Active/inert factorisation systems, operads and plus constructions

Clemens Berger

University of Côte d'Azur

Topology, geometry and higher structures –
a symposium in honor of Ieke Moerdijk

May 21, 2026

Citation (A theory of natural equivalences - Eilenberg-Mac Lane '1945)

"In a metamathematical sense, our theory provides concepts applicable to all branches of abstract mathematics, and so contributes to the current trend towards *uniform* treatment of different mathematical disciplines. In particular, it provides opportunities for the *comparison* of ... different branches of mathematics; in this way it may occasionally suggest *new* results by *analogy*."

Purpose of the talk

Starting from Segal's formalism of Γ-spaces, establish "structural analogies" between several simplicial presheaf categories for "operad-like structures".

Definition (Categories and cohomology theories - Segal '1974)

- objects of Γ are finite sets $\underline{n} = \{1, 2, \dots, n\}$, $n \geq 0$.
- $\phi \in \Gamma(\underline{m}, \underline{n})$ given by $(\phi(i) \subset \underline{n})_{i \in \underline{m}}$ sth. $\phi(i) \cap \phi(j) = \emptyset$ for $i \neq j$.

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Definition (active/inert morphisms - Higher Algebra, Lurie L12)

- $\phi : \underline{m} \to \underline{n}$ is *active* iff $\underline{n} = \phi(i) \sqcup \cdots \sqcup \phi(m)$ (partition)
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Proposition (Γ admits ...)

- an active/inert factorisation system
- a retractive/mono factorisation system (generalised Reedy, BM11)
- a retractive/active mono/inert mono *triple factorisation system*
- active mono=*inner face* and inert mono=*outer face*

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Definition (cf. algebraic pattern CH21, hypermoment category B22)

A *Segal pattern* is a category $\mathbb{C}$ endowed with

- an active/inert factorisation system $(\mathbb{C}_{act}, \mathbb{C}_{in})$;
- an augmentation $\gamma_{\mathbb{C}} : \mathbb{C} \to \Gamma$ preserving active/inert morphisms;
- essentially unique *inert lifts* $U_\alpha \to X$ for elements $\alpha : \underline{1} \to \gamma_{\mathbb{C}}(X)$;
- a *dense Segal core* $\mathbb{C}_{Seg} \subset \mathbb{C}_{in}$ spanned by *nilobjects* and *units* (i.e. objects in the fibre of $\underline{0}$ and in the fibre of $\underline{1}$).

Lemma (Reedy property satisfied by many Segal patterns)

If the units of $\mathbb{C}$ span a groupoid and $\mathbb{C} = (\mathbb{C}_{retr}, \mathbb{C}_{mono})$, then $\gamma_{\mathbb{C}} : \mathbb{C} \to \Gamma$ is conservative, and $\mathbb{C}$ is a generalised Reedy category.

Definition (homotopy theory for simplicial presheaves on a Segal pattern)

The inclusion $\mathbb{C}_{Seg} \subset \mathbb{C}_{in}$ defines for each representable presheaf a *spine inclusion*. We consider the *left Bousfield localisation* of the Reedy model structure on simplicial presheaves with respect to spine inclusions.

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Remark (Segal condition, cf. Se74, BF78)

- Density of $\mathbb{C}_{Seg} \subset \mathbb{C}_{in}$ means that each object is a *canonical* colimit of *nilobjects* and *units* or, equivalently, the inert part $\mathbb{C}_{in}$ fully faithfully embeds into presheaves on $\mathbb{C}_{Seg}$, so-called $\mathbb{C}$-*graphs*.
- A simplicial presheaf on $\mathbb{C}$ is said to satisfy the *Segal condition* if it takes these density colimits to homotopy limits in simplicial sets.
- *Segal-fibrancy* $\iff$ *Reedy-fibrancy* + *Segal condition*

Examples (some Segal patterns with their graphs and Segal presheaves)

$\mathbb{C}$	Δ	Θ_n	Ω	$\mathbb{U}_{cyc}$
$\mathbb{C}_{Seg}$	$[0] \rightrightarrows [1]$	n-globe	rooted corollas	corollas
$\mathbb{C}$-graph	graph	n-graph	coloured coll.	cyclic col. coll.
$\mathbb{C}$-model	$(\infty, 1)$-cat	(∞, n)-cat	∞-operad	cyclic ∞-op.
authors	EZ50&R01	J97&R10	MW07&CM13	HRY20

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Definition (dendrix category Ω – Moerdijk-Weiss MW07)

- the objects of Ω are dendrices, i.e. finite rooted trees.
- $\Omega(S,T) = \mathrm{SymOp}(\Omega(S_*),\Omega(T_*))$ where S_* is the Ω-graph of S and $\Omega(S_*)$ is the free coloured symmetric operad on S_*.
- inert monos $S \to T$: subdendrix inclusions.
- active monos $S \to T$: S-indexed partitions of T into subdendrices.
- retractive morphisms $S \to T$: removal of vertices with one input.
- $\gamma_\Omega : \Omega \to \Gamma : S \mapsto$ vertex set of S.
- a single nilobject : the free-living edge.
- for each $n \geq 0$ a unit : the rooted corolla with n leaves.
- density : each dendrix is a canonical colimit of corollas along edges.

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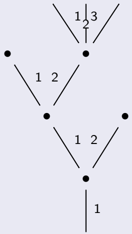

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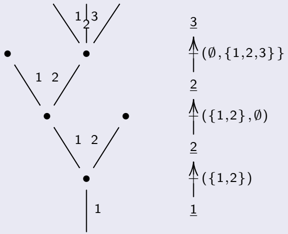

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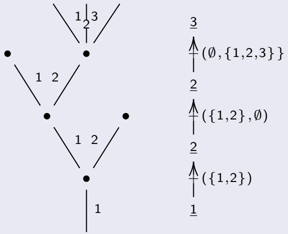

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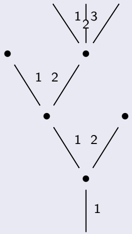

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Example (2-graph induced by the "sectors" of a 2-level tree)

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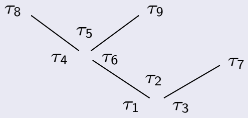

$$\begin{tikzcd}
\tau_1 \arrow[r, bend left=70, "\tau_4"{name=A}] \arrow[r, "\tau_5"{name=B, description}] \arrow[r, bend right=70, "\tau_6"'{name=C}] & \tau_2 \arrow[r, "\tau_7"] & \tau_3
\arrow[Rightarrow, from=A, to=B, "\tau_8"]
\arrow[Rightarrow, from=B, to=C, "\tau_9"]
\end{tikzcd}$$

Definition (unital Segal patterns, their operads and monoids)

- A *Segal pattern* $\mathbb{C}$ is called *unital* if every object receives an essentially unique active morphism from a unit.
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Proposition (B22)

For unital Segal patterns, the categories of $\mathbb{C}$-operads and of $\mathbb{C}^+$-monoids are equivalent. Segal presheaves on $\mathbb{C}^+$ are homotopical $\mathbb{C}$-operads.

Theorem (cf. B07, Bat08, Bar18; also BF78, BoM20 for $n = \infty$)

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Definition (BV-tensor product for ∞-operads BV73, L12, Bar18, BoM20)

There is a *leaf functor* $\lambda : \Gamma^+ \to \Gamma$ and a commutative diagram

$$\begin{tikzcd} \Gamma^+ \wr \Gamma^+ \arrow[r, "graft"] \arrow[d, "\lambda\wr\lambda"'] & \Gamma^+ \arrow[d, "\lambda"] \\ \Gamma \wr \Gamma \arrow[r, "\oplus"] & \Gamma \end{tikzcd}$$

inducing a Quillen bifunctor $\otimes_{BV} : s\mathcal{P}(\Gamma^+) \times s\mathcal{P}(\Gamma^+) \longrightarrow s\mathcal{P}(\Gamma^+)$

Theorem (Additivity theorem for E_n-operads, cf. L12, Bat18, Bar18)

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- the objects of $\mathbb{U}$ are finite connected "graphs".
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Lemma (cf. GK78, HRY20)

Discrete reduced Segal presheaves on $\mathbb{U}_{cyc}$ (resp. $\mathbb{U}$) are cyclic (resp. modular) operads. $\mathbb{U}_{cyc}$ is a Segal pattern, $\mathbb{U}$ is a *graded* Segal pattern.

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Remark (Exceptional features of the graded Segal pattern $\mathbb{U}$, cf. HRY20)

- free-living edge corepresents "oriented" edges.
- there are non-monic inerts increasing the "genus".
- among the units of $\mathbb{U}$ those belonging to $\mathbb{U}_{cyc}$ are also dense in $\mathbb{U}$.
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Proposition (operadic envelopes, cf. GK98, HRY20)

- Pushforward along $\Omega \longrightarrow \mathbb{U}_{cyc}$ computes the cyclic envelope of a symmetric operad.
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