The $\mathbf{A}^1$–homotopy type of Atiyah–Hitchin schemes I: the geometry of complex points

Christophe Cazanave

Given a smooth algebraic variety Y, we construct a family of new algebraic varieties $\mathcal{R}_n Y$ indexed by a positive integer n, which we baptize the *Atiyah–Hitchin schemes* of Y. This paper is the first of a series devoted to the study of the $\mathbf{A}^1$–homotopy type (in the sense of Morel and Voevodsky) of these schemes. The interest of the Atiyah–Hitchin schemes is that we conjecture that, as n tends to infinity, the sequence of spaces $\mathcal{R}_n Y$ converges, in a precise sense, to $\Omega^{\mathbf{P}^1}\Sigma^{\mathbf{P}^1}Y$, the free $\mathbf{P}^1$–loop space generated by Y.
This first paper focuses on the geometry of the schemes $\mathcal{R}_n Y$: the slogan is that $\mathcal{R}_n Y$ is a scheme-theoretic "completion" of the unordered configuration space of n distinct points in $\mathbf{A}^1$ with labels in Y. This makes $\mathcal{R}_n Y$ analogous—although in general different—to the May–Milgram model.

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1 Introduction

This is the first of a series of papers devoted to the introduction and the study of some interesting families of algebraic varieties which we baptize the *Atiyah–Hitchin schemes*. There is a family of Atiyah–Hitchin schemes attached to any given algebraic variety Y; they are indexed by a positive integer n and we denote them $\mathcal{R}_n Y$. The fundamental example at the source of the definition, corresponding to the case where Y is the affine line minus the origin, is given by the schemes of pointed degree n rational functions $(\mathcal{F}_n)_{n\geqslant 0}$ (see example 2.4–(3) for the precise definition of these schemes). The homotopy type of the topological space $\mathcal{F}_n(\mathbf{C})$ of complex pointed degree n rational functions has been studied by Graeme Segal in his seminal article [31]. Segal proved that "the sequence of topological spaces $\mathcal{F}_n(\mathbf{C})$ converges, as n tends to infinity, to the double loop space $\Omega^2 S^3$". (The meaning of this last statement is that the natural inclusion map of $\mathcal{F}_n(\mathbf{C})$ into the degree n component of the space of continuous pointed maps $\mathrm{map}_*(\mathbf{P}^1(\mathbf{C}), \mathbf{P}^1(\mathbf{C}))$—a space homotopy equivalent to $\Omega^2 S^3$—induces isomorphisms on the n first homotopy groups.) This result has been a large source of inspiration for many authors, among which C Boyer, F Cohen, R Cohen, M. Guest, J Hurtubise, S. Kallel, Y. Kamiyama, F Kirwan, A. Kozlowzki, B. Mann, J. Milgram, M Murayama, J. Mostovoy, D. Shimamoto, K. Yamaguchi [7, 8, 9, 16, 17, 18, 19, 20, 21, 22, 28, 34]. In particular, one can find in the literature several constructions inspired by Segal's paper [31] of successive finite-dimensional manifolds approximating certain mapping spaces.

The objective of this series of articles is twofold. First, we would like a unifying framework for the various topological results mentioned above. As a partial answer, we construct a general "machine" which, when fed with a smooth connected manifold $\mathcal{Y}$ (resp. with a "nice" topological space $\mathcal{Y}$), returns a family $\mathcal{R}_n\mathcal{Y}$ of smooth finite-dimensional manifolds (resp. topological spaces) which are successive approximations of $\Omega^2\Sigma^2\mathcal{Y}$, the double loop space freely generated by $\mathcal{Y}$. (The construction of $\mathcal{R}_n\mathcal{Y}$ is given in this article; that it does approximate $\Omega^2\Sigma^2\mathcal{Y}$ will be proved in [3].)
Our second aim, which is more speculative, explores a new direction of generalization: that of $\mathbf{A}^1$–homotopy theory, the homotopy theory of schemes developed by Morel and Voevodsky [27]. Indeed, our construction is algebraic: when fed with a smooth algebraic variety Y, the "machine" returns a family of smooth algebraic varieties, the Atiyah–Hitchin schemes $\mathcal{R}_nY$. In this series of articles, we will give evidence that when the algebraic variety Y is $\mathbf{A}^1$–connected then "the sequence of schemes $\mathcal{R}_nY$ converges (in the homotopy category) to $\Omega^{\mathbf{P}^1}\Sigma^{\mathbf{P}^1}Y$, the $\mathbf{P}^1$–loop space freely generated by Y". (As above, one can give a precise meaning to this statement.)

This first paper is devoted to the analysis of the geometry of the Atiyah–Hitchin schemes. The slogan is that for every integer $n \geqslant 1$, $\mathcal{R}_nY$ has to be thought of as a scheme-theoretic "completion" of the unordered configuration space of n distinct points in $\mathbf{A}^1$ with labels in Y. (Here by "completion" one should understand that a generic point of $(\mathcal{R}_nY)(\mathbf{C})$ belongs to the above configuration space but that there are also more *degenerate* configurations.) We illustrate this by describing the geometry of the complex manifold $(\mathcal{R}_nY)(\mathbf{C})$ associated to a complex algebraic variety Y.
The geometry of $(\mathcal{R}_nY)(\mathbf{C})$ is reminiscent of an other well-known approximation of the double loop space $\Omega^2\Sigma^2Y(\mathbf{C})$: the so-called May–Milgram model $\mathcal{C}_2Y(\mathbf{C})$ (see May's book [26, construction 2.4]). The space $(\mathcal{R}_nY)(\mathbf{C})$ is closely related to the n-th term $\mathrm{F}_n(\mathcal{C}_2Y(\mathbf{C}))$ in the canonical filtration of $\mathcal{C}_2Y(\mathbf{C})$. Indeed, we will prove in [3] (see also [6, chapitre 5]) that there is a *stable* homotopy equivalence

$$\Sigma^\infty(\mathcal{R}_nY)(\mathbf{C}) \approx \Sigma^\infty\mathrm{F}_n(\mathcal{C}_2Y(\mathbf{C}))$$

which is compatible with the Snaith splitting of $\Omega^2\Sigma^2Y(\mathbf{C})$. (In the special case of complex rational functions, this stable homotopy equivalence was first proved by F Cohen, R Cohen, B Mann and J Milgram [7, 8].) However, *unstably* the two spaces $(\mathcal{R}_n Y)(\mathbf{C})$ and $\mathrm{F}_n\mathcal{C}_2 Y(\mathbf{C})$ may differ: for an integer $d \geqslant 1$ and for $n > 1$, $(\mathcal{R}_n(\mathbf{A}^d - \{0\}))(\mathbf{C})$ and $\mathrm{F}_n\mathcal{C}_2(\mathbf{C}^d - \{0\})$ are homotopy equivalent if and only if $d > 1$ (see R Cohen and D Shimamoto [9] and Totaro [32]).

We leave to the next articles [3, 4] the study of the (unstable and stable) homotopy type of the Atiyah–Hitchin schemes and its relation to $\Omega^{\mathbf{P}^1}\Sigma^{\mathbf{P}^1}Y$. There is also an interesting connection with our previous work in [5] on the algebraic connected components of the schemes of pointed rational functions $\mathcal{F}_n$ which suggests the existence of a version of the little disks operad and of the group completion theorem in $\mathbf{A}^1$–homotopy theory.

Let us give a flavor of the results and techniques contained in the paper. Let Y be a fixed smooth complex algebraic variety and let n be a positive integer. A point in the space $(\mathcal{R}_n Y)(\mathbf{C})$ is a pair (A, B) where:

- $A = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ is a monic degree n polynomial with complex coefficients
- B corresponds to the datum, for each root α of the polynomial A, of a point in the total space of a certain vector bundle J_α over $Y(\mathbf{C})$. These vector bundles J_α are "jet-like" bundles over $Y(\mathbf{C})$ of order the multiplicity of the root α. For example, when α is a simple root J_α is the zero-dimensional vector bundle over $Y(\mathbf{C})$; when α has multiplicity two, J_α is the tangent bundle of $Y(\mathbf{C})$ and so on. (See the introduction of Section 3 for a more precise account about this.)

This description leads to a decomposition of the complex manifold $(\mathcal{R}_n Y)(\mathbf{C})$ *as a set* as a disjoint union of complex submanifolds, each one individually well understood. For example, the open stratum (corresponding to the locus where the first coordinate A has all its roots simple) is homeomorphic to the space of unordered configurations of n distinct points in $\mathbf{C}$ with labels in $Y(\mathbf{C})$. Note that this is exactly the same space which appears in the definition of $\mathrm{F}_n(\mathcal{C}_2 Y(\mathbf{C}))$, the n-th term of the canonical filtration of the May–Milgram model $\mathcal{C}_2 Y(\mathbf{C})$ for $\Omega^2\Sigma^2 Y(\mathbf{C})$. In general, each piece of $(\mathcal{R}_n Y)(\mathbf{C})$ is *up to homotopy* a space of configurations of a certain number of points in $\mathbf{C}$ with labels in $Y(\mathbf{C})$ (we leave vague here whether the configurations are ordered or not, see definition 3.1.1).

A full understanding of the geometry of the space $(\mathcal{R}_n Y)(\mathbf{C})$ requires also information about how the different pieces are glued together. We can provide this information by using the fact that the decomposition of $(\mathcal{R}_n Y)(\mathbf{C})$ satisfies a strong regularity condition: it is a so-called *Whitney stratification*. For such stratifications, the work of J

Mather in the 1970's (based on the former contribution of R Thom) provides a suitable notion of tubular neighbourhood of one stratum into another adjacent stratum, which we can be described. Although our description requires a combinatorial formalism which is a bit intricate, the two main underlying ideas are simple.

- The geometry of the strata of $(\mathcal{R}_n Y)(\mathbf{C})$ has the following property: if, in a configuration of points in $\mathbf{C}$ with labels in[1] $Y(\mathbf{C})$ (which represents a point in some stratum), two points $\alpha_1 \neq \alpha_2 \in \mathbf{C}$ have the same label, say $y_1 = y_2 = y \in Y(\mathbf{C})$, then, as the two points α_1 and α_2 tend to a common value, say $\alpha \in \mathbf{C}$, then the configuration $[(\alpha_1, y), (\alpha_2, y), (\alpha_i, y_i)]$ tends to the configuration $[(\alpha, y), (\alpha_i, y_i)]$ (which represents a point in a lower stratum). It turns out that this information suffices to capture the homotopy type of $(\mathcal{R}_n Y)(\mathbf{C})$.
- The stratification of the space of monic complex polynomials associated to the multiplicity of the roots is closely related to the stratification of $(\mathcal{R}_n Y)(\mathbf{C})$. (In fact, it corresponds to the special case $Y = \mathrm{pt}$.) Its strata are genuine configuration spaces of points in $\mathbf{C}$. And the attaching maps between the strata can be described by some versions of the structure maps in the little disks operad *on the level of configuration spaces of points*: embedding a little disk into another one is analogous to blowing up a multiple root into other roots of lower multiplicities.

A rough summary of our main result (theorem 3.4.3) is the following. There is a natural homotopy equivalence between $(\mathcal{R}_n Y)(\mathbf{C})$ and a space denoted $\Psi^n(Y(\mathbf{C}))$ obtained by attaching configuration spaces of little discs in $\mathbf{C}$ with labels in $Y(\mathbf{C})$ together *via* structure maps in the little discs operad and diagonals on the labels in $Y(\mathbf{C})$.
In particular, our description implies that the homotopy type of the topological space $(\mathcal{R}_n Y)(\mathbf{C})$ only depends on the homotopy type of $Y(\mathbf{C})$, which is not obvious from the definition of $\mathcal{R}_n Y$ (and which is convenient to use in [3]). Moreover, when the algebraic variety Y is defined over the field of real numbers $\mathbf{R}$, complex conjugation induces an involution on both $(\mathcal{R}_n Y)(\mathbf{C})$ and $\Psi^n(Y(\mathbf{C}))$. Our analysis of the stratification is compatible with these involutions: we have a homotopy equivalence $\Psi^n(Y(\mathbf{C})) \approx (\mathcal{R}_n Y)(\mathbf{C})$ as $\mathbf{Z}/2$–spaces. This suggests that the $\mathbf{A}^1$–homotopy type of the space $\mathcal{R}_n Y$ should only depend on the $\mathbf{A}^1$–homotopy type of Y, a result which we could not prove (see question 3.6.2).

Overview of the article

[1] Here, $Y(\mathbf{C})$ is identified with the zero sections of the "jet-like" bundles J_α.

- Section 2 is devoted to the definition of the Atiyah–Hitchin schemes. Their functor of points are easily defined and shown to be representable. Some basic properties are established. The algebraic geometry here is elementary and aimed at topologists (in particular, we avoid using the theory of Hilbert schemes). We also discuss examples we find illuminating.
- Section 3 describes completely the stratification of the space of complex points $(\mathcal{R}_nY)(\mathbf{C})$ when Y is defined over $\mathbf{C}$. In more details:
  - § 3.1 describes the topology of the strata of $(\mathcal{R}_nY)(\mathbf{C})$. These are the configuration spaces of points in $\mathbf{C}$ with labels in the "jet-like" bundles over $Y(\mathbf{C})$ alluded to in the introduction.
  - § 3.2 is a warm-up: we analyse in details the stratifications of $(\mathcal{R}_2Y)(\mathbf{C})$ and $(\mathcal{R}_3Y)(\mathbf{C})$. The study of $(\mathcal{R}_3Y)(\mathbf{C})$ requires already all the technical difficulty contained in the use of the Thom–Mather theory of controlled tubular neighbourhoods in Whitney stratifications. It is written in order to motivate and illustrate the general method.
  - To treat the general case, we first introduce in § 3.3 the required formalism to handle the combinatorics. This allows us to define in § 3.4 the functor $\Psi^n\colon \mathcal{T}op \longrightarrow \mathcal{T}op$ for any n and to state our main result (theorem 3.4.3): there is a natural homotopy equivalence between $(\mathcal{R}_nY)(\mathbf{C})$ and $\Psi^n(Y(\mathbf{C}))$. The proof is then given in § 3.5.
  - In § 3.6, we study the case when the variety Y is defined over the field of real numbers $\mathbf{R}$. In this case, complex conjugation induces an involution on both spaces $\Psi^n(Y(\mathbf{C}))$ and $(\mathcal{R}_nY)(\mathbf{C})$ and the homotopy equivalence $(\mathcal{R}_nY)(\mathbf{C}) \approx \Psi^n(Y(\mathbf{C}))$ is shown to be compatible with this action.
- Appendix A is a recollection of the necessary material on stratifications. In particular, we briefly present the Thom–Mather theory of controlled tubular neighbourhoods for Whitney stratifications, which is used as an important ingredient in our description of the homotopy type of $(\mathcal{R}_nY)(\mathbf{C})$.

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2 Atiyah–Hitchin schemes

Throughout this section, we work over a fixed base field k, although the construction would be valid over a more general base. By an algebraic variety, we mean a finite type separated k–scheme.

Let Y be a fixed algebraic variety. We introduce the Atiyah–Hitchin schemes attached to Y, which are our main object of study. These form a family $\mathcal{R}_n Y$ of algebraic varieties indexed by a positive integer n. The idea behind the construction that we give is due to M Atiyah and N Hitchin [1, chapter 5], which justifies our terminology. However, our presentation differs notably from the aforementioned reference for two reasons. The first one is that we don't want to use the deep theory of Hilbert schemes, as this text is aimed at topologists. The second reason is that we think that our viewpoint is more convenient for the applications we have in mind. The connection between our approach and that of Atiyah and Hitchin is explained in remark 2.5 below.

Definition 2.1 Let n be a positive integer. For any k–algebra R, let $(\mathfrak{R}_n Y)(R)$ be the set of pairs (A, B) where

- $A = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ is a monic degree n polynomial in $R[X]$
- B is a k–scheme morphism $B\colon$ Spec $R[X]/(A) \longrightarrow Y$. (In other words, B is an element of $Y\big(R[X]/(A)\big)$.)

The set $\mathfrak{R}_n Y(R)$ is natural in the k–algebra R; we have thus just defined a functor

$$\mathfrak{R}_n Y\colon \mathcal{A}lg_k \longrightarrow \mathcal{S}et$$

from the category of k–algebras to the category of sets.

The following proposition ensures that for reasonable schemes Y, the above functor $\mathfrak{R}_n Y$ is indeed the functor of points of a scheme $\mathcal{R}_n Y$.

Proposition 2.2 *Let n be a positive integer and Y be a quasi-projective algebraic variety*—ie *an open of some projective algebraic variety. Then the functor $\mathfrak{R}_n Y$ is representable by a quasi-projective algebraic variety, which we denote $\mathcal{R}_n Y$ and baptize the n*-th Atiyah–Hitchin scheme associated to Y.
Moreover, if Y is smooth of dimension d, then the scheme $\mathcal{R}_n Y$ is also smooth of dimension $n(d+1)$.

Nota bene: Even in the case when Y is projective, the scheme $\mathcal{R}_n Y$ is usually not projective. For example, one has a canonical isomorphism $\mathcal{R}_1 Y \simeq Y \times \mathbf{A}^1$, *cf* example 2.4–(1) below.

Proof of proposition 2.2 We first check that the functor $\mathfrak{R}_n Y$ is representable. One has to show that the functor $\mathfrak{R}_n Y$ is a sheaf in the Zariski topology and that it is covered by affine open subfunctors.
That $\mathfrak{R}_n Y$ is a sheaf in the Zariski topology is a consequence of the following fact. Let R be a k–algebra and $(f_i)_{i \in I}$ be a family of elements such that the opens $\operatorname{Spec} R[f_i^{-1}]$ cover $\operatorname{Spec} R$, then for any monic polynomial $A \in R[X]$, the schemes $\operatorname{Spec} R[f_i^{-1}][X]_{/(A)}$ form an open cover of $\operatorname{Spec} R[X]_{/(A)}$.
To check that $\mathfrak{R}_n Y$ is covered by affine open subfunctors, we observe the following facts.

Lemma 2.3 (1) *If U is an open subvariety of Y, then the functor $\mathfrak{R}_n U$ is an open subfunctor of $\mathfrak{R}_n Y$.*

(2) *If Y is an affine algebraic variety, then the functor $\mathfrak{R}_n Y$ is represented by an affine algebraic variety.*

Proof (1) This follows from the fact that the functor of points of U is an open subfunctor of the functor of points of Y.

(2) Let Y be a closed subscheme of some affine space $\mathbf{A}^N$ given by the vanishing of some family of polynomials $P_i(X_1, \ldots, X_N)$. For any k–algebra R and for any monic polynomial $A \in R[X]$ of degree n, an element of $R[X]_{/(A)}$ has a unique lifting as a polynomial of $R[X]_{\deg<n}$. The datum of $B \in Y(k[X]_{/(A)})$ is then equivalent to the datum of a family $(B_1, \ldots, B_N)$ of polynomials in $R[X]_{\deg<n}$ such that the rests of the $P_i(B_1, \ldots, B_N)$ in the Euclidean division by A vanish. This is given by polynomial equations on the coefficients of the polynomials $A, B_1, \ldots, B_N$, which gives the coordinates and the equations of the scheme representing $\mathfrak{R}_n Y$. □

We claim that the family of subfunctors $\mathfrak{R}_n U$ when U runs over all the affine open subschemes of Y forms a cover of $\mathfrak{R}_n Y$ by affine open subfunctors. This follows directly from the definition and from the fact that in a quasi-projective scheme Y any finite set of points in contained in an open subscheme $U \subset Y$ (*cf* [24, Proposition 3.36] for example).
This proves that the functor $\mathfrak{R}_n Y$ is representable by a scheme $\mathcal{R}_n Y$. For briefness,

we postpone the proof that the representing scheme $\mathcal{R}_n Y$ is quasi-projective until remark 2.5, where it's deduced from a general result of Grothendieck on the quasi-projectivity of Hilbert schemes.
We finish by checking that when Y is assumed to be smooth then $\mathcal{R}_n Y$ is also smooth. Since smoothness is a local condition, the previous discussion on open covers of $\mathcal{R}_n Y$ implies that it's enough to treat the case when Y is affine. One shows that $\mathcal{R}_n Y \otimes \overline{k}$ is non-singular at any point $(A, B) \in \mathcal{R}_n Y \otimes \overline{k}$ by checking a Jacobian criterion. (For this, observe that one can assume that A is irreducible, for if $A = A_1 A_2$ is a coprime decomposition, then $\mathcal{R}_n Y$ is, locally around (A, B), isomorphic to the product $\mathcal{R}_{n_1} Y \times \mathcal{R}_{n_2} Y$, with $n_i = \deg A_i$ for $i = 1, 2$. One can thus assume that $A = X^n$; note then that the "constant terms" of B gives a point of Y, say $b_0 \in Y(\overline{k})$. The Jacobian matrix of the equations of $\mathcal{R}_n Y$ at (X^n, B) is block triangular, with diagonal entries the Jacobian matrix of the equations of Y at b_0.) □

Nota bene: Note that the proof of the second assertion in proposition 2.2 shows that the canonical projection on the first factor

$$\begin{array}{rccc} \wp\colon & \mathcal{R}_n Y & \longrightarrow & \mathrm{Pol}_n \\ & (A, B) & \mapsto & A \end{array}$$

is smooth. In particular, this means that when Y is defined over $\mathbf{C}$, the map of complex manifolds $\wp\colon (\mathcal{R}_n Y)(\mathbf{C}) \longrightarrow \mathrm{Pol}_n(\mathbf{C})$ is a submersion.

Example 2.4 Let n be a positive integer.

(1) For every scheme Y, one has a canonical isomorphism of schemes

$$\mathcal{R}_1 Y \simeq Y \times \mathbf{A}^1 .$$

(2) Let $\mathrm{pt} := \operatorname{Spec} k$ be the "one point scheme" and let Pol_n be the scheme of monic degree n polynomials. (Pol_n is thus canonically isomorphic to the affine space $\mathbf{A}^n$.) One has a canonical isomorphism of schemes

$$\mathcal{R}_n \mathrm{pt} \simeq \mathrm{Pol}_n .$$

(3) Recall that we denote $\mathcal{F}_n$ the scheme of pointed degree n rational functions; $\mathcal{F}_n$ is thus the open subscheme of $\mathbf{A}^{2n} := \operatorname{Spec} k[a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}]$ complementary to the hypersurface given by the vanishing of the resultant

$$\mathrm{res}_{n,n}(X^n + a_{n-1} X^{n-1} + \cdots + a_0,\ b_{n-1} X^{n-1} + \cdots + b_0) .$$

One has a canonical isomorphism of schemes

$$\mathcal{R}_n(\mathbf{A}^1 - \{0\}) \simeq \mathcal{F}_n .$$

(4) More generally, for every positive integer $d \geqslant 1$, let $\mathcal{F}_n^d$ denote the Hom scheme of pointed degree n morphisms from $\mathbf{P}^1$ to $\mathbf{P}^d$. (For any field K, the set $\mathcal{F}_n^d(K)$ is thus in bijection with $(d+1)$–tuples of polynomials $(A, B_1, \ldots, B_d)$ in $K[X]$ with A monic of degree n, each B_i of degree $< n$ and such that there is no root common to all polynomials.) One has a canonical isomorphism of schemes

$$\mathcal{R}_n(\mathbf{A}^d - \{0\}) \simeq \mathcal{F}_n^d \, .$$

Proof (1) For every k–algebra R and every $a \in R$, one has a canonical isomorphism of k–algebras $R[X]_{/(X-a)} \simeq R$.

(2) For every k–algebra R and for any monic polynomial A, there is only one map of k–schemes from $\operatorname{Spec} R[X]_{/(A)}$ to $\operatorname{Spec} k$.

(3) For any k–algebra R and for any monic polynomial A in $R[X]$, $(\mathbf{A}^1 - \{0\})(R[X]_{/(A)})$ is the set of units of $R[X]_{/(A)}$. Since A is monic, any element $B \in R[X]_{/(A)}$ admits a *unique* lifting as a polynomial $\widetilde{B} \in R[X]$ of degree strictly less than n. The pair $(A, \widetilde{B})$ is then an element of $\mathcal{F}_n(R)$. This describes a natural isomorphism of functors $\mathcal{R}_n(\mathbf{A}^1 - \{0\}) \simeq \mathcal{F}_n$.

(4) The argument is similar to the previous one and is thus omitted. □

Remark 2.5 (1) Our presentation differs notably from that given by Atiyah and Hitchin in [1, chapter 5]. So we indicate here the link between the two approaches.

In general, given a smooth algebraic variety Y and a positive integer n, the punctual Hilbert scheme $Y^{[n]}$ is not a desingularization of the symmetric product $\mathrm{SP}^n(Y)$. However, Atiyah and Hitchin observed that one can construct a desingularization of $\mathrm{SP}^n(Y \times \mathbf{A}^1)$ by the following construction. Let $\widetilde{\mathfrak{R}}_n Y$ be the following subfunctor of $\mathrm{Hilb}^n(Y \times \mathbf{A}^1)$

$$\begin{array}{rccl} \widetilde{\mathfrak{R}}_n(Y): & \mathcal{A}lg_k & \longrightarrow & \mathcal{S}et \\ & R & \longmapsto & \left\{ \begin{array}{l} \text{Flat closed sub-}R\text{–schemes } Z \subset (Y \times \mathbf{A}^1)_R \\ \text{with Hilbert polynomial } P_Z = n \text{ and such that} \\ Z \hookrightarrow (Y \times \mathbf{A}^1)_R \xrightarrow{\mathrm{pr}} \mathbf{A}^1_R \text{ remains a closed} \\ \text{immersion.} \end{array} \right\} \end{array}$$

The datum of a length n closed immersion $Z \hookrightarrow \mathbf{A}^1$ is equivalent to the datum of a monic degree n polynomial, and one then has $Z \simeq \operatorname{Spec}(R[X]_{/(A)})$. The datum of the immersion $Z \hookrightarrow (Y \times \mathbf{A}^1)$ thus corresponds to the datum of the morphism $\operatorname{Spec}(R[X]_{/(A)}) \longrightarrow Y$. This induces an isomorphism of functors $\widetilde{\mathfrak{R}}_n Y \simeq \mathfrak{R}_n Y$.

(2) As promised in the proof of proposition 2.2, we indicate to the reader why the algebraic varieties $\mathcal{R}_n Y$ are quasi-projective. In the point of view of Atiyah and Hitchin, the functor $\mathfrak{R}_n Y$ is an open subfunctor of the functor of points of the punctual Hilbert scheme of $Y \times \mathbf{A}^1$. By a general result of Grothendieck [15, théorème 3.2 and § 4], $\mathcal{H}\mathrm{ilb}^n(Y \times \mathbf{A}^1)$ is quasi-projective thus so is $\mathcal{R}_n Y$.

3 The homotopy type of the space of complex points

In all Section 3, we fix once for all a positive integer n and a smooth algebraic variety Y defined over the field of complex numbers $\mathbf{C}$. For briefness, we denote $\mathcal{Y} := Y(\mathbf{C})$ the topological space of complex points.

Our analysis of the geometry of the complex manifold $(\mathcal{R}_n Y)(\mathbf{C})$ follows the idea, already present in the work of F Cohen, R Cohen, Mann and Milgram [7], of cutting the manifold into disjoint pieces, each one well understood separately. Let

$$\begin{array}{cccc} \wp : & (\mathcal{R}_n Y)(\mathbf{C}) & \longrightarrow & \mathrm{Pol}_n(\mathbf{C}) \\ & (A,B) & \longmapsto & A \end{array}$$

be the canonical projection on the first factor. The fiber of $\wp$ over a polynomial A is by definition $Y\left(\mathbf{C}[X]_{/(A)}\right)$. The topology of this space highly depends on the nature of the $\mathbf{C}$–algebra $\mathbf{C}[X]_{/(A)}$. Indeed, if α_i $(1 \leqslant i \leqslant r)$ denote the distinct complex roots of A, with multiplicity $n_i \geqslant 1$, the Chinese remainder theorem induces a homeomorphism

$$\wp^{-1}(A) \simeq \prod_{i=1}^{r} Y\left(\mathbf{C}[X]_{/(X^{n_i})}\right).$$

The spaces $Y\left(\mathbf{C}[X]_{/(X^{n_i})}\right)$ appearing above are the total spaces of vector bundles over $\mathcal{Y} := Y(\mathbf{C})$—these are the "jet-like" bundles alluded to in the introduction. For example, it is well-known that $Y\left(\mathbf{C}[X]_{/(X^2)}\right)$ identifies with the tangent bundle of $\mathcal{Y}$. Each space $Y\left(\mathbf{C}[X]_{/(X^{n_i})}\right)$ deformation retracts onto its zero-section $\mathcal{Y}$. Thus the fiber $\wp^{-1}(A)$ has the homotopy type of a product of r copies of $\mathcal{Y}$.

It follows from this discussion that the manifold $(\mathcal{R}_n Y)(\mathbf{C})$ naturally decomposes, *as a set*, into a disjoint union of submanifolds, whose homotopy type is a simple function in the homotopy type of $\mathcal{Y}$. For example, the open submanifold corresponding to the locus of points (A,B) where A has only simple roots is homeomorphic to the unordered space of configuration of n distinct points in $\mathbf{C}$ with labels in $\mathcal{Y}$ denoted $\mathbf{C}^{(n)} \underset{\mathfrak{S}_n}{\times} \mathcal{Y}^n$ (see definition 3.1.1 below for the explanation of the notation). The relevant notion here is that of *stratification*. (Appendix A gives a brief introduction to this notion.) The space of monic complex degree n polynomials $\mathrm{Pol}_n(\mathbf{C})$ admits a natural stratification $\mathcal{S}^{\mathrm{pt}}$ associated to the multiplicity of the roots, which lifts *via* the canonical submersion $\wp\colon (\mathcal{R}_n Y)(\mathbf{C}) \longrightarrow \mathrm{Pol}_n(\mathbf{C})$ to the stratification $\mathcal{S}^Y$ of $(\mathcal{R}_n Y)(\mathbf{C})$ which we consider. In the rest of this section, we give a complete description of the stratification $\mathcal{S}^Y$: we describe the topology of the strata and the attaching data.

3.1 Description of the strata

We start our analysis of the stratification $\mathcal{S}^Y$ of the space $(\mathcal{R}_n Y)(\mathbf{C})$ by describing the topology of the strata. As we will see, the set of strata is indexed by the partitions of the integer n and each of these has the homotopy type of a space of configuration of points in $\mathbf{C}$ with labels in $\mathcal{Y} := Y(\mathbf{C})$. Here, the term “configuration space” has to be taken in a slightly more general sense than usual as we do not infer anything about whether the configuration is ordered or not (see definition 3.1.1 below).

3.1.1 Configuration spaces and partitions of an integer

For every positive integer $e > 0$, let $\underline{e}$ denote the set $\{1, 2, \ldots, e\}$.

Definition 3.1.1 The *unordered* configuration space of n distinct points in $\mathbf{C}$, denoted $\mathbf{C}^{(n)}$, is the space of injective maps from $\underline{n}$ to $\mathbf{C}$. It comes with a natural action of the symmetric group $\mathfrak{S}_n$ by composition at the source.
For any topological space Z, we use the general terminology *configuration space of n distinct points in $\mathbf{C}$ with labels in Z* for a space $\mathbf{C}^{(n)} \underset{S}{\times} Z^n$ for some subgroup $S \subset \mathfrak{S}_n$.

Definition 3.1.2 (1) A partition of the integer n is the datum of a positive integer m and of an equivalence class of surjections $\pi\colon \underline{n} \twoheadrightarrow \underline{m}$ according to the following equivalence relation :

$$(\underline{n} \overset{\pi}{\twoheadrightarrow} \underline{m}) \sim (\underline{n} \overset{\pi'}{\twoheadrightarrow} \underline{m}) \iff \exists\, \alpha\colon \underline{n} \xrightarrow{\simeq} \underline{n} \text{ and } \beta\colon \underline{m} \xrightarrow{\simeq} \underline{m} \text{ such that } \pi' \circ \alpha = \beta \circ \pi\,.$$

We denote $[\pi]$ the equivalence class of the surjection π.
Let $\Pi(n)$ denote the set of partitions of the integer n. This set is equipped with the following partial order: $[\pi] \succeq [\pi']$ whenever there exist some representing elements π and π' of $[\pi]$ and $[\pi']$ and a surjection $\varphi\colon \underline{m} \twoheadrightarrow \underline{m'}$ such that $\pi = \varphi \circ \pi'$.

(2) For every surjection $\pi\colon \underline{n} \twoheadrightarrow \underline{m}$, let also $\overline{\pi}$ denote the class of π modulo composition at the target by an element of the symmetric group $\mathfrak{S}_m$. In other terms, one has $\overline{\pi} = \overline{\pi'}$ if and only if there exists $\beta \in \mathfrak{S}_m$ such that $\pi' = \beta \circ \pi$. The set $\widetilde{\Pi}(n)$ of classes of surjections $\overline{\pi}$ is equipped with the partial order analogous to that of $\Pi(n)$.

Example 3.1.3 Every integer n admits the following two partitions:

- The *generic* partition, denoted $[\pi_g]$, is the class of the identity map $\mathrm{id}_{\underline{n}}$. This is the maximal element of the set of partitions $\Pi(n)$.
- The *trivial* partition, denoted $[\pi_t]$, is the class of the unique map from $\underline{n}$ to $\underline{1}$. This is the minimal element of the set of partitions $\Pi(n)$.

3.1.2 The strata of $\mathrm{Pol}_n(\mathbf{C})$

We are now ready to describe the strata of the stratification $\mathcal{S}^{\mathrm{pt}}$ of the space $\mathrm{Pol}_n(\mathbf{C})$ of complex monic degree n polynomials associated to the multiplicity of the roots. It plays a major rôle in our analysis for two reasons: first, as we already discussed before, the stratification $\mathcal{S}^Y$ of $(\mathcal{R}_n Y)(\mathbf{C})$ we focus on is a lift of $\mathcal{S}^{\mathrm{pt}}$. Secondly, one should consider this as the particular case $Y = \mathrm{pt}$. Indeed, through the canonical isomorphism $\mathcal{R}_n\mathrm{pt} \simeq \mathrm{Pol}_n$ of example 2.4–(2), the stratification $\mathcal{S}^{Y=\mathrm{pt}}$ corresponds to $\mathcal{S}^{\mathrm{pt}}$ (which justifies our notation). As we will see, our strategy is to reduce the study of $\mathcal{S}^Y$ to that of $\mathcal{S}^{\mathrm{pt}}$.

Definition 3.1.4 Let $\mathcal{S}^{\mathrm{pt}}$ be the stratification of $\mathrm{Pol}_n(\mathbf{C})$ associated to the algebraic hypersurface defined by the vanishing of the discriminant (see example A.1.2–(3)). Alternatively, the stratification $\mathcal{S}^{\mathrm{pt}}$ is also the *orbit type* stratification under the identification $\mathrm{Pol}_n(\mathbf{C}) \simeq \mathbf{C}^n/\mathfrak{S}_n$ (see proposition–definition A.1.4).
Our convention here is to take the strata of $\mathcal{S}^{\mathrm{pt}}$ *path-connected*.

Identify the space $\mathrm{Pol}_n(\mathbf{C})$ with the symmetric product $\mathrm{SP}^n(\mathbf{C}) := \mathbf{C}^{\underline{n}}/\mathfrak{S}_n$. A polynomial in $\mathrm{Pol}_n(\mathbf{C})$ is equivalent to the set of its roots, which is though of as an orbit of maps from $\underline{n}$ to $\mathbf{C}$ under the natural action of the symmetric group $\mathfrak{S}_n$.
Let $q\colon \mathbf{C}^n \longrightarrow \mathrm{SP}^n(\mathbf{C})$ be the canonical projection. For technical reasons, it is also useful to analyse the ($\mathfrak{S}_n$–equivariant) stratification $\mathcal{E}^{\mathrm{pt}} := q^{-1}(\mathcal{S}^{\mathrm{pt}})$ of $\mathbf{C}^n$. The description of the strata is the following:

Proposition–definition 3.1.5 (1) *The strata* $S^{\mathrm{pt}}_{[\pi]}$ *of the stratification* $\mathcal{S}^{\mathrm{pt}}$ *of the space* $\mathrm{Pol}_n(\mathbf{C})$ *are indexed by the partitions* $[\pi]$ *of the integer* n. *By definition, the stratum* $S^{\mathrm{pt}}_{[\pi]}$ *attached to the partition* $[\pi]$ *is composed of the orbits (under the action of* $\mathfrak{S}_n$*) of maps* $x\colon \underline{n} \longrightarrow \mathbf{C}$ *such that there exists a representing element* $\pi\colon \underline{n} \twoheadrightarrow \underline{m}$ *of* $[\pi]$ *and a factorization* $x = \varphi \circ \pi$ *with* $\varphi\colon \underline{m} \hookrightarrow \mathbf{C}$ *injective.*

(2) *The strata* $E^{\mathrm{pt}}_{[\pi]}$ *of the stratification* $\mathcal{E}^{\mathrm{pt}}$ *of* $\mathbf{C}^n$ *are indexed by the partitions* $[\pi]$ *of the integer* n. *By definition, one has* $E^{\mathrm{pt}}_{[\pi]} = q^{-1}\big(S^{\mathrm{pt}}_{[\pi]}\big)$. *Note that the stratification* $\mathcal{E}^{\mathrm{pt}}$ *is* $\mathfrak{S}_n$*–equivariant in the sense that each stratum is stable under the action of* $\mathfrak{S}_n$.

(3) *The partial orders (*cf *definition A.1.1) on the sets of strata of* $\mathcal{S}^{\mathrm{pt}}$ *and* $\mathcal{E}^{\mathrm{pt}}$ *are given by*

$$S^{\mathrm{pt}}_{[\pi]} \succeq S^{\mathrm{pt}}_{[\pi']} \iff [\pi] \succeq [\pi'] \qquad \text{and} \qquad E^{\mathrm{pt}}_{[\pi]} \succeq E^{\mathrm{pt}}_{[\pi']} \iff [\pi] \geqslant [\pi'] \,.$$

(4) *Let* $[\pi]$ *be a partition of* n. *The connected components* $E^{\mathrm{pt}}_{\overline{\pi}}$ *of the stratum* $E^{\mathrm{pt}}_{[\pi]}$ *are indexed by the classes of surjections* $\overline{\pi} \in [\pi]$*:*

$$E^{\mathrm{pt}}_{[\pi]} = \coprod_{\overline{\pi}\in[\pi]} E^{\mathrm{pt}}_{\overline{\pi}} \,.$$

For every class of surjections $\overline{\pi}$, *the space* $E^{\mathrm{pt}}_{\overline{\pi}}$ *is the subspace of* $\mathbf{C}^n$ *of maps* $x\colon \underline{n} \longrightarrow \mathbf{C}$ *such that for every representing element* $\pi\colon \underline{n} \twoheadrightarrow \underline{m}$ *of* $\overline{\pi}$, *there exists a factorization* $x = \varphi \circ \pi$ *with* $\varphi\colon \underline{m} \hookrightarrow \mathbf{C}$ *injective.*
For each choice of a representing element π *of* $\overline{\pi}$, *the map from* $E^{\mathrm{pt}}_{\overline{\pi}}$ *to the space* $\mathbf{C}^{(n)}$ *of injective maps from* $\underline{n}$ *to* $\mathbf{C}$ *which associates to every* x *the (unique) map* φ *as above is a homeomorphism.*

(5) *The symmetric group* $\mathfrak{S}_n$ *acts on each stratum* $E^{\mathrm{pt}}_{[\pi]}$. *Its action exchanges the connected components as follows*

$$\forall \sigma \in \mathfrak{S}_n, \quad \sigma \cdot E^{\mathrm{pt}}_{\overline{\pi}} = E^{\mathrm{pt}}_{\overline{\pi\circ\sigma^{-1}}} \,.$$

(6) *Let* $\overline{\pi}$ *be a class of surjections and let* $\pi\colon \underline{n} \twoheadrightarrow \underline{m}$ *be a representing element of* $\overline{\pi}$. *The subgroup of* $\mathfrak{S}_n$ *of those permutations* σ *such that there exists a (non-necessarily unique) permutation* $\tau \in \mathfrak{S}_n$ *satisfying the relation* $\pi \circ \sigma = \tau \circ \pi$ *doesn't depend on the choice of* π. *This group is the stabilizer of the connected component* $E^{\mathrm{pt}}_{\overline{\pi}}$ *and we denote it* $\mathfrak{S}_{\overline{\pi}}$.

(7) *For every partition* $[\pi]$ *of* n *and for every element* $\overline{\pi} \in [\pi]$, *we have canonical homeomorphisms*

$$E^{\mathrm{pt}}_{[\pi]} /\!\!\!/ \mathfrak{S}_n \xrightarrow{\simeq} S^{\mathrm{pt}}_{[\pi]} \xleftarrow{\simeq} E^{\mathrm{pt}}_{\overline{\pi}} /\!\!\!/ \mathfrak{S}_{\overline{\pi}}$$

describing $S^{\mathrm{pt}}_{[\pi]}$ *as a space of configuration of points in* $\mathbf{C}$.

Example 3.1.6 (1) The stratum $S^{\mathrm{pt}}_{[\pi_g]}$ is the open subspace of $\mathrm{Pol}_n(\mathbf{C})$ of polynomials with all their roots simple. The stratum $E^{\mathrm{pt}}_{[\pi_g]}$ is here the subspace $\mathbf{C}^{(n)}$ of $\mathbf{C}^n$.

(2) The stratum $S^{\mathrm{pt}}_{[\pi_t]}$ is the space of polynomials with only one root (of order n). The stratum $E^{\mathrm{pt}}_{\overline{\pi_g}}$ corresponds to the diagonal $x_1 = \cdots = x_n$ of $\mathbf{C}^n$.

(3) When $n = 3$, let $\pi_1 \colon \underline{3} \twoheadrightarrow \underline{2}$ be the following surjection: $\begin{array}{ccc} 1 & \longrightarrow & 1 \\ 2 & \nearrow & \\ 3 & \longrightarrow & 2 \end{array}$. The stratum $S^{\mathrm{pt}}_{[\pi_1]}$ is the space of polynomials of degree 3 having exactly one double root. One has a homeomorphism $S^{\mathrm{pt}}_{[\pi_1]} \simeq \mathbf{C}^{(2)}$. (Recall that the notation $\mathbf{C}^{(2)}$ denotes the space of configuration of two distinct points in $\mathbf{C}$.)
The space $E_{\overline{\pi_1}} \subset \mathbf{C}^3$ is composed of the 3–tuples (x_1, x_2, x_3) such that $x_1 = x_2$ and $x_1 \neq x_3$. One has

$$E_{[\pi_1]} = E_{\overline{\pi_1}} \coprod (\tau_{1,3} \cdot E_{\overline{\pi_1}}) \coprod (\tau_{2,3} \cdot E_{\overline{\pi_1}}),$$

where $\tau_{i,j}$ is the transposition $(i,j) \in \mathfrak{S}_3$. The group $\mathfrak{S}_{\overline{\pi_1}}$ is the subgroup of $\mathfrak{S}_3$ generated by the transposition $\tau_{1,2}$. Here, it acts trivially on $E_{\overline{\pi_1}}$.

Remark 3.1.7 Here is a more intrinsic rephrasing of the homeomorphism (7) above. Let $[\pi]$ be a partition of n and $\mathfrak{F}_0([\pi])$ be the following category. The objects of $\mathfrak{F}_0([\pi])$ are the surjections $\underline{n} \overset{\pi_0}{\twoheadrightarrow} \underline{m_0}$ such that $[\pi_0] = [\pi]$. A morphism between two objects π_0 and π_0' is a commutative diagram

$$\begin{array}{ccc} \underline{n} & \xrightarrow[\simeq]{\alpha} & \underline{n} \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_0'} \\ \underline{m_0} & \xrightarrow[\simeq]{\alpha_0} & \underline{m_0} \end{array} .$$

Let $\mathbf{E}^{\mathrm{pt}}$ be the unique functor from $\mathfrak{F}_0([\pi])$ to $\mathcal{T}op$ such that $\mathbf{E}^{\mathrm{pt}}(\pi_0) = E^{\mathrm{pt}}_{\overline{\pi_0}}$ and such that for every morphism $(\alpha, \alpha_0) \colon \pi_0 \longrightarrow \pi_0'$ in $\mathfrak{F}_0([\pi])$, the morphism $\mathbf{E}^{\mathrm{pt}}(\alpha, \alpha_0)$ is induced by the action of the element $\alpha \in \mathfrak{S}_n$. The homeomorphism (7) in proposition–definition 3.1.5 can be reformulated as:

$$S^{\mathrm{pt}}_{[\pi]} \simeq \operatorname*{colim}_{\pi_0 \in \mathfrak{F}_0([\pi])} \mathbf{E}^{\mathrm{pt}}_0(\pi_0).$$

Proposition 3.1.8 *The stratifications $\mathcal{S}^{\mathrm{pt}}$ and $\mathcal{E}^{\mathrm{pt}}$ are Whitney-regular (this notion is explained in § A.1.2 of the appendix).*

Proof That the stratification $\mathcal{S}^{\mathrm{pt}}$ is Whitney-regular is the purpose of [10]. By proposition A.1.7, the inverse image of a Whitney regular stratification though a submersion is Whitney regular. The stratification $\mathcal{E}^{\mathrm{pt}}$ is thus Whitney-regular. One could also have deduced this directly from general results about *orbit-type* stratifications. (*cf* proposition–definition A.1.4 and proposition A.1.10). □

3.1.3 The strata of $(\mathcal{R}_n Y)(\mathbf{C})$

Recall that the stratification $\mathcal{S}^Y$ of $(\mathcal{R}_n Y)(\mathbf{C})$ is by definition the inverse image through the canonical projection $\wp\colon (\mathcal{R}_n Y)(\mathbf{C}) \longrightarrow \mathrm{Pol}_n(\mathbf{C})$ of the stratification $\mathcal{S}^{\mathrm{pt}}$. We deduce from the previous paragraph a description of the strata $S^Y_{[\pi]}$ of $\mathcal{S}^Y$ as a function of the strata $S^{\mathrm{pt}}_{[\pi]}$ of $\mathcal{S}^{\mathrm{pt}}$ and of $\mathcal{Y}$.
Let $(\widetilde{\mathcal{R}_n} Y)(\mathbf{C})$ be the canonical fiber product:

$$\begin{array}{ccc} (\widetilde{\mathcal{R}_n} Y)(\mathbf{C}) & \xrightarrow{\widetilde{q}} & (\mathcal{R}_n Y)(\mathbf{C}) \\ \downarrow{\scriptstyle \widetilde{\wp}} & & \downarrow{\scriptstyle \wp} \\ \mathbf{C}^n & \xrightarrow[q]{} & \mathrm{Pol}_n(\mathbf{C}) \end{array} .$$

For some technical reasons that will appear later, we also describe the strata of the stratification $\mathcal{E}^Y := (\wp \circ \widetilde{q})^{-1}(\mathcal{S}^{\mathrm{pt}})$ of $(\widetilde{\mathcal{R}_n} Y)(\mathbf{C})$.
By definition, the strata of $\mathcal{S}^Y$ and $\mathcal{E}^Y$ are again indexed by the partitions of the integer n. For every partition $[\pi]$ of the integer n, we set

$$S^Y_{[\pi]} = \wp^{-1}(S^{\mathrm{pt}}_{[\pi]}) \quad \text{and} \quad E^Y_{[\pi]} = (\wp \circ \widetilde{q})^{-1}(S^{\mathrm{pt}}_{[\pi]}) .$$

Note that the strata of $\mathcal{E}^Y$ are all stable under the action of $\mathfrak{S}_n$.
The next proposition gives the aforementioned description of the strata of $\mathcal{S}^Y$ as configuration spaces of points in $\mathbf{C}$ with labels in a vector bundle over $\mathcal{Y}$.

Proposition 3.1.9 (1) *Let $[\pi]\colon \underline{n} \longrightarrow \underline{m}$ be a partition of n. The connected components $E^Y_{\overline{\pi}}$ of the stratum $E^Y_{[\pi]}$ are indexed by classes of surjections $\overline{\pi} \in [\pi]$. For each representing element π of some class $\overline{\pi}$, one has a $\mathfrak{S}_{\overline{\pi}}$–equivariant homeomorphism:*

$$E^Y_{\overline{\pi}} \simeq E^{\mathrm{pt}}_{\overline{\pi}} \times \prod_{i=1}^{n} Y\Big(\mathbf{C}[X]\big/\big(X^{|\pi^{-1}(i)|}\big)\Big) .$$

Here, the group $\mathfrak{S}_{\overline{\pi}}$ acts on the product in the left hand side term by permuting $\underline{m}$, via the morphism $\mathfrak{S}_{\overline{\pi}} \longrightarrow \mathfrak{S}_n$ associated to the choice of a representing

element π.
Moreover, the stratum $E^Y_{\overline{\pi}}$ *deformation retracts* $\mathfrak{S}_{\overline{\pi}}$*–equivariantly on its subspace*

$$E^{\circ}_{\overline{\pi}} := E^{\mathrm{pt}}_{\overline{\pi}} \times \mathcal{Y}^{\underline{m}} \subset E^{\mathrm{pt}}_{\overline{\pi}} \times \prod_{i=1}^{n} Y\left(\mathbf{C}[X]/_{(X^{|\pi^{-1}(i)|})}\right).$$

(2) *For every partition* $[\pi]\colon \underline{n} \longrightarrow \underline{m}$, *and for every representing element* π *of* $[\pi]$, *one has a homeomorphism:*

$$S^Y_{[\pi]} \simeq E^{\mathrm{pt}}_{\overline{\pi}} \underset{\mathfrak{S}_{\overline{\pi}}}{\times} \prod_{i=1}^{n} Y\left(\mathbf{C}[X]/_{(X^{|\pi^{-1}(i)|})}\right) \tag{3–1}$$

independent of the choice of π.
Moreover, the stratum $S^Y_{[\pi]}$ *deformation retracts on its subspace*

$$S^{\circ}_{[\pi]} := E^{\mathrm{pt}}_{\overline{\pi}} \underset{\mathfrak{S}_{\overline{\pi}}}{\times} \mathcal{Y}^{\underline{m}} \subset E^{\mathrm{pt}}_{\overline{\pi}} \underset{\mathfrak{S}_{\overline{\pi}}}{\times} \prod_{i=1}^{n} Y\left(\mathbf{C}[X]/_{(X^{|\pi^{-1}(i)|})}\right).$$

We encourage the reader to rewrite formula (3–1) in a more intrinsic way, in the spirit of remark 3.1.7.

The regularity properties of the stratifications of proposition 3.1.8 naturally pass to $\mathcal{S}^{\mathrm{pt}}$ and $\mathcal{E}^{\mathrm{pt}}$. Precisely, one has:

Proposition 3.1.10 *The stratifications* $\mathcal{S}^Y$ *and* $\mathcal{E}^Y$ *are Whitney-regular.*

Proof It is a consequence of proposition 3.1.8 and proposition A.1.7. □

Remark 3.1.11 For every partition $[\pi] = [\underline{n} \twoheadrightarrow \underline{m}]$ of n, let $\Sigma^Y_{[\pi]}$ be the subspace of $(\mathcal{R}_n Y)(\mathbf{C})$ of pairs (A, B) such that:

- there exists a factorization $A = A_1 \cdots A_n$ with polynomials A_i monic of respective degrees $|\pi^{-1}(i)|$, and pairwise coprime;
- B belongs to $\mathcal{Y}^n \subset Y\left(\mathbf{C}[X]/_{(A_1)}\right) \times \cdots \times Y\left(\mathbf{C}[X]/_{(A_n)}\right)$.

In general, $\Sigma^Y_{[\pi]}$ isn't a submanifold of $(\mathcal{R}_n Y)(\mathbf{C})$. Let also $\widetilde{\Sigma}^Y_{[\pi]}$ be the subspace of $(\widetilde{\mathcal{R}_n} Y)(\mathbf{C})$

$$\widetilde{\Sigma}^Y_{[\pi]} := \wp^{-1}(\Sigma^Y_{[\pi]}).$$

Here, $\widetilde{\Sigma}^Y_{[\pi]}$ is a sub-$\mathfrak{S}_n$–manifold of $(\widetilde{\mathcal{R}_n} Y)(\mathbf{C})$, which is the (non-disjoint) union of $\mathfrak{S}_{\overline{\pi}}$–submanifolds $\widetilde{\Sigma}^Y_{\overline{\pi}}$ for $\overline{\pi} \in [\pi]$. One checks that $\widetilde{\Sigma}_{[\pi]}$ is transverse to the stratification $\mathcal{E}^Y$, *ie* that it is transverse to each of its stratum.

Example 3.1.12 (1) For the generic stratum, one has $\Sigma^Y_{[\pi_g]} = S^Y_{[\pi_g]}$ and $\widetilde{\Sigma}^Y_{[\pi_g]} = E^Y_{[\pi_g]}$.

(2) For the trivial stratum, one has $\Sigma^Y_{[\pi_t]} = \mathrm{Pol}_n(\mathbf{C}) \times Y$ and $\widetilde{\Sigma}^Y_{[\pi_t]} = \mathbf{C}^n \times Y$.

(3) When $Y = \mathrm{pt}$, for each partition $[\pi]$ of n, $\Sigma^{\mathrm{pt}}_{[\pi]}$ is the open subspace $\bigcup_{[\pi'] \geqslant [\pi]} S^{\mathrm{pt}}_{[\pi']} \subset \mathrm{Pol}_n(\mathbf{C})$.

(4) For every surjection $\pi\colon \underline{n} \twoheadrightarrow \underline{m}$, one has a $\mathfrak{S}_{\overline{\pi}}$–equivariant homeomorphism: $\widetilde{\Sigma}^Y_{\overline{\pi}} \simeq \widetilde{\Sigma}^{\mathrm{pt}}_{\overline{\pi}} \times \mathcal{Y}^n$.

3.2 The functors Ψ^2 and Ψ^3

The previous paragraph has given a description of the homotopy type of the strata of $\mathcal{S}^Y$. In order to complete the description of the stratification, one needs to describe how these spaces are glued together. One way to proceed is to thicken up every stratum by replacing it by a tubular neighbourhood and to describe all the possible multi-intersection *à la Cech* of these spaces and the maps between them.
Because the formalism becomes tedious, we illustrate in some details the analysis of the special case $n = 2$ and $n = 3$ for the reader's convenience. All the technical difficulty is already present in the study of the stratification of$(\mathcal{R}_3 Y)(\mathbf{C})$.

3.2.1 Analysis of the stratification of $(\mathcal{R}_2 Y)(\mathbf{C})$

The integer $n = 2$ has only two distinct partitions $[\pi_g] \succ [\pi_t]$. The stratification $\mathcal{S}^Y$ of $(\mathcal{R}_2 Y)(\mathbf{C})$ thus admits only the two strata

$$S^Y_{[\pi_g]} \simeq \mathbf{C}^{(2)} \underset{\mathfrak{S}_2}{\times} Y(\mathbf{C})^2 \quad \text{and} \quad S^Y_{[\pi_t]} \simeq \mathbf{C} \times Y\Big(\mathbf{C}[X]/(X^2)\Big) .$$

Note that $S^Y_{[\pi_t]}$ deformation retracts on its subspace $S^\circ_{[\pi_t]} := \mathbf{C} \times Y(\mathbf{C})$.
As subspaces of $(\mathcal{R}_2 Y)(\mathbf{C})$, $S_{[\pi_g]}$ and $S_{[\pi_t]}$ do not intersect. The stratum $S^Y_{[\pi_t]}$ is a submanifold of $(\mathcal{R}_2 Y)(\mathbf{C})$, so we can thicken up without changing its homotopy type by replacing it by an (open) tubular neighbourhood, say $T^Y_{[\pi_t]}$. (As the other stratum $S_{[\pi_g]}$ is open, it should be thought of as its own tubular neighbourhood.) The open subspaces $S^Y_{[\pi_g]}$ and $T^Y_{[\pi_t]}$ from an open cover of $(\mathcal{R}_2 Y)(\mathbf{C})$. In particular, the canonical map

$$\operatorname{hocolim} \left(\begin{array}{ccc} S^Y_{[\pi_g]} & & T^Y_{[\pi_t]} \\ & \nwarrow \quad \nearrow & \\ & S^Y_{[\pi_g]} \cap T^Y_{[\pi_t]} & \end{array} \right) \xrightarrow{\approx} (\mathcal{R}_2 Y)(\mathbf{C})$$

is a homotopy equivalence (*cf* [30, proposition 4.1] for example). Above, the use of a homotopy colimit rather than an ordinary colimit is motivated by the following lemma, which we will apply to make explicit the left hand side diagram.

Lemma 3.2.1 *Let $\mathcal{A}$ be a small category, let also F and G be two functors $\mathcal{A} \longrightarrow \mathcal{T}op$ and let $\varphi\colon F \longrightarrow G$ be a natural homotopy homotopy equivalence (that is to say a natural transformation such that, for each object $a \in \mathcal{A}$, the induced map $\varphi_a\colon F(a) \xrightarrow{\approx} G(a)$ is a homotopy equivalence). Then, the canonical induced map*

$$\varphi_*\colon \operatorname*{hocolim}_{\mathcal{A}} F \xrightarrow{\approx} \operatorname*{hocolim}_{\mathcal{A}} G$$

is a homotopy equivalence.

Reduction to the case $Y = \mathrm{pt}$ The space $S^Y_{[\pi_g]} \cap T^Y_{[\pi_t]}$ is a punctured (*ie* with its zero section removed) tubular neighbourhood of $S^Y_{[\pi_t]}$; we denote it $T^Y_{[\pi_t],[\pi_g]}$. The base $S^Y_{[\pi_t]}$ of this tube deformation retracts onto its subspace $S^\circ_{[\pi_t]} \simeq \mathbf{C} \times \mathcal{Y}$. By the following *restriction lemma*, this retraction induces a deformation retraction of the tube $T^Y_{[\pi_t]}$ onto its restriction to $S^\circ_{[\pi_t]}$, which we denote $T^\circ_{[\pi_t]}$. Moreover, restricting this retraction to the punctured tube gives a deformation retraction of $T^Y_{[\pi_t],[\pi_g]}$ onto its restriction $T^\circ_{[\pi_t],[\pi_g]}$ over $S^\circ_{[\pi_t]}$.

Lemma 3.2.2 (Restriction lemma) *Let M be a manifold, $S \subset M$ be a submanifold and T_S be a tubular neighbourhood of S. Assume moreover that S deformation retracts onto a submanifold S°. Then, up to shrinking, the tube T_S deformation retracts onto its restriction T°_S to S°. Moreover, when restricted to the punctured tube T^{pct}_S, this retraction gives a deformation retraction of T^{pct}_S onto its restriction $(T^{pct}_S)^\circ$ over S°.*

This lemma is proved in a more general context in A (see lemma A.2.7).

On the other hand, let $\Sigma^Y_{[\pi_t]}$ be the submanifold of $(\mathcal{R}_2 Y)(\mathbf{C})$ defined by

$$\Sigma^Y_{[\pi_t]} := \left\{(A,B) \in (\mathcal{R}_2 Y)(\mathbf{C}),\ B \in Y(\mathbf{C}) \subset Y\left(\mathbf{C}[X]_{/(A)}\right)\right\}$$

(see remark 3.1.11). There is a homeomorphism $\Sigma^Y_{[\pi_t]} \simeq \mathrm{Pol}_2(\mathbf{C}) \times \mathcal{Y}$. The submanifold $\Sigma^Y_{[\pi_t]}$ is transverse to $S^Y_{[\pi_t]}$ along their intersection $S^\circ_{[\pi_t]}$.

Remark 3.2.3 Note that the datum of a tubular neighbourhood $T^{\mathrm{pt}}_{[\pi_t]}$ of $S^{\mathrm{pt}}_{[\pi_t]}$ inside $\mathrm{Pol}_2(\mathbf{C})$ provides a tubular neighbourhood $T^\Sigma_{[\pi_t]}$ of $S^\circ_{[\pi_t]}$ inside $\Sigma^Y_{[\pi_t]}$ such that we have a homeomorphism

$$T^\Sigma_{[\pi_t]} \simeq T^{\mathrm{pt}}_{[\pi_t]} \times \mathcal{Y}$$

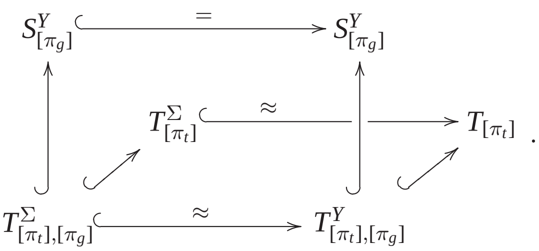
compatible to the above homeomorphism.

Let $T^{\mathrm{pt}}_{[\pi_t]}$ be *any* chosen tubular neighbourhood of $S^{\mathrm{pt}}_{[\pi_t]}$ inside $\mathrm{Pol}_2(\mathbf{C})$ and let $T^{\Sigma}_{[\pi_t]}$ be the corresponding tubular neighbourhood of $S^{\circ}_{[\pi_t]}$ inside $\Sigma^{Y}_{[\pi_t]}$ as above. Up to shrinking $T^{\mathrm{pt}}_{[\pi_t]}$, we can assume that $T^{\Sigma}_{[\pi_t]}$ lies inside $T^{Y}_{[\pi_t]}$. By the following result of uniqueness of tubular slices, there exists an isotopy inside $T^{Y}_{[\pi_t]}$ between (shrinkings of) $T^{\Sigma}_{[\pi_t]}$ and $T^{\circ}_{[\pi_t],[\pi_g]}$.

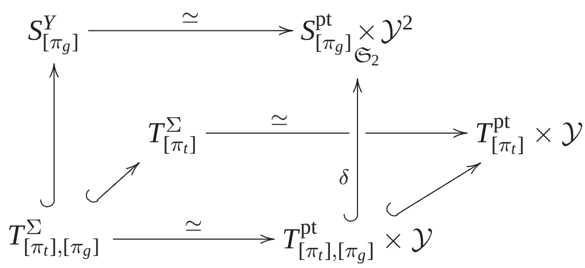
Lemma 3.2.4 (Uniqueness of tubular slices, *cf* theorem A.2.12) *Let M be a manifold, with two submanifolds S and Σ intersecting transversely along $S^{\circ} := S \cap \Sigma \neq \emptyset$. Let also T be a tubular neighbourhood of S inside M, T° be the restriction of T over S° and T^{Σ} be a tubular neighbourhood of S° inside Σ such that $T^{\Sigma} \subset T$. Then, there exists an isotopy inside T between (shrinkings of) T^{Σ} and T°.*

The previous discussion can be summarized in the following proposition.

Proposition 3.2.5 *In the following commutative diagram,*



the horizontal inclusion maps are homotopy equivalences. Moreover, by construction, we have a homeomorphism of diagrams



(Above, the map labelled δ is induced by a diagonal map $\mathcal{Y} \longrightarrow \mathcal{Y}^2$.) In particular, the analysis reduces to the study of the special case $Y = \mathrm{pt}$.

The case $Y = \mathrm{pt}$ In the previous step, we have made no assumption on the tubular neighbourhood $T_{[\pi_t]}$, so we need to describe the diagram of intersections for one particular choice of tubes (it follows from the uniqueness of tubular neighbourhoods that the description is in fact essentially independent of that choice). Our choice will be that given by the normal tube (for the euclidean metric). However, to prepare the generalization to the general case, we are going to describe the diagram of intersections associated to a $\mathfrak{S}_2$–equivariant tube in $\mathbf{C}^2$, quotienting by the action of $\mathfrak{S}_2$ only at the very end.

Notations:

- We recall from § 3.1.1 that $\mathbf{C}^{(2)}$ denotes the space of configuration of two distinct points in $\mathbf{C}$.
- Let $\mathrm{D} := \mathrm{D}(0,1) \subset \mathbf{C}$ be the unit disc.
- Let also $\widetilde{isq}^2$ be the space of pairs $(x_1, x_2) \in \mathrm{D}^2$ such that $x_1 + x_2 = 0$, endowed with its natural action of $\mathfrak{S}_2$. It is the fiber of normal tube around the diagonal $\delta \subset \mathbf{C}^2$.
- Let finally $\widetilde{\mathrm{D}}^{(2)}$ be the subspace of $\widetilde{\mathrm{D}}^2$ composed of pairs $(x_1, x_2) \in \mathrm{D}^2$ such that $x_1 + x_2 = 0$ and $x_1 \neq x_2$. It is the fiber of *punctured* normal tube around the diagonal δ.

In the case $n = 2$, $S^{\mathrm{pt}}_{\overline{\pi_g}}$ is the configuration space $\mathbf{C}^{(2)} \subset \mathbf{C}^2$ and $S^{\mathrm{pt}}_{\overline{\pi_t}}$ is the diagonal $\mathbf{C} \overset{\delta}{\hookrightarrow} \mathbf{C}^2$.
We take $\widetilde{T}_{\overline{\pi_t}}$ to be the normal $\mathfrak{S}_2$–equivariant tubular neighbourhood, that is to say:

$$\begin{array}{rccc} \widetilde{T}_{\overline{\pi_t}}: & \mathbf{C} \times \widetilde{\mathrm{D}}^2 & \xrightarrow{\widetilde{e}_{\overline{\pi_t}}} & \mathbf{C}^2 \\ & (x, (y_1, y_2)) & \mapsto & (x + y_1, x + y_2) \end{array}$$

The diagram of intersections is thus $\mathfrak{S}_2$–homeomorphic to:

$$\mathbf{C}^{(2)} \xleftarrow{\widetilde{e}_{\overline{\pi_t}}} \mathbf{C} \times \widetilde{\mathrm{D}}^{(2)} \hookrightarrow \mathbf{C} \times \widetilde{\mathrm{D}}^2 .$$

The analysis is essentially complete. However, for aesthetical reasons, we prefer to get rid of the contractible factor $\widetilde{D}^2$ and to make appear the little 2-disks operad $\mathcal{C}_2$ (for simplicity, we will often omit the subscript). Indeed, note how the inclusion map $\mathbf{C} \times \widetilde{D}^{(2)} \hookrightarrow \mathbf{C}^{(2)}$ is similar to the structure map in $\mathcal{C}_2$ associated to the surjection $\underline{2} \twoheadrightarrow \underline{1}$. There is a zig-zag of $\mathfrak{S}_2$–equivariant homotopy equivalences of diagrams to:

$$\mathcal{C}(\underline{2}) \xleftarrow{\mu} \mathcal{C}(\underline{2}) \times \mathcal{C}(\underline{1}) \xrightarrow{\mathrm{pr}} \mathcal{C}(\underline{1}) ,$$

(above, μ denotes the structure map in $\mathcal{C}$ and pr the canonical projection). The analysis finishes by quotienting this diagram by the action of $\mathfrak{S}_2$.

Conclusion The previous discussion summarizes as follows.

Definition 3.2.6 For every topological space Z, set $\psi_0^2(Z) := \mathcal{C}(2) \underset{\mathfrak{S}_2}{\times} Z^2 \coprod \mathcal{C}(1) \times Z$, $\psi_1^2(Z) := (\mathcal{C}(2)_{/\mathfrak{S}_2} \times \mathcal{C}(1)) \times Z$ and $\psi_i^2 = \emptyset$ for $i \geqslant 2$. Then

$$\Psi^2(Z) := |\psi_\bullet^2| := \operatorname{hocolim} \left(\begin{array}{ccc} \mathcal{C}(2) \underset{\mathfrak{S}_2}{\times} Z^2 & & \mathcal{C}(1) \times Z \\ & \nwarrow^{\mu \times \Delta} \quad \nearrow^{\mathrm{pr} \times \mathrm{id}} & \\ & \big(\mathcal{C}(2)_{/\mathfrak{S}_2} \times \mathcal{C}(1)\big) \times Z & \end{array} \right) \tag{3–2}$$

This defines a functor $\Psi^2 \colon \mathcal{T}op \longrightarrow \mathcal{T}op$.

Theorem 3.2.7 *For every tubular neighbourhood* $T^Y_{[\pi_t]}$ *of* $S^Y_{[\pi_t]}$ *inside* $(\mathcal{R}_2 Y)(\mathbf{C})$*, let* $\tau_\bullet$ *be the Cech* Δ*–space*

$$\tau_\bullet := \left(S^Y_{[\pi_g]} \coprod T^Y_{[\pi_t]} \right) \leftleftarrows T^Y_{[\pi_t],[\pi_g]}$$

(where $T^Y_{[\pi_t],[\pi_g]} := S^Y_{[\pi_g]} \cap T^Y_{[\pi_t]}$*). After possibly shrinking* $T^Y_{[\pi_t]}$*, there exists a zig-zag of homotopy equivalences of* Δ*–spaces between* $\psi_\bullet^2$ *and* $\tau_\bullet$ *which is natural in* Y*. In particular, one has a natural homotopy equivalence*

$$\Psi^2(\mathcal{Y}) \approx (\mathcal{R}_2 Y)(\mathbf{C}) .$$

3.2.2 Analysis of the stratification of $(\mathcal{R}_3 Y)(\mathbf{C})$

The case $n = 3$ is more intricate than the previous one. To fully describe the stratification $\mathcal{S}^Y$ we will use as a crucial ingredient the Thom–Mather theory of control of tubular neighbourhoods. All the technical difficulty is already present here.

The integer $n = 3$ admits the following three partitions:

$$[\pi_g] \succ [\pi_1] \succ [\pi_t] ,$$

where π_1 denotes the surjection $\underline{3} \twoheadrightarrow \underline{2}$ defined in example 3.1.6–(3). Thus the stratification $\mathcal{S}^Y$ of $(\mathcal{R}_3 Y)(\mathbf{C})$ is composed of three strata: the open stratum $S^Y_{[\pi_g]}$ homeomorphic to $\mathbf{C}^{(3)} \underset{\mathfrak{S}_3}{\times} \mathcal{Y}^3$, a submanifold of (complex) codimension one $S^Y_{[\pi_1]}$ homeomorphic to $\mathbf{C}^{(2)} \times Y\big(\mathbf{C}[X]/(X^2)\big) \times \mathcal{Y}$, and a submanifold of (complex) codimension two $S^Y_{[\pi_t]}$ homeomorphic to $\mathbf{C} \times Y\big(\mathbf{C}[X]/(X^3)\big)$. Note that $S^Y_{[\pi_1]}$ deformation retracts onto its subspace $S^\circ_{[\pi_1]} \simeq \mathbf{C}^{(2)} \times \mathcal{Y}^2$ and that $S^Y_{[\pi_t]}$ deformation retracts onto its subspace $S^\circ_{[\pi_t]} \simeq \mathbf{C} \times \mathcal{Y}$.

Let $\mathcal{T}^Y = \{T^Y_{[\pi_g]}, T^Y_{[\pi_1]}, T^Y_{[\pi_t]}\}$ be a system of tubular neighbourhoods of $\mathcal{S}^Y$, that is to say a family of tubular neighbourhoods for each stratum. This set of tubes forms an open cover of $(\mathcal{R}_3Y)(\mathbf{C})$ and we are going to give an explicit description of the associated *Cech* Δ*–space*:

$$\tau^Y_\bullet = \coprod_{i\in\{g,1,t\}} T^Y_{[\pi_i]} \leftleftarrows \coprod_{i\neq j\in\{g,1,t\}} T^Y_{[\pi_i],[\pi_j]} \leftleftarrows\!\!\!\leftarrow T^Y_{[\pi_g],[\pi_1],[\pi_t]} \ .$$

(Above, multiple subscripts denote multiple intersections; for example, $\widetilde{T}^Y_{[\pi_g],[\pi_t]}$ is a notation for the space $\widetilde{T}^Y_{[\pi_g]} \cap \widetilde{T}^Y_{[\pi_t]}$.) In order to do so, we need to restrict to families of tubes which are compatible one with the others. The precise notion is that of *controlled system of tubular neighbourhoods* developed by J N Mather in [25] (see also § A.2 of the appendix for a brief introduction to this notion). Among other properties, such a system of tubes satisfies the following regularity conditions. For every stratum S:

- The projection from the tube T_S to its base S is a locally trivial fibration *as stratified spaces*. By this, we mean that for every point $s \in S$, there exists a neighbourhood U of s inside M, a neighbourhood V of s inside S and a stratified space F with a homeomorphism of stratified spaces
$$U \simeq V \times \Gamma F \ .$$
(The notation Γ denotes the cone endofunctor of the category of stratified spaces.) See Thom's first isotopy lemma (*cf* theorem A.2.6) and proposition A.1.8.
- For every adjacent stratum $R \succ S$, the intersection $T_S \cap T_R$ is the restriction of the tube T_R over the open subspace $R \cap T_S$ of R.

*From now on, we assume that $\mathcal{T}^Y$ is a controlled system of tubular neighbourhoods of $\mathcal{S}^Y$. The existence of such a system is a consequence of $\mathcal{S}^Y$ being Whitney regular (*cf *proposition 3.1.8), see theorem A.2.5.*

In fact, it is more convenient to carry out a $\mathfrak{S}_3$*–equivariant* analysis of the induced stratification $\mathcal{E}^Y$ on the space $(\widetilde{\mathcal{R}_3Y})(\mathbf{C}) := (\mathcal{R}_3Y)(\mathbf{C}) \underset{\mathrm{Pol}_3(\mathbf{C})}{\times} \mathbf{C}^3$ and to quotient it at the end by the action of $\mathfrak{S}_3$. Indeed, giving a controlled system of tubular neighbourhoods $\mathfrak{T}^Y$ of $\mathcal{S}^Y$ is equivalent to giving a controlled system of $\mathfrak{S}_3$–equivariant tubular neighbourhoods $\widetilde{\mathfrak{T}}^Y$ of $\mathcal{E}^Y$. The Cech Δ–space associated to $\mathfrak{T}^Y$ is obtained from that associated to $\widetilde{\mathfrak{T}}^Y$ by quotienting by the action of $\mathfrak{S}_3$ (see proposition 3.5.1 below for a precise statement).

The $\mathfrak{S}_3$–stratification $\mathcal{E}^Y$ is also composed of three strata: $E^Y_{[\pi_t]}$ which is homeomorphic to $\mathbf{C}^{(3)} \times \mathcal{Y}^3$; $E^Y_{[\pi_1]}$ is the disjoint union of $E^Y_{\overline{\pi_1}} \simeq \mathbf{C}^{(2)} \times Y\Big(\mathbf{C}[X]_{/(X^2)}\Big) \times \mathcal{Y}$ and of its translations through the transpositions $(1,3)$ and $(2,3)$ of $\mathfrak{S}_3$; and $E^Y_{[\pi_t]}$ which is homeomorphic to $\mathbf{C} \times Y\Big(\mathbf{C}[X]_{/(X^3)}\Big)$. The isotropy group $\mathfrak{S}_{\overline{\pi_1}}$ of $E^Y_{\overline{\pi_1}}$ is the order 2 subgroup generated by the transposition $(1,2)$, acting trivially. Note that $E^Y_{\overline{\pi_1}}$ deformation retracts $\mathfrak{S}_{\overline{\pi_1}}$–equivariantly onto its subspace $E^\circ_{\overline{\pi_1}} \simeq \mathbf{C}^{(2)} \times \mathcal{Y}^2$. The isotropy group $\mathfrak{S}_{\overline{\pi_t}}$ of $E^Y_{\overline{\pi_t}}$ is $\mathfrak{S}_3$ and $E^Y_{\overline{\pi_t}}$ deformation retracts $\mathfrak{S}_{\overline{\pi_t}}$–equivariantly onto its subspace $E^\circ_{\overline{\pi_t}} \simeq \mathbf{C} \times \mathcal{Y}$.
We now come to the description of the $\mathfrak{S}_3$–equivariant Cech Δ–space associated to $\widetilde{T}^Y$:

$$\widetilde{\tau}^Y_\bullet := \coprod_{i\in\{g,1,t\}} \widetilde{T}^Y_{[\pi_i]} \leftleftarrows \coprod_{i\neq j\in\{g,1,t\}} \widetilde{T}^Y_{[\pi_i],[\pi_j]} \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; \widetilde{T}^Y_{[\pi_g],[\pi_1],[\pi_t]} \,.$$

As before, our starting point is that the canonical map from the geometric realization $|\widetilde{\tau}^Y_\bullet|$ (*ie* the homotopy colimit of the diagram of spaces) to $(\widetilde{\mathcal{R}_3 Y})(\mathbf{C})$ is a $\mathfrak{S}_3$–equivariant homotopy equivalence. The rest of the paragraph consists in making explicit the homotopy colimit by using the equivariant analogue of lemma 3.2.1.

Reduction to the case $Y = \mathrm{pt}$ In § 3.2.1, the two main tools used were the restriction lemma (lemma 3.2.2) and the uniqueness of tubular slices (lemma 3.2.4). Here are the appropriate analogues of these results in an equivariant and stratified context.

Let G be a finite group and let $(M, \mathcal{S})$ be a Whitney-regular G–stratified manifold.

Lemma 3.2.8 (Restriction lemma) *Let S be a G–stratum of M and let T_S be a G–tubular neighbourhood of S inside M. Assume that S G–equivariantly deformation retracts onto some sub-G–space S°. Then, this retraction induces a deformation retraction, which is G–equivariant and which respects the stratification, of (some shrinking) of the tube T_S onto its restriction over S°, say T°_S.*
In particular, for every adjacent G–stratum $R \succeq S$, the space $T_S \cap R$ deformation retracts G–equivariantly onto $T^\circ_S \cap R$.

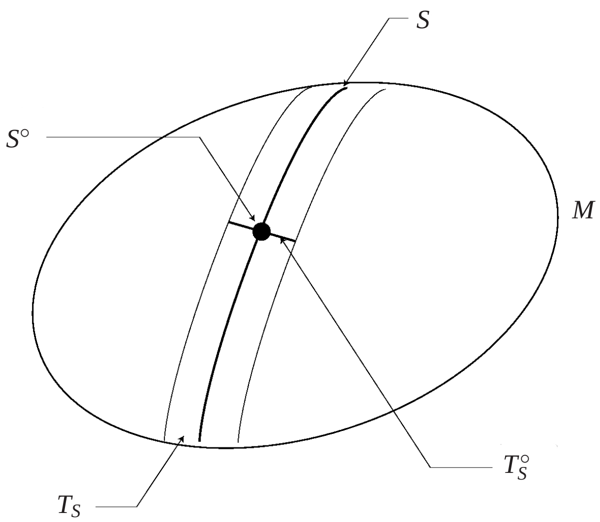


Figure 1: Restriction lemma

The restriction lemma is proved in lemma A.2.7 of the appendix.

Lemma 3.2.9 (Uniqueness of tubular slices) *Let S be a G–stratum of M and let Σ be a sub-G–manifold of M which is transverse to $\mathcal{S}$. We denote S° the intersection $S \cap \Sigma$. Let also T_S be a G–tubular neighbourhood of S inside M, T_S° be the restriction of T_S over S° and $T^\Sigma \subset T_S$ be a G–tubular neighbourhood of S° inside Σ. Then, the spaces T_S° and T^Σ inherit of Whitney stratifications and, after possibly shrinking them, there exists a G–equivariant stratified isotopy inside T_S between T_S° and T^Σ over S°.*

This lemma is proved in proposition A.2.13 of the appendix.

Remark 3.2.10 In the sequel, each time we apply the restriction lemma or the uniqueness of tubular slices, we will omit the possible shrinkings of tubes. It is important to note that there are only finitely many such shrinking and that a new shrinking at a given time does not affect what has been proved before.

The last ingredient in the analysis of the case $n = 2$ was the existence of the manifold $\Sigma^Y_{[\pi_t]}$. The higher dimensional generalisation of these manifolds were already introduced in remark 3.1.11: for every surjection $\pi\colon \underline{n} \longrightarrow \underline{m}$, we defined a submanifold $\widetilde{\Sigma}^Y_{[\pi]}$ of $(\widetilde{\mathcal{R}_n Y})(\mathbf{C})$. Here, using the notations of the aforementioned remark,

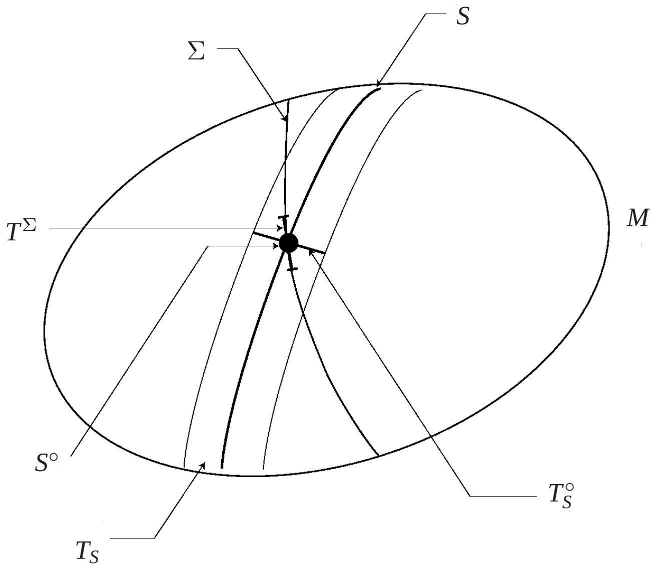


Figure 2: Uniqueness of tubular slices.

- We have a $\mathfrak{S}_{\overline{\pi_1}}$–equivariant homeomorphism $\widetilde{\Sigma}^Y_{\overline{\pi_1}} \simeq \widetilde{\Sigma}^{\mathrm{pt}}_{\overline{\pi_1}} \times \mathcal{Y}^2$, with $\widetilde{\Sigma}^{\mathrm{pt}}_{\overline{\pi_1}}$ the open subspace of $\mathbf{C}^3$ composed of those triples (x_1, x_2, x_3) satisfying $x_1 \neq x_3$ and $x_2 \neq x_3$ (the action of $\mathfrak{S}_{\overline{\pi_1}} \simeq \mathfrak{S}_2$ exchanges x_1 and x_2).
- We have a $\mathfrak{S}_{\overline{\pi_t}} \simeq \mathfrak{S}_3$–equivariant homeomorphism $\widetilde{\Sigma}^Y_{[\pi_t]} = \widetilde{\Sigma}^Y_{\overline{\pi_t}} \simeq \mathbf{C}^3 \times \mathcal{Y}$.

Remark 3.2.11 Let $\pi\colon \underline{3} \twoheadrightarrow \underline{m}$ be surjection. Note that the datum of a $\mathfrak{S}_3$–equivariant tubular neighbourhood of $E^{\mathrm{pt}}_{[\pi]}$ inside $\mathbf{C}^3$, say $\widetilde{T}^{\mathrm{pt}}_{[\pi]}$, provides a $\mathfrak{S}_3$–equivariant tubular neighbourhood of $E^\circ_{[\pi]}$ inside $\widetilde{\Sigma}^Y_{[\pi]}$, say $\widetilde{T}^{\widetilde{\Sigma}}_{[\pi]}$. By definition, one has a $\mathfrak{S}_3$–equivariant homeomorphism $\widetilde{T}^{\widetilde{\Sigma}}_{[\pi]} \simeq \widetilde{T}^{\mathrm{pt}}_{[\pi]} \times \mathcal{Y}^m$ compatible to that of example 3.1.12(4).

Let $\widetilde{\mathfrak{T}}^{\mathrm{pt}}$ be a controlled system of $\mathfrak{S}_3$–equivariant tubular neighbourhoods of the stratification $\mathcal{E}^{\mathrm{pt}}$ of $\mathbf{C}^3$. For every partition $[\pi]$ of the integer 3, we denote by $\widetilde{T}^{\widetilde{\Sigma}}_{[\pi]}$ the tubular neighbourhood of $E^\circ_{[\pi]}$ inside $\widetilde{\Sigma}^Y_{[\pi]}$ associated to the tube $\widetilde{T}^{\mathrm{pt}}_{[\pi]}$ as in the previous remark. Up to shrinking the tubes, we can assume having the inclusion $\widetilde{T}^{\widetilde{\Sigma}}_{[\pi]} \subset \widetilde{T}^Y_{[\pi]}$. By the uniqueness of tubular slices , we have a $\mathfrak{S}_3$–equivariant stratified isotopy inside $\widetilde{T}^Y_{[\pi]}$ between $\widetilde{T}^\circ_{[\pi]}$ and $\widetilde{T}^{\widetilde{\Sigma}}_{[\pi]}$.

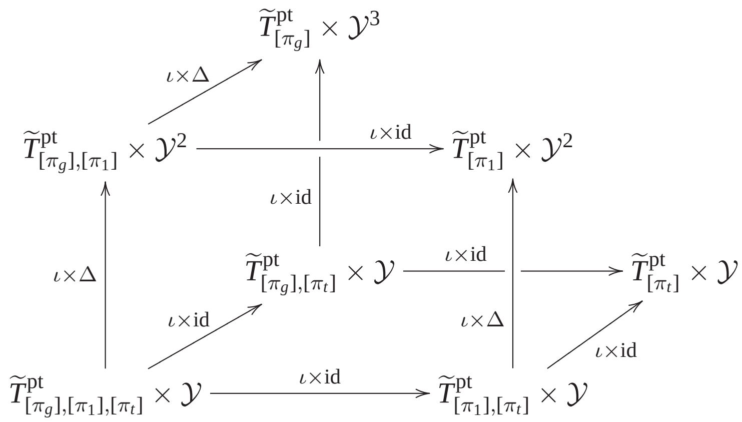
The following proposition reduces the study to the particular case $Y = \mathrm{pt}$.

Proposition 3.2.12 *Let* $\tau_\bullet^{\widetilde{\Sigma}}$ *be the* Cech Δ*–space associated to the open cover of* $(\mathcal{R}_3Y)(\mathbf{C})$*:*

$$\widetilde{\tau}_\bullet^{\widetilde{\Sigma}} := \coprod_{i\in\{g,1,t\}} \widetilde{T}^{\widetilde{\Sigma}}_{[\pi_i]} \leftleftarrows \coprod_{i\neq j\in\{g,1,t\}} \widetilde{T}^{\widetilde{\Sigma}}_{[\pi_i],[\pi_j]} \Lleftarrow \widetilde{T}^{\widetilde{\Sigma}}_{[\pi_g],[\pi_1],[\pi_t]} .$$

The canonical inclusion $\widetilde{\tau}_\bullet^{\widetilde{\Sigma}} \hookrightarrow \widetilde{\tau}_\bullet^{Y}$ *is a* $\mathfrak{S}_3$*–equivariant homotopy equivalence of* Δ*–spaces. By construction, we have a* $\mathfrak{S}_3$*–equivariant homeomorphism of* Δ*–spaces:*

$\tau_\bullet^{\widetilde{\Sigma}} \simeq$



where the maps labelled ι *are inclusions and those labelled* Δ *are diagonals on* $\mathcal{Y}$*. In particular, the homotopy type of* $(\mathcal{R}_3Y)(\mathbf{C})$ *depends only on* $\mathcal{Y}$*.*

Proof The proof is based on the use of the restriction lemma (lemma 3.2.2) and of the uniqueness of tubular slices (lemma 3.2.9). For example, we are going to show that the canonical inclusion $\widetilde{T}^{\widetilde{\Sigma}}_{[\pi_g],[\pi_1],[\pi_t]} \hookrightarrow \widetilde{T}^{Y}_{[\pi_g],[\pi_1],[\pi_t]}$ is a homotopy equivalence, leaving the other cases to the reader.

Since the system of tubular neighbourhoods $\widetilde{\mathfrak{T}}^Y$ is controlled, the space $\widetilde{T}^Y_{[\pi_1],[\pi_t]}$ is the restriction of $\widetilde{T}_{[\pi_1]}$ to $E^Y_{[\pi_1]} \cap \widetilde{T}^Y_{[\pi_t]}$. As $E^Y_{[\pi_t]}$ deformation retracts $\mathfrak{S}_3$–equivariantly onto $E^\circ_{[\pi_t]}$, the restriction lemma shows that $E^Y_{[\pi_1]} \cap \widetilde{T}^Y_{[\pi_t]}$ deformation retracts $\mathfrak{S}_3$–equivariantly onto $E^Y_{[\pi_1]} \cap \widetilde{T}^\circ_{[\pi_t]}$. By the uniqueness of tubular slices, we have $\mathfrak{S}_3$–equivariant isotopy between $E^Y_{[\pi_1]} \cap \widetilde{T}^\circ_{[\pi_t]}$ and $E^Y_{[\pi_1]} \cap \widetilde{T}^{\widetilde{\Sigma}}_{[\pi_t]}$. As we have the inclusion $E^Y_{[\pi_1]} \cap \widetilde{T}^{\widetilde{\Sigma}}_{[\pi_t]} \subset E^\circ_{[\pi_1]}$, by the uniqueness of tubular slices, we have a $\mathfrak{S}_3$–equivariant isotopy between $E^Y_{[\pi_g]} \cap \left(\widetilde{T}^Y_{[\pi_1]}\right)_{|E^Y_{[\pi_1]}\cap \widetilde{T}^{\widetilde{\Sigma}}_{[\pi_t]}}$ and $E^Y_{[\pi_g]} \cap \left(\widetilde{T}^{\widetilde{\Sigma}}_{[\pi_1]}\right)_{|E^Y_{[\pi_1]}\cap \widetilde{T}^{\widetilde{\Sigma}}_{[\pi_t]}} = \widetilde{T}^{\widetilde{\Sigma}}_{[\pi_g],[\pi_1],[\pi_t]}$. As a consequence the canonical inclusion $\widetilde{T}^{\widetilde{\Sigma}}_{[\pi_g],[\pi_1],[\pi_t]} \hookrightarrow \widetilde{T}^{Y}_{[\pi_g],[\pi_1],[\pi_t]}$ is a $\mathfrak{S}_3$–equivariant homotopy equivalence. □

Le cas $Y = \mathrm{pt}$ As for the analysis of the case $n = 2$, we chose to describe the Cech Δ–space associated to the particular system of $\mathfrak{S}_3$–equivariant tubes which are normal to the strata.

We recall briefly that: $E^{\mathrm{pt}}_{[\pi_t]} = \mathbf{C}^{(3)}$; $E^{\mathrm{pt}}_{[\pi_1]}$ is the disjoint union of $E^{\mathrm{pt}}_{\overline{\pi_1}} \simeq \mathbf{C}^{(\pi_1)} := \{(x_1, x_2, x_3) \in \mathbf{C}^3, \quad x_1 = x_2 \neq x_3\}$ and of its translates through the action of the transpositions $(1,3)$ and $(2,3)$; $E^{\mathrm{pt}}_{[\pi_t]}$ is the diagonal $\mathbf{C}^{(\pi_t)} := \mathbf{C} \subset \mathbf{C}^3$.

Notations: (1) $\mathrm{D} = \mathrm{D}(0,1)$ still denotes the unit disc in $\mathbf{C}$.

(2) Let $\widetilde{\mathrm{D}}^3 \subset \mathrm{D}^3$ be the space of triples (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 = 0$. It is "the" fibre of the tube normal to the diagonal $\mathbf{C} \subset \mathbf{C}^3$.

(3) Let $\widetilde{\mathrm{D}}^{(3)} := \widetilde{\mathrm{D}}^3 \cap \mathbf{C}^{(3)}$. It is the "fibre" of the punctured tube normal to the diagonal.

(4) Set $\widetilde{\mathrm{D}}^1 = \widetilde{\mathrm{D}}^{(1)} := \{0\} \subset \mathrm{D}$.

(5) Let $\widetilde{\mathrm{D}}^{(\pi_1)}$ be the subspace of $\widetilde{D}^3$:

$$\widetilde{\mathrm{D}}^{(\pi_1)} := \{(u_1, u_2, u_3) \in \mathrm{D}^3, \quad u_1 = u_2 \neq u_3 \text{ and } u_1 + u_2 + u_3 = 0\}$$

It is the trace in $\mathbf{C}^{(\pi_1)}$ of "the" fiber of a normal tube around the diagonal.

Let $\widetilde{T}_{\overline{\pi_1}}$ and $\widetilde{T}_{\overline{\pi_t}}$ be the following equivariant tubular neighbourhoods:

$$\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_1}} : \qquad \mathbf{C}^{(\overline{\pi_1})} \times \widetilde{\mathrm{D}}^2 \xrightarrow{\widetilde{e}_{\overline{\pi_1}}} \mathbf{C}^3$$
$$(x, x, x_3), (u_1, u_2) \longmapsto \big(x + \varepsilon_x u_1,\ x + \varepsilon_x u_2,\ x_3\big)$$
$$\text{where } \varepsilon_x = \frac{1}{10}|x - x_3|$$

$$\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_t}} : \qquad \mathbf{C}^{(\pi_t)} \times \widetilde{\mathrm{D}}^3 \xrightarrow{\widetilde{e}_{\overline{\pi_t}}} \mathbf{C}^3$$
$$(x, x, x), (u_1, u_2, u_3) \longmapsto (x_1 + u_1,\ x_2 + u_2,\ x_3 + u_3)$$

The (image of the) tube $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_t}}$ is thus the configuration space of three particles (x_1, x_2, x_3) which are all at distance $\leqslant 1$ from their barycenter $\frac{1}{3}(x_1 + x_2 + x_3)$. The (image of the) tube $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_1}}$ is the configuration space of three particles (x_1, x_2, x_3) such that the distance from x_1 and x_2 to their barycenter $\frac{1}{2}(x_1 + x_2)$ is much smaller (at most $\frac{1}{10}$) than the distance from $\frac{1}{2}(x_1 + x_2)$ to x_3.

Remark 3.2.13 (1) Note that by construction, the tube $\widetilde{T}_{\overline{\pi_1}}$ does not meet its translates through the transpositions $(1,3)$ and $(2,3)$, as required.

(2) Note also that the system of tubes satisfies the following needed condition that if $\overline{\pi} \leqslant \overline{\pi}'$, then $\widetilde{T}_{\overline{\pi},\overline{\pi}'}$ is a tube over $\widetilde{T}_{\overline{\pi}} \cap E_{\overline{\pi}'}$.

(3) Note that all the tubes are trivial since the strata have trivial normal bundles (these are opens of sub-vector spaces of $\mathbf{C}^n$).

(4) From this observation, the topology of the intersection spaces is determined: it is a product of spaces of configuration of points. For example, the space $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}}$ is a punctured tube over $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_t}} \cap E^{\mathrm{pt}}_{\overline{\pi_1}}$. In his turn, $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_t}} \cap E^{\mathrm{pt}}_{\overline{\pi_1}}$ is a punctured tube of $E_{\overline{\pi_t}}$ inside the closure of $E_{\overline{\pi_1}}$. Thus, we have an equivariant homeomorphism $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}} \simeq \mathbf{C} \times [\widetilde{\mathrm{D}}^{(\pi_1)}] \times [\widetilde{\mathrm{D}}^{(2)} \times \widetilde{\mathrm{D}}^{(1)}]$.

Finally, let us give a name to the opens of $\mathbf{C}^n$ images of the following homeomorphisms:

$$\mathbf{C}^{(\pi_t)} \times \widetilde{\mathrm{D}}^{(3)} \xrightarrow{\simeq} \widetilde{U}_{\overline{\pi_g},\overline{\pi_t}}$$
$$(x,\, x,\, x),\, (u_1,\, u_2,\, u_3) \longmapsto (x+u_1,\, x+u_2,\, x+u_3)$$
$$\text{with } \varepsilon_x = 1, \varepsilon_u = \tfrac{1}{10}\min_{i\neq j}|u_i - u_j|$$

$$\mathbf{C}^{(\pi_1)} \times [\widetilde{\mathrm{D}}^{(2)} \times \widetilde{\mathrm{D}}^{(1)}] \xrightarrow{\simeq} \widetilde{U}_{\overline{\pi_g},\overline{\pi_1}}$$
$$(x,\, x,\, x_3),\, [(u_1,\, u_2), 0] \longmapsto (x+u_1,\, x+u_2,\, x_3+0)$$
$$\text{with } \varepsilon_x = 1, \varepsilon_u = \tfrac{1}{10}|u_1 - u_2|$$

$$\mathbf{C}^{(\pi_t)} \times [\widetilde{\mathrm{D}}^{(\pi_1)}] \times [\widetilde{\mathrm{D}}^{2} \times \widetilde{\mathrm{D}}^{1}] \xrightarrow{\simeq} \widetilde{U}_{\overline{\pi_1},\overline{\pi_t}}$$
$$(x,\, x,\, x),\, [(u,\, u,\, u_2)],\, [(v_1,\, v_2), 0] \longmapsto (x+u+\varepsilon_u v_1,\, x+u+\varepsilon_u v_2,\, x+u_2+\varepsilon_u \cdot 0)$$
$$\text{with } \varepsilon_x = 1, \varepsilon_u = \tfrac{1}{10}|u - u_2|, \varepsilon_v = \tfrac{1}{10}|v_1 - v_2|$$

$$\mathbf{C}^{(\pi_t)} \times [\widetilde{\mathrm{D}}^{(\pi_1)}] \times [\widetilde{\mathrm{D}}^{(2)} \times \widetilde{\mathrm{D}}^{(1)}] \xrightarrow{\simeq} \widetilde{U}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}}$$
$$(x,\, x,\, x),\, [(u,\, u,\, u_3)],\, [(v_1,\, v_2), 0] \longmapsto (x+u+\varepsilon_u v_1,\, x+u+\varepsilon_u v_2,\, x_3+u_3+\varepsilon_u \cdot 0)$$
$$\text{with } \varepsilon_x = 1, \varepsilon_u = \tfrac{1}{10}|u - u_2|, \varepsilon_v = \tfrac{1}{10}|v_1 - v_2|$$

These spaces are configuration spaces of three points in $\mathbf{C}$ which can be obtained from one or two points by successively "blowing up in the normal direction" each point into multiple points in a ball of radius $\frac{1}{10}$ the minimal distance to the other points. For example, a point $\widetilde{U}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}}$ is obtained from a "triple" point $x \in \mathbf{C}$, by first blowing it up into two distinct points, one double $x+u$ and one simple $x+u_3$ and then blowing up $x+u$ into $x+u+\varepsilon_u v_1$ and $x+u+\varepsilon_u v_2$.

Lemma 3.2.14 *The four inclusion maps* $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_1},\overline{\pi_t}} \hookrightarrow \widetilde{U}_{\overline{\pi_1},\overline{\pi_t}}$, $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_g},\overline{\pi_1}} \hookrightarrow \widetilde{U}_{\overline{\pi_g},\overline{\pi_1}}$, $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_1},\overline{\pi_t}} \hookrightarrow \widetilde{U}_{\overline{\pi_g},\overline{\pi_t}}$ *and* $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}} \hookrightarrow \widetilde{U}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}}$ *are homotopy equivalences.*

Proof This follows from the uniqueness of tubular neighbourhoods and from remark 3.2.13 above. For example, the inclusion $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}} \hookrightarrow \widetilde{U}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}}$ is an equivariant homotopy equivalence since both $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}}$ and $\widetilde{U}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}}$ are punctured tubes over $\widetilde{T}^{\mathrm{pt}}_{\overline{\pi_t}} \cap E^{\mathrm{pt}}_{\overline{\pi_1}}$. □

Remark 3.2.15 (1) There are composition "pseudo-maps":

$$\widetilde{\mathrm{D}}^{(\pi_1)} \times [\widetilde{\mathrm{D}}^{(2)} \times \widetilde{\mathrm{D}}^{(1)}] \overset{\mu}{\dashrightarrow} \widetilde{\mathrm{D}}^{(3)}$$

$$(u,u,u_2),\ [(v_1,v_2),0] \longmapsto (u+\varepsilon_u v_1, u+\varepsilon_u v_2, u_2+\varepsilon_u \cdot 0)$$

$$\text{with } \varepsilon_u = \tfrac{1}{10}|u-u_2|, \varepsilon_v = \tfrac{1}{10}|v_1-v_2|$$

$$\mathbf{C}^{(\pi_t)} \times \widetilde{\mathrm{D}}^{(\pi_1)} \xrightarrow{\ \mu\ } \mathbf{C}^{(\pi_1)}$$

$$(x,x,x),\ (u_1,u_2,u_3) \longmapsto (x+u_1, x+u_2, x+u_3)$$

$$\text{with } \varepsilon_x = 1, \varepsilon_u = \tfrac{1}{10}\min_{i\neq j}|u_i-u_j|$$

The dashed arrow is not well defined, for its first two coordinates do not necessarily remain inside of the unit disk D. All the other linear conditions are fully satisfied.

(2) Note how close the above maps are from their analogues in the little disks operad. More precisely, there are sections from the four spaces $\widetilde{U}$ (seen as products of configuration spaces of points) to the product of spaces of configurations of little disks lying above them, such that the above composition maps between configuration spaces *are compatible with the structure maps in the little disks operad*. This compatibility is important: it insures that below our diagrams commutes on the nose and not only up to homotopy.
These sections are defined by taking radii equal to the values $\varepsilon_x, \varepsilon_u$ or ε_v defined in the formula for the composition map. For example, we mean

$$\begin{array}{rcl} \widetilde{U}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}} \simeq \mathbf{C}^{(\pi_t)} \times [\widetilde{\mathrm{D}}^{(\pi_1)}] \times [\widetilde{\mathrm{D}}^{(2)} \times \widetilde{\mathrm{D}}^{(1)}] & \longrightarrow & \mathcal{C}(1) \times [\mathcal{C}(2)] \times [\mathcal{C}(2) \times \mathcal{C}(1)] \\ (x,x,x),(u_1,u_2),[(v_1,v_2),0] & \mapsto & (r=1, r=\varepsilon_u, [r=\varepsilon_v, r=\varepsilon_v]) \end{array}$$

The following lemma is the final ingredient to replace the spaces of configurations of points with spaces of configurations of little disks.

Lemma 3.2.16 *Let $\ell > 0$ be a positive integer and let $r_1,\dots,r_\ell$ be a list of positive integers. Any section*

$$\sigma\colon\ \mathrm{D}^{(r_1)} \times \cdots \times \mathrm{D}^{(r_\ell)} \longrightarrow \mathcal{C}(r_1) \times \cdots \times \mathcal{C}(r_\ell)$$

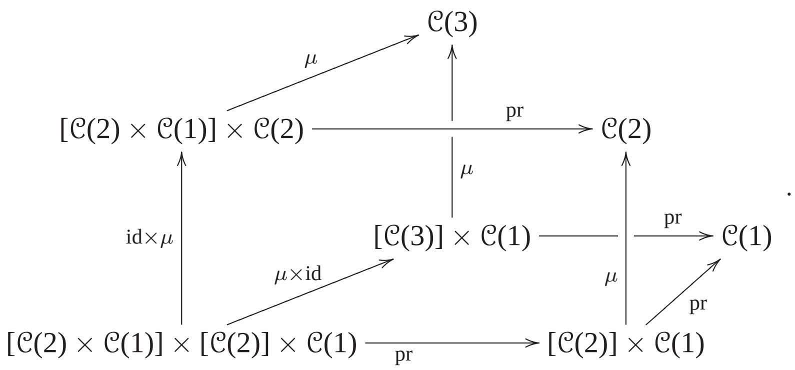
to the natural projection map is a $\mathfrak{S}_{r_1} \times \cdots \times \mathfrak{S}_{r_\ell}$–homotopy equivalence.

Proof The space of sections has the following partial order: we say that $\sigma \leqslant \sigma'$ if for every point x, the radii of all the disks in $\sigma(x)$ are smaller than those of $\sigma'(x)$. Note that two such sections $\sigma \leqslant \sigma'$ are $\mathfrak{S}_{r_1} \times \cdots \times \mathfrak{S}_{r_\ell}$ equivariantly homotopic.
It follows that any two sections σ and σ' are equivariantly homotopic, since they are both homotopic to $\min(\sigma, \sigma')$. We can thus assume that σ is a product map. This reduces the problem to the case $\ell = 1$. Then, the space of little disks deformation retracts onto the image of the section. □

The composition of the inclusions of the spaces $\widetilde{T}^{\mathrm{pt}}$ into the spaces $\widetilde{U}$ followed by the previous section gives an equivariant homotopy equivalence of Δ–spaces between

$$\coprod_{i\in\{g,1,t\}} \widetilde{T}^{\mathrm{pt}}_{\overline{\pi_i}} \leftleftarrows \coprod_{i\neq j\in\{g,1,t\}} \widetilde{T}^{\mathrm{pt}}_{\overline{\pi_i},\overline{\pi_j}} \Lleftarrow \widetilde{T}^{\mathrm{pt}}_{\overline{\pi_g},\overline{\pi_1},\overline{\pi_t}}$$

and



.

To conclude, we have to divide each term by the action of its isotropy subgroup.

Conclusion The previous discussion summarizes as follows.

Definition 3.2.17 For every topological space Z, define spaces $\psi^3_0(Z), \psi^3_1(Z)$ and $\psi^3_2(Z)$ as follows. Let

$$\psi^3_0(Z) := \mathcal{C}(3) \underset{\mathfrak{S}_3}{\times} Z^3 \coprod \mathcal{C}(2) \times Z^2 \coprod \mathcal{C}(1) \times Z$$

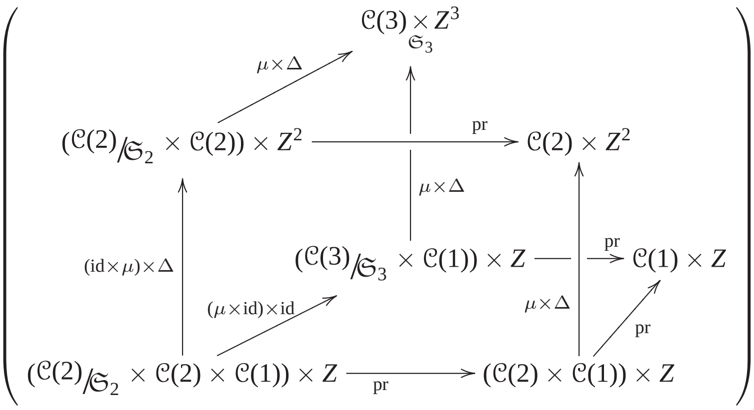
$$\psi^3_1(Z) := (\mathcal{C}(2)/_{\mathfrak{S}_2} \times \mathcal{C}(2)) \times Z^2 \coprod \mathcal{C}(3)/_{\mathfrak{S}_3} \times Z \coprod (\mathcal{C}(2) \times \mathcal{C}(1)) \times Z$$

and
$$\psi^3_2(Z) := (\mathcal{C}(2)/_{\mathfrak{S}_2} \times \mathcal{C}(2)) \times Z$$

The structure maps in the little disks operad (our general notation for these is a symbol μ) allow to turn these into a Δ–space $\psi^3_\bullet(Z) := \ \psi^3_0(Z) \leftleftarrows \psi^3_1(Z) \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} \psi^3_2(Z)$ (see the diagram below). Let $\Psi^3(Z)$ be the associated geometric realization:

(3–3)

$\Psi^3(Z) := |\psi^3_\bullet(Z)| :=$ hocolim



(Above, the maps labelled by a μ are induced by structure maps in the little disks operad and those labelled "pr" are projections.) The above construction defines a functor $\Psi^3\colon \mathcal{T}op \longrightarrow \mathcal{T}op$.

Theorem 3.2.18 *For every controlled system of tubular neighbourhoods of the stratification* $\mathcal{S}^Y$ *of* $(\mathcal{R}_3 Y)(\mathbf{C})$*, say* $\mathcal{T}^Y$*, let* $\tau_\bullet$ *be the associated Cech* Δ*–space*

$$\tau_\bullet := \coprod_{i\in\{g,1,t\}} T^Y_{[\pi_i]} \leftleftarrows \coprod_{i\neq j\in\{g,1,t\}} T^Y_{[\pi_i],[\pi_j]} \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} T^Y_{[\pi_g],[\pi_1],[\pi_t]} \ .$$

Then, up to shrinking the tubes of $\mathcal{T}^Y$*, there exists a zig-zag of homotopy equivalences of* Δ*–spaces between* $\tau_\bullet$ *and* $\psi^3_\bullet(\mathcal{Y})$*, which is natural in* Y*. In particular, one has a natural homotopy equivalence* $\Psi^3(\mathcal{Y}) \approx (\mathcal{R}_3 Y)(\mathbf{C})$*.*

3.3 Formalism

To extend the analysis of $\mathcal{S}^Y$ for a general n, we need to introduce some formalism to encode the combinatorics of the multiple intersections between all the tubular neighbourhoods of the strata. Of course, we can organise the information on n levels *à la Cech*: the tubes, the twofold intersections, the threefold intersections, etc. We get what is known as a Δ–space: a simplicial space without degeneracies. However, the description of the terms appearing in the k–fold intersections requires a combinatorial model for the datum of nested partitions $[\pi_0] \succeq \cdots \succeq [\pi_k]$, which we call a *k–flag*.

3.3.1 Δ–spaces

Definitions 3.3.1 (1) Let Δ be the simplicial category without the degeneracies. The objects of Δ are thus the positive integers. For each pair of positive integers $i < j$, $\mathrm{Hom}_\Delta(j, i)$ is the set of order-preserving maps from $\underline{i}$ to $\underline{j}$.

(2) A Δ–space is a functor $X_\bullet : \Delta \longrightarrow \mathcal{T}op$. As usual, we write X_i instead of $X_\bullet(i)$.

(3) Let $X_\bullet$ be a Δ–space. The geometric realization of $X_\bullet$, denoted $|X_\bullet|$, is the space

$$|X_\bullet| := \operatorname*{hocolim}_{i \in \Delta} X_i \,.$$

(4) Let $X_\bullet$ and $Y_\bullet$ be two Δ–spaces. A morphism $f : X_\bullet \longrightarrow Y_\bullet$ is said to be a homeomorphism (resp. a homotopy equivalence) if it induces such a map between X_i and Y_i for all i.

Our typical example of Δ–space is the Cech Δ–space associated to a partially-ordered open cover of a given topological space. The following proposition is useful.

Proposition 3.3.2 *Let $\mathcal{U}$ be a partially-ordered open cover of a paracompact space X. Then the canonical map $|\mathcal{U}_\bullet| \longrightarrow X$ is a homotopy equivalence.*

3.3.2 Operads

It turns out that the description of the stratification $\mathcal{S}^{\mathrm{pt}}$ of the space $\mathrm{Pol}_n(\mathbf{C})$ of degree n complex polynomials leads to a "configuration space model" for the little disks operad. In order to set up notations, we briefly recall the definition of an operad, and in particular of the little disks operad.

Notations: (1) Let $f\mathcal{S}et$ denote the category whose objects are finite sets and whose morphisms are bijections.

(2) Let also $\mathrm{Funct}(f\mathcal{S}et, \mathcal{T}op)$ denote the category of functors from $f\mathcal{S}et$ to $\mathcal{T}op$. This category has a monoidal structure [13, § 1.1.3].

Definition 3.3.3 A (topological) operad $\mathcal{O}$ is a monoid in the category $\mathrm{Funct}(f\mathcal{S}et, \mathcal{T}op)$. More concretely — and this is the relevant way to think of an operad for our purposes — an operad is a functor from $\mathcal{S}et^f$ to $\mathcal{T}op$ such that:

- $\mathcal{O}(\emptyset) = \mathrm{pt}$

- For every surjection $p : E \twoheadrightarrow F$, there is a map
$$\mathcal{O}(F) \times \prod_{f\in F} \mathcal{O}(p^{-1}(f)) \xrightarrow{\mu_p} \mathcal{O}(E)$$
which is natural in p (that is to say such that for every commutative diagram $\begin{array}{ccc} E & \xrightarrow{p} & F \\ \wr| & & \wr| \\ E' & \xrightarrow{p'} & F' \end{array}$, one has the adequate compatibility relation)

- For every composition of surjections $E \overset{p}{\twoheadrightarrow} F \overset{q}{\twoheadrightarrow} G$ of finite sets, one has the following associativity condition:
$$\begin{array}{ccc} \mathcal{O}(G) \times \prod_{g\in G} \mathcal{O}\big((q\circ p)^{-1}(g)\big) & \xrightarrow{\mu_{q\circ p}} & \mathcal{O}(E) \\ \uparrow \prod_{g\in G} \mu_{(q\circ p)^{-1}(g)\twoheadrightarrow q^{-1}(g)} & & \uparrow \mu_p \\ \mathcal{O}(G) \times \prod_{g\in G} \left[\mathcal{O}(q^{-1}(g)) \times \prod_{h\in q^{-1}(g)} \mathcal{O}(p^{-1}(h))\right] & \xrightarrow{\mu_q} & \mathcal{O}(F) \times \prod_{f\in F} \mathcal{O}(p^{-1}(f)) \end{array}$$

Example 3.3.4 Let $\mathrm{D} := \mathrm{D}(0,1)$ be the (closed) unit disk of the complex plane $\mathbf{C}$. The little disks operad is the following functor $\mathcal{P} : f\mathcal{S}et \longrightarrow \mathcal{T}op$. For each (non-empty) finite set E, $\mathcal{P}(E)$ is the space of *disjoint embeddings* $E \times \mathrm{D} \hookrightarrow \mathrm{D}$ of the form
$$(e,z) \mapsto \varphi_e(z) := \lambda_e z + c_e$$
with $\lambda_e \in \mathbf{R}_+^\times$ and $c_e \in \mathrm{D}$. The structure morphism associated to a surjection $p : E \twoheadrightarrow F$ is induced by composition of embeddings, *ie*for all $e \in E$ we have
$$\varphi_e = \varphi_{p(e)} \circ \varphi_{e\in p^{-1}(p(e))} \,.$$

3.3.3 Categories of flags

Definition 3.3.5 For each integer $0 \leqslant k \leqslant n-1$, we define a category $\mathfrak{F}_k$ of k*–flags* as follows.

- The objects of $\mathfrak{F}_k$ are sequences of surjections
$$\underline{n} \overset{p_0}{\twoheadrightarrow} \underline{m_0} \overset{p_1}{\twoheadrightarrow} \dots \overset{p_k}{\twoheadrightarrow} \underline{m_k}$$
where $m_0 > m_1 > \cdots > m_k > 0$ is a *strictly decreasing* sequence of integers.

- A morphism $\underline{\alpha} = (\alpha, \alpha_0, \ldots, \alpha_k)$ from the k–flag $\mathfrak{f} = (p_0, \ldots, p_k)$ to the k–flag $\mathfrak{f}' = (p'_0, \ldots, p'_k)$ is a commutative diagram:

$$\begin{array}{ccccccc}
\underline{n} & \xrightarrow{p_0} & \underline{m_0} & \xrightarrow{p_1} & \cdots & \xrightarrow{p_k} & \underline{m_k} \\
\simeq \downarrow \alpha & & \simeq \downarrow \alpha_0 & & & & \simeq \downarrow \alpha_k \\
\underline{n} & \xrightarrow{p'_0} & \underline{m'_0} & \xrightarrow{p'_1} & \cdots & \xrightarrow{p'_k} & \underline{m'_k}
\end{array}$$

  where all the vertical maps are bijections.

Remarks 3.3.6 (1) Note that we demand the sequence (m_i) to be strictly decreasing, but *not* that $n > m_0$.

(2) Let $\mathfrak{f}$ be a k–flag. For each integer $0 \leqslant i \leqslant k$, we write $\pi_i = p_i \circ \cdots \circ p_0$. The datum of the list $(p_0, \ldots, p_k)$ is equivalent to the datum of the list $(\pi_0, \ldots, \pi_k)$. In the text, we allow ourself to use one notation or the other according to the context.

(3) Note that the category $\mathfrak{F}_k$ is a groupoid. For every k–flag $\mathfrak{f} = (p_0, \ldots, p_k)$, we denote aut($\mathfrak{f}$) the group of automorphisms of $\mathfrak{f}$ in $\mathfrak{F}_k$. Sending an automorphism $(\alpha, \alpha_0, \ldots, \alpha_k)$ to the permutation $\alpha \in \mathfrak{S}_n$ defines a group injection $\mathrm{aut}(\mathfrak{f}) \hookrightarrow \mathfrak{S}_n$. In the sequel, aut($\mathfrak{f}$) is thus considered as a subgroup of $\mathfrak{S}_n$.

As k varies, the family of categories $(\mathfrak{F}_k)_{0 \leqslant k \leqslant m-1}$ forms a "Δ–category". Indeed, for each integer $1 \leqslant k \leqslant m-1$ and for each integer $0 \leqslant i \leqslant k$, let $d_{i,k} \colon \mathfrak{F}_k \longrightarrow \mathfrak{F}_{k-1}$ be the forgetful functor such that $d_{i,k}(\pi_0, \ldots, \pi_k) = (\pi_0, \ldots, \widehat{\pi_i}, \ldots, \pi_k)$. The functorial identity $d_{i,k-1} \circ d_{j,k} = d_{j-1,k-1} \circ d_{i,k}$ is satisfied for all $i < j$.

Definition 3.3.7 (1) Let $\mathcal{A}_\bullet = (\mathcal{A}_k)_{0 \leqslant k \leqslant m-1}$ be a family of functors $\mathcal{A}_k \colon \mathfrak{F}_k \longrightarrow \mathcal{T}op$. We say that $\mathcal{A}_\bullet$ is compatible with the Δ–structure, or that $\mathcal{A}_\bullet$ is a Δ–functor for short, if there exist, for each k, natural transformations $\delta_{i,k} \colon \mathcal{A}_k \longrightarrow \mathcal{A}_{k-1} \circ d_{i,k}$ $(0 \leqslant i \leqslant k)$ such that

$$\delta_{i,k-1} \circ \delta_{j,k} = \delta_{j-1,k-1} \circ \delta_{i,k} \,.$$

For such a Δ–functor, note that the natural transformations $\delta_{i,k}$ induce maps $\operatorname*{colim}_{x \in \mathfrak{F}_k} \mathcal{A}_k(x) \longrightarrow \operatorname*{colim}_{x \in \mathfrak{F}_{k-1}} \mathcal{A}_{k-1}(x)$, which endow the family of topological spaces $\big(\operatorname*{colim}_{x \in \mathfrak{F}_k} \mathcal{A}_k(x)\big)_k$ with a Δ–space structure.

(2) Let $\mathcal{A}_\bullet$ and $\mathcal{B}_\bullet$ be two Δ–functors. A Δ–natural transformation $f \colon \mathcal{A}_\bullet \longrightarrow \mathcal{B}_\bullet$ is a family of natural transformations $f_k \colon \mathcal{A}_k \longrightarrow \mathcal{B}_k$, $0 \leqslant k \leqslant m-1$ such

that for each integer $1 \leqslant i \leqslant k$ and for each flag $\mathfrak{f} \in \mathfrak{F}_k$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{A}_k(\mathfrak{f}) & \xrightarrow{f_k} & \mathcal{B}_k(\mathfrak{f}) \\ \Big\downarrow{\scriptstyle d_{i,k}} & & \Big\downarrow{\scriptstyle d_{i,k}} \\ \mathcal{A}_{k-1}(d_{i,k}(\mathfrak{f})) & \xrightarrow{f_{k-1}} & \mathcal{B}_{k-1}(d_{i,k}(\mathfrak{f})) \end{array} .$$

Such an f is called a Δ–natural homotopy equivalence (resp. a homeomorphism) when for each flag $\mathfrak{f} \in \mathfrak{F}_k$ the map $\mathcal{A}_k(\mathfrak{f}) \longrightarrow \mathcal{B}_k(\mathfrak{f})$ is a homotopy equivalence (resp. a homeomorphism).
Moreover, a Δ–natural homotopy equivalence is said *equivariant* when for each flag $\mathfrak{f} \in \mathfrak{F}_k$ the map $\mathcal{A}_k(\mathfrak{f}) \longrightarrow \mathcal{B}_k(\mathfrak{f})$ is an $\mathrm{aut}(\mathfrak{f})$–equivariant homotopy equivalence.

Using this formalism, here is the analogue of lemma 3.2.1.

Lemma 3.3.8 *Let $\mathcal{A}_\bullet$ et $\mathcal{B}_\bullet$ be two Δ–functors and $f\colon \mathcal{A}_\bullet \longrightarrow \mathcal{B}_\bullet$ be a Δ–natural transformation. Then:*

(1) *The natural transformation f induces a morphism of Δ–spaces $f_*\colon \operatorname*{colim}_{\mathfrak{F}_k} \mathcal{A}_k \longrightarrow \operatorname*{colim}_{\mathfrak{F}_k} \mathcal{B}_k$.*

(2) *When f is an equivariant Δ–natural homotopy equivalence, the above morphism f_* is a Δ–homotopy equivalence of spaces. In particular, the map induced on geometric realizations*

$$|\operatorname*{colim}_{\mathfrak{F}_k} \mathcal{A}_k| \xrightarrow{\approx} |\operatorname*{colim}_{\mathfrak{F}_k} \mathcal{B}_k|$$

is a homotopy equivalence.

Proof (1) This point has already been noticed in definition 3.3.7.

(2) The categories $\mathfrak{F}_k$ are groupoids. The colimit of a functor $\mathcal{F}$ defined on a groupoid $\mathcal{G}$ identifies with the disjoint union over the isomorphism classes of objects $[x]$ of $\mathcal{F}(x)_{/\mathrm{aut}(x)}$. Here, the induced maps $\mathcal{A}_k(\mathfrak{f})_{/\mathrm{aut}(\mathfrak{f})} \longrightarrow \mathcal{B}_k(\mathfrak{f})_{/\mathrm{aut}(\mathfrak{f})}$ are homotopy-equivalences since f is equivariant.
The last claim is a consequence of lemma 3.2.1. □

Examples 3.3.9 (1) For every topological space Z and for every integer $0 \leqslant k \leqslant m-1$, let $\mathcal{P}^Z_k\colon \mathfrak{F}_k \longrightarrow \mathcal{T}op$ denote the functor such that for each morphism $\underline{\alpha}\colon \mathfrak{f} \longrightarrow \mathfrak{f}'$, one has

$$\begin{array}{ccc} \mathcal{P}^Z_k(\mathfrak{f}) & = & Z^{\underline{m_k}} \\ {\scriptstyle \mathcal{P}^Z_k(\underline{\alpha})}\downarrow & & \downarrow{\scriptstyle Z^{\alpha_k}} \\ \mathcal{P}^Z_k(\mathfrak{f}') & = & Z^{\underline{m_k}'} \end{array}$$

The natural transformations $\delta_{i,k}\colon \mathcal{P}^Z_k \longrightarrow \mathcal{P}^Z_{k-1}$ given by

$$\delta_{i,k}(p_0,\ldots,p_k) = \begin{cases} \mathrm{id}\colon Z^{\underline{m_k}} \longrightarrow Z^{\underline{m_k}} & \text{if } 0 \leqslant i \leqslant k-1 \\ p_k^*\colon Z^{\underline{m_k}} \longrightarrow Z^{\underline{m_{k-1}}} & \text{if } i = k \end{cases}$$

endow $\mathcal{P}^Z_\bullet$ with a Δ–functor structure.

(2) Let $\mathcal{O}$ be a topological operad.
For each integer $0 \leqslant k \leqslant m-1$, let $\Theta^{\mathcal{O}}_k\colon \mathfrak{F}_k \longrightarrow \mathcal{T}op$ be the unique functor such that on each morphism $\underline{\alpha}\colon \mathfrak{f} \longrightarrow \mathfrak{f}'$ in $\mathfrak{F}_k$ one has

$$\begin{array}{ccccccccc} \Theta^{\mathcal{O}}_k(\mathfrak{f}) & = & \mathcal{O}(\underline{m_k}) & \times & \prod_{x_k\in\underline{m_k}} \mathcal{O}(p_k^{-1}(x_k)) & \times & \ldots & \times & \prod_{x_1\in\underline{m_1}} \mathcal{O}(p_1^{-1}(x_1)) \\ {\scriptstyle \Theta^{\mathcal{O}}_k(\underline{\alpha})}\downarrow & & \downarrow{\scriptstyle \mathcal{O}(\alpha_k)} & & \downarrow{\scriptstyle \mathcal{O}(\alpha_{k-1})} & & & & \downarrow{\scriptstyle \mathcal{O}(\alpha_0)} \\ \Theta^{\mathcal{O}}_k(\mathfrak{f}') & = & \mathcal{O}(\underline{m_k}') & \times & \prod_{y_k\in\underline{m_k}'} \mathcal{O}(p_k'^{-1}(y_k)) & \times & \ldots & \times & \prod_{y_1\in\underline{m_1}'} \mathcal{O}(p_1'^{-1}(y_1)) \end{array}$$

Note that the vertical arrows are well defined since for each $1 \leqslant i \leqslant k$ and for each $x_i \in \underline{m_i}$, the map α_{i-1} induce a bijection $p_i^{-1}(x_i) \simeq p_i'^{-1}(\alpha_i(x_i))$.
One defines a natural transformation $\delta_{0,k}\colon \Theta^{\mathcal{O}}_k \longrightarrow \Theta^{\mathcal{O}}_{k-1} \circ d_{0,k}$ using the canonical projection.
For each $1 \leqslant i \leqslant k-1$, the operadic structure maps of $\mathcal{O}$

$$\mathcal{O}\Big(p_{i+1}^{-1}(x_{i+1})\Big) \times \prod_{x_i\in p_{i+1}^{-1}(x_{i+1})} \mathcal{O}\Big(p_i^{-1}(x_i)\Big) \longrightarrow \mathcal{O}\big(p_i^{-1}(p_{i+1}^{-1}(x_{i+1}))\big)$$

associated to the surjections $p_i^{-1}(p_{i+1}^{-1}(x_{i+1})) \twoheadrightarrow p_{i+1}^{-1}(x_{i+1})$ pour tout $x_{i+1} \in \underline{m_{i+1}}$, induce a map

$$\prod_{x_{i+1}\in\underline{m_{i+1}}} \mathcal{O}\Big(p_{i+1}^{-1}(x_{i+1})\Big) \times \prod_{x_i\in\underline{m_i}} \mathcal{O}\Big(p_i^{-1}(x_i)\Big) \longrightarrow \prod_{x_{i+1}\in\underline{m_{i+1}}} \mathcal{O}\Big((p_{i+1}\circ p_i)^{-1}(x_{i+1})\Big)$$

This leads to a natural transformation $\delta_{i,k}\colon \Theta^{\mathcal{O}}_k \longrightarrow \Theta^{\mathcal{O}}_{k-1} \circ d_{i,k}$.
For $i = k$, the operadic structure map

$$\mathcal{O}(\underline{m_k}) \times \prod_{x_k\in\underline{m_k}} \mathcal{O}\Big(p_k^{-1}(x_k)\Big) \longrightarrow \mathcal{O}(\underline{m_{k-1}})$$

associated to the surjection $\underline{m_{k-1}} \twoheadrightarrow \underline{m_k}$ gives a natural transformation $\delta_{k,k}\colon \Theta_k^{\mathcal{O}} \longrightarrow \Theta_{k-1}^{\mathcal{O}} \circ d_{k,k}$.
The family $\delta_{i,k}$, $0 \leqslant i \leqslant k$, endows $\Theta_\bullet^{\mathcal{O}}$ with a Δ–structure.

3.4 Statement of the main result

We are now ready to analyse the stratification $\mathcal{S}^{\mathrm{pt}}$ in the general case. We will define for each positive integer n a functor $\Psi^n\colon \mathcal{T}op \longrightarrow \mathcal{T}op$, generalizing the previous functors Ψ^2 and Ψ^3, with a natural homotopy equivalence

$$\Psi^n(Y(\mathbf{C})) \approx (\mathcal{R}_n Y)(\mathbf{C})\,.$$

The functor Ψ^n is completely explicit: it is defined as the homotopy colimit of a diagram, in which the spaces are products of configuration spaces of little disks (the spaces of the lite disks operad) with some labels in $\mathcal{Y}$ related by maps defined using composition maps of the little disks operad and diagonals maps on $\mathcal{Y}$. However, the diagram defining Ψ^n becomes rapidly huge and complicated (it contains in particular implicitly the combinatorics of the set of partitions of the integer n), which makes Ψ^n not so easy to handle for practical purposes. One consequence of the existence of such a functor Ψ^n is that the homotopy type of the space $(\mathcal{R}_n Y)(\mathbf{C})$ depends only on the homotopy type of $\mathcal{Y} := Y(\mathbf{C})$.

Definition 3.4.1 We use the notations introduced in examples 3.3.9.

(1) For every topological space Z, for every topological operad $\mathcal{O}$ and for every integer $0 \leqslant k \leqslant n-1$, let

$$\psi_k^n(\mathcal{O}, Z) := \operatorname*{colim}_{\mathfrak{F}_k} \Theta_k^{\mathcal{O}} \times \mathcal{P}_k^Z\,.$$

The family $\psi_\bullet^n(\mathcal{O}, Z)$ together with the maps $\delta_{i,k}$ forms a Δ–space of depth k.

(2) Associating to every topological space the space

$$\Psi^n(Z) := |\psi_\bullet^n(\mathcal{P}, Z)| := \operatorname*{hocolim}_{k} \psi_k^n(\mathcal{P}, Z)\,,$$

defines a functor $\Psi^n\colon \mathcal{T}op \longrightarrow \mathcal{T}op$. (Recall that the notation $\mathcal{P}$ stands for the (2–dimensional) little disks operad, see example 3.3.4.)

Example 3.4.2 One checks that the previous functors Ψ^2 and Ψ^3 introduced in Equation 3–2 and Equation 3–3 correspond to the particular cases $n = 2$ and $n = 3$.

Our main result about the functor Ψ^n is that for each "sufficiently small" controlled system of tubular neighbourhoods of the stratification $\mathcal{S}^Y$ of $(\mathcal{R}_n Y)(\mathbf{C})$, say $\mathcal{T}^Y$, then the associated Cech Δ–space $\tau(\mathcal{T}^Y)$ is homotopically equivalent to $\psi^n_\bullet(\mathcal{P}, Y(\mathbf{C}))$. More precisely, one has the following theorem.

Theorem 3.4.3 *Let $\mathcal{T}^Y$ be a controlled system of tubular neighborhoods of $\mathcal{S}^Y$ and $\tau_\bullet$ be the associated Cech Δ–space. Then, up to shrinking the tubes in $\mathcal{T}^Y$, there exists a canonical zig-zag of homotopy equivalences of Δ–spaces between $\tau_\bullet$ and $\psi^n_\bullet(\mathcal{P}, \mathcal{Y})$ which is natural in Y. In particular, we have a homotopy equivalence natural in Y:*

$$\Psi^n(\mathcal{Y}) \approx (\mathcal{R}_n Y)(\mathbf{C}),$$

describing the homotopy type of $(\mathcal{R}_n Y)(\mathbf{C})$ as an explicit functor in $\mathcal{Y} := Y(\mathbf{C})$.

Remark 3.4.4 When the algebraic Y is defined over the field of real numbers $\mathbf{R}$, complex conjugation acts on all the spaces above. The result of theorem 3.4.3 can be made compatible with the action of this involution: the Δ–homotopy equivalences are $\mathbf{Z}/2$–equivariant, see theorem 3.6.1.

3.5 Proof of the main result

The proof parallels the analysis of the stratification of $(\mathcal{R}_3 Y)(\mathbf{C})$ which was detailed in § 3.2.2. We analyse the Δ–space associated to the system of tubular neighbourhoods $\mathcal{T}^Y$ by analysing the lifted controlled system of $\mathfrak{S}_n$–equivariant tubular neighbourhoods $\widetilde{\mathcal{T}}^Y$ of $\mathcal{E}^Y$ and by taking a quotient at the end.

The plan of the proof is the following. We first reformulate in § 3.5.1 the problem in terms of the lifted stratification $\mathcal{E}^Y$. Then, in § 3.5.2, we show that the problem reduces to the study of the special case $Y = \mathrm{pt}$. This case is finally analized in § 3.5.3.

3.5.1 Preliminaries

We remind that $\widetilde{q}$ denotes the canonical map $\widetilde{q}\colon (\widetilde{\mathcal{R}_n} Y)(\mathbf{C}) \longrightarrow (\mathcal{R}_n Y)(\mathbf{C})$ (*cf* § 3.1.3).

Proposition 3.5.1 *Let $\mathcal{T}^Y$ be a system of tubular neighbourhoods of $\mathcal{S}^Y$. Then, the family of spaces $\widetilde{q}^{-1}(T^Y_{[\pi]})$ when $[\pi]$ runs into the set of partitions of n form a system $\widetilde{\mathcal{T}}^Y$ of $\mathfrak{S}_n$–equivariant tubular neighbourhoods of $\mathcal{E}^Y$.*

Conversely, for every system $\widetilde{\mathcal{T}}^Y$ of $\mathfrak{S}_n$–equivariant tubular neighbourhoods of $\mathcal{E}^Y$, the family of subspaces $T^Y_{[\pi]}$ of $(\mathcal{R}_n Y)(\mathbf{C})$ defined by $T^Y_{[\pi]} = \widetilde{T}^Y_{[\pi]}/\mathfrak{S}_n$ when $[\pi]$ runs into the set of partitions of n forms a system of tubular neighbourhoods of $\mathcal{S}^Y$. Moreover, the following properties hold:

(1) *The system of tubular neighbourhoods $\mathcal{T}^Y$ is controlled if and only if the system $\widetilde{\mathcal{T}}^Y$ is.*

(2) *The Δ–space $\tau_\bullet(\mathcal{T}^Y)$ is obtained by quotienting the Δ–space $\tau_\bullet(\widetilde{\mathcal{T}}^Y)$ by $\mathfrak{S}_n$. In particular, for each $(k+1)$–tuple of partitions $([\pi_0], \dots, [\pi_k])$, one has a homeomorphism*

$$\left(\widetilde{T}^Y_{[\pi_0],\dots,[\pi_k]}\right)/\mathfrak{S}_n \xrightarrow{\simeq} T^Y_{[\pi_0],\dots,[\pi_k]} .$$

The preceding proposition is a special case of the following more general proposition, which is true *orbit type* stratifications (*cf* proposition–definition A.1.4 in the appendix).

Proposition 3.5.2 *Let G be a finite group, let M be a G–manifold and $q\colon M \longrightarrow M/G$ be the canonical projection. We denote by $\mathcal{S}$ the* orbit type *stratification of M/G. Let $\mathcal{T}$ be a system of tubes of $\mathcal{S}$. The family of spaces $\widetilde{q}^{-1}(T_S)$ when S runs over the set of strata of $\mathcal{S}$ forms a system $\widetilde{\mathcal{T}}$ of G–equivariant tubular neighbourhoods of $\mathcal{E}$. Conversely, for every system $\widetilde{\mathcal{T}}$ of G–equivariant tubular neighbourhoods of $\mathcal{E}$, the family of subspaces $T_S \subset M/G$ given by $T_S = \widetilde{T}_S/G$ when S runs over the strata of $\mathcal{S}$ forms a system of tubes of $\mathcal{S}$. Moreover, the following properties hold:*

(1) *The system of tubes $\mathcal{T}$ is controlled if and only if the system $\widetilde{\mathcal{T}}$ is controlled.*

(2) *The Δ–space $\tau_\bullet(\mathcal{T})$ is obtained by quotienting $\tau_\bullet(\widetilde{\mathcal{T}})$ by the action of G. In particular, for each $(k+1)$–tuple of strata $(S_0, \dots, S_k)$, one has a homeomorphism*

$$\left(\widetilde{T}_{S_0,\dots,S_k}\right)/G \xrightarrow{\simeq} T_{S_0,\dots,S_k} .$$

From now on, we fix[2] *a a controlled system $\mathcal{T}^Y$ of tubular neighbourhoods of $\mathcal{S}^Y$. We denote $\widetilde{\mathcal{T}}^Y$ the controlled system of $\mathfrak{S}_n$–equivariant tubular neighbourhoods induced on $\mathcal{E}^Y$.*

For every partition $[\pi]$ of n, the stratum $E^Y_{[\pi]} \subset (\widetilde{\mathcal{R}_n Y})(\mathbf{C})$ has as connected components the spaces $E^Y_{\overline{\pi}}$. The tube $\widetilde{T}^Y_{[\pi]}$ is thus the disjoint union of its restrictions $\widetilde{T}^Y_{\overline{\pi}}$ to $E^Y_{\overline{\pi}}$. The space $\widetilde{T}^Y_{\overline{\pi}}$ is a $\mathfrak{S}_{\overline{\pi}}$–equivariant tubular neighbourhood of $E^Y_{\overline{\pi}}$ and one can reconstruct all $\widetilde{T}^Y_{[\pi]}$ from $\widetilde{T}^Y_{\overline{\pi}}$ by letting $\mathfrak{S}_n$ act. The understanding of the Δ–space $\tau_\bullet(\widetilde{\mathcal{T}}^Y)$ and $\tau_\bullet(\mathcal{T}^Y)$ reduces to the understanding of the Δ–functor associating to each k–flag $(\pi_0, \dots, \pi_k)$ the space $\widetilde{T}^Y_{\overline{\pi_0,\dots,\pi_k}} := \widetilde{T}^Y_{\overline{\pi_0}} \cap \dots \cap \widetilde{T}^Y_{\overline{\pi_k}}$.

[2] We will allow ourself to shrink the tubes, see remark 3.2.10.

Proposition–definition 3.5.3 (1) *For each integer* $0 \leqslant k \leqslant m-1$*, there exists a functor* $\widetilde{\mathbf{T}}^Y_k \colon \mathfrak{F}_k \longrightarrow \mathcal{T}op$ *taking on a* k*–flag* $\mathfrak{f} = (\pi_0, \ldots, \pi_k)$ *the value:*

$$\widetilde{\mathbf{T}}^Y_k(\mathfrak{f}) = \widetilde{T}^Y_{\overline{\pi_0},\ldots,\overline{\pi_k}} := \widetilde{T}^Y_{\overline{\pi_0}} \cap \cdots \cap \widetilde{T}^Y_{\overline{\pi_k}}\,.$$

(2) *The family of functors* $\widetilde{\mathbf{T}}^Y_\bullet$ *is a* Δ*–functor (*cf definition 3.3.7*).* *Thus the spaces* $\operatorname*{colim}_{\mathfrak{F}_k} \widetilde{\mathbf{T}}^Y_k$ *for* $0 \leqslant k \leqslant m-1$ *form a* Δ *space, denoted* $\operatorname*{colim}_{\mathfrak{F}_\bullet} \widetilde{\mathbf{T}}^Y_\bullet$*.*

(3) *The quotient map* $\widetilde{q}$ *induces a homeomorphism of* Δ*–spaces*

$$\operatorname*{colim}_{\mathfrak{F}_\bullet} \widetilde{\mathbf{T}}^Y_\bullet \longrightarrow \tau_\bullet(\mathcal{T}^Y)\,.$$

Proof (1) To characterize the functor $\widetilde{\mathbf{T}}^Y_k$, one should define the image of morphisms in $\mathfrak{F}_k$.
Let $\underline{\alpha} = (\alpha, \alpha_0, \ldots, \alpha_k) \colon \mathfrak{f} \longrightarrow \mathfrak{f}'$ be a morphism of k–flags. By definition, for each integer $0 \leqslant i \leqslant k$, we have the identity $\overline{\pi_i} = \overline{\pi'_i \circ \alpha}$. Thus, the action of the permutation $\alpha \in \mathfrak{S}_n$ maps the subspace $\widetilde{T}^Y_{\overline{\pi_0},\ldots,\overline{\pi_k}}$ of $(\widetilde{\mathcal{R}_n}Y)(\mathbf{C})$ on $\widetilde{T}^Y_{\overline{\pi'_0},\ldots,\overline{\pi'_k}}$. As a consequence, it is legitimate to define $\widetilde{\mathbf{T}}^Y_k(\underline{\alpha})$ as the action of the permutation $\alpha \in \mathfrak{S}_n$. One checks that this defines indeed a functor.

(2) For each $0 \leqslant i \leqslant k$, the canonical inclusions $\widetilde{T}^Y_{\overline{\pi_0},\ldots,\overline{\pi_k}} \hookrightarrow \widetilde{T}^Y_{\overline{\pi_0},\ldots,\widehat{\overline{\pi_i}},\ldots,\overline{\pi_k}}$ (the notation $\widehat{\overline{\pi_i}}$ stands for the omission of the subscript $\overline{\pi_i}$) giving the natural transformations $\delta_{i,k}$.

(3) For each flag $\mathfrak{f} \in \mathfrak{F}_k$, the application $\widetilde{q} \colon (\widetilde{\mathcal{R}_n}Y)(\mathbf{C}) \longrightarrow (\mathcal{R}_nY)(\mathbf{C})$ induces a map $\widetilde{q}_{\mathfrak{f}} \colon \widetilde{\mathbf{T}}^Y_k(\mathfrak{f}) \longrightarrow T^Y_{[\pi_0],\ldots,[\pi_k]}$. These maps are compatible with the action of the symmetric group $\mathfrak{S}_n$: whenever the action of $\alpha \in \mathfrak{S}_n$ on $(\widetilde{\mathcal{R}_n}Y)(\mathbf{C})$ exchanges $\widetilde{\mathbf{T}}^Y_k(\mathfrak{f})$ and $\widetilde{\mathbf{T}}^Y_k(\mathfrak{f}')$, then the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathbf{T}}^Y_k(\mathfrak{f}) & \xrightarrow{\quad \alpha\cdot \quad} & \widetilde{\mathbf{T}}^Y_k(\mathfrak{f}') \\ & {\scriptstyle \widetilde{q}_{\mathfrak{f}}} \searrow \quad \swarrow {\scriptstyle \widetilde{q}_{\mathfrak{f}'}} & \\ & T^Y_{[\pi_0],\ldots,[\pi_k]} & \end{array}\,.$$

In particular, the map $\widetilde{q}$ induces a morphism of Δ–spaces $\operatorname*{colim}_{\mathfrak{F}_\bullet} \widetilde{\mathbf{T}}^{\mathbf{Y}}_\bullet \longrightarrow \tau_\bullet$. That this map indeed induces a homeomorphism of Δ–spaces is a rephrasing of the homeomorphism $T^Y_{[\pi_0],\ldots,[\pi_k]} \simeq \widetilde{T}^Y_{[\pi_0],\ldots,[\pi_k]}/\mathfrak{S}_n$, taking into account the decompositions $\widetilde{T}^Y_{[\pi_i]} = \coprod_{\overline{\pi_i} \in [\pi_i]} \widetilde{T}^Y_{\overline{\pi_i}}$ and the fact that $\mathfrak{F}_k$ is a groupoid. □

3.5.2 Reduction to the case $Y = \mathrm{pt}$

As in the case $n = 3$, the first step of the proof consists in using the restriction lemma (lemma 3.2.8) and the uniqueness of tubular slices (lemma 3.2.9) to reduce the analysis to the special case $Y = \mathrm{pt}$.

From now on, we fix a controlled system of $\mathfrak{S}_n$–equivariant tubular neighbourhoods of $\mathcal{E}^{\mathrm{pt}}$, say $\widetilde{\mathcal{T}}^{\mathrm{pt}}$.

Notations: (1) Recall that we have defined for each surjection $\pi\colon \underline{n} \twoheadrightarrow \underline{m}$ sub-$\mathfrak{S}_n$–manifolds $\widetilde{\Sigma}_{\overline{\pi}}$ of $(\widetilde{\mathcal{R}_n Y})(\mathbf{C})$ in remark 3.1.11. Using $\widetilde{\mathcal{T}}^{\mathrm{pt}}$, one defines a family $(\widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi}})$ of $\mathfrak{S}_{\overline{\pi}}$–equivariant tubular neighbourhoods of $E^{\circ}_{\overline{\pi}}$ into $\widetilde{\Sigma}_{\overline{\pi}}$. By definition, one has a $\mathfrak{S}_{\overline{\pi}}$–equivariant homeomorphism

$$\widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi}} \simeq \widetilde{T}^{\mathrm{pt}}_{\overline{\pi}} \times \mathcal{Y}^n \tag{3–4}$$

compatible with the homeomorphisms $E^{\circ}_{\overline{\pi}} \simeq E^{\mathrm{pt}}_{\overline{\pi}} \times \mathcal{Y}^n$ of proposition 3.1.9–(1) and $\widetilde{\Sigma}^{Y}_{\overline{\pi}} \simeq \widetilde{\Sigma}^{\mathrm{pt}}_{\overline{\pi}} \times \mathcal{Y}^n$ of example 3.1.12–(4).

(2) Up to shrinking the tubes, we can assume having for each π the inclusion $\widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi}} \subset \widetilde{T}^{Y}_{\overline{\pi}}$. An application of the uniqueness of tubular slices (lemma 3.2.9) gives an equivariant stratified isotopy between $\widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi}}$ and $\widetilde{T}^{\circ}_{\overline{\pi}}$ inside $\widetilde{T}^{Y}_{\overline{\pi}}$.

(3) For each k–flag $\mathfrak{f} = (\pi_0, \ldots, \pi_k)$, let $\widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi_0}, \ldots, \overline{\pi_k}} := \widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi_0}} \cap \cdots \cap \widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi_k}}$. One has a $\mathfrak{S}_{\overline{\pi_0}, \ldots, \overline{\pi_k}}$–equivariant homeomorphism

$$\widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi_0}, \ldots, \overline{\pi_k}} \simeq \widetilde{T}^{\mathrm{pt}}_{\overline{\pi_0}, \ldots, \overline{\pi_k}} \times \mathcal{Y}^{\underline{m_k}}, \tag{3–5}$$

compatible to the above homeomorphism (3–4) (considering $\mathcal{Y}^{\underline{m_k}}$ as the diagonal subspace of $\mathcal{Y}^{\underline{m_i}}$ *via* the surjection $\underline{m_i} \twoheadrightarrow \underline{m_k}$).

Proposition–definition 3.5.4 *For each integer $0 \leqslant k \leqslant n-1$, there exists a functor $\widetilde{\mathbf{T}}^{\widetilde{\Sigma}}_k \colon \mathfrak{F}_k \longrightarrow \mathcal{T}op$ taking on each k–flag $\mathfrak{f} = (\pi_0, \ldots, \pi_k)$ the value:*

$$\widetilde{\mathbf{T}}^{\widetilde{\Sigma}}_k(\mathfrak{f}) = \widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi_0}, \ldots, \overline{\pi_k}}.$$

The family of functors $\widetilde{\mathbf{T}}^{\widetilde{\Sigma}}_\bullet$ is a Δ–functor and one has an equivariant homeomorphism of Δ–functors

$$\widetilde{\mathbf{T}}^{\widetilde{\Sigma}}_\bullet \simeq \widetilde{\mathbf{T}}^{\mathrm{pt}}_\bullet \times \mathcal{P}^Y_\bullet.$$

(The notation $\mathcal{P}^Y_\bullet$ has been introduced in examples 3.3.9–(1).)

Proof As for the proof of proposition–definition 3.5.3, the image through $\widetilde{\mathbf{T}}_k^{\widetilde{\Sigma}}$ of a morphism $\underline{\alpha} = (\alpha, \alpha_0, \dots, \alpha_k)$ is given by the action of the permutation $\alpha \in \mathfrak{S}_n$ and the natural transformations $\delta_{i,k}$ come from inclusions.
The previously described homeomorphism (3–5) shows that we have for each integer k a homeomorphism of functors $\widetilde{\mathbf{T}}_k^{\widetilde{\Sigma}} \simeq \widetilde{\mathbf{T}}_k^{\mathrm{pt}} \times \mathcal{P}_k^Y$. □

The next proposition is analogous to proposition 3.2.12. It reduces the problem to the study of the case $Y = \mathrm{pt}$.

Proposition 3.5.5 *The canonical inclusion* $\mathbf{i}_\bullet \colon \widetilde{\mathbf{T}}_\bullet^{\widetilde{\Sigma}} \hookrightarrow \widetilde{\mathbf{T}}_\bullet$ *is an equivariant homotopy equivalence of* Δ*–functors. In particular, as one has by construction the* $\mathfrak{S}_n$*–equivariant homotopy equivalence*

$$(\mathcal{R}_n Y)(\mathbf{C}) \approx |\operatorname*{colim}_{\mathfrak{F}_\bullet} \widetilde{\mathbf{T}}_\bullet^{\mathrm{pt}} \times \mathcal{P}_\bullet^Y|,$$

the homotopy type of the space $(\mathcal{R}_n Y)(\mathbf{C})$ *depends only on that of* $\mathcal{Y} := Y(\mathbf{C})$*.*

Proof Let $0 \leqslant k \leqslant n-1$ be an integer and let $\mathfrak{f} = (\pi_0, \dots, \pi_k) \in \mathfrak{F}_k$ be a k–flag. We are going to show that the inclusion $\widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi_0}, \dots, \overline{\pi_k}} \hookrightarrow \widetilde{T}^{Y}_{\overline{\pi_0}, \dots, \overline{\pi_k}}$ is a homotopy equivalence.

For every surjection π, recall that $\widetilde{T}^{\circ}_{\overline{\pi}}$ stands for the restriction of $\widetilde{T}^{Y}_{\overline{\pi}}$ to the open $E^{\circ}_{\overline{\pi}} \subset E^{Y}_{\overline{\pi}}$. Then one defines recursively the space $\widetilde{T}^{\circ}_{\overline{\pi_0}, \dots, \overline{\pi_k}}$ as the restriction of $\widetilde{T}^{Y}_{\overline{\pi_0}}$ to the open subspace $E^{Y}_{\overline{\pi_0}} \cap \widetilde{T}^{\circ}_{\overline{\pi_1}, \dots, \overline{\pi_k}} \subset E^{Y}_{\overline{\pi_0}}$. A repeated application of the restriction lemma 3.2.8 shows that $\widetilde{T}^{Y}_{\overline{\pi_0}, \dots, \overline{\pi_k}}$ $\mathfrak{S}_n$–equivariantly deformation retracts onto $\widetilde{T}^{\circ}_{\overline{\pi_0}, \dots, \overline{\pi_k}}$. On the other hand, the uniqueness of tubular slices (lemma 3.2.9) gives an equivariant stratified isotopy between $\widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi_0}, \dots, \overline{\pi_k}}$ and $\widetilde{T}^{\circ}_{\overline{\pi_0}, \dots, \overline{\pi_k}}$ inside $\widetilde{T}^{Y}_{\overline{\pi_0}, \dots, \overline{\pi_k}}$. It follows that the canonical inclusion $\widetilde{T}^{\widetilde{\Sigma}}_{\overline{\pi_0}, \dots, \overline{\pi_k}} \hookrightarrow \widetilde{T}^{Y}_{\overline{\pi_0}, \dots, \overline{\pi_k}}$ is a $\mathfrak{S}_{\overline{\pi_0}, \dots, \overline{\pi_k}}$–equivariant homotopy equivalence. □

Remark 3.5.6 The proof of proposition 3.5.5 did not require the system of tubes $\widetilde{\mathfrak{T}}^{\mathrm{pt}}$ to be controlled *stricto sensu*, but only that, for each $\overline{\pi} \succeq \overline{\pi}' \in \widetilde{\Pi}(n)$, the intersection $\widetilde{T}_{\overline{\pi}, \overline{\pi}'}$ is (a shrinking) of the restriction of $\widetilde{T}_{\overline{\pi}'}$ to $\widetilde{T}_{\overline{\pi}} \cap E_{\overline{\pi}'}$.

3.5.3 The case $Y = \mathrm{pt}$

As for the case $n = 3$, the previous reduction shows that for every controlled system of $\mathfrak{S}_n$–equivariant tubular neighbourhoods $\widetilde{\mathfrak{T}}^{\mathrm{pt}}$ of $\mathbf{C}^n$ "sufficiently small", the corresponding Δ–functors $\widetilde{\mathbf{T}}_\bullet^{\mathrm{pt}}$ are homotopically equivalent. We are going to give the

explicit description of one of them.
We choose “the” system of equivariant tubes normal to the strata $E_{\overline{\pi}}^{\mathrm{pt}}$ and with sufficiently small radii. The spaces and the maps that appear in the Δ–space associated to this system of tubes are intimately related to what might be called the *configuration space operad*.

Notations: Let E and F be two finite sets and let $\pi\colon E \twoheadrightarrow F$ be a surjection.

(1) We remind the reader that the space $\mathbf{C}^{(E)} \subset \mathbf{C}^E$ is the space of injective maps $\varphi\colon E \hookrightarrow \mathbf{C}$.

(2) The space $\mathbf{C}^\pi$ denotes the subspace of $\mathbf{C}^E$ whose elements are those maps $\varphi\colon E \longrightarrow \mathbf{C}$ which factorize as $\varphi = \varphi' \circ \pi$.
We also denote $\mathbf{C}^{(\pi)}$ the subspace of $\mathbf{C}^\pi$ for which the above map $\varphi'\colon F \hookrightarrow \mathbf{C}$ is injective. The space $\mathbf{C}^{(\pi)}$ is thus equivariantly homeomorphic to the space of configuration of points $\mathbf{C}^{(F)}$.

(3) We also need the reduced version of the previous notation, that is we denote $\widetilde{\mathbf{C}}^\pi$ (resp. $\widetilde{\mathbf{C}}^{(\pi)}$) the subspace of $\mathbf{C}^\pi$ (resp. of $\mathbf{C}^{(\pi)}$) of those maps $\varphi\colon E \longrightarrow \mathbf{C}$ which satisfy in addition:
$$\sum_{e\in E} \varphi(e) = 0\,.$$
Note that $\widetilde{\mathbf{C}}^\pi$ (resp. $\widetilde{\mathbf{C}}^{(\pi)}$) is the fibre of the normal tube (resp. of the punctured normal tube) over $E_{\overline{\pi}}$.

(4) We need also need the relative version of these notations to describe the trace of the normal tube over $E_{\overline{\pi}}$ with the adjacent strata. For every subset $A \subset F$, let $\pi_A\colon \pi^{-1}(A) \twoheadrightarrow A$ denote the restriction of π. We then set:
$$\widetilde{\mathbf{C}}_A^{(\pi)} := \widetilde{\mathbf{C}}^{(\pi_A)}\,.$$

(5) Let $\mathrm{D} = \mathrm{D}(0,1)$ be the unit disc in $\mathbf{C}$. We use the notations $\widetilde{\mathrm{D}}^\pi, \widetilde{\mathrm{D}}^{(\pi)}, \widetilde{\mathrm{D}}_A^{(\pi)}$ for the subspaces of D^E analogous to the previous ones.

(6) For every $\varphi\colon E \longrightarrow \mathbf{C}$, we set:
$$\varepsilon_\varphi := \begin{cases} \dfrac{1}{10} \inf\limits_{\substack{e,f\in E\\ \varphi(f)\neq\varphi(e)}} |\varphi(e)-\varphi(f)| & \text{if } \mathrm{Card}(\mathrm{Im}(\varphi)) \neq 1 \\ 1 & \text{if } \mathrm{Card}(\mathrm{Im}(\varphi)) = 1 \end{cases}$$

We now define our chosen system of tubular neighbourhoods $\widetilde{\mathfrak{T}}^{\mathrm{pt}}$.

Definition 3.5.7 For every surjection $\pi\colon \underline{n} \longrightarrow \underline{m}$, let $\widetilde{T}_{\overline{\pi}}$ be the following normal (for the euclidean metric) $\mathfrak{S}_{\overline{\pi}}$–equivariant tubular neighbourhood of $E^{\mathrm{pt}}_{\overline{\pi}} \simeq \mathbf{C}^{(\pi)}$ inside $\mathbf{C}^n$:

$$\begin{array}{rccc} \widetilde{T}_{\overline{\pi}}\colon & \mathbf{C}^{(\pi)} \times \prod_{x\in\underline{m}} \widetilde{\mathrm{D}}^{\pi^{-1}(x)}_{\{x\}} & \xrightarrow{\widetilde{e}_{\overline{\pi}}} & \mathbf{C}^n \\ & (\varphi, (\varphi_x)_{x\in\underline{m}}) & \mapsto & \left(i \mapsto \varphi(i) + \varepsilon_\varphi \cdot \varphi_{\pi(i)}(i)\right) \end{array}$$

Note that by construction, the radius ε_φ are taken sufficiently small so that the tube $\widetilde{T}_{\overline{\pi}}$ only meets the strata $E_{\overline{\pi}'}$ with $\overline{\pi}' \succeq \overline{\pi}$ and so that for every $\sigma \in \mathfrak{S}_n$ such that $\overline{\pi} \neq \overline{\pi\cdot\sigma}$, the tubes $\widetilde{T}_{\overline{\pi}}$ and $\sigma\cdot\widetilde{T}_{\overline{\pi}}$ are disjoint.

As in remark 3.2.13, note that all the tubes are trivial as stratified sets. The multiple intersections of these tubes are thus homeomorphic to products of the relative punctured fibers, which are exactly configurations spaces of points.
Here is a variant of examples 3.3.9–(2) which describes the homeomorphism type of the intersections. It should be thought of as the special case where $\mathcal{O} = \mathscr{C}$ is the operad of configuration spaces.

Definition 3.5.8 Let $\widetilde{\Theta}^{\mathscr{C}}_k\colon \mathfrak{F}_k \longrightarrow \mathcal{T}op$ be the unique functor such that for every morphism $\underline{\alpha}\colon \mathfrak{f} \longrightarrow \mathfrak{f}'$ we have

$$\begin{array}{ccccccccccc} \widetilde{\Theta}^{\mathscr{C}}_k(\mathfrak{f}) = & \mathbf{C}^{(\pi_k)} & \times & \prod_{x_k\in\underline{m_k}} \widetilde{\mathrm{D}}^{(\pi_{k-1})}_{p_k^{-1}(x_k)} & \times & \cdots & \times & \prod_{x_1\in\underline{m_1}} \widetilde{\mathrm{D}}^{(\pi_0)}_{p_1^{-1}(x_1)} & \times & \prod_{x_0\in\underline{m_0}} \widetilde{\mathrm{D}}^{p_0^{-1}(x_0)} \\ \Big\downarrow \widetilde{\Theta}^{\mathscr{C}}_k(\underline{\alpha}) & \Big\downarrow (\alpha_k)_* & & \Big\downarrow (\alpha_{k-1})_* & & & & \Big\downarrow (\alpha_0)_* & & \Big\downarrow \alpha_* \\ \widetilde{\Theta}^{\mathscr{C}}_k(\mathfrak{f}') = & \mathbf{C}^{(\pi'_k)} & \times & \prod_{y_k\in\underline{m_k}'} \widetilde{\mathrm{D}}^{(\pi'_{k-1})}_{p_k'^{-1}(y_k)} & \times & \cdots & \times & \prod_{y_1\in\underline{m_1}'} \widetilde{\mathrm{D}}^{(\pi'_0)}_{p_1'^{-1}(y_1)} & \times & \prod_{y_0\in\underline{m_0}'} \widetilde{\mathrm{D}}^{p_0'^{-1}(y_0)} \end{array}$$

(Note that the last factor in the right hand term has no parenthesis in its exponent, whereas the other terms have.)

The spaces $\widetilde{\Theta}^{\mathscr{C}}_k(\mathfrak{f})$ above are good approximations, but are not exactly equal to the multiple intersections.

Proposition–definition 3.5.9 *For every k–flag $\mathfrak{f}$, let $\widetilde{U}(\mathfrak{f})$ be the image in $\mathbf{C}^n$ of the following homeomorphism:*

$$\begin{array}{ccl} \widetilde{\Theta}^{\mathscr{C}}_k(\mathfrak{f}) & \longrightarrow & \widetilde{U}(\mathfrak{f}) \subset \mathbf{C}^n \\ \left(\varphi, (\varphi_{x_k}), \ldots, (\varphi_{x_1}), (\varphi_{x_0})\right) & \mapsto & i \mapsto \varphi(i) + \varepsilon_\varphi \left[\varphi_{\pi_k(i)}(i) + (\min_{x_k\in\underline{m_k}} \varepsilon_{\varphi_{x_k}}) \cdot \left[\varphi_{\pi_{k-1}(i)}(i) + \cdots + (\min_{x_1\in\underline{m_1}} \varepsilon_{\varphi_{x_1}}) \cdot \varphi_{\pi_0(i)}(i)\right] \ldots\right] \end{array}$$

The natural inclusions $\widetilde{\mathbf{T}}^{\mathrm{pt}}(\mathfrak{f}) \hookrightarrow \widetilde{U}(\mathfrak{f})$ are $\mathrm{aut}(\mathfrak{f})$–equivariant homotopy equivalences.

The space $\widetilde{U}(\mathfrak{f})$ is thus the space of configuration of n points in $\mathbf{C}$ which can be obtained in the following way: start with m_k distinct points x_i in $\mathbf{C}$ labelled by $\underline{m_k}$ and "blow up in the normal direction" each of these points, with as many points as the cardinal of the inverse image of its label through p_k^{-1}, in a disc of radius $\frac{1}{10}$ the minimal distance between the x_i and repeat the blowing-up process until $k = 0$, taking at each step a radius of ball equal to the minimal distance between any two points (not in subfamilies).

Proof The proof is analogous to that of lemma 3.2.14. We make an induction on k. The inductive step is based on a the use of the uniqueness of tubular neighbourhoods, as was illustrated in the proof of lemma 3.2.14. □

So far, we didn't mention any Δ–structure. For every integer $0 \leqslant i \leqslant k-1$, we have a "pseudo-map" defined by

$$\begin{array}{ccc} \prod\limits_{x_{i+1} \in \underline{m_{i+1}}} \widetilde{D}^{(\pi_i)}_{p_{i+1}^{-1}(x_{i+1})} \times \prod\limits_{x_i \in \underline{m_i}} \widetilde{D}^{(\pi_{i-1})}_{p_i^{-1}(x_i)} & \dashrightarrow & \prod\limits_{x_{i+1} \in \underline{m_{i+1}}} \widetilde{D}^{(\pi_{i-1})}_{(p_{i+1} \circ p_i)^{-1}(x_{i+1})} \\ (\varphi_{x_{i+1}}), (\varphi_{x_i}) & \longmapsto & e \mapsto \varphi_{\pi_i(e)}(e) + (\min\limits_{x_i \in \underline{m_i}} \varepsilon_{\varphi_{x_i}}) \cdot \varphi_{\pi_{i-1}(e)}(e) \end{array}$$

Here, by "pseudo-map", we mean, as before, that the image of the map defined by the formula satisfy all the linear conditions but some coordinates in the image may lie a little outside of D. These lead to a "pseudo-natural" transformation $\delta_{i,k} \colon \widetilde{\Theta}^{\mathscr{C}}_k \longrightarrow \widetilde{\Theta}^{\mathscr{C}}_{k-1} \circ d_{i,k}$, which is defined exactly on the subspace $\widetilde{\mathbf{T}}^{\mathrm{pt}}_k$ and compatible with the natural transformation $\widetilde{\mathbf{T}}^{\mathrm{pt}}_k \longrightarrow \widetilde{\mathbf{T}}^{\mathrm{pt}}_{k-1} \circ d_{i,k}$.

There are natural sections from $\widetilde{\Theta}^{\mathscr{C}}_\bullet(\mathfrak{f})$ to $\Theta^{\mathfrak{C}}_\bullet(\mathfrak{f})$ which are *compatible with the natural transformations* $\delta_{i,k}$. These are given by taking radii

$$\begin{array}{ccc} \prod\limits_{x_i \in \underline{m_i}} \widetilde{D}^{(\pi_{i-1})}_{p_i^{-1}(x_i)} & \longrightarrow & \prod\limits_{x_i \in \underline{m_i}} \mathfrak{C}(p_i^{-1}(x_i)) \\ (\varphi_{x_i}) & \mapsto & r = \min\limits_{x_i \in \underline{m_i}} \varepsilon_{\varphi_{x_i}} \end{array}$$

Using lemma 3.2.16, we conclude that the composite $\widetilde{\mathbf{T}}^{\mathrm{pt}}_\bullet \hookrightarrow \widetilde{U}_\bullet \simeq \widetilde{\Theta}^{\mathscr{C}}_\bullet \xrightarrow{\sigma} \Theta^{\mathfrak{C}}_\bullet$ is an equivariant homotopy equivalence of Δ–spaces, which completes the proof of theorem 3.4.3. □

3.6 The homotopy type of the space of real points

When the algebraic variety Y is defined over the field of real numbers, the spaces $\Psi^n(Y(\mathbf{C}))$ and $(\mathcal{R}_n Y)(\mathbf{C})$ are both equipped with involutions induced by the action of complex conjugation. The stratification $\mathcal{S}^Y$ of $(\mathcal{R}_n Y)(\mathbf{C})$ is stable under this involution and our previous description can be made equivariant.

Theorem 3.6.1 *Let $\mathcal{T}^Y$ be a controlled system of $\mathbf{Z}/2$–tubular neighborhoods of $\mathcal{S}^Y$ and $\tau_\bullet$ be the associated Cech Δ–space. Then, up to shrinking the tubes in $\mathcal{T}^Y$, there exists a canonical zig-zag of $\mathbf{Z}/2$–homotopy equivalences of Δ–spaces between $\tau_\bullet$ and $\psi^n_\bullet(\mathcal{P},\mathcal{Y})$ which is natural in Y. In particular, we have a $\mathbf{Z}/2$–homotopy equivalence natural in Y:*

$$\Psi^n(\mathcal{Y}) \xrightarrow{\approx} (\mathcal{R}_n Y)(\mathbf{C})\,.$$

As a consequence, the homotopy type of the space $(\mathcal{R}_n Y)(\mathbf{R})$ depends only on the homotopy type of $\mathcal{Y} := Y(\mathbf{C})$ and $\mathcal{Y}^{\mathbf{Z}/2} := Y(\mathbf{R})$.

Proof The proof is similar to that of theorem 3.4.3. We carry a $\mathfrak{S}_n \times \mathbf{Z}/2$–equivariant analysis of the stratification $\mathcal{E}^Y$ of $(\widetilde{\mathcal{R}_n}Y)(\mathbf{C})$. All the steps in the proof of theorem 3.4.3 translate with little changes. □

The result of theorem 3.6.1 suggests that the answer to the following algebraic question is affirmative, a fact which we were not able to prove.

Question 3.6.2 Does the $\mathbf{A}^1$–homotopy type of the space $\mathcal{R}_n Y$ only depend on the $\mathbf{A}^1$–homotopy type of Y?

A Thom–Mather theory of controlled tubular neighbourhoods

We present a survey on the theory of stratifications and in particular of the theory of controlled tubular neighbourhoods, as developed by Mather in [25]. This theory is an important ingredient in the proof of theorem 3.4.3 in § 3.4.
More complete treatment on this material are [29, chapters 1, 3 and 4] and [14, chapters I and II]. Our treatment of the subject is as elementary as possible. In particular, we restrict ourself to stratifications of manifolds.

In this appendix, all manifolds are assumed smooth Riemannian.

A.1 Stratifications

A.1.1 Definition

Definition A.1.1 Let M be a manifold. A finite[3] stratification $\mathcal{S}$ of M is the datum of a finite set I and a partition of M by disjoint smooth submanifolds $S_i \subset M$ $(i \in I)$, called the strata of $\mathcal{S}$,

$$M = \coprod_{i \in I} S_i$$

and satisfying the following condition. For every pair $(i, j) \in I^2$ for which the intersection $S_i \cap \overline{S_j}$ is non empty, one has the inclusion $S_i \subset \overline{S_j}$.
The family of strata $(S_i)_{i \in I}$ has a natural partial ordering: we write $S_i \prec S_j$ whenever the inclusion $S_i \subset \overline{S_j}$ holds.

Example A.1.2 We list here some examples to keep in mind.

(1) Let M be a smooth manifold and let N be a smooth submanifold. The partition $M = N \coprod (M - N)$ induces a stratification of M. In general, specializing the results of this appendix to this example recovers some well known results in differential geometry.

(2) Let n be a positive integer and let $V \subset \mathbf{C}^n$ be an algebraic set, that is to say the set of zeros of a family of polynomials in $\mathbf{C}[X_1, \ldots, X_n]$. The singular locus of V is again an algebraic set (possibly empty) denoted $\mathrm{Sing}(V)$. The smooth locus $V \setminus \mathrm{Sing}(V)$ is dense and open in V; it is in particular a smooth submanifold of $\mathbf{C}^n$. Setting $V_0 = V$ and $S_0 = \mathbf{C}^n \setminus V$, we define inductively for an integer $i \geqslant 0$, $V_{i+1} = \mathrm{Sing}(V_i)$ and $S_{i+1} = V_i \setminus V_{i+1}$ until we end up with $V_N = \emptyset$ for some integer N. By construction, the family $(S_i)_{0 \leqslant i \leqslant N}$ induces a stratification of $\mathbf{C}^n$. For every integer $0 \leqslant i \leqslant N-1$, we have $S_i \succ S_{i+1}$.

(3) An important special case for us is the following. Let $\mathrm{Pol}_n(\mathbf{C})$ be the space of complex monic degree n polynomials $P \in \mathbf{C}[X]$, canonically identified with $\mathbf{C}^n$. The stratification $\mathcal{S}^{\mathrm{pt}}$ of $\mathrm{Pol}_n(\mathbf{C})$ induced by the algebraic set associated to the vanishing of the discriminant is described in detail in § 3.1.

The following proposition allows to produce new stratifications.

[3]For simplicity, we restrict ourself to *finite* stratifications, that is to say such that the set of strata is finite. However, most the results presented in the appendix would hold under the hypothesis that the stratification is locally finite, that is that every point in M has a neighbourhood which meets only finitely many strata.

Proposition A.1.3 *Let $(M, \mathcal{S})$ be a stratified manifold.*

(1) *Let N be a manifold and $f\colon N \longrightarrow M$ be a map transverse to $\mathcal{S}$, that is to say transverse to each stratum in $\mathcal{S}$. Then, the family of submanifolds $f^{-1}(S_i) \subset f^{-1}(M)$, $i \in I$ defines a stratification of $f^{-1}(M)$, denoted $f^{-1}(\mathcal{S})$.*

(2) *In particular, a submanifold $N \subset M$ transverse to $\mathcal{S}$ comes with an induced stratification.*

Another important example of stratification appears in the context of equivariant manifolds (see [29, theorem 4.3.2 and corollary 4.3.11] for example).

Proposition–definition A.1.4 *Let G be a finite group and M a G–manifold. For every point $x \in M$, let G_x denote the isotropy subgroup of x.*

(1) *For every subgroup $H \subset G$, let $M_{(H)}$ be the subset of M consisting of points x with isotropy group G_x conjugated to H. It is a sub-G–manifold of M.*

(2) *The decomposition $M = \bigcup_{H \subset G} M_{(H)}$ induces a stratification of M. It is classically referred to as the* stratification by orbit type.

(3) *The quotient of the stratification by orbit type of M induces a stratification of the orbit space M/G.*

A.1.2 The Whitney conditions

Let $(M, \mathcal{S})$ be a stratified manifold. H. Whitney has defined in [33] some nice regularity conditions (the so-called (A) and (B) conditions) on the stratification $\mathcal{S}$ insuring a locally nice topological behaviour. We recall below these conditions and some of their consequences.

Definition A.1.5 Let $(M, \mathcal{S})$ be a stratified manifold and $R \preceq S$ be two strata of $\mathcal{S}$.

(1) We say that S is (A)-Whitney regular with respect to R when

(A) For every point r in $R \cap \overline{S}$ and for every sequence (s_i) of points in S which converges to r and such that the sequence of tangent spaces $\mathrm{T}_{s_i}S$ converges[1] to a space T, then the inclusion $\mathrm{T}_rR \subset T$ holds.

When this condition holds for every pair of strata $R \preceq S$, we say that the stratification $\mathcal{S}$ of M is (A)-Whitney regular.

[1] Convergence is well defined in a chart around r. Indeed, one can use convergence in a Grassmannian of subspaces of $\mathbf{R}^m$, where $m = \dim M$. Checking that the notion of convergence is independent of the choice of a chart makes it intrinsic.

(2) We say that S is (B)-Whitney regular with respect to R when

(B) For every point r in $R \cap \overline{S}$ and for every sequences (r_i) and (s_i) of points of R and S both converging to r and such that the sequence of tangent spaces T_{s_i} converges[1] to a space T and such that the sequence of lines $(x_i y_i)$ converges[1] to a line L, then the inclusion $L \subset T$ holds.

When this condition is satisfied for every pair of strata $R \preceq S$, we say that the stratification $\mathcal{S}$ of M is (B)-Whitney regular.

Condition (B) implies condition (A). When the two conditions hold, we say that the stratification is Whitney-regular.

Example A.1.6 (1) For every submanifold $N \subset M$, the stratification $M = N \coprod M - N$ of example A.1.2–(2) is Whitney-regular.

(2) In general, the natural stratification $\mathcal{S}_V$ of an algebraic set V is not Whitney-regular. However, in his seminal article [33], H. Whitney showed that there is a canonical stratification which is finer than $\mathcal{S}_V$ and Whitney-regular. For the important case of the stratification induced by the discriminantal hypersurface, see example A.1.2–(3), A Dimca and R Rosian prove in [10] that it is Whitney-regular.

The following proposition corresponds to [14, proposition 1.4].

Proposition A.1.7 *The inverse image of a Whitney stratification by a transverse map (*cf *proposition A.1.3–(1)) is again Whitney-regular.*
In particular, the stratification induced on a transverse submanifold of a Whitney stratified manifold is again Whitney-regular.

The Whitney conditions insure that the strata are glued together is not too pathological way. In particular, a Whitney-regular stratification is locally trivial in the following sense ([29, corollary 3.9.3]).

Proposition A.1.8 *Let $(M, \mathcal{S})$ be a Whitney-regular stratified manifold, let S be a stratum and x be a point in S. Then, there exist an open neighborhood of x in M, say U, an open neighborhood of x in S, say V, and a stratified space F such that one has a* stratified homeomorphism

$$U = V \times \Gamma F\,.$$

Above, the notation ΓF stands for the "cone" of F with its canonical stratification.

Remark A.1.9 In the previous statement, one cannot replace the word "homeomorphism" by the word "diffeomorphism".

For the application that we have in view, the interest of the topological local triviality is that one then has a good notion of a tubular neighbourhood of a stratum into an other adjacent stratum.

Proposition A.1.10 (*cf* Pflaum [29], theorem 4.3.7) *Let G be a finite group and M be a G–manifold. Then the stratification of M by orbit type (*cf *proposition–definition A.1.4) is Whitney-regular.*

A.2 Tubular neighbourhoods

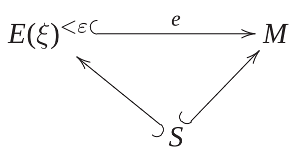
In this section, we focus on the notion of a tubular neighbourhood. After briefly recalling the definition, we present the notion of controlled systems of tubular neighbourhood in the context of (equivariant) Whitney-regular stratifications.

Definition A.2.1 Let M be a manifold and S be a submanifold. A tubular neighbourhood of S inside M is a triple (ξ, ε, e) with:

- ξ is a vector bundle over S; we denote $E(\xi)$ the total space of this bundle, pr the canonical projection $E(\xi) \xrightarrow{\text{pr}} S$ and $\rho\colon E(\xi) \longrightarrow \mathbf{R}^+$ the Riemannian norm.
- $\varepsilon\colon S \longrightarrow \mathbf{R}^+ - \{0\}$ is a continuous function. (For every point $s \in S$, $\varepsilon(s)$ is the radius of the tube at s.)
- e is an embedding of the subspace

  $$E(\xi)^{<\varepsilon} := \{x \in E(\xi), \quad \rho(x) < \varepsilon(\text{pr}(x))\} \subset E(\xi)$$

  inside M such that the following diagram commutes



We say that a tubular neighbourhood (ξ', ε', e') is a *shrinking* of (ξ, ε, e) when

- $\xi' = \xi$
- For every $s \in S$, $\varepsilon'(s) \leqslant \varepsilon(s)$
- The embedding e' is the restriction of the embedding e to $E(\xi')^{<\varepsilon'} \subset E(\xi)^{<\varepsilon}$

Remark A.2.2 We often identify a tubular neighbourhood with the associated subspace $T := e\left(E(\xi)^{<\varepsilon}\right)$ of M. We still denote $\mathrm{pr}\colon T \longrightarrow S$, $\varepsilon\colon T \longrightarrow \mathbf{R}^+ - \{0\}$, $\rho\colon T \longrightarrow \mathbf{R}^+$ the images through e of the previous maps pr, ε and ρ.

The theory for tubular neighbourhoods, in particular existence and uniqueness, is classical and can be found in [23] for example. The theory extends in the equivariant stratified case, on which we focus next.

This theory was developed in the 70's by John Mather in his work on topological stability [25], following ideas of Thom. (For the equivariant case, see [29, chapter 4] or [11].) An important notion uncovered is that of a system of controlled tubular neighbourhoods. Given a stratified manifold M, these are families of tubes of the strata which are compatible one with the others.

Definition A.2.3 Let $(M, \mathcal{S})$ be a stratified manifold.

(1) A system of tubular neighbourhoods of $\mathcal{S}$, say $\mathcal{T}$, is the datum for each stratum $S \in \mathcal{S}$ of a tubular neighbourhood (each stratum is a genuine submanifold of M) T_S of S inside M.

(2) One says that $\mathcal{T}$ is controlled when the following holds (compare with [29, § 3.6.4 and proposition 3.6.7])

- For all strata R and S in $\mathcal{S}$, one has
$$(T_R \cap T_S \neq \emptyset) \implies (R \prec S \text{ or } R = S \text{ or } R \succ S)\,.$$
- For every pair of strata $R \succ S$ in $\mathcal{S}$,
$$\begin{aligned} x \in T_R \cap T_S &\implies \mathrm{pr}_R(x) \in T_S \\ \mathrm{pr}_S \circ \mathrm{pr}_R(x) &= \mathrm{pr}_R(x) \\ \text{and} \quad \rho_S \circ \pi_R(x) &= \rho_S(x) \end{aligned}$$
  The space $T_R \cap T_S$ is then the restriction of the tube T_R to the open subspace $T_S \cap R \subset R$ and the space $T_S \cap \overline{R}$ can be thought of as a tubular neighbourhood of the stratum S inside the adjacent stratum R.

(3) When $(M, \mathcal{S})$ admits such a system of tubes, we say that $(M, \mathcal{S})$ is controllable.

Definition A.2.4 Let $(M, \mathcal{S})$ and $(M', \mathcal{S}')$ be two controlled stratified manifolds. A stratified map $f\colon M \longrightarrow M'$ is said to be controlled if for any connected component S_0 of a stratum $S \in \mathcal{S}$, there exists a stratum $R_{S_0} \in \mathcal{S}'$ such that:

- $f(T_{S|S_0}) \subset T_{R_{S_0}}$

- For all x in T_{S_0}, $(f \circ \mathrm{pr}_S)(x) = (\mathrm{pr}_{R_{S_0}} \circ f)(x)$
- For all x in T_{S_0}, $\rho_S(x) = (\rho_{R_{S_0}} \circ f)(x)$

In general, it is not *a priori* clear that such systems of tubes exist. The following theorem, due to Mather, ensures the existence for Whitney-regular stratifications. (It is a special case of [29, theorem 3.6.9].)

Theorem A.2.5 *Any Whitney-regular stratified manifold is controllable.*

The following theorem is lemma 3.9.2 in [29]. With the same proof, it admits an equivariant version. See [12, proof of lemma 5.3, p. 443], and [2, proof of theorem 1.5, p. 465].

Theorem A.2.6 (Thom's first isotopy lemma) *Let $(M, \mathcal{S})$ be a controlled space, N be a manifold with trivial stratification. Then, a proper controlled submersion $f\colon M \longrightarrow N$ is a locally trivial fibration (of stratified spaces).*

The following lemma is an *ad hoc* result which is useful for our analysis of the stratification $\mathcal{S}^Y$ of $(\mathcal{R}_n Y)(\mathbf{C})$ in section 3.

Lemma A.2.7 (Restriction lemma, *cf* lemma 3.2.8) *Let G be a finite group, $(M, \mathcal{S})$ be stratified Whitney regular G–manifold, S a G–stratum and T_S a G–tubular neighbourhood of S inside M. Assume that S G–equivariantly deformation retracts onto a sub-G–manifold S°. Then, up to shrinking, the tube T_S deformation retracts G–equivariantly and compatibly with the stratification onto its restriction over S°, say T_S°. In particular, for all G–stratum $R \succeq S$, the space $T_R \cap S$ deformation retracts G–equivariantly onto $T_S^\circ \cap R$.*

Proof For lack of a precise reference, we indicate the proof of this result. For simplicity, assume $G = 1$. Let $\rho\colon S \longrightarrow S^\circ$ be the deformation retraction of S onto S° and let $i\colon S^\circ \hookrightarrow S$ be the inclusion. By definition, there exists a homotopy $\widetilde{\rho}\colon S \times I \longrightarrow S$ between $\widetilde{\rho}_0 = \mathrm{id}_S$ and $\widetilde{\rho}_1 = i \circ \rho$. Let then T be the fibre product of $S \times I$ and T_S over S:

$$\begin{array}{ccc} T & \longrightarrow & T_S \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_R} \\ S \times I & \xrightarrow{\ \widetilde{\rho}\ } & S \end{array}$$

The map $T \longrightarrow T_S$ is a submersion and induces a Whitney-regular stratification of T, lifting that of T_S. Thom's first isotopy lemma (*cf* theorem A.2.6) implies that the canonical projection $p\colon\ T \longrightarrow S\times I$ is a locally trivial fibration. Thus, for all x in $S\times I$, there exists an open neighbourhood U of x and a stratified space F with a homeomorphism of stratified spaces:

$$p^{-1}(U)\simeq U\times F\,.$$

Since a locally trivial fibration over a base of the form $B = X\times I$ is trivial, we have a homeomorphism of stratified spaces $\varphi\colon\ T_S\times I\simeq T$. The composite map $T_S \xrightarrow{\simeq} T_{|S\times\{1\}} = \rho^*(T_S^\circ) \longrightarrow T_S^\circ$ gives a deformation retraction of T_S onto T_S° which respects the stratification. □

Definition A.2.8 Let $(M,\mathcal{S})$ be a stratified manifold.

(1) A stratified vector field V on M is the datum, for every stratum $S\in\mathcal{S}$, of a (smooth) vector field V_S on S.

(2) A stratified vector field V on M is *controlled* when there exists a controlled system of tubular neighbourhoods of $\mathcal{S}$ such that for every pair of strata $R\succ S$

$$T\mathrm{pr}_S\circ V_{T_S\cap R} = V\circ(\mathrm{pr}_S)_{|T_S\cap R} \quad\text{and}\quad T_{\rho_S}\circ V_{|T_S\cap R} = 0\,.$$

Remark A.2.9 Note that in general a stratified vector field V does not induce a continuous vector field globally on M. However, when V is controlled, the associated flow is continuous.

The following proposition, which is proved in [29, theorem 3.7.3 and theorem 3.7.6], is useful to construct stratified vector fields.

Proposition A.2.10 *Let M be a controlled stratified manifold, B be a manifold (endowed with its trivial stratification) and let $f\colon\ M\longrightarrow B$ be a controlled submersion. Then, for every vector field $W\colon\ B\longrightarrow \mathrm{T}B$ on the base, there exists a controlled stratified vector field $V\colon\ M\longrightarrow \mathrm{T}M$ such that*

$$\mathrm{T}f\circ V = W\circ f\,.$$

Moreover, when f is assumed to be proper and W to be integrable, then V is also integrable.

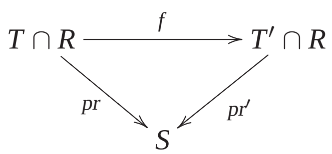
Definition A.2.11 Let $(M,\mathcal{S})$ be a stratified manifold, M_1 and M_2 be two sub-stratified-manifolds of M and let $\iota_i\colon\ M_i\hookrightarrow M$ denote the canonical inclusions $(i=1,2)$. A stratified isotopy between M_1 and M_2 inside M is a map $F\colon\ M_1\times I\longrightarrow M$ satisfying:

- For every $t \in [0,1]$, the map $F_t = f_{|M_1 \times \{t\}}$ is a stratified homeomorphism from M_1 onto its image in M.
- $F_0 = \iota_1$
- There exists a stratified homeomorphism $\varphi\colon\ M_1 \simeq M_2$ such that $F_1 = \iota_2 \circ \varphi$.

We define similarly the notion of a stratified isotopy in the relative case over a base B.

With this notion, we can state the uniqueness of tubular neighbourhoods in the (equivariant) Whitney-regular stratified context.

Theorem A.2.12 (Uniqueness of tubular neighbourhoods) *Let G be a finite group. Let $(M, \mathcal{S})$ be a Whitney-regular G–stratified manifold, S be G–stratum and T and T' be two G–equivariant tubular neighbourhoods of S inside M. The images of T and T' inherit Whitney-regular G–stratified manifold structures. Then, up to shrinking the tubes T and T', there exists a G–stratified isotopy between T and T' over the base S. In particular, for every adjacent G–stratum $R \succeq S$, after possibly shrinking T and T', there exists a G–equivariant homeomorphism $f\colon\ T \cap R \simeq T' \cap R$ such that the following diagram commutes*



Proof One can assume, up to shrinking T, that $T \subset T'$. Let then $\iota\colon\ T \times I \hookrightarrow T' \times I$ be an isotopy between T and T' given by the classical uniqueness of tubular neighbourhoods. So, *a priori*, the isotopy ι does not respect the stratification. The image through ι of the vector field $(0, \frac{\partial}{\partial t})$ is a vector field, say V, on $T' \times I$. By construction, V is integrable and the associated flow maps $\iota_1(T)$ on $\iota_0(T)$.
Consider the space $T' \underset{S}{\times} T'$, with its two projections pr_1 and pr_2 on T'. This space $T' \underset{S}{\times} T'$ is endowed with the inverse image stratification induced by pr_1. By proposition A.1.7, this stratification is Whitney-regular. Its set of strata is in bijection with that of the stratification of T': a point $(x_1, x_2) \in T' \underset{S}{\times} T'$ belongs to a stratum R if and only if, the point $x_1 \in T'$ is in R .
By proposition A.2.10 applied to the projection $p_2 \times \mathrm{id}$, there exists a controlled stratified vector field W on $T' \underset{S}{\times} T' \times I$, which is integrable and such that $(p_2 \times \mathrm{id})_*(W) = V \circ (p_2 \times \mathrm{id})$. Let $D \simeq T' \times I \hookrightarrow T' \underset{S}{\times} T' \times I$ be the "diagonal" subspace. The image

through $(p_1 \times \mathrm{id})_*$ of the restriction of W to D gives a controlled stratified vector field $\widetilde{V}$ on $T' \times I$, which is integrable. By construction, for every point (x,t) in $T' \times I$, we have

$$\mathrm{pr}'_*(V(x,t)) = \mathrm{pr}'_*(\widetilde{V}(x,t))$$

(pr' denoting here the canonical projection from T' onto S).
Integrating this flow gives a stratified isotopy between T and (a shrinking of) T'. □

Note that the same techniques as in the proof of theorem A.2.12 leads to the following *ad hoc* result, needed for our analysis of the stratification $\mathcal{S}^Y$ of $(\mathcal{R}_n Y)(\mathbf{C})$.

Proposition A.2.13 (Uniqueness of slices, *cf* lemma 3.2.9) *Let G be a finite group, $(M, \mathcal{S})$ be a Whitney-regular G–stratified manifold, S be a G–stratum, and Σ be a sub-G–manifold of M transverse to $\mathcal{S}$ (that is to say transverse to any stratum of $\mathcal{S}$). In particular, $\mathcal{S}$ induces a G–stratification of Σ which is still Whitney-regular.*
We denote S° the intersection $S \cap \Sigma$. Let also T be a G–tubular neighbourhood of S inside M, T° be the restriction of T over S° and T^Σ be a G–tubular neighbourhood of S° inside Σ. Then, the spaces T° and T^Σ inherit Whitney-regular G–stratified manifold structures and, after possibly shrinking them, there exists a G–equivariant stratified isotopy between T° and T^Σ over the base S°.

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Hausdorff center for mathematics, Endenicher Allee 60, Bonn, Germany.

cazanave@math.uni-bonn.de