

Catalan monoids & Dold-Kan correspondence

Clemens Berger

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Algebraic Combinatorics and Finite Groups IV

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Definition (simplex category Δ and Catalan monoid $\mathcal{C}_n$)

- objects of Δ are finite ordinals $[n] = \{0, 1, \dots, n\}$, $n \geq 0$;
- morphisms of Δ are order-preserving maps;
- the Catalan monoid $\mathcal{C}_n$ is the submonoid of $\Delta([n], [n])$ consisting of expansive $\phi : [n] \rightarrow [n]$, i.e. $\phi(k) \geq k$ for all k .

Theorem (Norton 1979, Solomon 1996, DHST 2011)

- $\mathcal{C}_n$ is an ordered monoid with 2^n idempotents;
- The unit of $\mathbb{Z}[\mathcal{C}_n]$ is a sum of 2^n pairwise orth. idempotents.

Theorem (Dold 1958, Kan 1958)

(simplicial abelian groups) $\simeq$ (chain complexes).

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Purpose of the talk

- establish a link between the two theorems
- explain the following scheme



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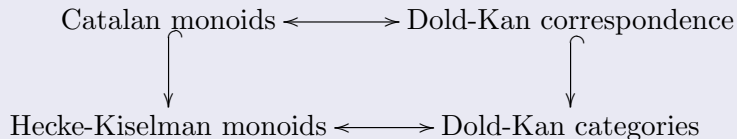


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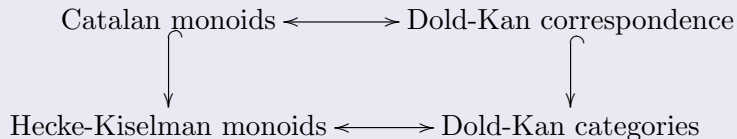


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Lemma (generating system of morphisms for Δ)

- *face operators* $\epsilon_i^n : [n-1] \rightarrow [n] \rightsquigarrow$ omit i
- *degeneracy operators* $\eta_i^n : [n+1] \rightarrow [n] \rightsquigarrow$ duplicate i
- every $\phi : [m] \rightarrow [n]$ factors as $[m] \xrightarrow{\eta} [p] \xrightarrow{\epsilon} [n]$
- if $m = n$ then $\phi = \epsilon_{i_k} \cdots \epsilon_{i_1} \eta_{j_1} \cdots \eta_{j_k} = (\epsilon_{i_k} \eta_{j_k}) \cdots (\epsilon_{i_1} \eta_{j_1})$
for $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$
- ϕ is expansive if and only if $i_1 \leq j_1, i_2 \leq j_2, \dots, i_k \leq j_k$.

Proposition (presentation of the Catalan monoids)

C_n admits n idempotent generators $x_i = \epsilon_{i-1}^n \eta_{i-1}^{n-1}$, $1 \leq i \leq n$, subject to the following commutation relations:

- $x_i x_{i+1} x_i = x_{i+1} x_i = x_{i+1} x_i x_{i+1}$ for $1 \leq i \leq n-1$;
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Lemma (Catalan numbers)

$$\#C_n = \frac{1}{n+2} \binom{2n+2}{n+1}$$

C_2	$[2] \xrightarrow{\phi} [2]$	$\omega(\phi)$	$Dyck(\phi)$	$Tamari(\phi)$
1	012	1	XYXYXY	$(a(b(cd)))$
x_1	112	1	XXYYXY	$((ab)(cd))$
x_2	022	1	XYXXYY	$(a((bc)d))$
$x_1 x_2$	122	2	XXYXYX	$((a(bc))d)$
$x_2 \cdot x_1$	222	1	XXXYYY	$((ab)c)d$

C_3	$[3] \xrightarrow{\phi} [3]$	$\omega(\phi)$	$Dyck(\phi)$	$Tamari(\phi)$
1	0123	1	XYXYXYXY	$(a(b(c(de))))$
x_1	1123	1	XXYYXYXY	$((ab)(c(de)))$
x_2	0223	1	XYXXYYXY	$(a((bc)(de)))$
x_3	0133	1	XYXYXXYY	$(a(b((cd)e)))$
$x_1 x_2$	1223	2	XXYXYXYX	$((a(bc))(de))$
$x_2 x_3$	0233	2	XYXXYXYX	$(a((b(cd))e))$
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$x_3 \cdot x_1$	1133	1	XXYYXXYY	$((ab)((cd)e))$
$x_3 \cdot x_2$	0333	1	XYXXXYYY	$(a(((bc)d)e))$
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$x_2 x_3 \cdot x_1 x_2$	2333	2	XXXYYYYY	$((a(bc))d)e$
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Lemma (Catalan numbers)

$$\#C_n = \frac{1}{n+2} \binom{2n+2}{n+1}$$

C_2	$[2] \xrightarrow{\phi} [2]$	$\omega(\phi)$	$Dyck(\phi)$	$Tamari(\phi)$
1	012	1	XYXYXY	$(a(b(cd)))$
x_1	112	1	XXYYXY	$((ab)(cd))$
x_2	022	1	XYXXYY	$(a((bc)d))$
$x_1 x_2$	122	2	XXYXYX	$((a(bc))d)$
$x_2 \cdot x_1$	222	1	XXXYYY	$((ab)c)d$

C_3	$[3] \xrightarrow{\phi} [3]$	$\omega(\phi)$	$Dyck(\phi)$	$Tamari(\phi)$
1	0123	1	XYXYXYXY	$(a(b(c(de))))$
x_1	1123	1	XXYYXYXY	$((ab)(c(de)))$
x_2	0223	1	XYXXYYXY	$(a((bc)(de)))$
x_3	0133	1	XYXYXXYY	$(a(b((cd)e)))$
$x_1 x_2$	1223	2	XXYXYXYX	$((a(bc))(de))$
$x_2 x_3$	0233	2	XYXXYYXY	$(a((b(cd))e))$
$x_2 \cdot x_1$	2223	1	XXXYYYXY	$((ab)c)(de)$
$x_3 \cdot x_1$	1133	1	XXYYXXYY	$((ab)((cd)e))$
$x_3 \cdot x_2$	0333	1	XYXXXYYY	$(a(((bc)d)e))$
$x_1 x_2 x_3$	1233	3	XXYXYXYX	$((a(b(cd)))e)$
$x_3 \cdot x_1 x_2$	1333	2	XXYXXYYY	$((a((bc)d))e)$
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Proposition ($\mathcal{J}$ -triviality)

$\mathcal{C}_n$ is a submonoid of the monoid U_{n+1} of (lower) unitriangular Boolean matrices. Both monoids are ordered, hence $\mathcal{J}$ -trivial.

Proof.

Send $x_k \in \mathcal{C}_n$ to the matrix $X^{(k)} \in U_{n+1}$ with 1's on the diagonal, $X_{k+1,k}^{(k)} = 1$ all other coefficients being zero.

This defines an injective monoid morphism.

U_{n+1} is ordered by the product order on the matrix coefficients. $\square$

Proposition

$\mathcal{C}_n$ has 2^n idempotents $x_{i_k} x_{i_{k-1}} \cdots x_{i_1}$, one for each $I \subset \{1, \dots, n\}$.

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Let $(x_i)_{1 \leq i \leq n}$ be idempotent elements of a ring R such that $x_i x_j x_i = x_j x_i = x_j x_i x_j$ for $i < j$. Then

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where $\{1, \dots, n\} = \{i_1 < \cdots < i_k\} \sqcup \{j_1 < \cdots < j_{n-k}\}$ and the summands are pairwise orthogonal idempotents of R .

Corollary (BCW)

If $R = \text{End}(X)$ for an abelian group X then we get

$$X = \bigoplus_{\{i_1, \dots, i_k\} \subset \{1, \dots, n\}} \text{im}(x_{i_k}) \cap \cdots \cap \text{im}(x_{i_1}) \cap \ker(x_{j_1}) \cap \cdots \cap \ker(x_{j_{n-k}})$$

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Lemma

In Δ there are exactly 2^n epimorphisms η with domain $[n]$. Each such has a unique maximal section ϵ . The composite endomorphism $\epsilon\eta : [n] \rightarrow [n]$ is an idempotent of $\mathcal{C}_n$.

Proof.

An epi $\eta : [n] \twoheadrightarrow [m]$ is uniquely determined by a partition of $[n]$ into segments. Such a partition corresponds to $I \subset \{1, \dots, n\}$. $\square$

Theorem (Dold-Kan correspondence revisited, BCW)

For a simplicial abelian group $X : \Delta^{\text{op}} \rightarrow \underline{\mathbf{Ab}}$ define $M_n^X = \bigcap_{i=0}^{n-1} \ker(X(\epsilon_i^n))$. Then there is a “simplicial” isomorphism $X([n]) \cong \bigoplus_{\eta : [n] \rightarrow [m]} X(\eta)^*(M_m^X)$ induced by $\mathcal{C}_n \subset \Delta([n], [n])$.

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Let Γ be a directed graph with vertices $1, \dots, n$ so that only edges $i \rightarrow j$ with $i < j$ are allowed. The *Hecke-Kiselman monoid* HK_Γ is generated by n idempotent elements x_i subject to the relations:

- $x_i x_j x_i = x_j x_i = x_j x_i x_j$ if $i \rightarrow j$ is an edge of Γ ;
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Remark (BCW)

- for the linear graph L_n on n vertices: $HK_{L_n} = \mathcal{C}_n$;
- for the complete graph K_n on n vertices: $HK_{K_n} = \mathcal{K}_n$ (*Kiselman monoid*);
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Any idempotent-generated $\mathcal{J}$ -trivial monoid is completely determined by a *quiver* whose

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- The Kiselman monoid $\mathcal{K}_n$ is finite and has 2^n idempotents;
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Proposition (BCW)

For each object A of a DK -category $\mathcal{D}$, the primitive $\mathcal{E}$ -projectors of A generate a *Hecke-Kiselman submonoid* of $\mathcal{D}(A, A)$.

Definition (inessential $\mathcal{M}$ -maps)

An $\mathcal{M}$ -map $m : A \rightarrow B$ is called *inessential* if there is a non-identity $\mathcal{E}$ -projector of B fixing m .

A DK -category $\mathcal{D} = (\mathcal{E}, \mathcal{M}, (-)^*)$ is *special* if inessential $\mathcal{M}$ -maps form an *ideal* in $\mathcal{M}$.

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$$\mathcal{M}/\mathcal{M}_{iness} = [0] \xrightarrow{\quad} [1] \xrightarrow{\quad} [2] \xrightarrow{\quad} [3] \xrightarrow{\quad} [4] \cdots$$

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In particular: $[(\mathcal{M}/\mathcal{M}_{iness})^{\text{op}}, \underline{\text{Ab}}]_* = \text{Ch}(\mathbb{Z})$.

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Theorem (generalised Dold-Kan correspondence, BCW)

For each special Dold-Kan category $\mathcal{D} = (\mathcal{E}, \mathcal{M}, (-)^*)$ there is an adjoint equivalence $[\mathcal{D}^{\text{op}}, \underline{\text{Ab}}] \simeq [(\mathcal{M}/\mathcal{M}_{\text{iness}})^{\text{op}}, \underline{\text{Ab}}]_*$

Examples

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