covering  $k : \mathfrak{F}_{mod} \rightarrow \mathfrak{F}_{ctd}$  where  $\mathfrak{F}_{mod}$  is the Feynman category for modular operads.

## Corollary (B-K)

The  $\mathfrak{F}_{ctd}$ -algebra  $h_!(1)$  counts the “number of loops”.

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### Corollary (B-K)

The  $\mathfrak{F}_{ctd}$ -algebra  $h_!(1)$  counts the “number of loops”.

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The Feynman functor  $h : \mathfrak{F}_{cyc} \rightarrow \mathfrak{F}_{ctd}$  factors as connected functor  $j : \mathfrak{F}_{cyc} \rightarrow \mathfrak{F}_{mod}$  followed by a covering  $k : \mathfrak{F}_{mod} \rightarrow \mathfrak{F}_{ctd}$  where  $\mathfrak{F}_{mod}$  is the Feynman category for modular operads.

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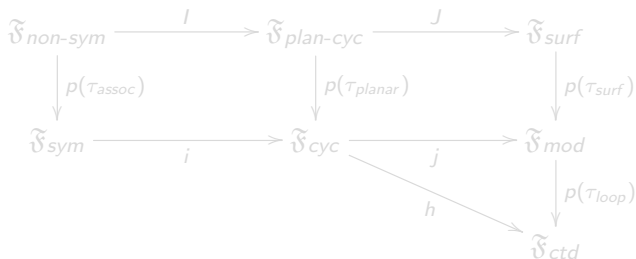
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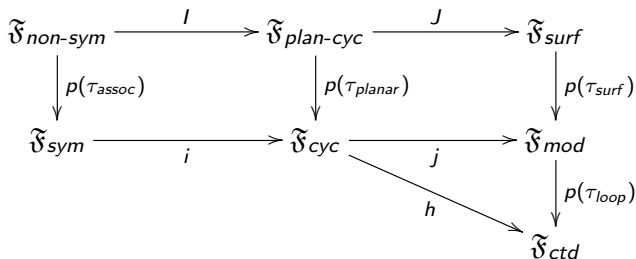
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- $\tau_{planar}$  is the  $\mathfrak{F}_{cyc}$ -algebra for planar structures
- $i^*(\tau_{planar}) = \tau_{assoc}$  ( $\tau_{planar}$  is the “cyclic” version of  $\tau_{assoc}$ )
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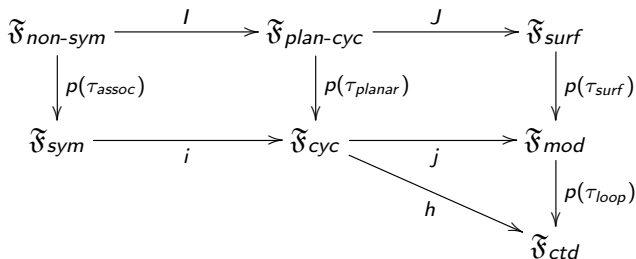
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The set  $j_!(\tau_{planar})(\gamma, s)$  is in bijection with either

- equ. cl. of one-vertex ribbon graphs with  $\gamma$  loops and  $s$  flags
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- equ. cl. of bordered oriented surfaces of genus  $g$  with  $n$  punctures and  $S$ -marked boundary such that  $1 - 2g = \nu(S) + |S| - \gamma$ .

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The morphisms of the Feynman category  $\mathfrak{F}_{surf}$  are  $\gamma/n$ -labeled polycyclic graphs.

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A Feynman category  $\mathfrak{F}$  is *cubical* if there is a degree function  $\deg : \text{Mor}(\mathfrak{F}) \rightarrow \mathbb{N}_0$  such that

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In the non-unital case without constants, the Feynman categories  $\mathfrak{F}_{\text{sym}}$ ,  $\mathfrak{F}_{\text{cyc}}$ ,  $\mathfrak{F}_{\text{mod}}$ ,  $\mathfrak{F}_{\text{non-sym}}$ ,  $\mathfrak{F}_{\text{plan-cyc}}$ ,  $\mathfrak{F}_{\text{surf}}$  are cubical. The degree of  $\phi$  is the number of edges of the representing graph  $\Gamma_\phi$ .

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Let  $P$  be an operad over a cubical Feynman category  $\mathfrak{F}$ . Put

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where identifications are on faces of  $[0, 1]^{\text{deg}(\phi)}$  according to coarser factorisations of  $\phi$ .

### Proposition (Kaufmann-Ward, cf. Boardman-Vogt, B-Moerdijk)

For any cubical Feynman category  $\mathfrak{F}$ , the category of topological  $\mathfrak{F}$ -algebras admits a *transferred model structure*. If  $P$  has an underlying cofibrant  $\mathcal{V}$ -collection then  $W_{\mathfrak{F}}P$  is a *cofibrant  $\mathfrak{F}$ -algebra*.

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## Example (cubically subdivided convex polytopes)

- $W_{\text{sym}}(\mathcal{T}_{\text{assoc}})(\text{rooted corolla}) = \text{associahedron}$
- $W_{\text{cyc}}(\mathcal{T}_{\text{planar}})(\text{corolla}) = \text{cyclohedron}$

## Proposition (Igusa-flow)

Let  $\phi : \mathfrak{F} \rightarrow \mathfrak{F}'$  be a functor of cubical Feynman categories.

- $(W_{\mathfrak{F}\mathbf{1}})(B) \simeq |\text{nerve}(\mathfrak{F} \downarrow B)|$
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$$J_!(W_{\text{plan-cyc}\mathbf{1}})(g, n, S) \simeq |\text{rb}_{g,n}^S| \simeq \mathcal{M}_{g,n}^S$$

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Let  $\phi : \mathfrak{F} \rightarrow \mathfrak{F}'$  be a functor of cubical Feynman categories.

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## Theorem (B-K)

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no for transferred *projective* model structure, but yes for transferred *equivariant* model structure, cf. Vogt.
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