Catalan monoids & Dold-Kan correspondence

Clemens Berger

Cetraro, 6 - 10 juillet 2026

Algebraic Combinatorics and Finite Groups IV

Definition (simplex category Δ and Catalan monoid $\mathcal{C}_n$)

- objects of Δ are finite ordinals $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$;
- morphisms of Δ are order-preserving maps;
- the Catalan monoid $\mathcal{C}_n$ is the submonoid of $\Delta([n], [n])$ consisting of expansive $\phi : [n] \to [n]$, i.e. $\phi(k) \geq k$ for all k.

Theorem (Norton 1979, Solomon 1996, DHST 2011)

- $\mathcal{C}_n$ is an ordered monoid with 2^n idempotents;
- The unit of $\mathbb{Z}[\mathcal{C}_n]$ is a sum of 2^n pairwise orth. idempotents.

Theorem (Dold 1958, Kan 1958)

$$(\text{simplicial abelian groups}) \simeq (\text{chain complexes}).$$

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Purpose of the talk

- establish a link between the two theorems
- explain the following scheme

$$\begin{tikzcd}
\text{Catalan monoids} \arrow[r, leftrightarrow] \arrow[d, hook] & \text{Dold-Kan correspondence} \arrow[d, hook] \\
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\end{tikzcd}$$

Main sources

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Lemma (generating system of morphisms for Δ)

- *face operators* $\epsilon_i^n : [n-1] \to [n]$ $\rightsquigarrow$ omit i
- *degeneracy operators* $\eta_i^n : [n+1] \to [n]$ $\rightsquigarrow$ duplicate i
- every $\phi : [m] \to [n]$ factors as $[m] \xrightarrow{\eta} \mathrel{\mkern-14mu}\rightarrow [p] \rightarrowtail \mathrel{\mkern-14mu}\xrightarrow{\epsilon} [n]$
- if $m = n$ then $\phi = \epsilon_{i_k} \cdots \epsilon_{i_1} \eta_{j_1} \cdots \eta_{j_k} = (\epsilon_{i_k} \eta_{j_k}) \cdots (\epsilon_{i_1} \eta_{j_1})$ for $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$
- ϕ is expansive if and only if $i_1 \leq j_1,\ i_2 \leq j_2, \ldots, i_k \leq j_k$.

Proposition (presentation of the Catalan monoids)

C_n admits n idempotent generators $x_i = \epsilon_{i-1}^n \eta_{i-1}^{n-1}$, $1 \leq i \leq n$, subject to the following commutation relations:

- $x_i x_{i+1} x_i = x_{i+1} x_i = x_{i+1} x_i x_{i+1}$ for $1 \leq i \leq n-1$;
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Lemma (Catalan numbers)

$$\#\mathcal{C}_n = \frac{1}{n+2}\binom{2n+2}{n+1}$$

$\mathcal{C}_2$	$[2] \xrightarrow{\phi} [2]$	$\omega(\phi)$	$Dyck(\phi)$	$Tamari(\phi)$
1	012	1	XYXYXY	$(a(b(cd)))$
x_1	112	1	XXYYXY	((ab)(cd))
x_2	022	1	XYXXYY	(a((bc)d))
x_1x_2	122	2	XXYXYY	((a(bc))d)
$x_2 \cdot x_1$	222	1	XXXYYY	(((ab)c)d)

$\mathcal{C}_3$	$[3] \xrightarrow{\phi} [3]$	$\omega(\phi)$	$Dyck(\phi)$	$Tamari(\phi)$
1	0123	1	XYXYXYXY	$(a(b(c(de))))$
x_1	1123	1	XXYYXYXY	((ab)(c(de)))
x_2	0223	1	XYXXYYXY	(a((bc)(de)))
x_3	0133	1	XYXYXXYY	(a(b((cd)e)))
x_1x_2	1223	2	XXYXYYXY	((a(bc))(de))
x_2x_3	0233	2	XYXXYXYY	(a((b(cd))e))
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$x_3 \cdot x_1$	1133	1	XXYYXXYY	((ab)((cd)e))
$x_3 \cdot x_2$	0333	1	XYXXXYYY	(a(((bc)d)e))
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x_3	0133	1	XYXYXXYY	(a(b((cd)e)))
x_1x_2	1223	2	XXYXYYXY	((a(bc))(de))
x_2x_3	0233	2	XYXXYXYY	(a((b(cd))e))
$x_2 \cdot x_1$	2223	1	XXXYYYXY	(((ab)c)(de))
$x_3 \cdot x_1$	1133	1	XXYYXXYY	((ab)((cd)e))
$x_3 \cdot x_2$	0333	1	XYXXXYYY	(a(((bc)d)e))
$x_1x_2x_3$	1233	3	XXYXYXYY	((a(b(cd)))e)
$x_3 \cdot x_1x_2$	1333	2	XXYXXYYY	((a((bc)d))e)
$x_2x_3 \cdot x_1$	2233	2	XXXYYXYY	(((ab)(cd))e)
$x_2x_3 \cdot x_1x_2$	2333	2	XXXYXYYY	(((a(bc))d)e)
$x_3 \cdot x_2 \cdot x_1$	3333	1	XXXXYYYY	((((ab)c)d)e)

Lemma (Catalan numbers)

$$\#\mathcal{C}_n = \frac{1}{n+2}\binom{2n+2}{n+1}$$

$\mathcal{C}_2$	$[2] \xrightarrow{\phi} [2]$	$\omega(\phi)$	$Dyck(\phi)$	$Tamari(\phi)$
1	012	1	XYXYXY	$(a(b(cd)))$
x_1	112	1	XXYYXY	((ab)(cd))
x_2	022	1	XYXXYY	(a((bc)d))
x_1x_2	122	2	XXYXYY	((a(bc))d)
$x_2 \cdot x_1$	222	1	XXXYYY	(((ab)c)d)

$\mathcal{C}_3$	$[3] \xrightarrow{\phi} [3]$	$\omega(\phi)$	$Dyck(\phi)$	$Tamari(\phi)$
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Proposition ($\mathcal{J}$-triviality)

$\mathcal{C}_n$ is a submonoid of the monoid U_{n+1} of (lower) unitriangular Boolean matrices. Both monoids are ordered, hence $\mathcal{J}$-trivial.

Proof.

Send $x_k \in \mathcal{C}_n$ to the matrix $X^{(k)} \in U_{n+1}$ with $1'$s on the diagonal, $X^{(k)}_{k+1,k} = 1$ all other coefficients being zero.
This defines an injective monoid morphism.
U_{n+1} is ordered by the product order on the matrix coefficients. $\square$

Proposition

$\mathcal{C}_n$ has 2^n idempotents $x_{i_k} x_{i_{k-1}} \cdots x_{i_1}$, one for each $I \subset \{1, \ldots, n\}$.

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Proposition (Kudryavtseva-Mazorchuk)

Let $(x_i)_{1\leq i\leq n}$ be idempotent elements of a ring R such that $x_ix_jx_i = x_jx_i = x_jx_ix_j$ for $i<j$. Then

$$1 = \sum_{\{i_1,\dots,i_k\}\subset\{1,\dots,n\}} x_{i_k}\cdots x_{i_2}x_{i_1}(1-x_{j_1})(1-x_{j_2})\cdots(1-x_{j_{n-k}})$$

where $\{1,\dots,n\} = \{i_1<\cdots<i_k\}\sqcup\{j_1<\cdots<j_{n-k}\}$ and the summands are pairwise orthogonal idempotents of R.

Corollary (BCW)

If $R=\mathrm{End}(X)$ for an abelian group X then we get

$$X = \bigoplus_{\{i_1,\dots,i_k\}\subset\{1,\dots,n\}} \mathrm{im}(x_{i_k})\cap\cdots\cap\mathrm{im}(x_{i_1})\cap\ker(x_{j_1})\cap\cdots\cap\ker(x_{j_{n-k}})$$

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Lemma

In Δ there are exactly 2^n epimorphisms η with domain $[n]$. Each such has a unique maximal section ϵ. The composite endomorphism $\epsilon\eta : [n] \to [n]$ is an idempotent of $\mathcal{C}_n$.

Proof.

An epi $\eta : [n] \twoheadrightarrow [m]$ is uniquely determined by a partition of $[n]$ into segments. Such a partition corresponds to $I \subset \{1, \ldots, n\}$. $\square$

Theorem (Dold-Kan correspondence revisited, BCW)

For a simplicial abelian group $X : \Delta^{\mathrm{op}} \to \underline{\mathrm{Ab}}$ define $M_n^X = \bigcap_{i=0}^{n-1} \ker(X(\epsilon_i^n))$. Then there is a "simplicial" isomorphism $X([n]) \cong \bigoplus_{\eta:[n]\twoheadrightarrow[m]} X(\eta)^*(M_m^X)$ induced by $\mathcal{C}_n \subset \Delta([n],[n])$.

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Definition (Hecke-Kiselman monoid, cf. Ganyushkin-Mazorchuk)

Let Γ by a directed graph with vertices $1, \ldots, n$ so that only edges $i \to j$ with $i < j$ are allowed. The *Hecke-Kiselman monoid* HK_Γ is generated by n idempotent elements x_i subject to the relations:

- $x_i x_j x_i = x_j x_i = x_j x_i x_j$ if $i \to j$ is an edge of Γ;
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Remark (BCW)

- for the linear graph L_n on n vertices: $HK_{L_n} = C_n$;
- for the complete graph K_n on n vertices: $HK_{K_n} = K_n$ (*Kiselman monoid*);
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- if $\Gamma \subset \Gamma'$ then there is a quotient map $HK_{\Gamma'} \twoheadrightarrow HK_\Gamma$ inducing a bijection between the sets of idempotents.

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Proposition (DHST)

Any idempotent-generated $\mathcal{J}$-trivial monoid is completely determined by a *quiver* whose

- vertex set consists of its idempotents;
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Any idempotent-generated $\mathcal{J}$-trivial monoid is completely determined by a *quiver* whose

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A DK-category $\mathcal{D} = (\mathcal{E}, \mathcal{M}, (-)^*)$ is a category equipped with an epi/mono factorisation system $(\mathcal{E}, \mathcal{M})$ and a faithful identity-on-objects functor $(-)^* : \mathcal{E}^{\mathrm{op}} \to \mathcal{M}$ sth.

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An epi is called primitive if $e = e_2e_1$ implies e_1 or e_2 is invertible.

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Proposition (BCW)

For each object A of a *DK*-category $\mathcal{D}$, the primitive $\mathcal{E}$-projectors of A generate a *Hecke-Kiselman submonoid* of $\mathcal{D}(A, A)$.

Definition (inessential $\mathcal{M}$-maps)

An $\mathcal{M}$-map $m : A \to B$ is called *inessential* if there is a non-identity $\mathcal{E}$-projector of B fixing m.
A *DK*-category $\mathcal{D} = (\mathcal{E}, \mathcal{M}, (-)^*)$ is *special* if inessential $\mathcal{M}$-maps form an *ideal* in $\mathcal{M}$.

Remark (Δ is a special *DK*-category)

$$\mathcal{M}/\mathcal{M}_{iness} = \begin{tikzcd} {[0]} \arrow[r] \arrow[rr, bend left, "0"] & {[1]} \arrow[r] \arrow[rr, bend left, "0"] & {[2]} \arrow[r] \arrow[rr, bend left, "0"] & {[3]} \arrow[r] & {[4]} \cdots \end{tikzcd}$$

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Theorem (generalised Dold-Kan correspondence, BCW)

For each special Dold-Kan category $\mathcal{D} = (\mathcal{E}, \mathcal{M}, (-)^*)$ there is an adjoint equivalence $[\mathcal{D}^{\mathrm{op}}, \underline{\mathrm{Ab}}] \simeq [(\mathcal{M}/\mathcal{M}_{iness})^{\mathrm{op}}, \underline{\mathrm{Ab}}]_*$

Examples

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