On the Groups J(X)

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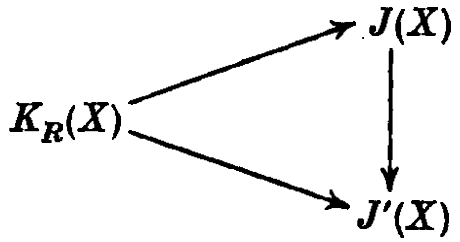
1. Introduction

In another lecture [2], I have explained that certain problems of algebraic topology can be reduced to the calculation of the groups $J(X)$ defined by Atiyah [4]. In the same lecture I have tried to show that the best method which we possess at present to calculate the groups $J(X)$ is to try and sandwich them between two further groups $J'(X)$, $J''(X)$. All these groups will be defined below; but we may say at once that we shall define them as quotients of $K_R(X)$, where $K_\Lambda(X)$ denotes the "extraordinary cohomology theory" of Grothendieck–Atiyah–Hirzebruch [1, 6, 7]. That is, we shall give definitions of the form

$$J(X) = K_R(X)/U(X)$$
$$J'(X) = K_R(X)/V(X)$$
$$J''(X) = K_R(X)/W(X)$$

where $U(X)$, $V(X)$ and $W(X)$ are suitably defined subgroups of $K_R(X)$.

The group $J'(X)$ is a lower bound for $J(X)$, in the sense that $U(X) \subset V(X)$, so that the quotient map $K_R(X) \to J'(X)$ factors through the quotient map $K_R(X) \to J(X)$; this yields the following commutative diagram of epimorphisms.



We hope to prove that $J''(X)$ is an upper bound for $J(X)$, in a similar sense; and this we have proved in some cases, notably those in which X is a complex projective space CP^n, or a sphere S^m with $m \not\equiv 0 \bmod 8$.

The main object of this paper is to prove the following result.

THEOREM 1.1. For each finite CW-complex X we have $J'(X) = J''(X)$.

The precise sense in which the groups $J'(X)$ and $J''(X)$ are "equal" is the following. We shall prove that the subgroups $V(X)$ and $W(X)$ of $K_R(X)$, although differently defined, are in fact the same. Therefore the corresponding quotient groups $J'(X)$ and $J''(X)$ are (as a matter of logic) identical; that is, they are one and the same object.

The proof to be presented is extracted from a more complete account of these matters now in preparation [3], to which the reader is referred for a fuller exposition of the fundamentals of the theory, for the results of certain calculations (Lemmas 4.6 and 4.8 of this paper), and for a certain piece of number-theory (Lemma 4.7 of this paper). With these exceptions, it is hoped that the proof is as complete as can be expected in the present limited space.

The arrangement of this paper is as follows. Theorem 1.1 is proved in §4. The proof is completely dependent on the existence of a certain commutative diagram (Diagram 3.1), which is established in §3. In §2 we dispose of various necessary preliminaries.

2. Preliminaries

We have said that we shall give definitions of the following form:

$$J(X) = K_R(X)/U(X)$$
$$J'(X) = K_R(X)/V(X)$$
$$J''(X) = K_R(X)/W(X).$$

We define $U(X)$ to be the subgroup of $K_R(X)$ generated by elements of the form $\{\xi\} - \{\eta\}$, where ξ and η are real vector-bundles whose associated sphere-bundles are fibre homotopy equivalent.

It takes only a little longer to define the subgroup $W(X)$. The following definition is good if X is a finite CW-complex. Let $e(k)$ be a function which assigns to each integer k a non-negative integer $e(k)$. Let $W(e; X)$ be the subgroup of $K_R(X)$ generated by elements of the form

$$k^{e(k)}((\Psi^k x) - x);$$

here k runs over all integers, x runs over all elements of $K_R(X)$, and Ψ^k is the K-cohomology operation introduced in [1].

It is clear that if $e(k) \geq f(k)$ for all k, then $W(e; X) \subset W(f; X)$. We define

$$W(X) = \bigcap_e W(e; X);$$

here the intersection is taken as e runs over all functions $e(k)$.

At this point it is convenient to introduce some further notions. Let X be a finite-dimensional CW-complex; then by considering $SO(n)$-bundles over X (for $n = 1, 2, \ldots$) we can construct a Grothendieck group based on orientable bundles, which we may write $K_{SO}(X)$. The group $K_{SO}(X)$ is embedded in $K_R(X)$ as the subgroup of elements x whose first Stiefel–Whitney class $w_1(x)$ is zero. We may call such elements x "orientable elements."

The notation for each of our Grothendieck groups will contain the letter K; we agree that by substituting $\tilde{K}$ for K we indicate the corresponding subgroup of elements of virtual dimension zero. For example, we have

$$\tilde{K}_{SO}(X) = \tilde{K}_R(X) \cap K_{SO}(X).$$

In order to define the subgroup $V(X)$, we require the "cannibalistic characteristic classes" of Bott [8, 9] and Atiyah [private communication]. We shall call them ρ^k, although Bott uses the letter θ. For the moment we need only recall the following points.

(1) ρ^k assigns to suitable bundles over X a characteristic class lying in $K_R(X)$.

(2) ρ^k can be defined on suitable virtual bundles at the price of introducing denominators. More precisely, one defines Q_k to be the additive group of fractions of the form a/k^b, where a and b are integers. Then if $x \in K_R(X)$ is an orientable element whose virtual dimension is even, one defines $\rho^k(x)$ as an element of $K_R(X) \otimes Q_k$.

(3) ρ^k is "homomorphic from addition to multiplication," or more shortly "exponential," in the sense that

$$\rho^k(x + y) = \rho^k(x) \otimes \rho^k(y).$$

(4) If x has virtual dimension $2n$, then the virtual dimension of $\rho^k(x)$ is k^n.

(5) ρ^k bears the same relation to the K-cohomology operation Ψ^k that the total Stiefel–Whitney class $\sum_0^\infty w_i$ does to the total Steenrod square $\sum_0^\infty Sq^i$ in ordinary cohomology. In particular, we can carry over the proof that the Stiefel–Whitney classes are fibre-homotopy invariants; we find that ρ^k is invariant up to multiplication by a factor of the form

$$\frac{\Psi^k(1 + y)}{1 + y},$$

where $y \in \tilde{K}_R(X)$. (Note that for such y, $1 + y$ is invertible.)

These points are treated more fully in [3, 8, 9], and below. For the moment, we hope that they are sufficient to provide a heuristic justification for the following definition. We will place an element x of $K_R(X)$ in $V(X)$ if:

(1) x is an orientable element whose virtual dimension is even, and

(2) there exists $y \in \tilde{K}_R(X)$ (independent of k) such that

$$\rho^k(x) = \frac{\Psi^k(1+y)}{1+y}$$

in $K_R(X) \otimes Q_k$, for all k.

We note that $V(X) \subset \tilde{K}_{SO}(X)$. In fact, suppose we have

$$\rho^k(x) = \frac{\Psi^k(1+y)}{1+y};$$

since the right-hand side has virtual dimension 1, x must have virtual dimension 0.

We will check that $V(X)$ is a subgroup. In fact, let $1 + \tilde{K}_R(X)$ be the multiplicative group of elements of virtual dimension 1 in $K_R(X)$; and let $1 + \tilde{K}_R(X) \otimes Q_k$ be the multiplicative group of elements of virtual dimension 1 in $K_R(X) \otimes Q_k$.

Let us define

$$\delta^k \colon 1 + \tilde{K}_R(X) \to 1 + \tilde{K}_R(X) \otimes Q_k$$

by

$$\delta^k(1+y) = \frac{\Psi^k(1+y)}{1+y};$$

then δ^k is a multiplicative homomorphism. We can now consider the following diagram of groups and homomorphisms; it will appear again later as part of a larger diagram.

$$\begin{array}{ccc} & & \tilde{K}_{SO}(X) \\ & & \Big\downarrow \prod_k \rho \\ 1 + \tilde{K}_R(X) & \xrightarrow{\prod_k \delta^k} & \prod_k (1 + \tilde{K}_R(X) \otimes Q_k) \end{array}$$

We have

$$V(X) = (\prod_k \rho^k)^{-1}(\prod_k \delta^k)(1 + \tilde{K}_R(X)).$$

This displays $V(X)$ as a subgroup of $\tilde{K}_{SO}(X)$.

In the present paper, we shall obtain our hold on the operations ρ^k by using representation-theory. If G is a compact Lie group we shall write $K'_\Lambda(G)$ for the Grothendieck ring of virtual representations of G over Λ. Our proof will involve establishing identities between composite operations such as $\rho^l\Psi^k$, $\Psi^l\rho^k$. We aim to establish these identities as

universal formulae before we apply them to $K_\Lambda(X)$ for any particular X; that is, we aim to establish these identities as equations between virtual representations. Unfortunately, the correct formal setting for these equations is not the Grothendieck ring $K'_\Lambda(G)$. Just as we are forced to introduce denominators in defining ρ^k on virtual bundles, so here we are forced to introduce denominators in order to define ρ^k on virtual representations; this leads us to consider rings $K'_\Lambda(G) \otimes Q_k$. But actually one must go further and introduce the completion of a representation ring [cf. 5, 7]. Let S be a subring (containing 1) of the complex numbers; and let I be the ideal $\tilde{K}'_\Lambda(G) \otimes S$ in $K'_\Lambda(G) \otimes S$ (that is, the ideal of elements of virtual dimension zero). We define

$$\text{Comp}\,(K'_\Lambda(G) \otimes S) = \text{Inv}\lim_{n\to\infty} \frac{K'_\Lambda(G) \otimes S}{I^n}.$$

Suppose that θ is a virtual representation of G which can be written as a linear combination of representations into groups $SO(2n)$ for $n = 1, 2, \ldots$; then it is possible to define $\rho^k(\theta)$ as an element of $\text{Comp}\,(K'_R(G) \otimes Q_k)$.

We note also that if ξ is a G-bundle over a finite-dimensional CW-complex X, and if $\hat{\rho}$ is an element of $\text{Comp}\,(K'_\Lambda(G) \otimes S)$, then it is possible to define $\hat{\rho}(\xi)$ as an element of $K'_\Lambda(X) \otimes S$; for I^n will annihilate ξ if $n > \dim X$ [cf. 7].

One calculates with Grothendieck rings $K'_\Lambda(G)$ by using characters. We next explain the use of characters in calculating with $\text{Comp}\,(K'_\Lambda(G) \otimes S)$ (where S is a subring of C). We shall follow Atiyah and Hirzebruch [7], except that we require more detailed information about a less general situation. In fact, we shall be almost exclusively concerned with the groups $G = SO(2n)$ and $G = SO(2n') \times SO(2n'')$.

We shall suppose that a maximal torus T is chosen in our compact connected Lie group G; the points of T are to be given by complex coordinates $(z_1, z_2, \ldots, z_n)$ with $|z_r| = 1$ for each r. For example, in $U(n)$ we consider the usual maximal torus consisting of the diagonal matrices with diagonal entries $z_1, z_2, \ldots, z_n$; by the usual embedding $U(n) \to SO(2n)$ we obtain a maximal torus in $SO(2n)$. The maximal torus in

$$SO(2n') \times SO(2n'')$$

is to be the product of those in $SO(2n')$ and $SO(2n'')$; we may give its points by complex coordinates $(z_1, \ldots, z_{n'}, z_{n'+1}, \ldots, z_{n'+n''})$.

The character $\chi(\theta)$ of an element θ in $K'_\Lambda(G) \otimes S$ is given by a polynomial in the letters z_r, z_r^{-1} $(r = 1, 2, \ldots, n)$; the coefficients of the polynomial lie in S. By writing $z_r = 1 + \zeta_r$, we can expand each such polynomial as a power-series in $\zeta_1, \zeta_2, \ldots, \zeta_n$ which is convergent in a

neighbourhood of the origin. Let J be the ideal of power-series with zero constant term; we see that if $\theta \in I = \tilde{K}'_\Lambda(G) \otimes S$, then $\chi(\theta) \in J$. It follows that $\chi(I^m) \subset J^m$ for each m. Thus the character defines a map

$$\chi\colon \mathrm{Comp}\,(K'_\Lambda(G) \otimes S) \to C[[\zeta_1, \zeta_2, \ldots, \zeta_n]],$$

where $C[[\zeta_1, \zeta_2, \ldots, \zeta_n]]$ is the ring of formal power-series in the variables $\zeta_1, \zeta_2, \ldots, \zeta_n$ (with complex coefficients). We shall use the letter J to denote also the ideal of formal power-series with zero constant term.

In the following lemma, we consider χ as defined on $K'_\Lambda(G) \otimes S$, so that $\chi^{-1}(J^r)$ means the set of elements $\theta \in K'_\Lambda(G) \otimes S$ whose characters are small of the nth order at the origin.

LEMMA 2.1 Suppose that $G = SO(2n)$ or $G = SO(2n') \times SO(2n'')$, and $\Lambda = R$ or C. Then given q there exists $r = r(n, q)$ or $r = r(n', n'', q)$ such that $\chi^{-1}(J^r) \subset I^q$.

PROOF. We must first describe the ring $K'_\Lambda(G) \otimes S$ in convenient terms. We will simply quote the results, which are well known in representation-theory. The description involves a choice; we choose each generator so that it lies in $\chi^{-1}(J^r)$ for as large an r as possible.

Consider first the case $G = SO(2n)$, $\Lambda = C$. Then there is an element τ_i of $K'_C(SO(2n))$ whose character is the ith elementary symmetric function of the expressions $(z_r - 2 + z_r^{-1})$; this element is in fact a polynomial in the exterior powers, and therefore real, i.e., in $K'_R(SO(2n))$. There is also an element ν of $K'_C(SO(2n))$ whose character is

$$\tfrac{1}{2}\Big[\prod_{1 \le r \le n} (z_r - 2 + z_r^{-1}) + \prod_{1 \le r \le n} (z_r - z_r^{-1})\Big].$$

(Note that this is a polynomial in the variables z_r, z_r^{-1} with integral coefficients.) The element ν is real if n is even. The ring $K'_C(SO(2n)) \otimes S$ is a free module on two generators $1, \nu$ over the polynomial ring $S[\tau_1, \tau_2, \ldots, \tau_n]$.

We now consider the case $\Lambda = R$. If n is even then

$$K'_R(SO(2n)) \otimes S = K'_C(SO(2n)) \otimes S.$$

If n is odd then

$$K'_R(SO(2n)) \otimes S = S[\tau_1, \tau_2, \ldots, \tau_n].$$

A similar description is valid for the case $G = SO(2n') \times SO(2n'')$. The (virtual) representations τ_i, ν of the two factors give representations of G which we may write τ'_i, ν', τ''_i, ν'' (with an obvious notation). By using the usual embedding of $SO(2n') \times SO(2n'')$ in $SO(2n' + 2n'')$, the representation ν of $SO(2n' + 2n'')$ yields a representation of G, which we will also write ν. These representations satisfy a relation:

$$\nu = 2\nu'\nu'' - \tau'_{n'}\nu'' - \nu'\tau''_{n''} + \tau'_{n'}\tau''_{n''}.$$

The ring $K'_C(G) \otimes S$ is a free module on four generators $1, \nu', \nu'', \nu'\nu''$ over the polynomial ring $S[\tau'_1, \ldots, \tau'_{n'}, \tau''_1, \ldots, \tau''_{n''}]$.

We now consider the case $\Lambda = R$. If n' and n'' are even then

$$K'_R(G) \otimes S = K'_C(G) \otimes S.$$

If n' is even and n'' is odd then $K'_R(G) \otimes S$ is a free module on two generators $1, \nu'$ over the polynomial ring $S[\tau'_1, \ldots, \tau'_{n'}, \tau''_1, \ldots, \tau''_{n''}]$; similarly if n' is odd and n'' is even. If both n' and n'' are odd then $K'_R(G) \otimes S$ is a free module over the same polynomial ring on the two generators $1, \nu$.

We have thus shown that in each of the cases to be considered the free S-module $K'_\Lambda(G) \otimes S$ admits an S-base consisting of certain monomials; for example, if $G = SO(2n)$ and $\Lambda = C$ we have the monomials

$$\tau_1^{a_1}\tau_2^{a_2} \cdots \tau_n^{a_n}\nu^\varepsilon,$$

where the a_i are non-negative integers and $\varepsilon = 0$ or 1.

We will next assign a weight to each of these monomials. We assign to τ_i, τ'_i and τ''_i the weight $2i$; we assign to ν, ν' and ν'' the weights n, n' and n''. Thus, for example, the monomial

$$\tau_1^{a_1}\tau_2^{a_2} \cdots \tau_n^{a_n}\nu^\varepsilon$$

is assigned the weight

$$2a_1 + 4a_2 + \cdots + 2na_n + \varepsilon_n.$$

We now reach the point of these proceedings. The character $\chi(\tau_i)$ is the ith elementary symmetric function of the variables $\zeta_1^2, \zeta_2^2, \ldots, \zeta_n^2$, plus a term in J^{2i+1}; similarly for the characters $\chi(\tau'_i)$ and $\chi(\tau''_i)$, using the variables $\zeta_1^2, \ldots, \zeta_{n'}^2$ or $\zeta_{n'+1}^2, \ldots, \zeta_{n'+n''}^2$. Similarly,

$$\begin{aligned}
\chi(\nu) &= 2^{n-1}\zeta_1\zeta_2 \cdots \zeta_n \bmod J^{n+1},\\
\chi(\nu') &= 2^{n'-1}\zeta_1\zeta_2 \cdots \zeta_{n'} \bmod J^{n'+1},\\
\chi(\nu'') &= 2^{n''-1}\zeta_{n'+1}\zeta_{n'+2} \cdots \zeta_{n'+n''} \bmod J^{n''+1}.
\end{aligned}$$

It follows that in each case, the monomials of weight exactly w map under χ into linearly independent elements of J^w/J^{w+1}. Therefore an element θ of $\chi^{-1}(J^r)$ can be expressed as a linear combination of monomials of weight $\geq r$. If $G = SO(2n)$ the maximum weight of a generator is $2n$; it follows that such an element θ lies in I^q, provided $r \geq 2nq$. Similarly for $G = SO(2n') \times SO(2n'')$; in this case the maximum weight of a generator is Max $(2n', 2n'')$.

This completes the proof of Lemma 2.1.

COROLLARY 2.2. Suppose that $G = SO(2n)$ or $G = SO(2n') \times SO(2n'')$ and $\Lambda = R$ or C; then the map

$$\chi\colon \mathrm{Comp}\,(K'_\Lambda(G) \otimes S) \to C[[\zeta_1, \zeta_2, \ldots, \zeta_n]]$$

is monomorphic.

This follows immediately from Lemma 2.1. It is this corollary which enables us to handle elements of Comp $(K'_\Lambda(G) \otimes S)$ by means of their characters.

In order to state the next corollary, suppose given two subrings $S \subset T \subset C$. Then we have ideals

$$I_S = \tilde{K}'_\Lambda(G) \otimes S \subset K'_\Lambda(G) \otimes S$$
$$I_T = \tilde{K}'_\Lambda(G) \otimes T \subset K'_\Lambda(G) \otimes T.$$

COROLLARY 2.3. Suppose that $G = SO(2n)$ or $G = SO(2n') \times SO(2n'')$ and $\Lambda = R$ or C. Then given q, there exists $r = r(n, q)$ or $r = r(n', n'', q)$ such that

$$(K'_\Lambda(G) \otimes S) \cap (I_T)^r \subset (I_S)^q.$$

This corollary follows immediately from Lemma 2.1. It is essential to the arguments of §3.

We will now introduce the virtual representations ρ^k. In the first instance these are representations of spinor groups. We take the maximal torus $\tilde{T}$ in Spin $(2n)$ to be the counterimage of our maximal torus T in $SO(2n)$. The symbol $(z_r)^{\frac{1}{2}}$ will indicate a function defined on a certain covering torus T_r of T, and taking the value $+1$ at the identity. Certain combinations of these symbols, such as $\prod_{1 \le r \le n} (z_r)^{\frac{1}{2}}$, indicate functions defined on $\tilde{T}$.

In the first instance, then, we define $(\rho^k)_n$ to be the virtual representation of Spin $(2n)$ whose character is

$$\prod_{1 \le r \le n} \frac{(z_r)^{\frac{1}{2}k} - (z_r)^{-\frac{1}{2}k}}{(z_r)^{\frac{1}{2}} - (z_r)^{-\frac{1}{2}}} = \prod_{1 \le r \le n} (z_r)^{\frac{1}{2}(k-1)} + (z_r)^{\frac{1}{2}(k-3)} + \cdots + (z_r)^{-\frac{1}{2}(k-1)}).$$

Two cases now arise. Consider first the case in which k is odd. In this case the given character is expressed in terms of integral powers of z_r; therefore it defines a virtual representation of $SO(2n)$. If n is even this representation is real, because $K'_R(SO(2n)) = K'_C(SO(2n))$. If $n = 2m - 1$ then we see that the restriction of $(\rho^k)_{2m}$ to $SO(4m - 2)$ is $k(\rho^k)_{2m-1}$. Now representation-theory shows that if k is odd and $k\sigma$ is real, then σ is real. It follows that $(\rho^k)_{2m-1}$ is real. When k is odd, then, we have

$$(\rho^k)_n \in K'_R(SO(2n)).$$

We consider next the case in which k is even. In this case we begin by remarking that the representation

$$2: SO(2n) \to SO(4n)$$

can be lifted into Spin $(4n)$. We can therefore form the composite

$$\rho^k.2 \in K'_C(SO(2n)).$$

This is a virtual representation whose character is

$$\prod_{1 \le r \le n} \left(\frac{(z_r)^{\frac{1}{2}k} - (z_r)^{-\frac{1}{2}k}}{(z_r)^{\frac{1}{2}} - (z_r)^{-\frac{1}{2}}} \right)^2.$$

Moreover, the virtual representation $\rho^k.2$ coincides with its complex conjugate; therefore

$$2(\rho^k.2) \in K'_R(SO(2n))$$

and

$$\rho^k.2 \in K'_R(SO(2n)) \otimes Q_k$$

(since k is divisible by 2). Since we expect ρ^k to be exponential we now seek to define ρ^k on $SO(2n)$ by extracting the square root of $(\rho^k.2)$. Since k is even, we can extract the square root of $\rho^k.2$ as an element of

$$\text{Comp } (K'_R(SO(2n)) \otimes Q_k);$$

and the result is unique, providing we specify that the virtual dimension of the square root is k^n rather than $-k^n$. When k is even, then, we can define $(\rho^k)_n$ as an element of Comp $(K'_R(SO(2n)) \otimes Q_k)$.

The character of $(\rho^k)_n$ is

$$\prod_{1 \le r \le n} \frac{(z_r)^{\frac{1}{2}k} - (z_r)^{-\frac{1}{2}k}}{(z_r)^{\frac{1}{2}} - (z_r)^{-\frac{1}{2}}};$$

of course this formula has to be interpreted as an element of

$$C[[\zeta_1, \zeta_2, \ldots, \zeta_n]],$$

by expanding the fractional powers of $z_r = 1 + \zeta_r$ as binomial series in ζ_r.

We will now discuss the sense in which the sequence of virtual representations $(\rho^k)_n$ is "homomorphic from addition to multiplication," or more shortly, "exponential." Let

$$\pi: SO(2n) \times SO(2m) \to SO(2n)$$

$$\tilde{\omega}: SO(2n) \times SO(2m) \to SO(2m)$$

be the projections of $SO(2n) \times SO(2m)$ onto its two factors. Let (σ_n) be a sequence of elements $\sigma_n \in \mathrm{Comp}\,(K'_\Lambda(SO(2n)) \otimes S)$. We will say that (σ_n) is "exponential" if we have

$$\sigma_{n+m}.(\pi \oplus \tilde{\omega}) = (\sigma_n.\pi) \otimes (\sigma_m.\tilde{\omega})$$

in $\mathrm{Comp}\,(K'_\Lambda(SO(2n) \times SO(2m)) \otimes S)$ for all n, m. (Cf. [1], pp. 607, 609.)

PROPOSITION 2.4. The sequence of elements

$$(\rho^k)_n \in \mathrm{Comp}\,(K'_R(SO(2n)) \otimes Q_k)$$

is exponential.

This follows immediately from the characters given above; one has only to apply Corollary 2.2.

PROPOSITION 2.5. If the sequence $\sigma_n \in \mathrm{Comp}\,(K'_\Lambda(SO(2n)) \otimes S)$ is exponential, then for any two representations $\alpha: G \to SO(2n)$, $\beta: G \to SO(2m)$ we have

$$\sigma_{n+m}.(\alpha \otimes \beta) = (\sigma_n.\alpha) \otimes (\sigma_m.\beta)$$

in $\mathrm{Comp}\,(K'_\Lambda(G) \otimes S)$. Moreover, for any two bundles ξ, η over a finite-dimensional CW-complex X, with groups $SO(2n)$, $SO(2m)$, we have

$$\sigma_{n+m}.(\xi \oplus \eta) = (\sigma_n.\xi) \otimes (\sigma_m.\eta)$$

in $K_\Lambda(X) \otimes S$.

This result is strictly analogous to those of [1], pp. 607–609, and so is its proof.

Given these propositions, one can define ρ^k on virtual representations so as to secure the following two properties.

(1) If the virtual representation is actually a representation α, then $\rho^k(\alpha)$ is the composite in the obvious sense.

(2) ρ^k is exponential, in the sense that

$$\rho^k(\theta + \phi) = (\rho^k\theta) \otimes (\rho^k\phi).$$

One can also define ρ^k on virtual bundles so as to enjoy similar properties. The details will be omitted from this paper.

3. An identity between virtual representations

The object of this section is to prove Theorem 3.2, which is vital to the proof of Theorem 1.1. The most important part of this theorem will state the commutativity of the following diagram.

(3.1)
$$\begin{array}{ccc} \tilde{K}_{SO}(X) & \xrightarrow{k^e(\Psi^k - 1)} & \tilde{K}_{SO}(X) \\ \downarrow{\scriptstyle \theta^k} & & \downarrow{\scriptstyle \rho^l} \\ 1 + \tilde{K}_{SO}(X) & \xrightarrow{\delta^l} & 1 + \tilde{K}_{SO}(X) \otimes Q^l \end{array}$$

Here the groups are as in §2; those in the top row are additive, and those in the bottom row are multiplicative. The additive homomorphism $k^e(\Psi^k - 1)$ is defined by

$$k^e(\Psi^k - 1)x = k^e((\Psi^k x) - x).$$

(Thus the 1 in $\Psi^k - 1$ means the identity function.)

The existence of the homomorphism θ^k will be asserted as part of the theorem. The homomorphism ρ^l is as in §2. The multiplicative homomorphism δ^l is defined by

$$\delta^l(1 + y) = \frac{\Psi^l(1 + y)}{1 + y},$$

as in §2.

THEOREM 3.2 *Given integers q, k and a sufficiently large integer e (viz. $e \geq e_0(q, k)$) there exists a function*

$$\theta^k = \theta^k(q, e) : \tilde{K}_{SO}(X) \to 1 + \tilde{K}_{SO}(X)$$

defined for CW-complexes of dimension $< q$ and having the following properties.

(1) *θ^k is homomorphic from addition to multiplication, that is, exponential.*
(2) *θ^k is natural for maps of X.*
(3) *The image of $\theta^k(x)$ in $1 + \tilde{K}_{SO}(X) \otimes Q_k$ is $\rho^k(k^e x)$.*
(4) *Diagram 3.1 is commutative.*

We will now give a heuristic plausibility argument for Theorem 3.2. For this purpose we will abandon the real K-theory and work instead with the complex K-theory, for simplicity. We now argue that in some suitable formal setting we may hope to have

$$(\rho^l \Psi^k) \otimes \rho^k = \rho^{kl} = (\Psi^l \rho^k) \otimes \rho^l. \tag{3.3}$$

In fact, all three expressions are "exponential," that is, homomorphic from addition to multiplication; therefore it is presumably sufficient to check the result for a complex line bundle ξ. Here we have

$$\Psi^k \xi = \xi^k, \qquad \rho^k \xi = \frac{\xi^k - 1}{\xi - 1}.$$

Therefore

$$(\rho^l \Psi^k \xi) \otimes (\rho^k \xi) = \frac{\xi^{kl} - 1}{\xi^k - 1} \cdot \frac{\xi^k - 1}{\xi - 1},$$

$$\rho^{kl} \xi = \frac{\xi^{kl} - 1}{\xi - 1},$$

$$(\Psi^l \rho^k \xi) \otimes (\rho^l \xi) = \frac{\xi^{kl} - 1}{\xi^l - 1} \cdot \frac{\xi^l - 1}{\xi - 1}.$$

(The last line uses the fact that Ψ^l preserves both addition and multiplication.) The three results are equal. We may therefore agree to suspend disbelief in Equation 3.3. Rewriting this equation, we obtain

$$\rho^l(\Psi^k - 1) = \frac{\Psi^l\rho^k}{\rho^k}.$$

That is, if $x \in \tilde{K}_C(X)$, then $1 + y = \rho^k x$ is a formal solution of the equation

$$\rho^l(\Psi^k - 1)x = \frac{\Psi^l(1 + y)}{1 + y}.$$

Now this formal solution involves denominators, that is, coefficients in Q_k. However, we can remove these denominators, up to dimension q, by considering

$$1 + z = (1 + y)^{k^e},$$

where e is suitably large. Raising the equation

$$\rho^l(\Psi^k - 1)x = \frac{\Psi^l(1 + y)}{1 + y}$$

to the power k^e, we find

$$\rho^l k^e(\Psi^k - 1)x = \frac{\Psi^l(1 + z)}{1 + z}.$$

This completes the plausibility argument.

We turn now to a rigorous version of this argument for the real case. In order to state the first lemma, let $\alpha_n, \beta_n \in \mathrm{Comp}\,(K'_\Lambda(SO(2n) \otimes S)$ be two exponential sequences.

LEMMA 3.4. *If $\chi(\alpha_1) = \chi(\beta_1)$, then $\alpha_n = \beta_n$ for all n.*

PROOF. Suppose that $\chi(\alpha_1) = \chi(\beta_1) = \phi(\zeta_1)$, where $\phi(\zeta_1)$ is a formal power series in ζ_1. Then by the exponential law,

$$\chi(\alpha_n) = \phi(\zeta_1)\phi(\zeta_2) \cdots \phi(\zeta_n) = \chi(\beta_n).$$

The result now follows from Corollary 2.2.

We will now apply this lemma. First we observe that (with an obvious notation)

$$K'_{SO}(SO(n)) = K'_R(SO(n)).$$

In fact, any representation $\gamma: SO(n) \to O(m)$ must map into $SO(m)$, since $SO(n)$ is connected. In particular, $(\Psi^k - 1) \in \tilde{K}'_{SO}(SO(2n))$. Thus

$\rho^l.(\Psi^k - 1)$ is defined, and lies in Comp $(K'_R(SO(2n)) \otimes Q_l)$. Again, since Ψ^l: $K'_R(G) \to K'_R(G)$ is a ring homomorphism, it induces

$$\Psi^l\colon K'_R(G) \otimes S \to K'_R(G) \otimes S$$

and

$$\Psi^l\colon \text{Comp } (K'_R(G) \otimes S) \to \text{Comp } (K'_R(G) \otimes S).$$

Thus $\Psi^l.\rho^k$ is defined and lies in Comp $(K'_R(SO(2n)) \otimes Q_k)$. Since ρ^k is invertible in this ring, $\delta^l.\rho^k = (\Psi^l.\rho^k)/\rho^k$ is defined.

LEMMA 3.5. *Consider the sequences of elements*

$$\alpha_n = \rho^l.(\Psi^k - 1)$$
$$\beta_n = \delta^l.\rho^k$$

in

$$\text{Comp } (K'_R(SO(2n)) \otimes Q_{kl}).$$

These sequences are exponential, and satisfy $\chi(\alpha_1) = \chi(\beta_1)$.

PROOF. It is easy to check that the sequences are exponential. In fact, α_n is exponential because $(\Psi^k - 1)$ is additive and ρ^l is exponential; β_n is exponential because ρ^k is exponential and δ^l is multiplicative.

It remains to check that $\chi(\alpha_1) = \chi(\beta_1)$. The virtual representation Ψ^k of $SO(2)$ is the representation $z \to z^k$. By definition $\rho^l.(\Psi^k - 1)$ means the element $(\rho^l.\Psi^k)/\rho^l$ of Comp $(K'_R(SO(2) \otimes Q_l)$. Its character is given by

$$\chi(\alpha_1) = \frac{(z_1)^{\frac{1}{2}kl} - (z_1)^{-\frac{1}{2}kl}}{(z_1)^{\frac{1}{2}k} - (z_1)^{-\frac{1}{2}k}} \cdot \frac{(z_1)^{\frac{1}{2}} - (z_1)^{-\frac{1}{2}}}{(z_1)^{\frac{1}{2}l} - (z_1)^{-\frac{1}{2}l}}.$$

(Of course, this expression is interpreted as an element of $C[[\zeta_1]]$.)

For any virtual representation θ of G, the character of $\Psi^l.\theta$ is given by $\chi(\Psi^l.\theta)g = \chi(\theta)g^l$ [1, p. 611]. Evidently this equation remains true when θ is replaced by an element of $K'_\Lambda(G) \otimes S$ or Comp $(K'_\Lambda(G) \otimes S)$. Therefore the character of $\delta^l.\rho^k = (\Psi^l.\rho^k)/\rho^k$ is given by

$$\chi(\beta_1) = \frac{(z_1)^{\frac{1}{2}kl} - (z_1)^{-\frac{1}{2}kl}}{(z_1)^{\frac{1}{2}l} - (z_1)^{-\frac{1}{2}l}} \cdot \frac{(z_1)^{\frac{1}{2}} - (z_1)^{-\frac{1}{2}}}{(z_1)^{\frac{1}{2}k} - (z_1)^{-\frac{1}{2}k}}.$$

(Of course, this expression also is interpreted as an element of $C[[\zeta_1]]$.) We have $\chi(\alpha_1) = \chi(\beta_1)$. This completes the proof of Lemma 3.5.

PROPOSITION 3.6. In Comp $(K'_R(SO(2n)) \otimes Q_{kl})$ we have

$$\rho^l.(\Psi^k - 1) = \delta^l.\rho^k$$

and

$$\rho^l.(\Psi^k - 1) = \frac{\Psi^l(k^{-n}\rho^k)}{k^{-n}\rho^k}.$$

PROOF. The first assertion follows immediately from Lemmas 3.4, 3.5. The second follows by rewriting the first.

The element $k^{-n}\rho^k$ lies in Comp $(K'_R(SO(2n)) \otimes Q_k)$, and has virtual dimension 1.

LEMMA 3.7. *Given integers n, k, r and sufficiently large e*

$$(viz.,\ e \geq e_0(n, k, r))$$

we have

$$(k^{-n}\rho^k)^{k^e} = 1 + x \qquad \text{in} \qquad (K'_R(SO(2n)) \otimes Q_k)/I^r,$$

where $x \in \tilde{K}'_R(SO(2n))$.

PROOF. It is clear that if the conclusion holds for one value of e, then it holds for all larger values of e. We now proceed by induction over r. Suppose that we have found e such that

$$(k^{-n}\rho^k)^{k^e} = 1 + x \qquad \text{in} \qquad (K'_R(SO(2n)) \otimes Q_k)/I^r,$$

where $x \in \tilde{K}'_R(SO(2n))$. (The induction starts with $r = 1$.) Then in

$$(K'_R(SO(2n)) \otimes Q_k)/I^{2r}$$

we can write

$$(k^{-n}\rho^k)^{k^e} = 1 + x + k^{-f}y,$$

where

$$y \in K'_R(SO(2n)) \cap I^r.$$

Now we have

$$\begin{aligned}(k^{-n}\rho^k)^{k^{e+f}} &= (1 + x + k^{-f}y)^{k^f} \\ &= (1 + x)^{k^f} + y(1 + x)^{k^f - 1} \bmod I^{2r} \\ &= 1 + z \bmod I^{2r}\end{aligned}$$

where $z \in \tilde{K}'_R(SO(2n))$. This completes the induction.

PROOF OF THEOREM 3.2. In what follows, X will always be a CW-complex of dimension $<q$. We can thus determine $n = n(q)$ so that there is a (1–1) correspondence between homotopy classes of maps $f: X \to BSO$ and homotopy classes of maps $f: X \to BSO(2n)$. That is, there is a (1–1) correspondence between the isomorphism classes of $SO(2n)$-bundles ξ over X and the elements of $\tilde{K}_{SO}(X)$; the correspondence is given by $\xi \to \{\xi\} - 2n$.

We will now invoke Lemma 3.7, and for this purpose we define an integer r depending only on q and n. (Thus r depends ultimately only on q.) We set

$$r = \text{Max}\,(q, r(n, q), r(n, n, q)),$$

where the functions $r(n, q)$ and $r(n', n'', q)$ are as in Corollary 2.3. We now employ Lemma 3.7 to choose elements

$$(\theta^k)_n \in K'_R(SO(2n))$$
$$(\theta^k)_{2n} \in K'_R(SO(4n))$$

such that the images of $(\theta^k)_n$, $(\theta^k)_{2n}$ in

$$(K'_R(SO(2n)) \otimes Q_k)/I^r,$$
$$(K'_R(SO(4n)) \otimes Q_k)/I^r$$

are $(k^{-n}\rho^k)^{k^e}$, $(k^{-2n}\rho^k)^{k^e}$. We can do this for all sufficiently large e (viz. for $e \geq e_0(q, k)$).

We shall use the element $(\theta^k)_n$ to define the function

$$\theta^k : \tilde{K}_{SO}(X) \to 1 + \tilde{K}_{SO}(X).$$

More precisely, we define

$$\theta^k(\{\xi\} - 2n) = (\theta^k)_n.\{\xi\},$$

where ξ runs over the $SO(2n)$-bundles. It is clear that this does define a function, which is natural for maps of X. Since $r \geq q$, it also follows that the image of $\theta^k(x)$ in $1 + \tilde{K}_{SO}(X) \otimes Q_k$ is $\rho^k(k^e x)$.

We shall use the element $(\theta^k)_{2n}$ to prove that the function θ^k is exponential. More precisely, let π, $\tilde{\omega}$ be the projections of $SO(2n) \times SO(2n)$ onto its first and second factors. Then in $(K'_R(SO(2n) \times SO(2n)) \otimes Q_k)/I^r$ we have the following equations.

$$(\theta^k)_{2n}.(\pi \oplus \tilde{\omega}) = (k^{-2n}\rho^k)^{k^e}.(\pi \oplus \tilde{\omega}),$$
$$(k^{-2n}\rho^k)^{k^e}.(\pi \oplus \tilde{\omega}) = (k^{-n}\rho^k)^{k^e}.\pi \otimes (k^{-n}\rho^k)^{k^e}.\tilde{\omega},$$
$$(k^{-n}\rho^k)^{k^e}.\pi \otimes (k^{-n}\rho^k)^{k^e}.\tilde{\omega} = (\theta^k)_n.\pi \otimes (\theta^k)_n.\tilde{\omega}.$$

We may now apply Corollary 2.3 with $G = SO(2n) \times SO(2n)$, $S = Z$, $T = Q_k$. The element

$$(\theta^k)_{2n}.(\pi \oplus \tilde{\omega}) - (\theta^k)_n.\pi \otimes (\theta^k)_n.\tilde{\omega}$$

lies in

$$[K'_R(SO(2n) \times SO(2n)) \otimes S] \cap (I_T)^r.$$

Since $r \geq r(n, n, q)$, Corollary 2.3 shows that

$$(\theta^k)_{2n}.(\pi \oplus \tilde{\omega}) - (\theta^k)_n.\pi \otimes (\theta^k)_n.\tilde{\omega}$$

lies in $(I_S)^q$. This shows that whenever ξ, η are $SO(2n)$-bundles over X, we have the equation

$$(\theta^k)_{2n}.(\xi \oplus \eta) = (\theta^k)_n.\xi \otimes (\theta^k)_n.\eta$$

in $K_R(X)$. It follows that the function

$$\theta^k\colon \tilde{K}_{SO}(X) \to 1 + \tilde{K}_{SO}(X)$$

is exponential.

We argue similarly to show that Diagram 3.1 is commutative. Proposition 3.6 states that

$$\rho^l.(\Psi^k - 1) = \frac{\Psi^{l}(k^{-n}\rho^k)}{k^{-n}\rho^k}$$

in Comp $(K'_R(SO(2n)) \otimes Q_{kl})$. Raising this equation to the power k^e, we have

$$\rho^l.k^e(\Psi^k - 1) = \frac{\Psi^{l}(k^{-n}\rho^k)^{k^e}}{(k^{-n}\rho^k)^{k^e}}.$$

Thus

$$\rho^l.k^e(\Psi^k - 1) = \frac{\Psi^{l}.\theta^k}{\theta^k}$$

in

$$(K'_R(SO(2n)) \otimes Q_{kl})/I^r.$$

We will now apply Corollary 2.3 with $G = SO(2n)$, $S = Q_l$, $T = Q_{kl}$. Let us take a representative

$$y \in K'_R(SO(2n)) \otimes Q_l$$

for the element $\rho^l.k^e(\Psi^k - 1)$ in

$$(K'_R(SO(2n)) \otimes Q_l)/(I_S)^r.$$

Then we have

$$y \in K'_R(SO(2n)) \otimes S,$$

$$\frac{\Psi^{l}.\theta^k}{\theta^k} \in K'_R(SO(2n)),$$

and

$$y - \frac{\Psi^{l}.\theta^k}{\theta^k} \in (K'_R(SO(2n)) \otimes S) \cap (I_T)^r.$$

Since $r \geq r(n, q)$, Corollary 2.3 shows that

$$y - \frac{\Psi^{l}.\theta^k}{\theta^k} \in (I_S)^q.$$

That is, we have

$$\rho^l.k^e(\Psi^k - 1) = \frac{\Psi^{l}.\theta^k}{\theta^k}$$

in $(K'_R(SO(2n)) \otimes Q_l)/(I_S)^q$. It follows that Diagram 3.1 is commutative.

This completes the proof of Theorem 3.2.

4. Proof of the main theorem

In proving Theorem 1.1, we shall have to work with square diagrams like Diagram 3.1. By a "square" S, we shall mean a commutative diagram of groups and homomorphisms which has the following form.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{i} & D \end{array}$$

We shall call a square "special" if it has the following property: given $b \in B$ and $c \in C$ such that $hb = ic$, there exists $a \in A$ such that $fa = b$ and $ga = c$. This is equivalent to demanding the exactnesss of the following sequence:

$$A \xrightarrow{(f,g)} B \oplus C \xrightarrow{(h,-i)} D$$

By a "short exact sequence $0 \to S' \to S \to S'' \to 0$" of squares, we shall mean a commutative diagram composed of three squares S', S, S'' and four short exact sequences

$$0 \to A' \to A \to A'' \to 0$$
$$0 \to B' \to B \to B'' \to 0, \text{ etc.}$$

LEMMA 4.1. *Suppose that* $0 \to S' \to S \to S'' \to 0$ *is a short exact sequence of squares in which* S' *and* S'' *are special; then* S *is special.*

PROOF. Each square determines a sequence

$$A \xrightarrow{(f,g)} B \oplus C \xrightarrow{(h,-i)} D,$$

which we may regard as a chain complex. We now have a short exact sequence of chain complexes. This yields an exact homology sequence, which leads immediately to the required result.

Alternatively, one may give a direct proof by routine diagram-chasing.

Let X be a finite CW-complex, say of dimension $<q$. The main part of the proof of Theorem 1.1 will proceed by filtering X. Let us define F_r to be the image of $\tilde{K}_R(X/X^{r-1})$ in $K_R(X)$; then for $r \geq 2$ and sufficiently large e, Theorem 3.2 provides us with the following commutative square.

$$\begin{array}{ccc} F_r & \xrightarrow{k^e(\Psi^k-1)} & F_r \\ {\scriptstyle \theta^k}\downarrow & & \downarrow{\scriptstyle \rho^l} \\ 1 + F_r & \xrightarrow{\delta^l} & 1 + F_r \otimes Q_l \end{array}$$

(Since Q_l is torsion-free, $\otimes Q_l$ is an exact functor, and the image of $\tilde{K}_R(X/X^{r-1}) \otimes Q_l$ in $K_R(X) \otimes Q_l$ is $F_r \otimes Q_l$.) If we pass to a (restricted) direct sum over k and an (unrestricted) direct product over l, we obtain the following commutative square S_r (for $r \geq 2$).

(4.2)
$$\begin{array}{ccc} \sum_k F_r & \xrightarrow{\sum_k k^{e(k)}(\Psi^k - 1)} & F_r \\ \downarrow{\scriptstyle \sum_k \theta^k} & & \downarrow{\scriptstyle \prod_l \rho^l} \\ 1 + F_r & \xrightarrow{\prod_l \delta^l} & \prod_l 1 + F_r \otimes Q_l \end{array}$$

More precisely, we obtain this commutative square whenever the function $e(k)$ is sufficiently large, viz. for $e(k) \geq e_0(q, k)$, where q is our fixed upper bound for the dimension of the complexes X considered.

THEOREM 4.3. *The square S_r displayed in Diagram* 4.2 *is special.*

PROOF. We shall prove this result by downwards induction over r. For $r = q$ the result is trivial. Let us therefore assume as an inductive hypothesis that the square S_{r+1} is special, and prove that S_r is special. For this purpose, by Lemma 4.1, it is sufficient to construct a short exact sequence of squares

$$0 \to S_{r+1} \to S_r \to S_r/S_{r+1} \to 0$$

and prove that the square S_{r+1}/S_r is special. Since the square S_{r+1} is embedded in S_r, it is clear that the required quotient square exists. In order to establish its structure, let us recall that $F_r.F_s \subset F_{r+s}$; thus F_{r+1} is an ideal in F_r, and F_r/F_{r+1} is a ring in which the product is zero (assuming $r \geq 1$). Besides the short exact sequence of additive groups

$$0 \to F_{r+1} \to F_r \to F_r/F_{r+1} \to 0,$$

we have also a short exact sequence of multiplicative groups:

$$1 \longrightarrow 1 + F_{r+1} \longrightarrow 1 + F_r \longrightarrow 1 + (F_r/F_{r+1}) \longrightarrow 1.$$

Here the product in the last group is given by

$$(1 + a)(1 + b) = 1 + (a + b).$$

Similar results hold for the following short exact sequence of multiplicative groups.

$$1 \to 1 + F_{r+1} \otimes Q_l \to 1 + F_r \otimes Q_l \to 1 + (F_r/F_{r+1}) \otimes Q_l \to 1.$$

This shows that the square S_r/S_{r+1} has the following form.

$$\begin{array}{ccc} \sum_k F_r/F_{r+1} & \xrightarrow{\sum_k k^{e(k)}(\Psi^k-1)} & F_r/F_{r+1} \\ \downarrow{\scriptstyle \sum_k \theta^k} & & \downarrow{\scriptstyle \prod_l \rho^l} \\ 1+(F_r/F_{r+1}) & \xrightarrow{\prod_l \delta^l} & \prod_l 1+(F_r/F_{r+1})\otimes Q_l \end{array} \tag{4.4}$$

The maps of the square S_r/S_{r+1} are induced from those of the square S_r.

Theorem 4.3 will thus be proved once we have established the following lemma.

LEMMA 4.5. *The square displayed in Diagram* 4.4 *is special.*

PROOF. Consider the following diagram of exact sequences.

$$\begin{array}{ccccc} \tilde{K}_R(X/X^r) & \longrightarrow & \tilde{K}_R(X/X^{r-1}) & \longrightarrow & \tilde{K}_R(X^r/X^{r-1}) \\ \downarrow & & \downarrow & & \\ F_{r+1} & \longrightarrow & F_r & \longrightarrow & F_r/F_{r+1} \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

This shows that the quotient group F_r/F_{r+1} of Diagram 4.4 is isomorphic to a quotient group of a subgroup of

$$K_R(X^r/X^{r-1}) = K_R(\bigvee S^r).$$

Since this isomorphism is given by induced maps, the operations Ψ^k, ρ^k and θ^k on F_r/F_{r+1} are given by the same formulae that hold in S^r;

$$\Psi^k y = a(k, r)\, y$$

$$\rho^k y = 1 + b(k, r)\, y$$

$$\theta^k y = 1 + c(k, r, e)\, y$$

Here the coefficients a, b and c do not depend on X or y. To give the coefficients, we have to divide the cases. Unless $r \equiv 0, 1, 2$ or $4 \bmod 8$ the group F_r/F_{r+1} is zero and there is nothing to prove. Consider first the case $r \equiv 0$ or $4 \bmod 8$; say $r = 4t$, $t > 0$.

LEMMA 4.6. *We have*

$$\Psi^k y = k^{2t} y$$

$$\rho^k y = 1 + \frac{n(t)}{m(t)} (k^{2t} - 1)y$$

$$\theta^k y = 1 + \frac{n(t)}{m(t)} k^{e(k)}(k^{2t} - 1)y.$$

Here $n(t)/m(t)$ is a fraction in its lowest terms, of which the denominator $m(t)$ has the following property.

LEMMA 4.7. *For any function $e(k)$ the highest common factor of the numbers*

$$k^{e(k)}(k^{2t} - 1) \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

divides $m(t)$. For sufficiently large functions $e(k)$ this highest common factor is exactly $m(t)$.

These two lemmas are quoted from [3].

We will now check that the square (4.4) is special for $r = 4t$. Suppose given $u, v \in F_r/F_{r+1}$ such that

$$\rho^l u = \frac{\Psi^l(1 + v)}{1 + v}$$

for all l. This yields

$$\frac{n(t)}{m(t)} (l^{2t} - 1)u = (l^{2t} - 1)v$$

in $(F_r/F_{r+1}) \otimes Q_l$ for all l. This means that for some exponent $f(l)$ we have

$$\frac{n(t)}{m(t)} l^{f(l)}(l^{2t} - 1)u = l^{f(l)}(l^{2t} - 1)v$$

in F_r/F_{r+1} (for each l). According to Lemma 4.7, by taking a suitable linear combination of these equations, we obtain

$$n(t)u = m(t)v.$$

Since the number $n(t)$, $m(t)$ are coprime we can choose $a(t), b(t)$ so that

$$a(t)m(t) + b(t)n(t) = 1;$$

now define

$$w = a(t)u + b(t)v \in F_r/F_{r+1};$$

this ensures that

$$m(t)\, w = u, \qquad n(t)\, w = v.$$

Using Lemma 4.7 again, let us choose integers $c(k)$, zero except for a finite number of k, such that

$$\sum_k c(k)k^{e(k)}(k^{2t}-1) = m(t).$$

Let σ be the element of $\Sigma_k F_r/F_{r+1}$ with components $c(k)w$. Then

$$\begin{aligned}(\sum_k k^{e(k)}(\Psi^k - 1))\sigma &= \sum_k c(k)k^{e(k)}(k^{2t}-1)w \\ &= m(t)w \\ &= u.\end{aligned}$$

Again,

$$\begin{aligned}(\sum_k \theta^k)\sigma &= 1 + \sum_k \frac{n(t)}{m(t)} c(k)k^{e(k)}(k^{2t}-1)w \\ &= 1 + n(t)w \\ &= 1 + v.\end{aligned}$$

Therefore the square (4.4) is special if $r \equiv 0$ or $4 \bmod 8$.

We turn now to the case $r \equiv 1$ or $2 \bmod 8$, $r \geq 2$. In this case every element of F_r/F_{r+1} has order 2, so that

$$(F_r/F_{r+1}) \otimes Q_l \cong \begin{cases} F_r/F_{r+1} & (l \text{ odd}) \\ 0 & (l \text{ even}). \end{cases}$$

LEMMA 4.8. *We have*

$$\Psi^k y = \begin{cases} 0 & (k \text{ even}) \\ y & (k \text{ odd}) \end{cases}$$

$$\left.\begin{aligned}\rho^k y &= 1 + y \\ \theta^k y &= 1 + y\end{aligned}\right\} (k \equiv \pm 3 \bmod 8).$$

This lemma is quoted from [3].

It follows from this lemma that the map $\Sigma_k k^{e(k)}(\Psi^k - 1)$ of Diagram 4.4 is zero (at least for $e(k) \geq 1$). The map $\Pi_l \delta^l$ is also trivial (with image 1). The map $\Sigma_k \theta^k$ is epimorphic. The map $\Pi_l \rho^l$ is monomorphic. It follow immediately that the square (4.4) is special.

This completes the proof of Lemma 4.5, and therefore of Theorem 4.3.

COROLLARY 4.9. The following square is special

(4.10)
$$\begin{array}{ccc} \sum_k \tilde{K}_{SO}(X) & \xrightarrow{\sum_k k^{e(k)}(\Psi^k - 1)} & \tilde{K}_{SO}(X) \\ \downarrow{\scriptstyle \sum_k \theta^k} & & \downarrow{\scriptstyle \prod_l \rho^l} \\ 1 + \tilde{K}_{SO}(X) & \xrightarrow{\prod_l \delta^l} & \prod_l 1 + \tilde{K}_{SO}(X) \otimes Q_l \end{array}$$

This follows immediately from Theorem 4.3, by setting $r = 2$.

PROOF OF THEOREM 1.1. In proving Theorem 1.1, we may assume without loss of generality that X is connected. With the notation of §2, we require to prove that $V(X) = W(X)$.

We have defined $W(X) = \cap_e W(e, X)$, where $W(e, X)$ is the subgroup of $K_R(X)$ generated by elements of the form $k^{e(k)}(\Psi^k - 1)x$ as k runs over all integers and x runs over $K_R(X)$. I claim that for $e(k)$ sufficiently large (viz. for $e(k) \geq e_0(q)$) we can obtain the same subgroup $W(e, X)$ by letting x run over $\tilde{K}_{SO}(X)$. In fact, any element $x \in K_R(X)$ can be written as $x = y + z$, where $y \in \tilde{K}_{SO}(X)$ and z is a linear combination of real line bundles. A real line bundle ζ can be induced by a map $f\colon X \to RP^q$; from this we see that

$$2^{e_0(q)}(\Psi^k - 1)\zeta = 0 \quad (k \text{ even})$$
$$(\Psi^k - 1)\zeta = 0 \quad (k \text{ odd}).$$

Thus

$$k^{e(k)}(\Psi^k - 1)z = 0$$

if $e(k) \geq e_0(q)$, and hence

$$k^{e(k)}(\Psi^k - 1)x = k^{e(k)}(\Psi^k - 1)y.$$

We have thus shown that for each sufficiently large function $e(k)$ the group $W(e, k)$ is the image of the map $\Sigma_k k^{e(k)}(\Psi^k - 1)$ appearing in Diagram 4.10.

Again, we have defined

$$V(X) = (\prod_l \rho^l)^{-1}(\prod_l \delta^l)(1 + \tilde{K}_R(X)).$$

I claim that we can obtain the same subgroup by taking

$$(\prod_l \rho^l)^{-1}(\prod_l \delta^l)(1 + \tilde{K}_{SO}(X)).$$

In fact, any element $(1 + x) \in (1 + K_R(X))$ can be written in the form $(1 + y)\zeta$, where $(1 + y) \in (1 + \tilde{K}_{SO}(X))$ and ζ is a real line bundle over X. By dividing the cases "l even" and "l odd," as above, we see that

$$\frac{\Psi^l\zeta}{\zeta} = 1 \qquad \text{in} \qquad 1 + \tilde{K}_R(X) \otimes Q_l$$

for all l. Hence

$$(\prod_l \delta^l)(1 + x) = (\prod_l \delta^l)(1 + y)$$

and

$$(\prod_l \delta^l)(1 + \tilde{K}_R(X)) = (\prod_l \delta^l)(1 + \tilde{K}_{SO}(X)).$$

We have thus shown that the group $V(X)$ is the group

$$(\prod_i \rho^i)^{-1}(\prod \delta^i)(1 + \tilde{K}_{SO}(X)).$$

of Diagram 4.10.

The fact that Diagram 4.10 is commutative now shows that $W(e, X) \subset V(X)$ for sufficiently large functions $e(k)$, so that $W(X) \subset V(X)$. The fact that Diagram 4.10 is special shows that $V(X) \subset W(e, X)$ for sufficiently large functions $e(k)$, so that $V(X) \subset W(X)$. This completes the proof of Theorem 1.1.

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