Quantum optimal control
Application to Bose-Einstein Condensates in an optical lattice

Nice, October 15 2024.
QuantAzur days

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Quantum control

Manipulating the quantum dynamics of atoms, molecules and spins with external electromagnetic fields.

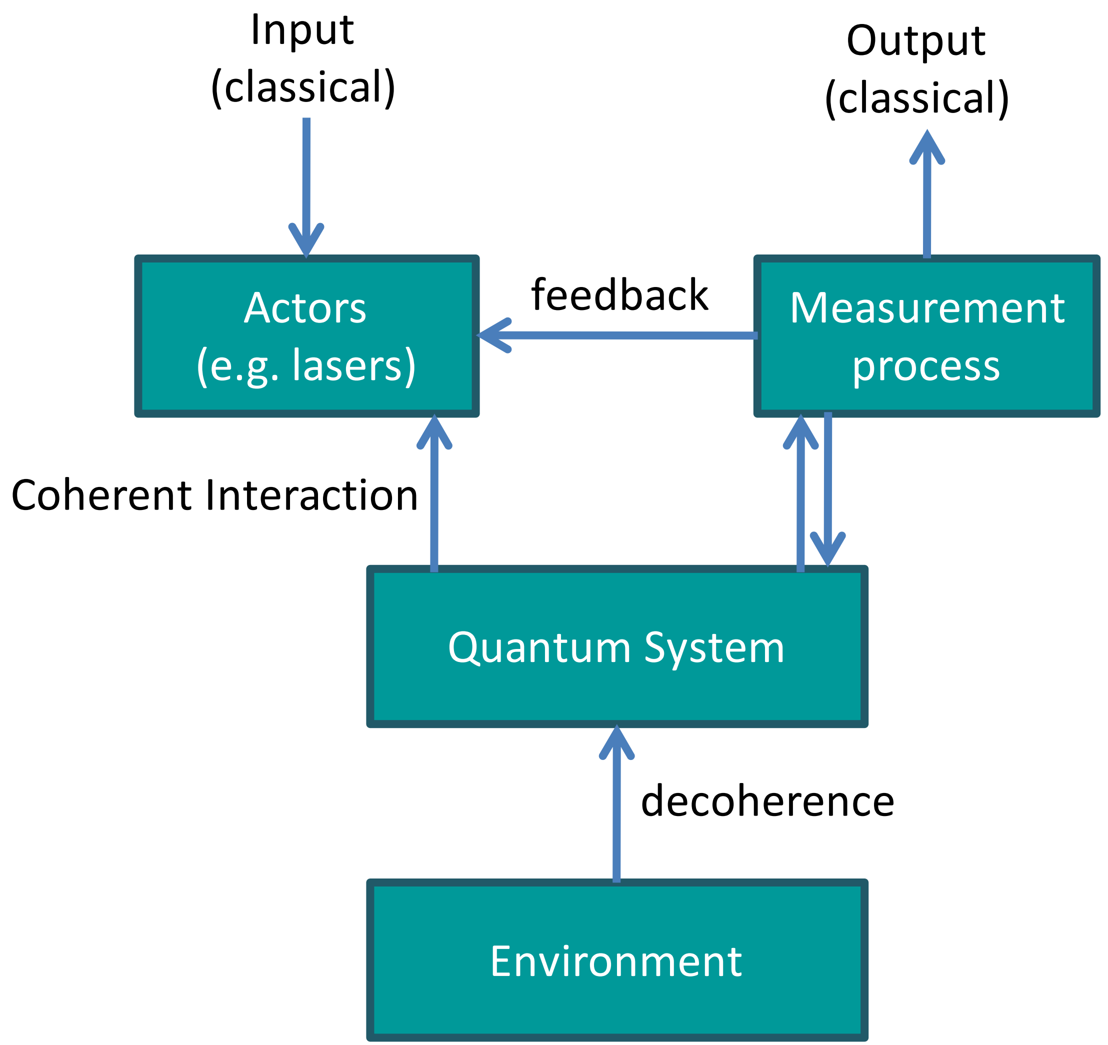
→ Design of specific electric or magnetic fields

→ Application of tools of control theory (Optimal control theory) to quantum physics

Control theory: Realization of basic operations (with a criterion)

Collaboration between Mathematicians, physicists, chemists, engineers…

Quantum control



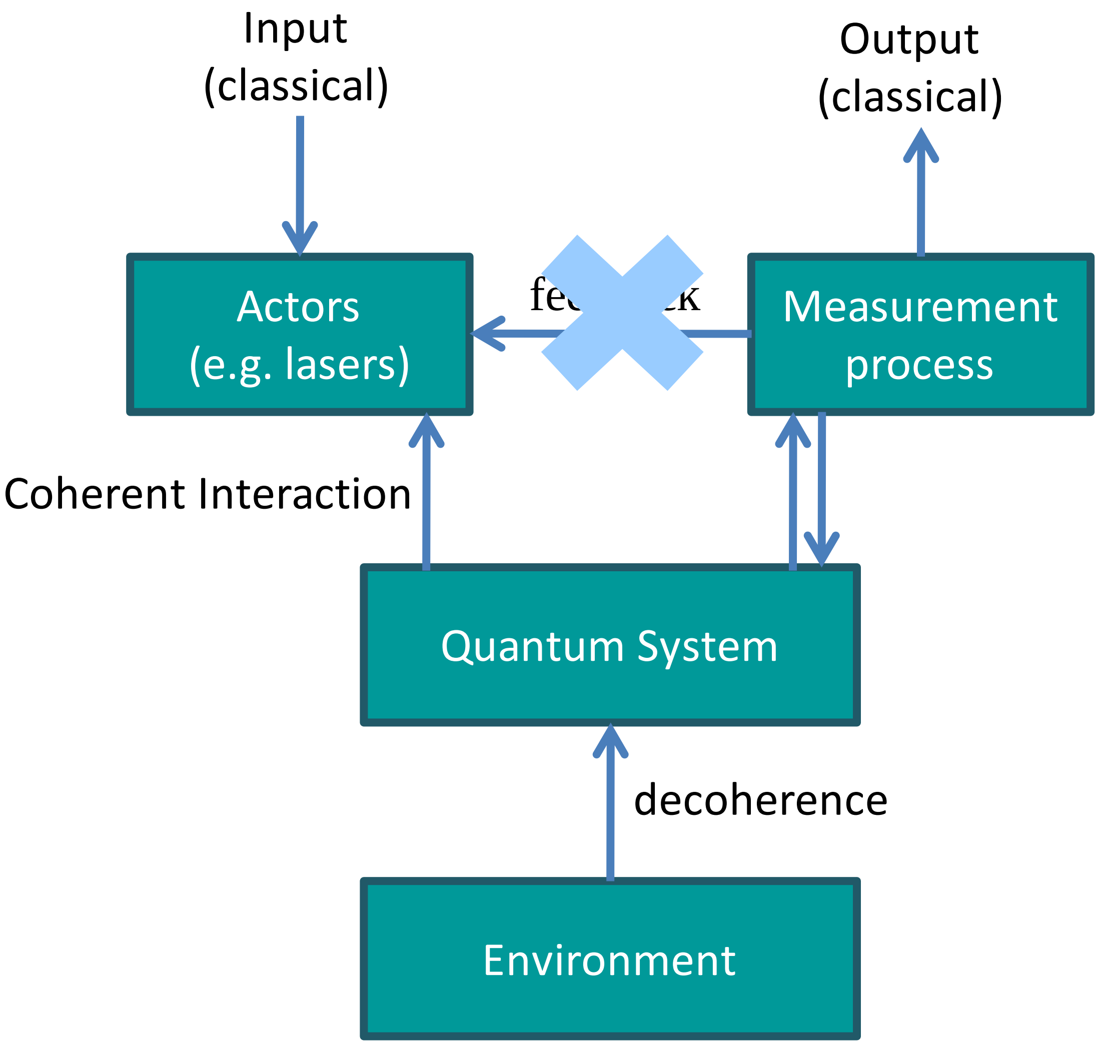
General description in the semi-classical case:

- Quantum system
- Classical control field

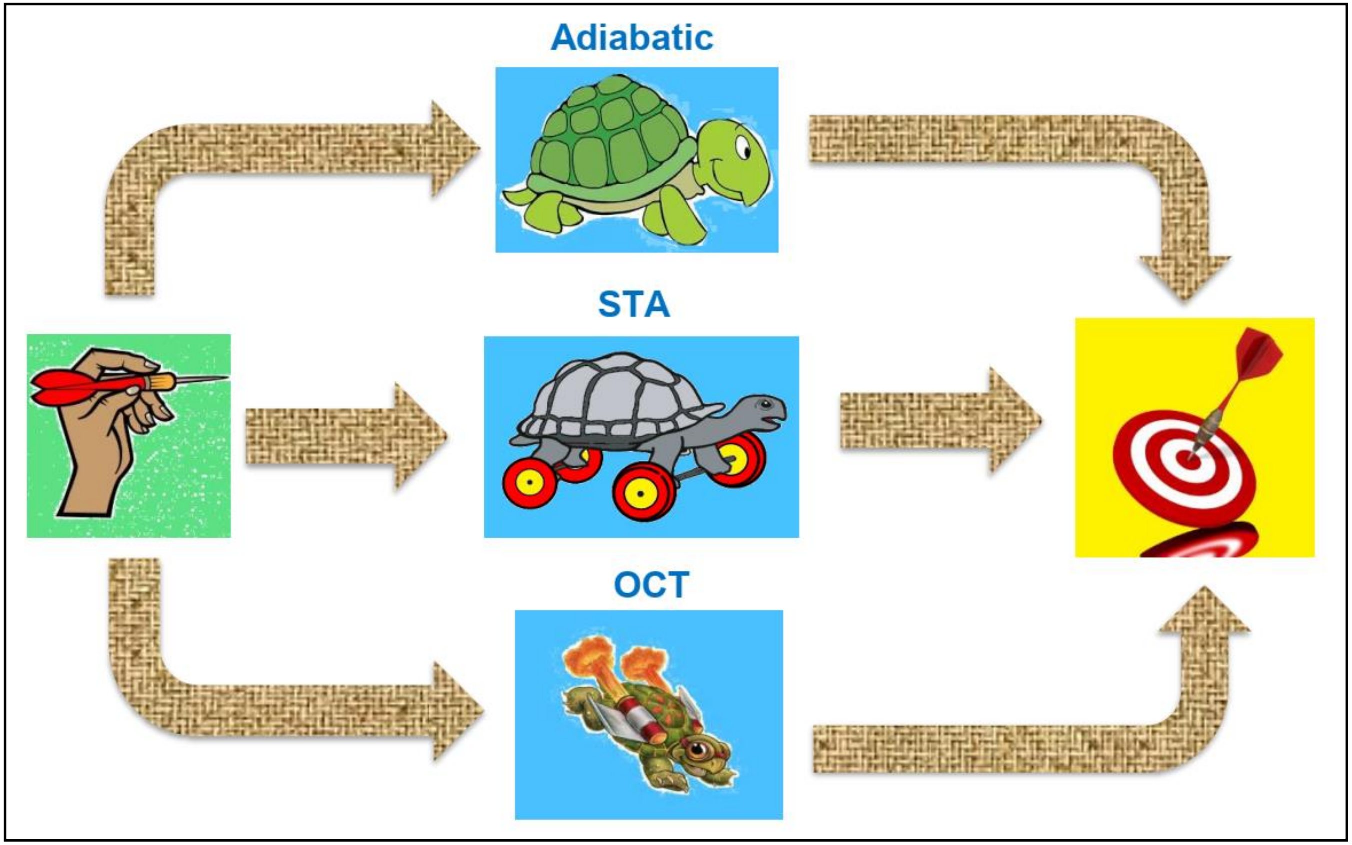
Quantum control

Open-loop control process:

- Modeling
- Calibration of unkonwn parameters



Quantum open-loop control: Different methods



- ➢ Adiabatic passage (N. Vitanov et al, RMP 2017)
- ➢ Shortcut to Adiabaticity (D. Guéry-Odelin et al, RMP 2019)
- ➢ Optimal control theory

✓ Different properties: control time, robustness, analytical expressions

References about Quantum Control

Optimal control theory will be a basic building block of quantum technologies

REVIEW

Quantum optimal control in quantum technologies. Strategic report on current status, visions and goals for research in Europe

Christiane P. Koch[1*], Ugo Boscain[2], Tommaso Calarco[3], Gunther Dirr[4], Stefan Filipp[5], Steffen Glaser[6,7], Ronnie Kosloff[8], Simone Montangero[9], Thomas Schulte-Herbrüggen[6,7], Dominique Sugny[10] and Frank K. Wilhelm[3,11]

C. P. Koch et al., EPJ Quantum Technol. 9, 19 (2022), 650 references

- Controllability (open quantum systems)
- Machine learning techniques
- STA and Quantum Speed Limit
- Experimental implementation (constraint)

- Quantum computing
- Quantum sensing
- Quantum simulation

✓ Optimal control allows us to improve the capabilities of experimental devices

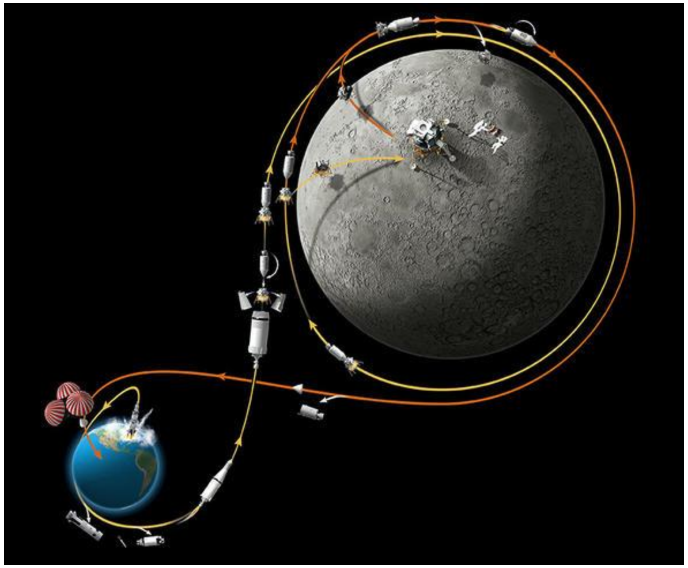
The modern history of optimal control: The PMP

➢ **Pontryagin Maximum Principle**: Generalization of the Euler-Lagrange principle for controlled systems (1960)

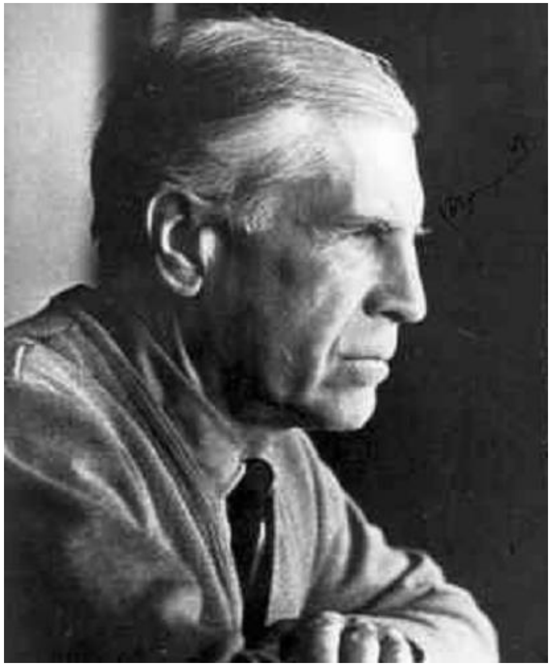
Principle : Design of a control field allowing to reach a target state while minimizing (or maximizing) a cost functional (time, energy).

Applications : Many different applications since 1960 (Electronics, space mechanics, economy…)

Some famous examples:



Apollo and Smart I



Lev Pontryagin 1908-1988

The weak PMP: Calculus of variations

- Definition of the optimal control problem: Modeling, cost functional

$$\dot{X}(t) = F(X(t), u(t), t),$$

State

Control

$$\mathcal{C} = G(X(t_f), t_f) + \int_0^{t_f} F_0(X(t), u(t), t)\,\mathrm{d}t,$$

Terminal cost

Running cost

Optimal control action

$$S = G(X(t_f), t_f) + \int_0^{t_f} \mathrm{d}t \underbrace{\left[F_0(X(t), u(t), t) + \Lambda(t) \cdot \left(\dot{X}(t) - F(X(t), u(t), t)\right)\right]}_{\mathcal{L}(X, \dot{X}, u, t, \Lambda)}.$$

Adjoint state

The weak PMP: Calculus of variations

- The functional derivative of the action is zero.

$$\delta S = \frac{\partial G}{\partial X(t_f)}\delta X(t_f) + \int_0^{t_f} \left[\frac{\partial F_0}{\partial X}\delta X + \frac{\partial F_0}{\partial u}\delta u + \delta\Lambda \cdot \left(\dot{X} - F\right) + \Lambda \cdot \left(\delta\dot{X} - \frac{\partial F}{\partial X}\delta X - \frac{\partial F}{\partial u}\delta u \right) \right].$$

Integrating by part, we obtain

$$\delta S = \left(\frac{\partial G}{\partial X(t_f)} + \Lambda(t_f) \right) \delta X(t_f) + \Lambda(0)\,\delta X(0) + \int_0^{t_f} dt \left(\left[\frac{\partial F_0}{\partial X} - \dot{\Lambda} - \Lambda \frac{\partial F}{\partial X} \right] \delta X + \left[\dot{X} - F \right] \delta\Lambda + \left[\frac{\partial F_0}{\partial u} - \Lambda \frac{\partial F}{\partial u} \right] \delta u \right).$$

Necessary conditions can be derived.

The weak PMP: Calculus of variations

- Necessary conditions for optimality

$$\dot{\Lambda} = \frac{\partial F_0}{\partial X} - \Lambda \frac{\partial F}{\partial X}, \; \Lambda(t_f) = -\frac{\partial G}{\partial X(t_f)},$$
$$\dot{X} = F, \; X(0) = X_0,$$
$$\Lambda \frac{\partial F}{\partial u} - \frac{\partial F_0}{\partial u} = 0.$$

Such conditions can be written with a Hamiltonian formulation

$$H_\mathrm{P} = \Lambda \cdot \dot{X} - \mathcal{L} = \Lambda \cdot F - F_0$$

If the state and adjoint state satisfy the extremal equations then

$$\delta S = \int_0^{t_f} \left(\frac{\partial F_0}{\partial u} - \Lambda \cdot \frac{\partial F}{\partial u} \right) \cdot \delta u(t) \, \mathrm{d}t \qquad \delta S = -\int_0^{t_f} \frac{\partial H_\mathrm{P}}{\partial u} \cdot \delta u(t) \, \mathrm{d}t.$$

Q. Ansel et al., *Introduction to theoretical and experimental aspects of quantum optimal control*, J. Phys B, 57, 133001 (2024)

Towards the Pontryagin Maximum Principle

We consider a general optimal control problem (formulation in real coordinates).

Definition of the control problem:

$$\dot{x}(t) = f[x(t),u(t)]$$

$$x(0) = x_0;\ x(t_f) = x_f$$

$$Min_{u(.)} \int_0^{t_f} f_0[x(t),u(t)]dt$$

Pontryagin Maximum Principle: A necessary condition with a set of coupled differential equations.

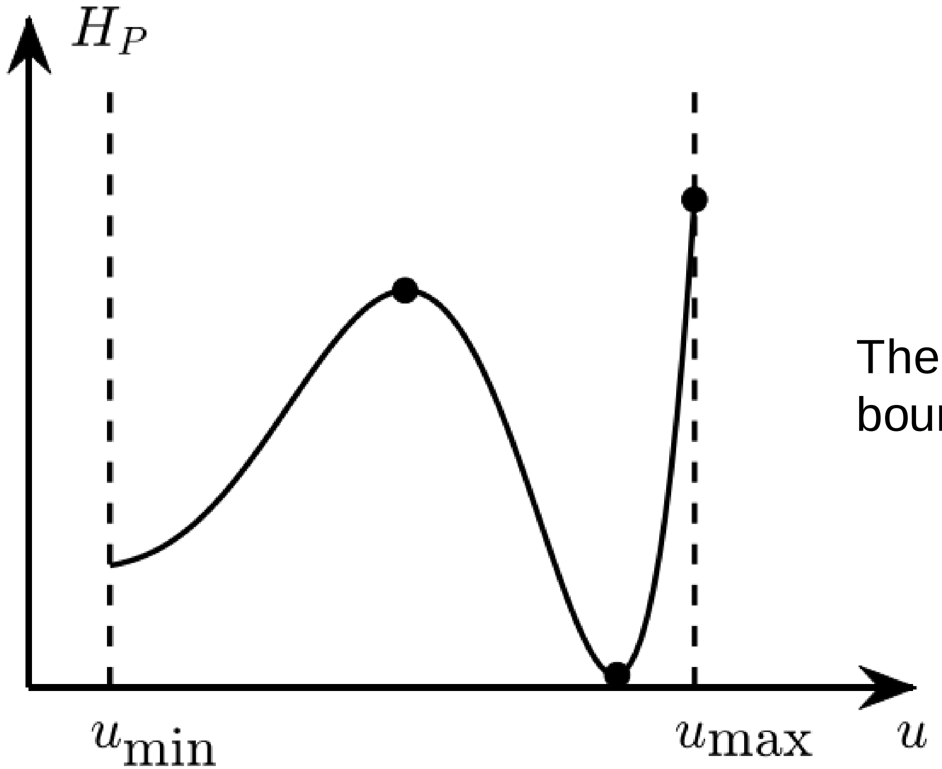
$$H_P = p \cdot f[x(t),u(t)] + p_0 f_0[x(t),u(t)]$$

$$\dot{x} = \frac{\partial H_P}{\partial p};\ \dot{p} = -\frac{\partial H_P}{\partial x}$$

$$Max_{u(t)} H_P[x(t), p(t), u(t)]$$

Comments about the PMP

- Constraints on the control can be taken into account



The maximum belongs to the boundary of the interval

- Abnormal extremals with the additional adjoint coordinate.

Optimal control of quantum systems

How to solve an optimal control problem ?

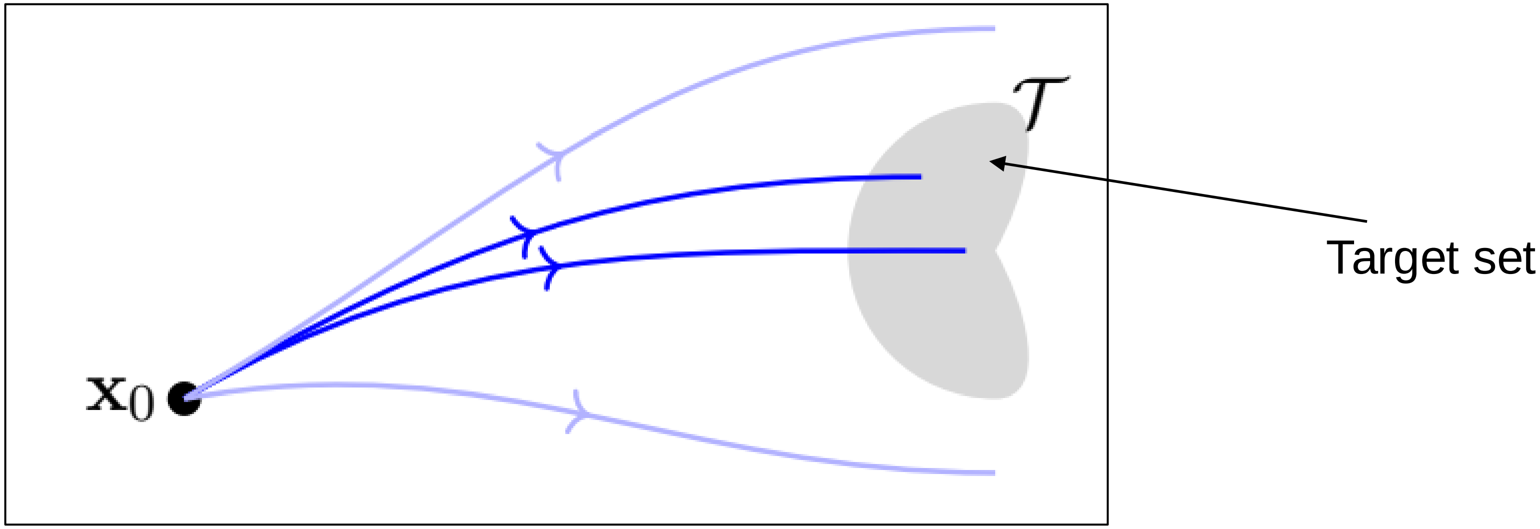
A rigorous mathematical framework: Pontryagin Maximum Principle

Different steps:

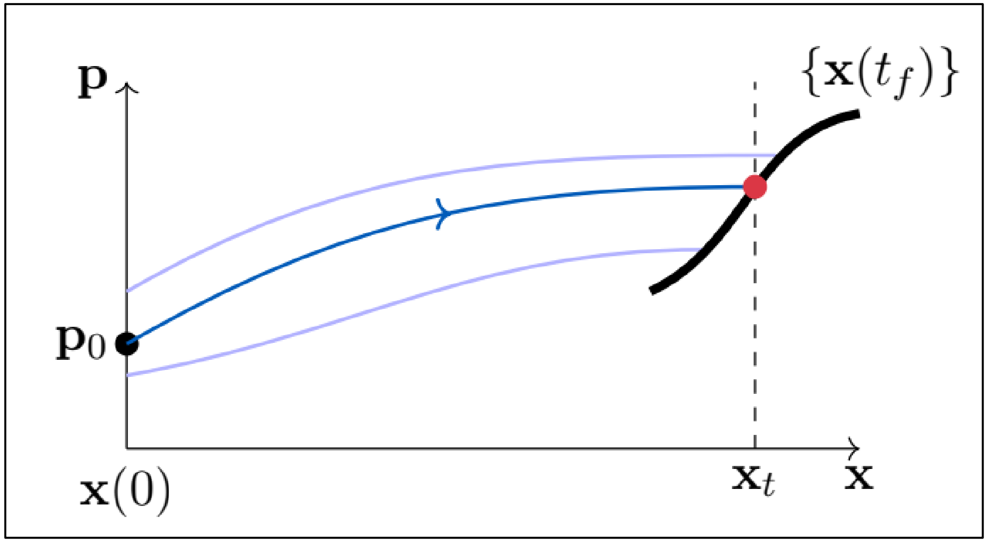
- **S1** - Definition of the task to realize (and of the optimal way)
- **S2** - Controllability: Is it possible to realize this task ?
- **S3** - How to deal with the constraints ?
- **S4** - Solution of the control problem: Global or local optimum, analytic or numeric

Qualitative description of the PMP

The system has different trajectories according to the control. What is the best one?



The PMP transforms the optimal control problem into a pseudo-Hamiltonian problem: **Finite-dimensional** optimization problem



Shooting algorithm: Find the initial adjoint state that reaches the target.

A brief history of quantum control

Theoretical and experimental progress was made in the eighties.

- Brumer-Shapiro: Role of quantum inferferences
- Tannor-Rice: Pump-dump control
- Bergmann: STIRAP and adiabatic techniques

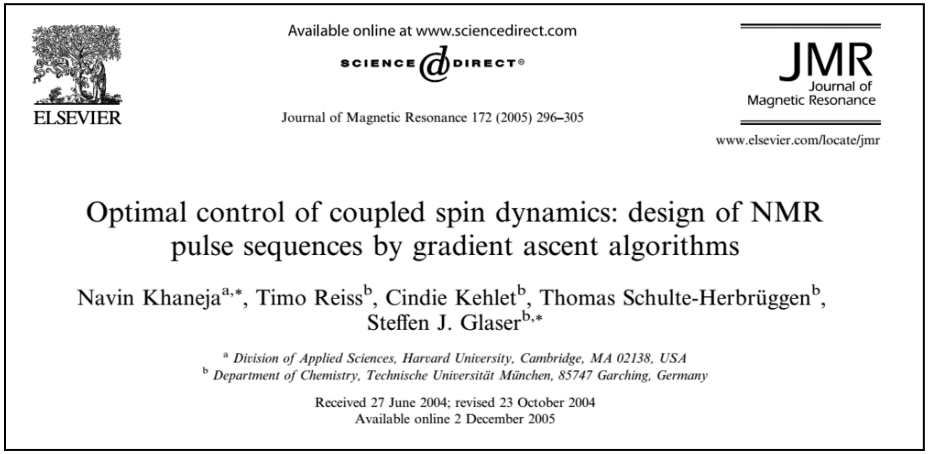
Intuitive methods with a limited efficiency: Optimization with one or several parameters.

At the end of the eighties: Tannor, Kosloff, Rabitz….

Introduction of more general approaches: Optimal control theory
Applications in molecular physics and Nuclear Magnetic Resonance.

Ref.: GRAPE algorithm

A renewal of the interest in the 2000s with the advent of mathematical and numerical tools.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Optimal control of coupled spin dynamics: design of NMR pulse sequences by gradient ascent algorithms

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Quantum optimal control theory

Development of optimal control techniques in quantum physics

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Tutorial

Introduction to the Pontryagin Maximum Principle for Quantum Optimal Control

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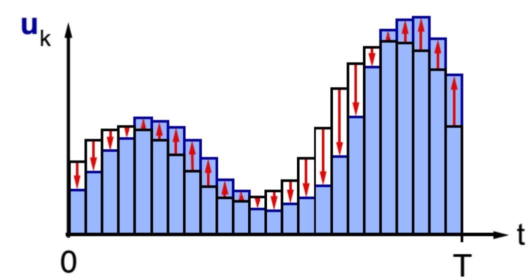
According to the dimension of the control problem, analytical or numerical solutions can be used.

✓ **Analytical solution**:
Global solution, analytical expressions, systems with parameters, specific constraints…

✓ **Numerical solution**:
Local solution, flexibility, experimental limitations and constraints…

Numerical optimization:
A piecewise constant pulse



Application of the PMP in quantum control

We consider the case of a pure state described by the Schrödinger equation (regular case and no constraint on the control).

Definition of the control problem:

$$\hat{H}(t) = \hat{H}_0 + \sum_{k=1}^{m} u_k(t)\hat{H}_k,$$

$$\frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t} = -\imath\hat{H}(t)|\psi\rangle,$$

$$S = G(|\psi(t_f)\rangle) + \int_0^{t_f} \mathrm{d}t\left(F_0(|\psi\rangle, u, t) + \Re\left(\langle\chi|\dot{\psi}\rangle + \imath\langle\chi|\hat{H}(t)|\psi\rangle\right)\right).$$

Application of the PMP:

$$H_\mathrm{P} = \Re\left(\langle\chi|\dot{\psi}\rangle\right) + \chi_0 F_0(|\psi\rangle, u, t),$$

$$H_\mathrm{P} = \Im\left(\langle\chi|\hat{H}(t)|\psi\rangle\right) + \chi_0 F_0(|\psi\rangle, u, t)$$

$$|\dot{\psi}\rangle = 2\frac{\partial H_\mathrm{P}}{\partial\langle\chi|} = -\imath\hat{H}|\psi\rangle,$$

$$\langle\dot{\chi}| = -2\frac{\partial H_\mathrm{P}}{\partial|\psi\rangle} = \imath\langle\chi|\hat{H} - 2\chi_0\frac{\partial F_0}{\partial|\psi\rangle},$$

$$\langle\chi(t_f)| = 2\chi_0\frac{\partial G}{\partial|\psi(t_f)\rangle}$$

$$\frac{\partial H_P}{\partial u_k} = \Im\left(\langle\chi|\hat{H}_k(t)|\psi\rangle\right) + \chi_0\frac{\partial F_0}{\partial u_k}(|\psi\rangle, u, t) = 0.$$

The same can be done for mixed states and for the evolution operator.

How to solve the PMP: Analytical or numerical solution

According to the dimension of the control problem, analytical or numerical solutions can be used.

✓ **Analytical solution**:
Global solution, analytical expressions, systems with parameters, specific constraints…

✓ **Numerical solution**:
Local solution, flexibility, experimental limitations and constraints…

Gradient-based optimization algorithms:

- Weak formulation of the PMP (no constraint on the control)
- First order variation of the cost: $\delta S = -\epsilon \int_0^{t_f} \left(\frac{\partial H_{\rm P}}{\partial u} \right)^2 \mathrm{d}t, \quad \delta u = \epsilon \frac{\partial H_{\rm P}}{\partial u}$
- An iterative optimization process can be defined.

Q. Ansel et al., *Introduction to theoretical and experimental aspects of quantum optimal control*, J. Phys B, 57, 133001 (2024)

Gradient-based optimization algorithm

(0) Choice of a guess field:

$$u_0$$

(i) Forward and backward propagations:

$$|\chi_k(t)\rangle \leftarrow |\chi_f\rangle$$

$$|\psi(0)\rangle \rightarrow |\psi_k(t)\rangle$$

(ii) Computation of the new field:

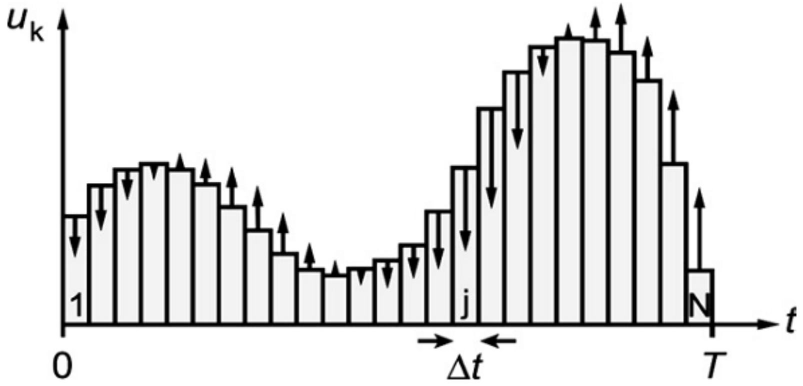
$$u_{k+1}(t) = u_k(t) + \varepsilon \left.\frac{\partial \mathrm{H_p}}{\partial u(t)}\right|_{u_k(t)}$$

Gradient-based optimization algorithm in quantum control

Quantum dynamics

$$i\dot{\hat{U}}(t) = [\hat{H}_0 + u(t)\hat{H}_1]\hat{U}(t)$$

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$$



We assume that the control *u* is a piecewise constant function with *N* time steps.

Time step $\Delta t = t_f / N$

Control discretization

$$n \in \{1, \cdots, N\}$$

$$u = (u_1, u_2, \cdots, u_N)$$

The cost to minimize is not a **functional** but a **function** of *N* variables that can be optimized using a standard gradient method.

Gradient-based optimization algorithm in quantum control

Cost functional to minimize: $\mathcal{C}(u)$

Construction of an iterative algorithm to find a local minimum: $\mathcal{C}(u') < \mathcal{C}(u)$

Hyp.: $$u'_n = u_n - \epsilon \frac{\partial \mathcal{C}(u)}{\partial u_n}$$

$$\epsilon > 0$$

Proof: $$\begin{aligned} \mathcal{C}(u') &= \mathcal{C}\left(u - \epsilon \frac{\partial \mathcal{C}(u)}{\partial u}\right) \\ &= \mathcal{C}(u) - \epsilon \left(\frac{\partial \mathcal{C}(u)}{\partial u}\right)^2 + O(\epsilon^2) \\ &\leq \mathcal{C}(u). \end{aligned}$$

The difficult point is to calculate the derivative of the cost functional with respect to the control.

Gradient-based optimization algorithm in quantum control

We introduce the following notations:

Propagator for a time step:

$$\hat{U}_n = \hat{U}(n\Delta t, (n-1)\Delta t) = e^{-i\Delta t(\hat{H}_0 + u_n \hat{H}_1)}$$

Discretization of the time evolution of the wave function:

$$|\psi(t_f)\rangle = \hat{U}_N \hat{U}_{N-1} \cdots \hat{U}_1 |\psi_0\rangle$$

$$|\psi_n\rangle = \hat{U}_n \cdots \hat{U}_1 |\psi_0\rangle$$

We consider a special cost functional corresponding to a terminal cost:

$$\mathcal{C} = G(|\psi(t_f)\rangle) = G(\hat{U}_N \hat{U}_{N-1} \cdots \hat{U}_1 |\psi_0\rangle)$$

The derivative can be expressed as:

$$\frac{\partial G}{\partial u_n} = \frac{\partial G}{\partial |\psi(t_f)\rangle} \frac{\partial |\psi(t_f)\rangle}{\partial u_n} + \frac{\partial \langle\psi(t_f)|}{\partial u_n} \frac{\partial G}{\partial \langle\psi(t_f)|}$$

Gradient-based optimization algorithm in quantum control

The derivative can be expressed as:

$$\frac{\partial G}{\partial u_n} = \frac{\partial G}{\partial |\psi(t_f)\rangle} \frac{\partial |\psi(t_f)\rangle}{\partial u_n} + \frac{\partial \langle \psi(t_f)|}{\partial u_n} \frac{\partial G}{\partial \langle \psi(t_f)|}$$

Expanding the wave function, we have:

$$\frac{\partial G}{\partial u_n} = \frac{\partial G}{\partial |\psi(t_f)\rangle} \hat{U}_N \cdots \frac{\partial \hat{U}_n}{\partial u_n} \cdots \hat{U}_1 |\psi_0\rangle + \langle \psi_0 | \hat{U}_1^\dagger \cdots \frac{\partial \hat{U}_n^\dagger}{\partial u_n} \cdots \hat{U}_N^\dagger \frac{\partial G}{\partial \langle \psi(t_f)|}$$

The only factor to calculate is thus: $\frac{\partial \hat{U}_n}{\partial u_n}$

The calculation is not trivial since

$$[\hat{H}_0, \hat{H}_1] \neq 0$$

A formal expression can be given to this derivative by using the **Wilcox formula.**

Gradient-based optimization algorithm in quantum control

Wilcox formula:

$$\frac{\partial e^{tA}}{\partial \theta} = e^{tA} \int_0^t e^{-t'A} \frac{\partial A}{\partial \theta} e^{t'A} dt'$$

Using this formula for the evolution operator, we have:

$$\frac{\partial \hat{U}_n}{\partial u_n} = -i\Delta t \hat{U}_n \overline{\hat{H}}_1$$

Time average of the interaction Hamiltonian:

$$\overline{\hat{H}}_1 = \frac{1}{\Delta t} \int_0^{\Delta t} e^{it'(\hat{H}_0 + u_n \hat{H}_1)} dt' \hat{H}_1 e^{-it'(\hat{H}_0 + u_n \hat{H}_1)} dt'$$

For a sufficient small time step:

$$\frac{\partial \hat{U}_n}{\partial u_n} \simeq -i\Delta t \hat{H}_1 \hat{U}_n$$

Gradient-based optimization algorithm in quantum control

We introduce the adjoint state: $|\chi\rangle$

The adjoint state is defined from a backward propagation:

$$\langle\chi_n| = \langle\chi_N|\hat{U}_N \cdots \hat{U}_n$$

The initial state is given by the gradient of the cost

$$\langle\chi(t_f)| = \langle\chi_N| = \frac{\partial G}{\partial|\psi(t_f)\rangle}$$

We obtain

$$\frac{\partial G}{\partial u_n} = -i\Delta t(\langle\chi_n|\hat{H}_1|\psi_n\rangle - \langle\psi_n|\hat{H}_1|\chi_n\rangle)$$

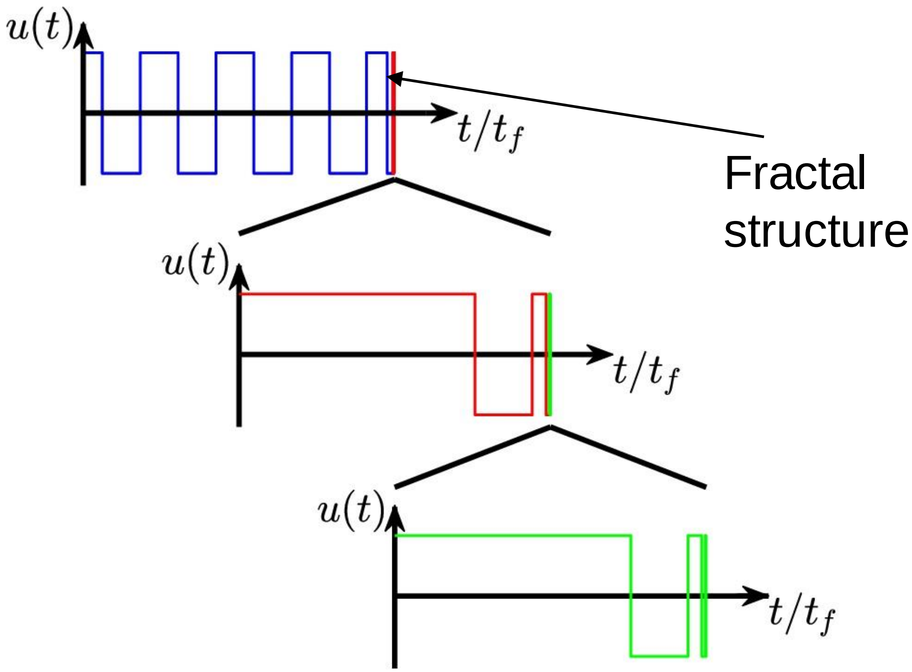
$$\frac{\partial G}{\partial u_n} = 2\Delta t\Im(\langle\chi_n|\hat{H}_1|\psi_n\rangle)$$

The correction to the control:

$$\boxed{\delta u_n = u_n' - u_n = -\epsilon\Im(\langle\chi_n|\hat{H}_1|\psi_n\rangle)}$$

Application of Optimal Control Theory: Chattering process

- ✓ Optimal control as a function of time is generally a piecewise function.
- ✓ In 1961, A. T. Fuller found an example (electronics) with an infinite number of switchings on a finite-time interval: **Chattering process**
- ✓ In 1990, I. Kupka showed that this phenomenon is generic.



Is it possible to observe this phenomenon in quantum control?

R. Robin, U. Boscain, M. Sigalotti and D. Sugny, *Chattering phenomenon in Quantum Optimal control*, New J. Phys. 24 (2022) 123037

Application of Optimal Control Theory: Chattering process

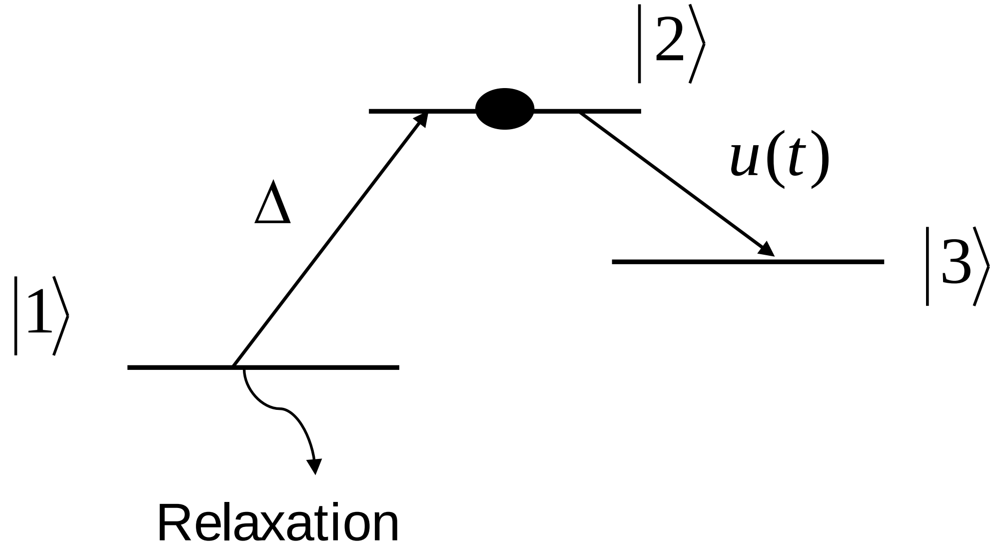
Is it possible to observe this phenomenon in quantum control?

- Three-level quantum system with a fixed control and a time-dependent one.

The goal is to transfer the system from the level |2> to the level |3> while minimizing the population in level |1> (relaxation process).

The control time is free and $|u(t)| \leq 1$

$$\min_{u(t)} \int_0^{t_f} \left|\langle 1|\psi(t)\rangle\right|^2 dt$$



R. Robin, U. Boscain, M. Sigalotti and D. Sugny, *Chattering phenomenon in Quantum Optimal control*, New J. Phys. 24 (2022) 123037

Application of Optimal Control Theory: Chattering process

- The model system: Dynamics on the sphere

$$\dot{\mathbf{X}} = (\Delta\Omega_3 + u(t)\Omega_1)\mathbf{X},$$

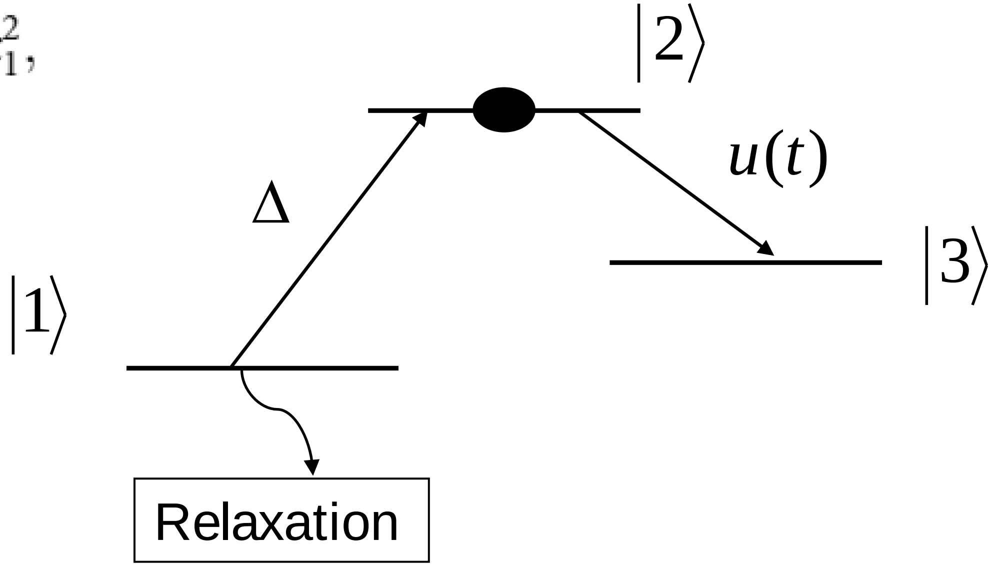
$$\Omega_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Omega_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Constraint on the control

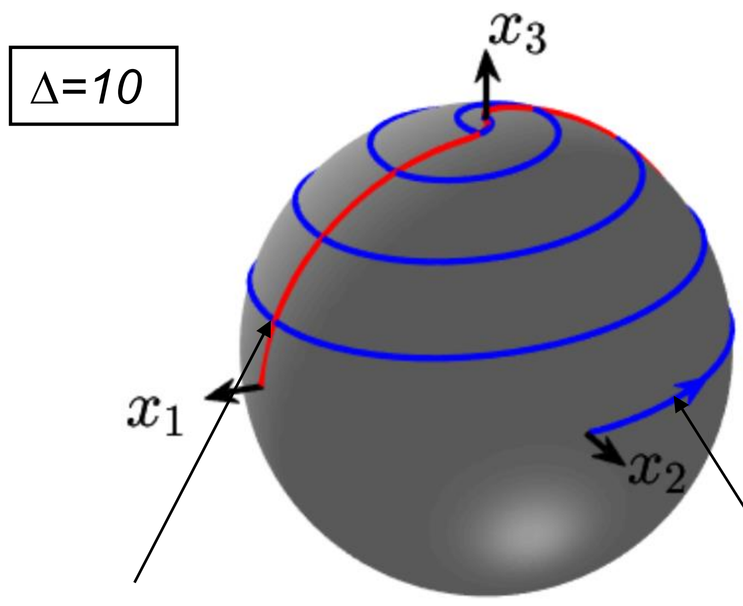
$$|u(t)| \leqslant u_0$$

- Application of the PMP: Bang-bang controls

$$H_P = \mathbf{P} \cdot (\Delta\Omega_3 + u\Omega_1)\mathbf{X} + p^0 x_1^2,$$

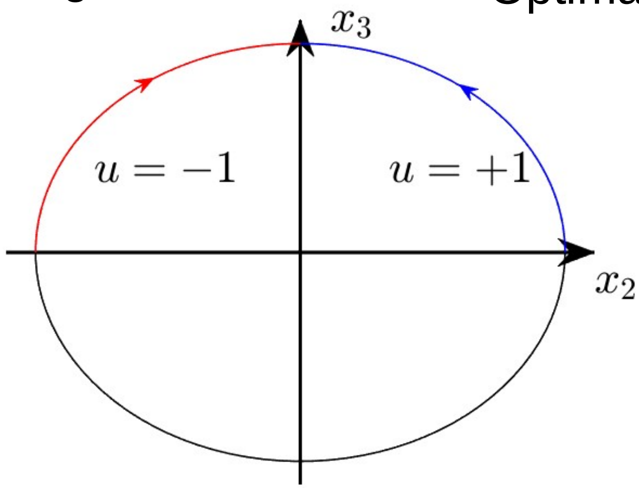


Application of Optimal Control Theory: Chattering process

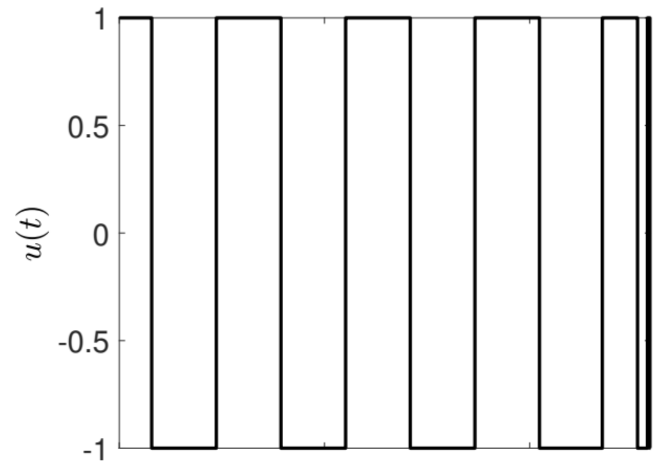


Switching curves

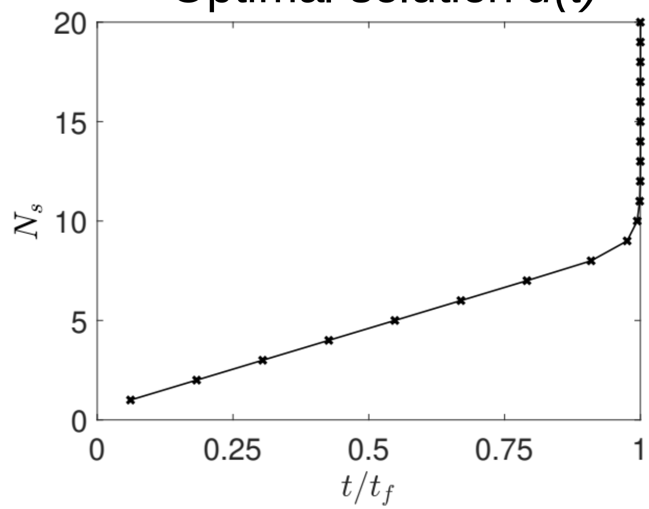
Optimal trajectory



Control Mechanism



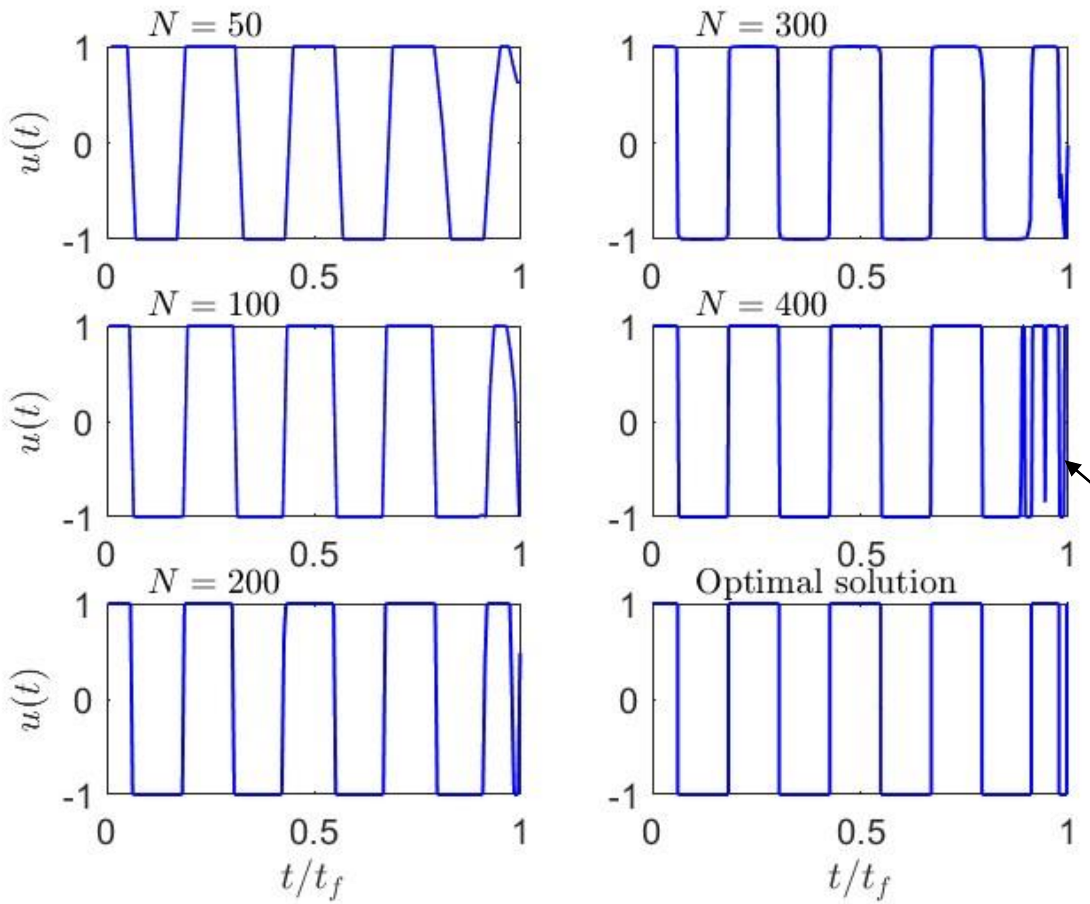
Optimal solution *u(t)*



✓ A bang-bang optimal control with an infinite number of switchings.

Application of Optimal Control Theory: Chattering process

✓ This structure leads to instabilities in the numerical optimization process.



Numerical instabilities

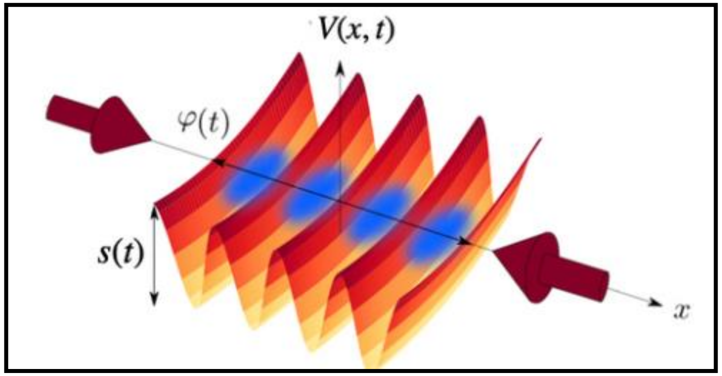
Numerical optimization with a direct optimal method

BEC: A powerful platform for quantum technologies

Manipulating the quantum dynamics of BEC for quantum simulation, quantum computing and quantum sensing applications.

- Potentials can be shaped with light (optical lattice)
- Atom interaction can be controlled
- Quantum state can be measured with precision

Optimal Control: A shaken optical lattice



Joint work with the experimental groups of D. Guéry-Odelin and A. Gauguet (Toulouse, France).

Q. Ansel et al., *Introduction to theoretical and experimental aspects of quantum optimal control*, J. Phys B, 57, 133001 (2024)

Matter waves under optimal control

Horizontal configuration of a BEC in a one-dimensional optical lattice (we neglect atomic interactions, magnetic potential): **A quantum planar pendulum**

$$\imath\hbar\frac{d|\psi(t)\rangle}{dt} = \left(\frac{\hat{p}^2}{2m} - \frac{s(t)E_L}{2}\cos\left(k_L\hat{x} + \varphi(t)\right)\right)|\psi(t)\rangle$$

In dimensionless coordinates:

$$\imath\frac{d|\psi(t)\rangle}{dt} \equiv \hat{H}(t)|\psi(t)\rangle = \left(\hat{p}^2 - \frac{s}{2}\cos\left(\hat{x} + \varphi(t)\right)\right)|\psi(t)\rangle$$

Bloch's theorem: The eigenvectors of the momentum operator can be expressed as $\phi_\alpha(x) = \frac{1}{\sqrt{2\pi}}e^{\imath\alpha x}$ $\quad \alpha = n + q$

n: a relative integer
q: quasi-momentum (a real number, constant of motion)

Numerical optimal control

We consider a subspace of the Hilbert space with a fixed quasi-momentum.
We expand the wave function of the system on the plane wave basis.

$$|\psi\rangle = \sum_{n\in\mathbb{Z}} c_{q,n} \left|\phi_{q+n}\right\rangle$$

$$|\psi(t)\rangle = \begin{pmatrix} \vdots \\ c_{q,n-1} \\ c_{q,n} \\ c_{q,n+1} \\ \vdots \end{pmatrix}$$

The Schrödinger equation can be written in terms of the coefficients of the wave function.

$$\imath \dot{c}_{q,n} = (n+q)^2 c_{q,n} - \frac{s}{4}\left(e^{\imath\varphi(t)} c_{q,n-1} + e^{-\imath\varphi(t)} c_{q,n+1}\right)$$

In matrix form, we have:

$$\imath \frac{d\,|\psi(t)\rangle}{dt} = \hat{H}\,|\psi(t)\rangle = \left(\hat{H}_0 + \cos(\varphi(t))\,\hat{H}_1 + \sin(\varphi(t))\,\hat{H}_2\right)|\psi(t)\rangle$$

Numerical optimal control

In matrix form, we have:

$$\imath \frac{d\left|\psi(t)\right\rangle}{dt} = \hat{H}\left|\psi(t)\right\rangle = \left(\hat{H}_0 + \cos\left(\varphi(t)\right)\hat{H}_1 + \sin\left(\varphi(t)\right)\hat{H}_2\right)\left|\psi(t)\right\rangle$$

$$\hat{H}_0 = \begin{pmatrix} & \ddots & & & & & \\ \dots & 0 & ((n-1)+q)^2 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & (n+q)^2 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & ((n+1)+q)^2 & 0 & \dots \\ & & & & & \ddots & \end{pmatrix}$$

Numerically, we truncate the infinite dimensional Hilbert space to define matrices.
Here, *n*<11.

$$\hat{H}_1 = \begin{pmatrix} \ddots & & \ddots & & & & \\ \dots & -\frac{s}{4} & 0 & -\frac{s}{4} & 0 & 0 & \dots \\ \dots & 0 & -\frac{s}{4} & 0 & -\frac{s}{4} & 0 & \dots \\ \dots & 0 & 0 & -\frac{s}{4} & 0 & -\frac{s}{4} & \dots \\ & & & & \ddots & & \ddots \end{pmatrix}, \quad \hat{H}_2 = \begin{pmatrix} \ddots & & \ddots & & & & \\ \dots & -\imath\frac{s}{4} & 0 & \imath\frac{s}{4} & 0 & 0 & \dots \\ \dots & 0 & -\imath\frac{s}{4} & 0 & \imath\frac{s}{4} & 0 & \dots \\ \dots & 0 & 0 & -\imath\frac{s}{4} & 0 & \imath\frac{s}{4} & \dots \\ & & & & \ddots & & \ddots \end{pmatrix}$$

Numerical optimal control

Fidelity:

$$F_1 = 1 - |\langle \psi(t_f)|\psi_t\rangle|^2$$

Definition of the Pontryagin Hamiltonian:

$$H_p = \Im(\langle \chi(t)|\hat{H}|\psi(t)\rangle)$$

Iterative algorithm:

$$\varphi_n' = \varphi_n + \epsilon \Im \left(\langle \chi(t_n)| \left(-\sin(\varphi_n)\hat{H}_1 + \cos(\varphi_n)\hat{H}_2 \right) |\psi(t_n)\rangle \right)$$

Q. Ansel et al., *Introduction to theoretical and experimental aspects of quantum optimal control*, J. Phys B, 57, 133001 (2024)

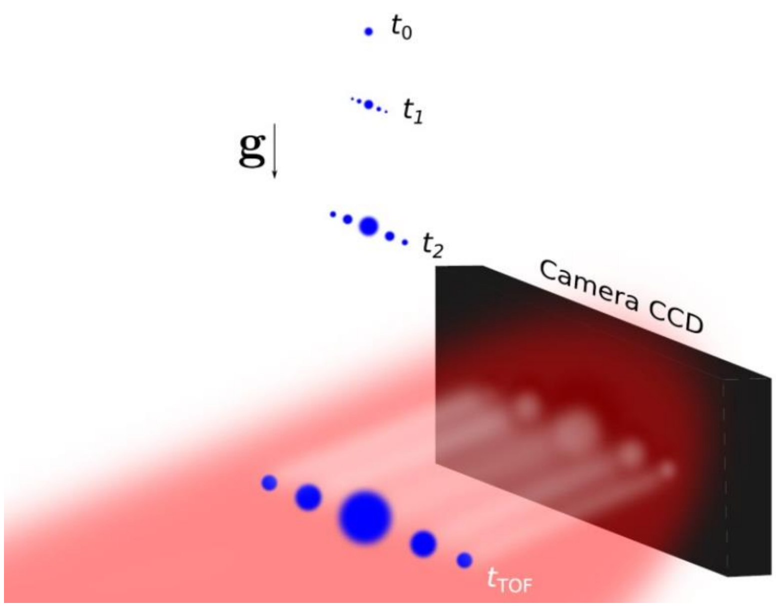
Control of BEC in an optical lattice

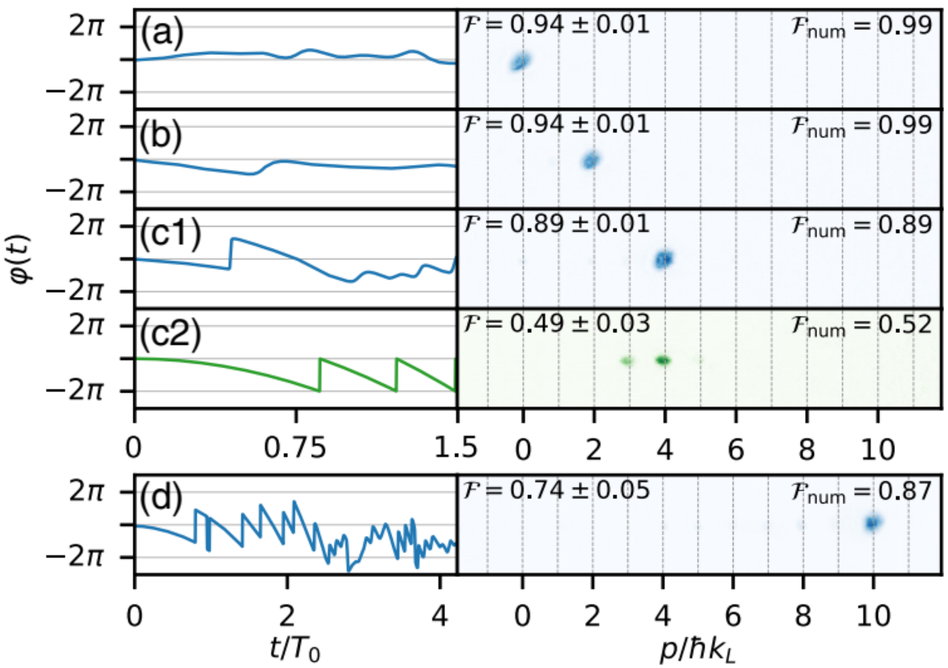
Complete experimental control of the wave function with the phase of the periodic lattice: A shaken optical lattice

$$|\psi\rangle = \sum_{n\in\mathbb{Z}} c_{q,n} |\phi_{q+n}\rangle$$

Eigenstate of the momentum operator

Measure of the momentum distribution by releasing the atoms from the lattice



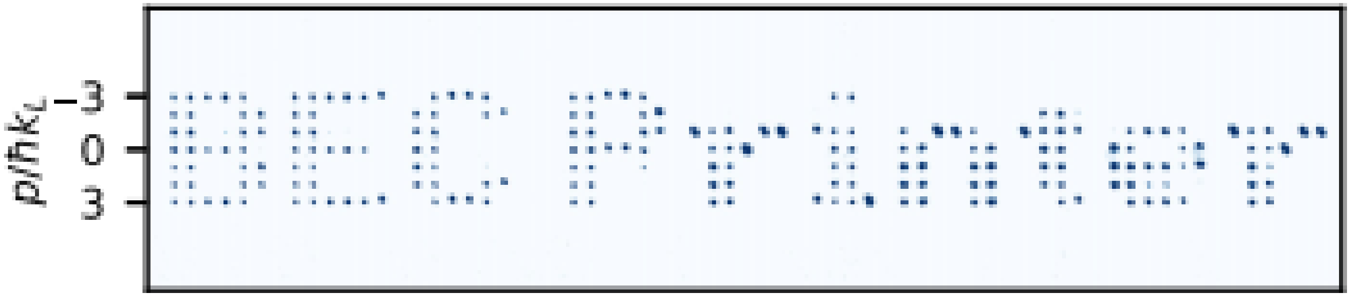


N. Dupont et al., PRX Quantum 2, 040303 (2021), image: B. Peaudecerf

Control of BEC in an optical lattice

Complete experimental control of the wave function with the phase of the periode lattice.

This very good control performance allows us to create an experimental BEC Printer.



- Preparation of an arbitrary number of momentum states
- Control of the phase of a momentum superposition
- Preparation of lattice eigenstates (with a non-zero quasi-momentum)

Other states and Husimi distribution

A classical description of the quantum state: **Husimi distribution**

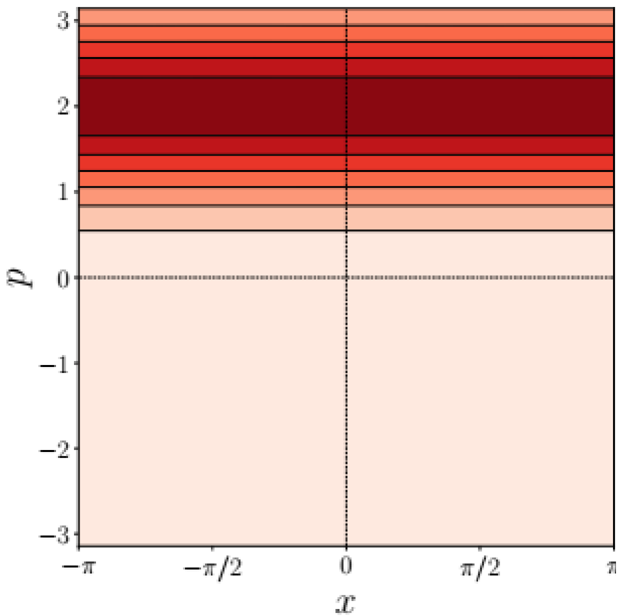
A classical phase space:

$$|\langle g(x,p)|\psi\rangle|^2$$

Gaussian states:

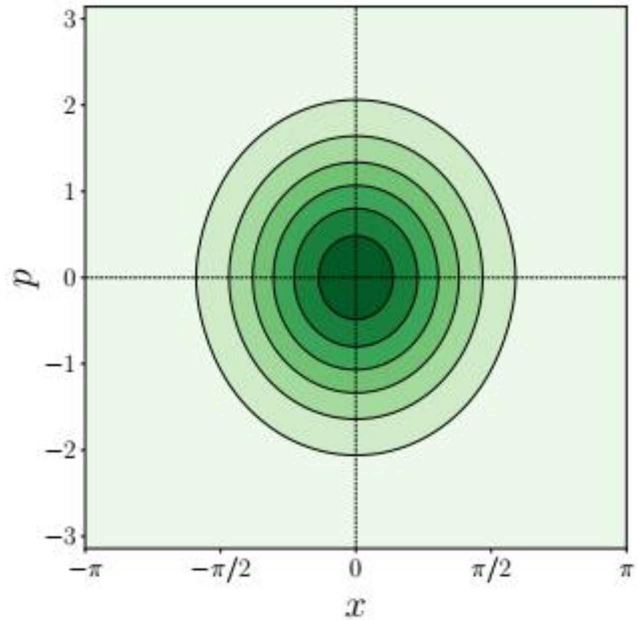
$$c_\ell(u,v) = \left(\frac{2}{\pi\sqrt{s}}\right)^{1/4} e^{iuv/2} e^{-i\ell u} e^{-(\ell-v)^2/\sqrt{s}}$$

Eigenstate of the momentum operator



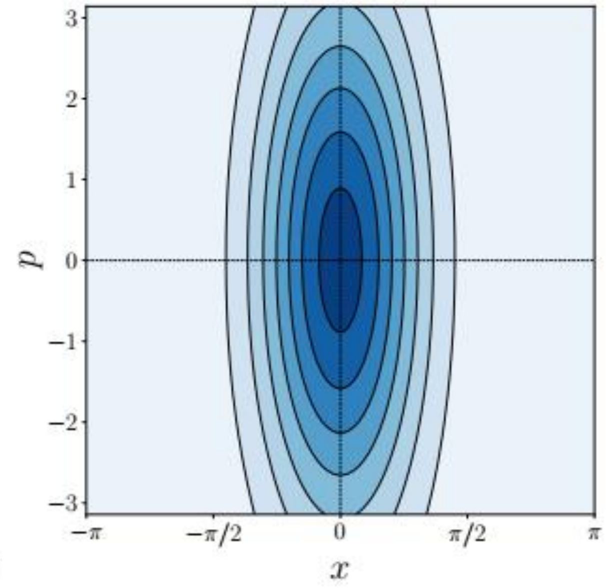
Gaussian state

$$|g(x_c,p_c)\rangle$$

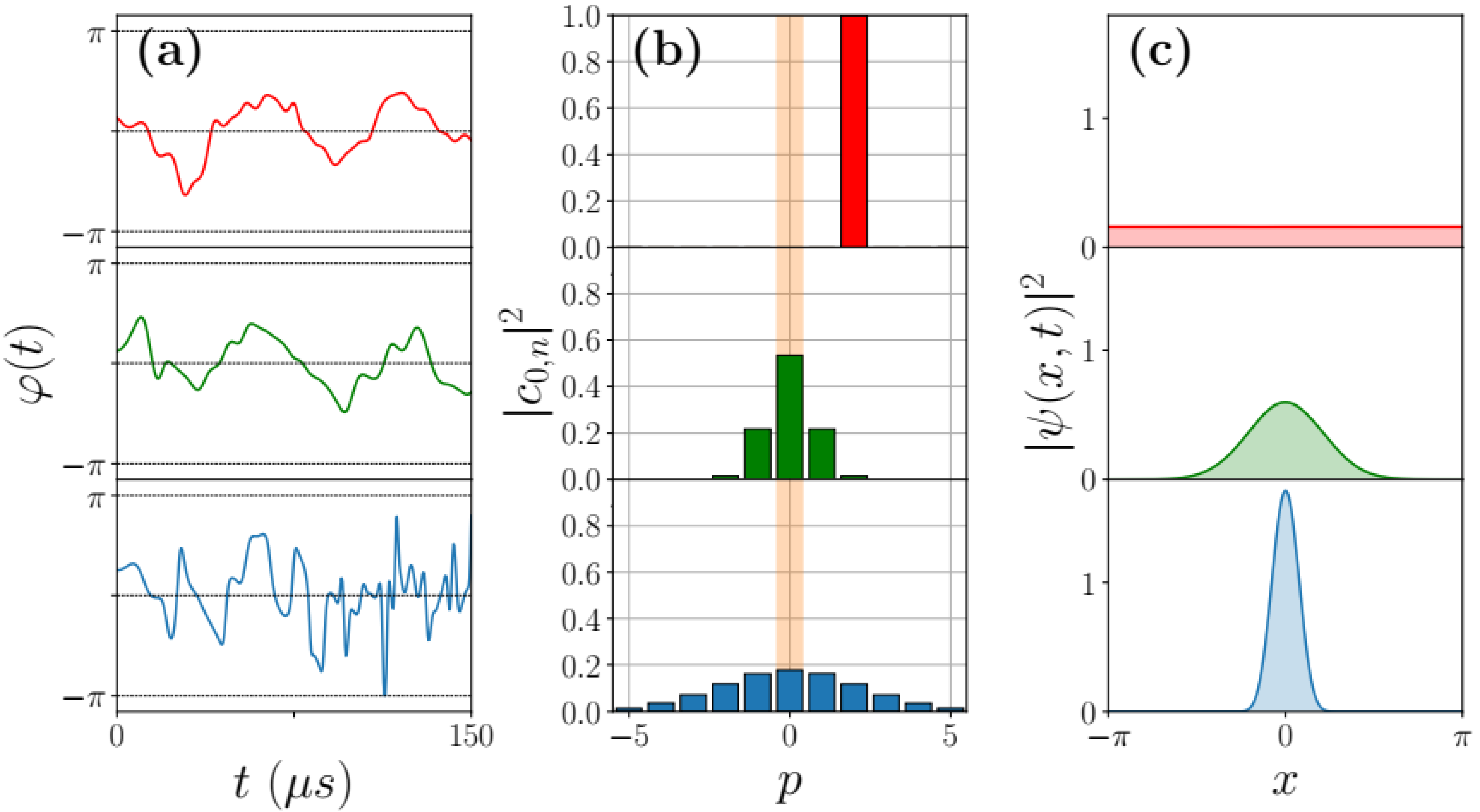


Squeezed state

$$|g(x_c,p_c,\xi)\rangle$$



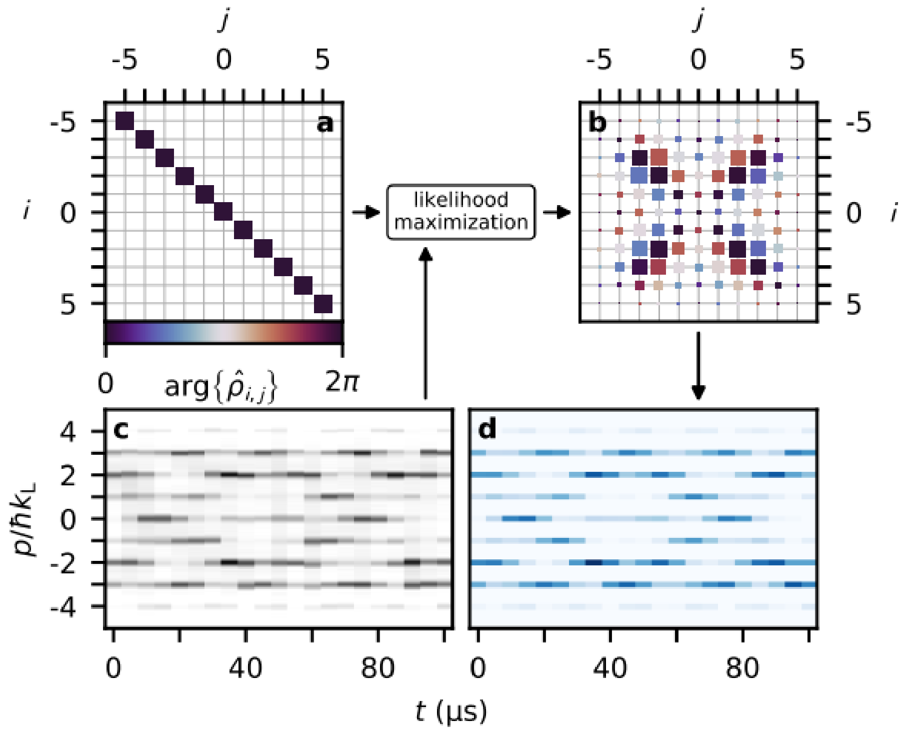
Numerical controlability of the quantum system



The phase of the lattice can be shaped to generate an eigenstate, a Gaussian or a squeezed state

Measure of a Gaussian state

A systematic approach to find the quantum state of the system from a large dataset.



Quantum state reconstruction: Likelihood Maximization approach

Finding the best state corresponding to the experimental data set.

$$\pi_j = \mathrm{tr}\{\hat{\rho}\hat{E}_j\}$$

$$f_j = f_{\ell,t} = \frac{1}{N_t}|c_\ell(t)|^2$$

$$\hat{E}_j = \hat{E}_{\ell,t} = \frac{1}{N_t}\hat{U}^\dagger(t,t_c)\,|\chi_\ell\rangle\langle\chi_\ell|\,\hat{U}(t,t_c)$$

$$\hat{\rho}_{\mathrm{ML}} = \arg\max\{\mathcal{L}[\hat{\rho}]\} \quad \text{with} \quad \mathcal{L}[\hat{\rho}] = \prod_j \pi_j^{f_j},$$

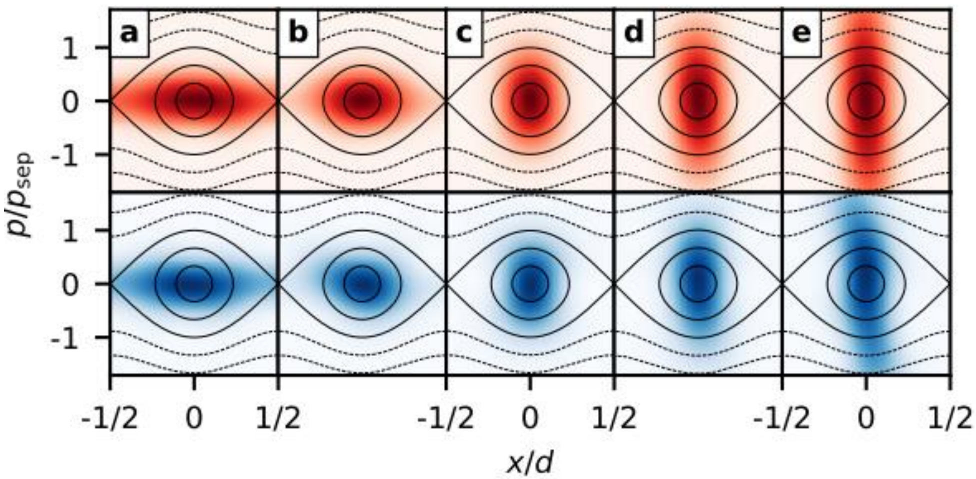
Theoretical probability

Exp.

Squeezed states

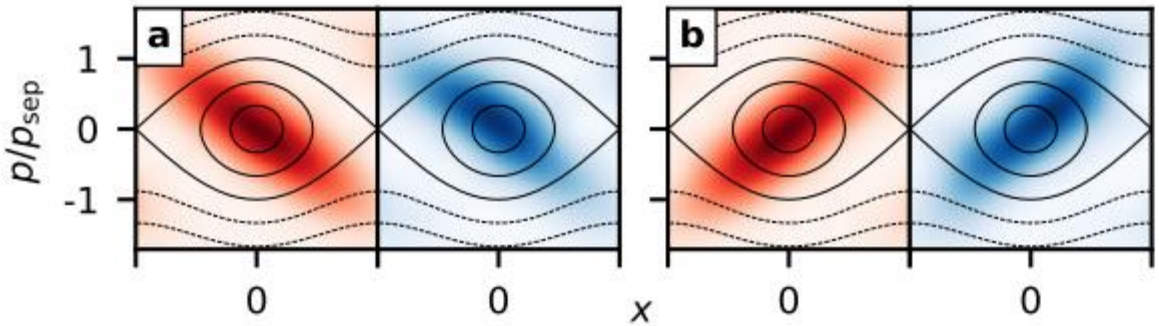
Squeezed states

$$c_\ell^{(\xi)}(u,v) = \left(\frac{2\xi^2}{\pi\sqrt{s}}\right)^{1/4} e^{iuv/2} e^{-i\ell u} e^{-\xi^2(\ell-v)^2/\sqrt{s}}$$



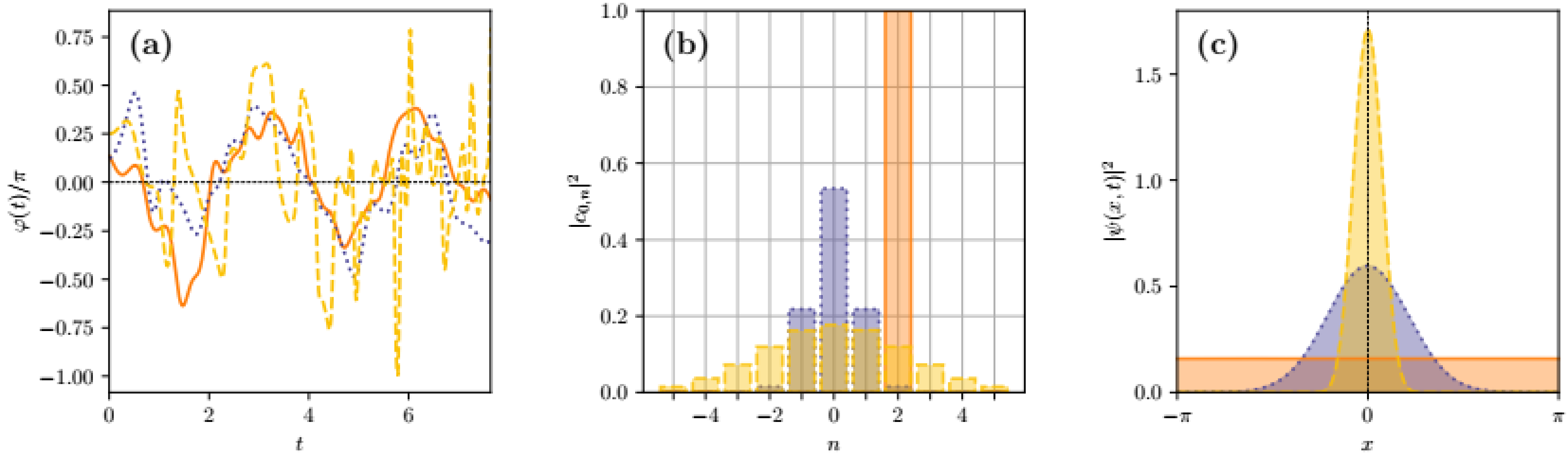
A very good agreement between theory and experiment

Rotated squeezed states

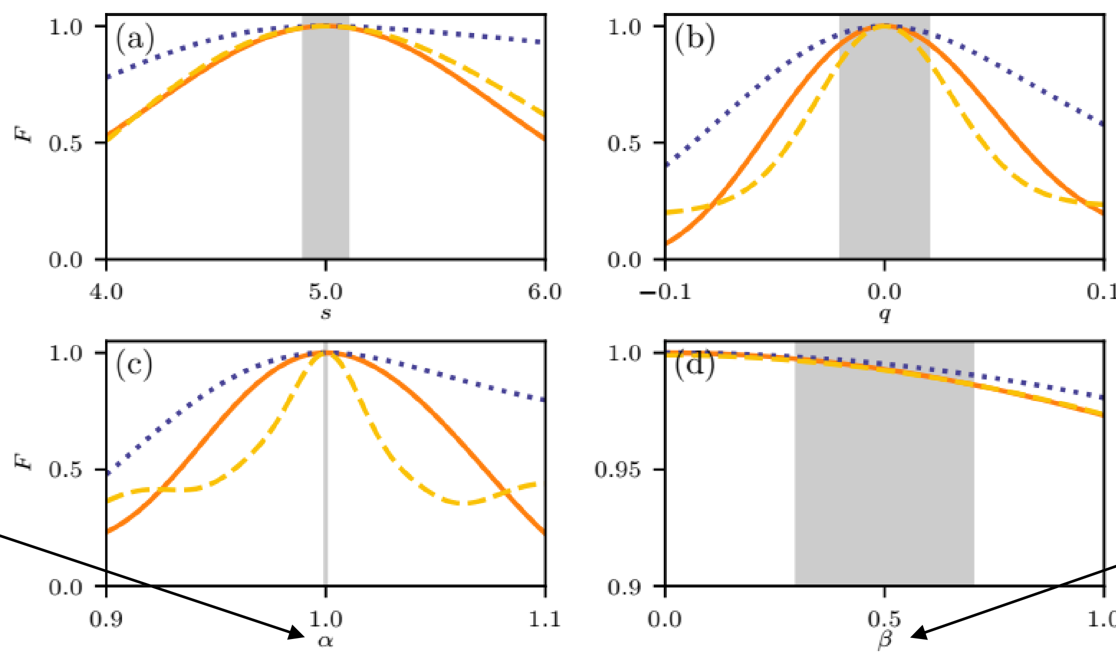


N. Dupont et al., NJP 25, 013012 (2023)

Numerical results



Open-loop control: Good estimation of the Hamiltonian parameters (grey area)



Wavelength

Non-linear parameter

Floquet theory

Floquet theory: We assume that the Hamiltonian is a time-periodic operator due to a time-periodic pulse.

$$\boxed{H(t+T) = H(t)}$$

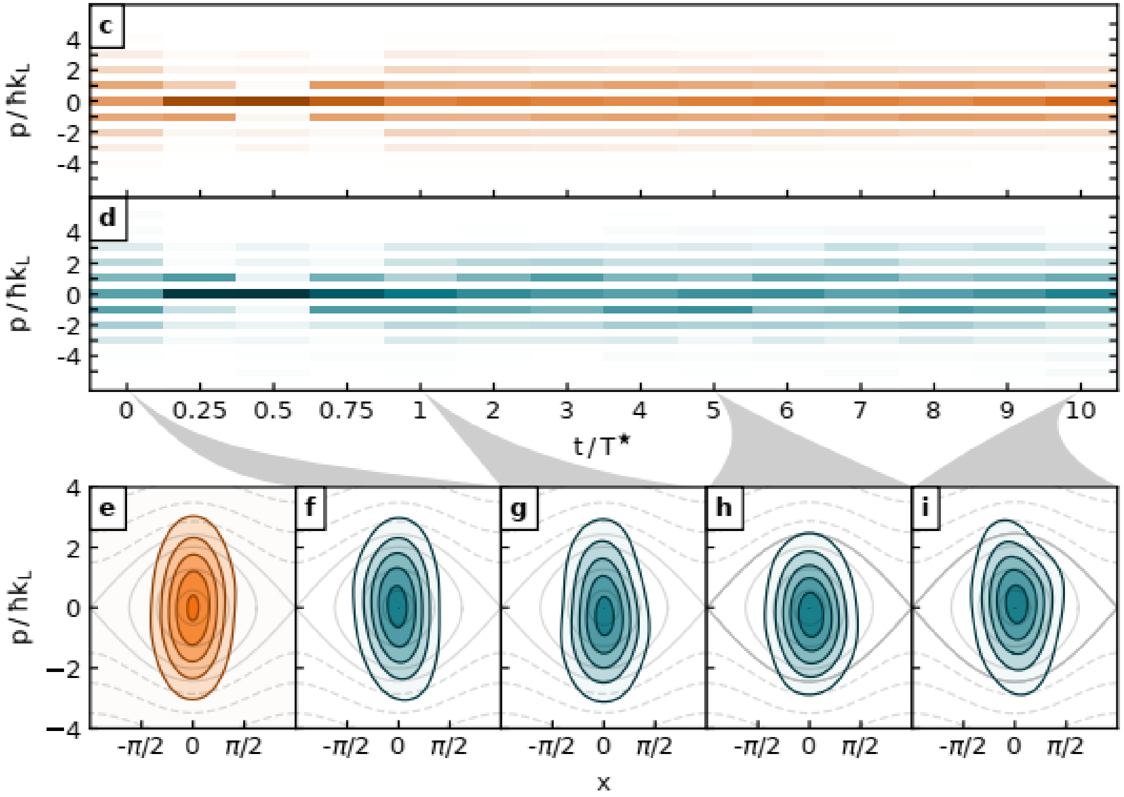
We introduce the Floquet states that are the eigenstates of the evolution operator.

$$\boxed{U(T,0)|\omega_n\rangle = e^{-i\epsilon_n T}|\omega_n\rangle}$$

Stroboscopic stabilization if we consider a Floquet state. $|\psi(0)\rangle = |\omega_k\rangle$

$$\boxed{|\psi(mT)\rangle = e^{-im\epsilon_k T}|\omega_k\rangle}$$

Stroboscopic stabilization



Idea for the optimal control: The **target state** is the same as the **initial state**.

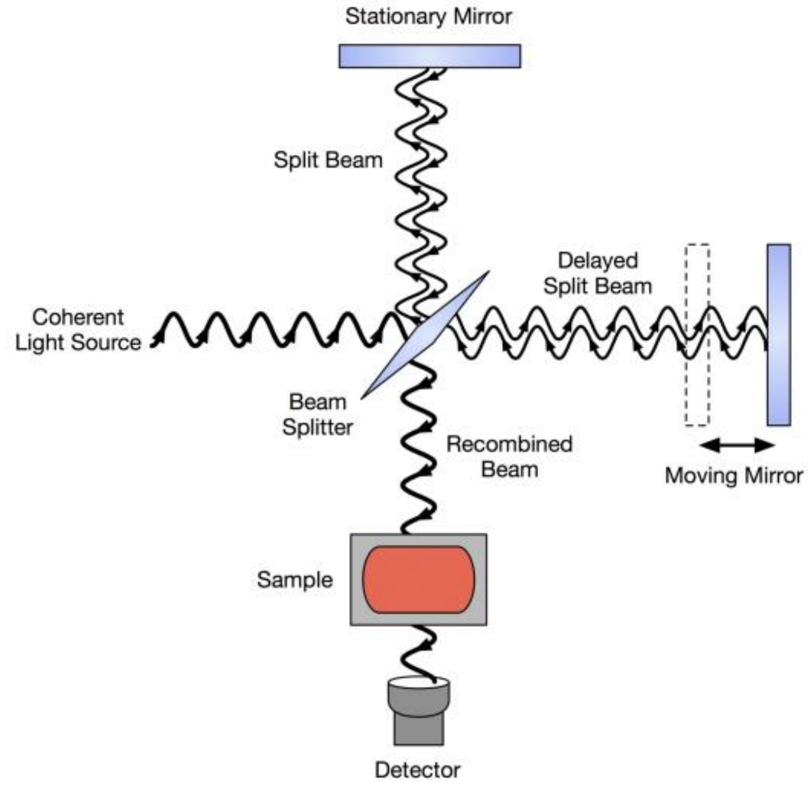
The goal is to find the control pulse such that the initial state corresponds to a Floquet state.

Experimental stroboscopic stabilization of a squeezed state over 10 periods.

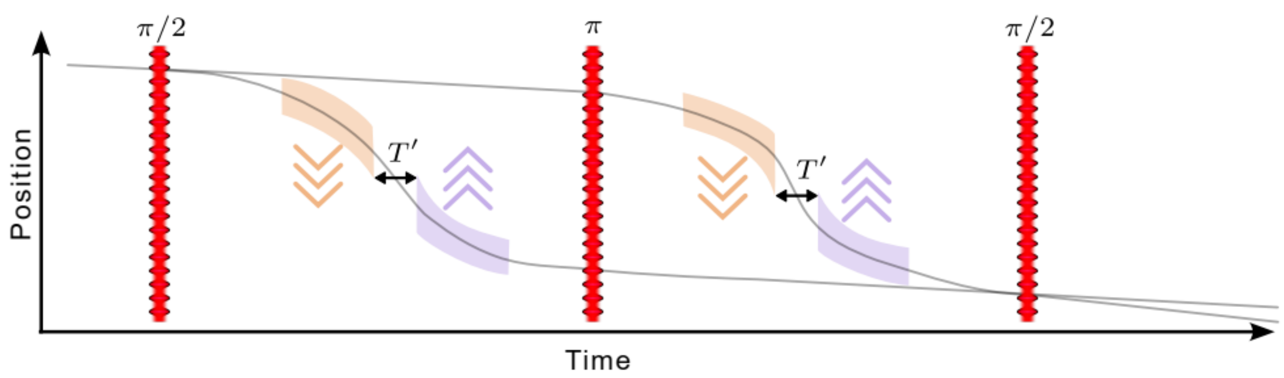
F. Arrouas et al., C. R. Acad. Sci.24, 173 (2023)
Floquet operator engineering for quantum state stroboscopic stabilization

Atom interferometry

A standard interferometer



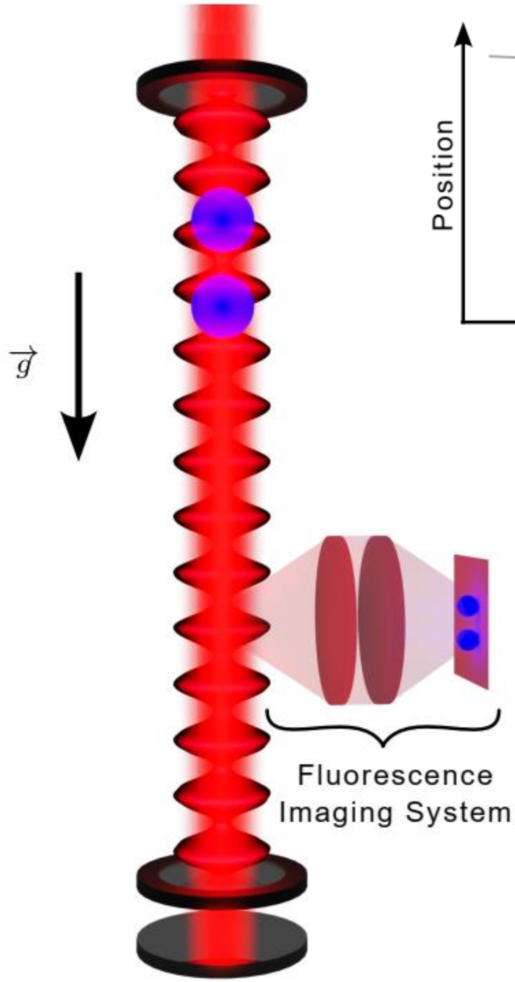
Atom interferometer

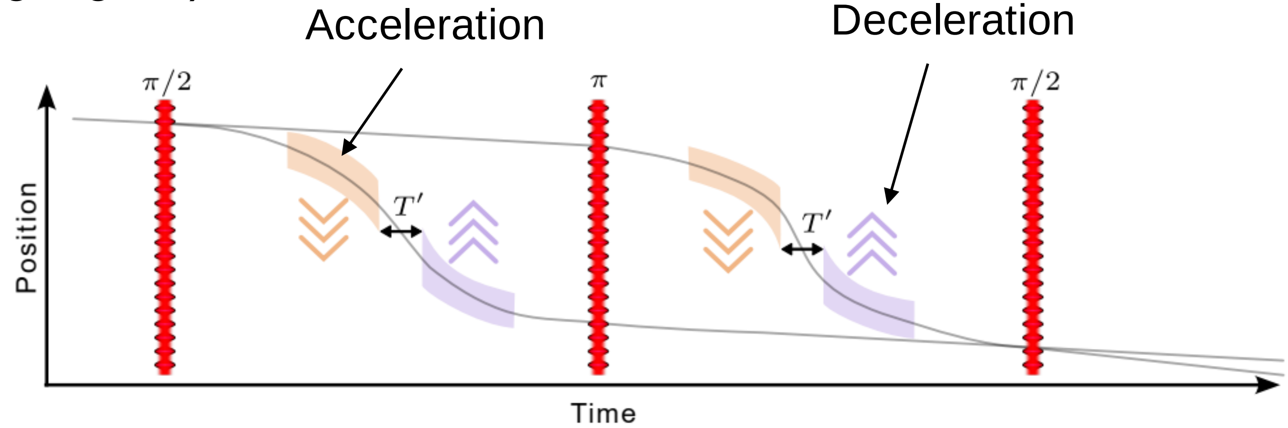


The roles of light and matter are reversed.

Optimal control for Atom interferometry

Vertical configuration of a BEC in a one-dimensional optical lattice: Mach-Zehnder interferometer (Bragg regime).





In a semi-classical framework, the phase difference between the two arms is proportional to the space-time area:

$$\phi_I = \frac{\lambda}{\hbar} \oint x dt = \frac{\lambda}{\hbar} \mathcal{A}_I$$

External force

Goal: Maximize the space-time area to reach the best sensitivity, i.e. maximize the momentum transfer.

The model system

Schrödinger equation for the BEC (in the laboratory frame):

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle = \left[\frac{\hat{p}^2}{2M} - \frac{V_0(t)}{2}\cos\left(2k\hat{z} - \varphi(t)\right) + Mg\hat{z}\right]|\psi(t)\rangle$$

Schrödinger equation in dimensionless coordinates in the free fall frame:

$$i\frac{d|\psi_2(t)\rangle}{dt} = \left[\left(\frac{\hat{p}}{2} - \frac{\omega(t)}{2}\right)^2 - 2\gamma(t)\cos(2\hat{z})\right]|\psi_2(t)\rangle$$

two control parameters

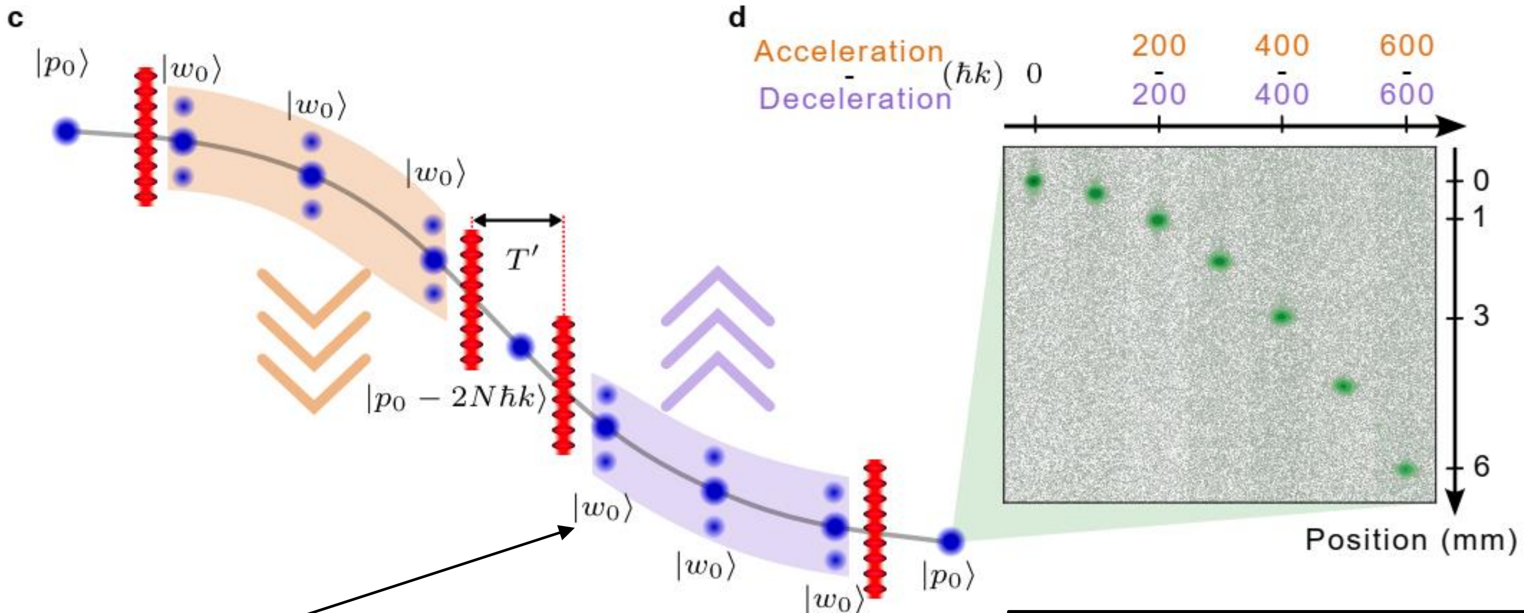
Thermal distribution (Gaussian) of the initial momentum:

$$f(p_0) = \frac{1}{\sigma_p\sqrt{2\pi}}\exp[-\frac{1}{2}(\frac{p_0}{\sigma_p})^2]$$

→ The control protocol must be robust to initial variations in momentum.

Optimization of the acceleration and deceleration processes

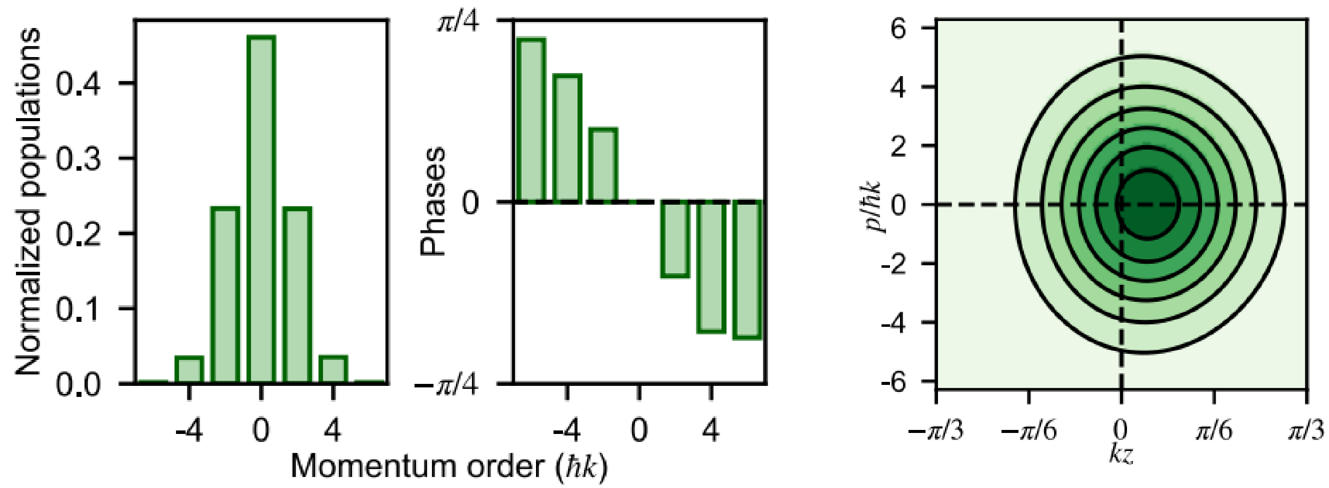
A very large and robust momentum transfer in a very short time



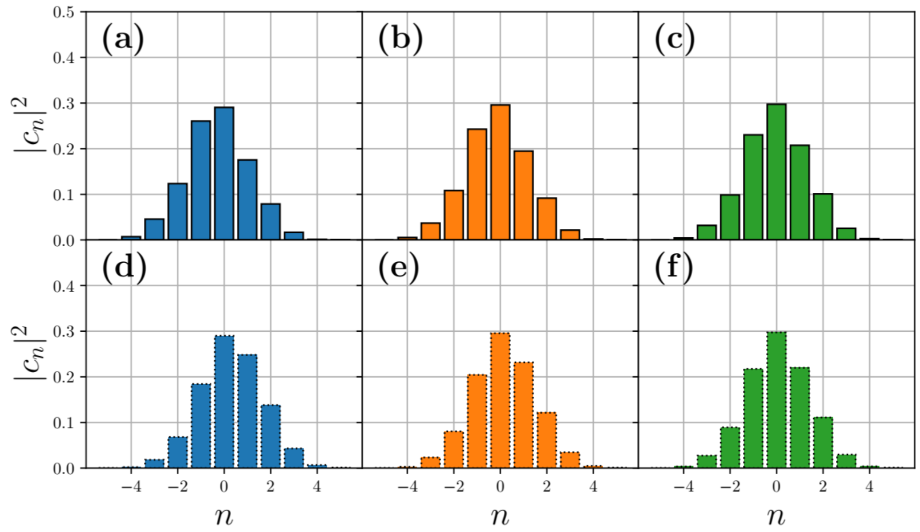
Idea: **Floquet state** in an accelerated frame

$$e^{2i\hat{z}}\hat{U}_{\mathrm{ff}}(\tau,0)|w_n\rangle = e^{-i\epsilon_n\tau}|w_n\rangle,$$

Characteristics of the Floquet state



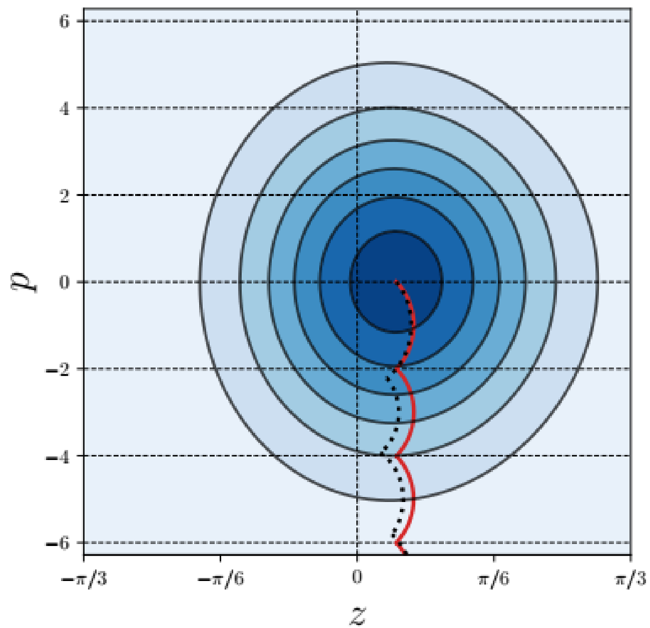
Floquet state in the accelerated frame



A Floquet state is defined for each value of the momentum p_0

$$e^{2i\hat{z}}\hat{U}_{\text{ff}}(\tau,0)|w_n\rangle = e^{-i\epsilon_n\tau}|w_n\rangle,$$

The Floquet state is very close to a squeezed state with a classical trajectory.

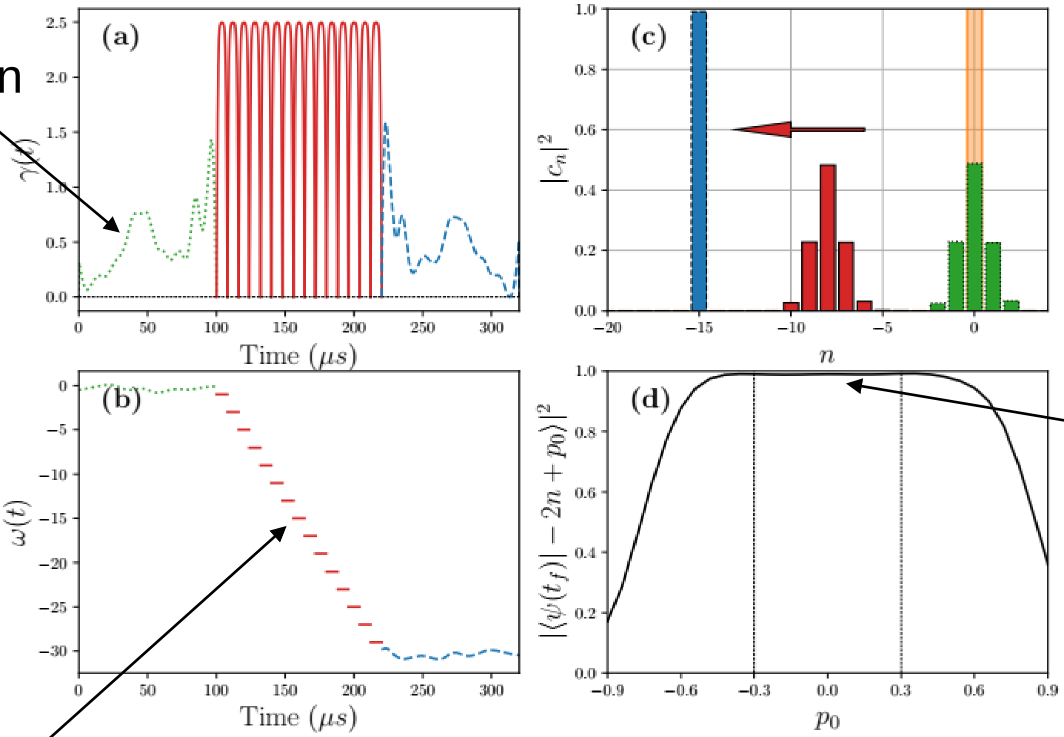


Balance between the inertial force and the restoring force of the lattice.

$$-Ma_l \quad / \quad M\omega_{\text{osc}}^2\delta z$$

Numerical results

Optimal preparation



Acceleration process

A very good robustness is achieved

Resonant π- pulse

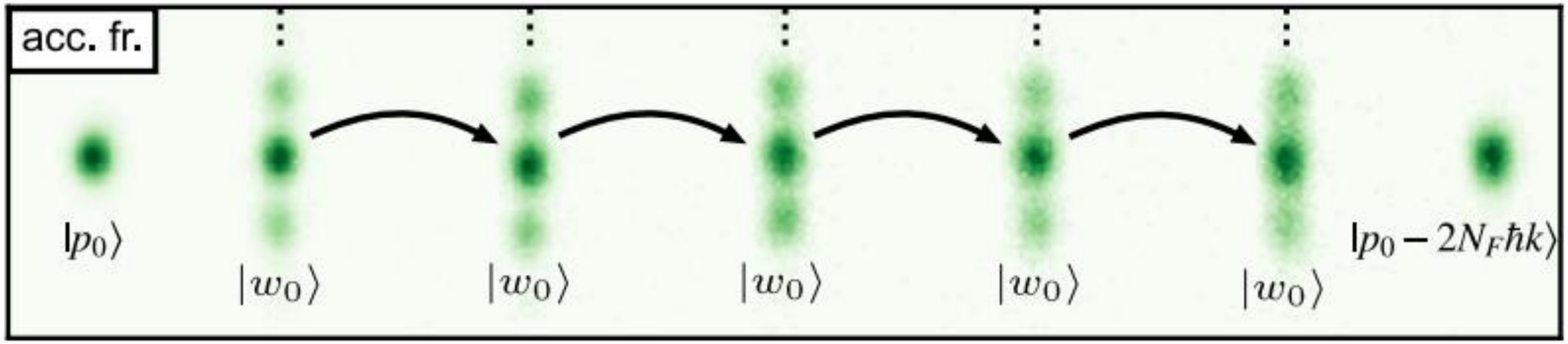
Figure of merit to reach the Floquet states

$$F_1^{(p_0)} = \int_{-\infty}^{+\infty} |\langle\psi(\tau_c)|w_0(p_0)\rangle|^2 f(p_0) dp_0,$$

$$F_2^{(p_0)} = \int_{-\infty}^{+\infty} \Re[\langle\psi(\tau_c)|w_0(p_0)\rangle] f(p_0) dp_0,$$

Experimental results

Acceleration process



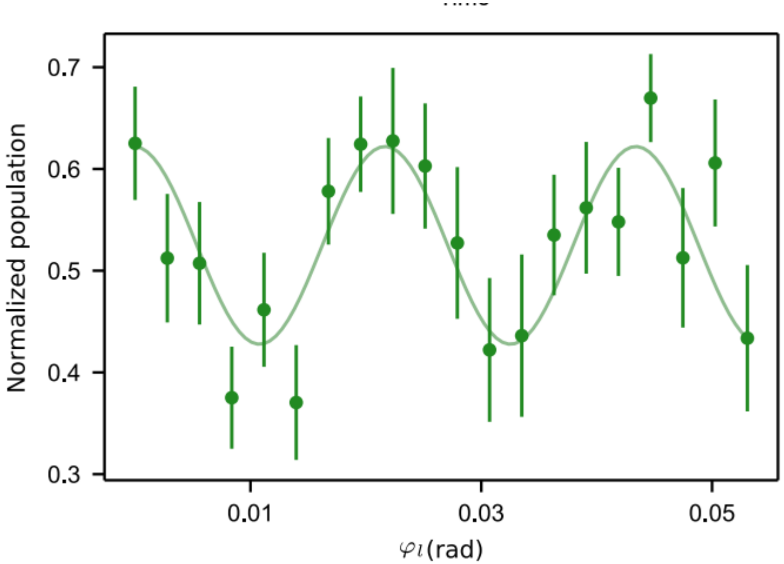
Optimal Floquet Engineering for Atom interferometers (current world record !):

Experimental fringes for the LMT interferometer

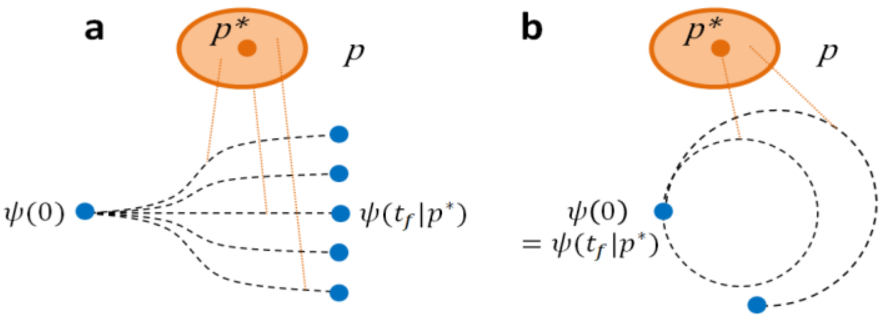
Signal: $S = N_0/(N_2 + N_0)$

Experimental signal (visibility of 20%):

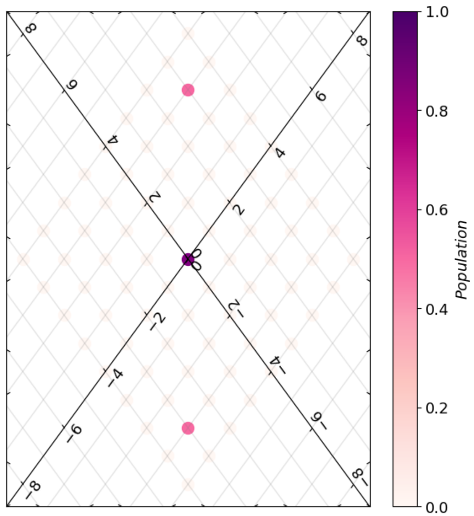
$$S(\phi) = A(1 + V\sin(\phi))$$

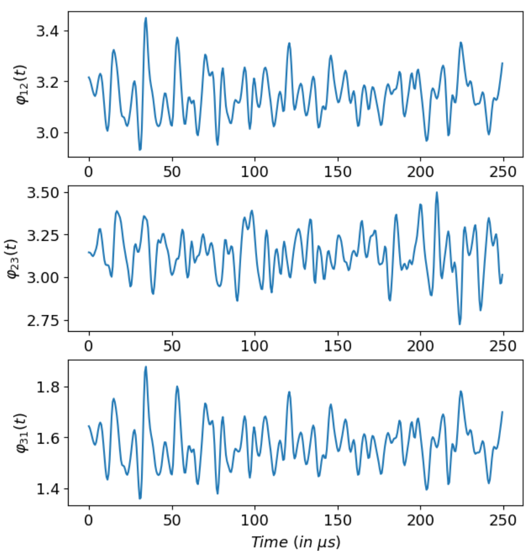


Preliminary theoretical results



- ✓ Quantum sensing: Optimal estimation of Hamiltonian parameters with the Quantum Fisher Information





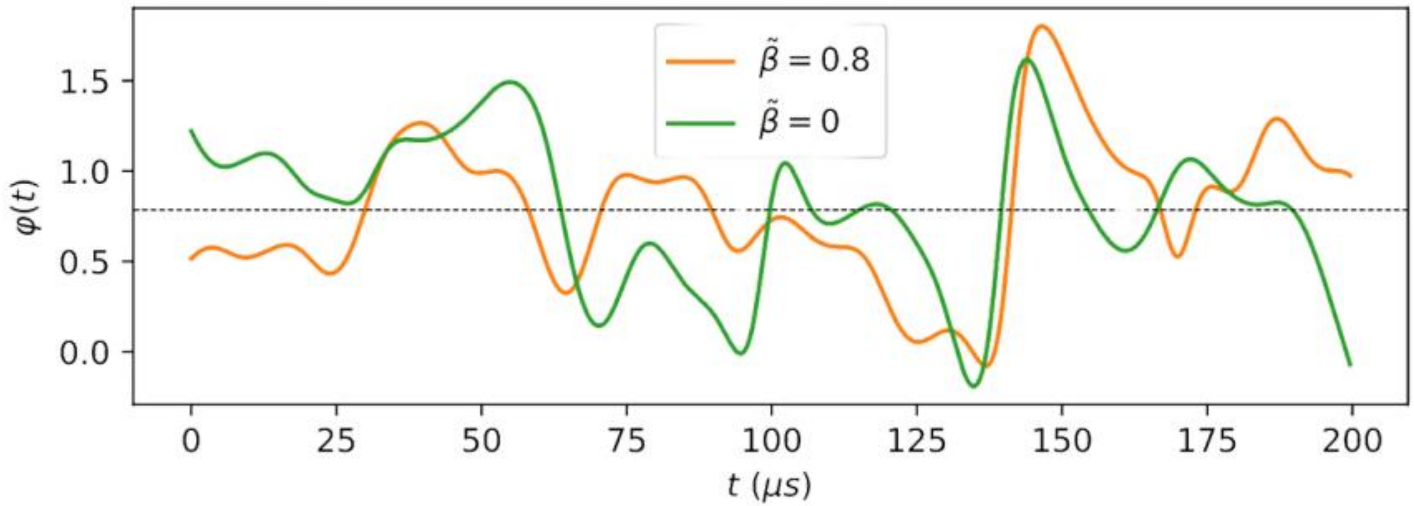
- ✓ Optimal control in a two-dimensional triangular lattice

- ✓ Quantum computing: Optimal generation of quantum gates for qudits

Preliminary theoretical results

✓ Optimal control in the non-linear case

$$i\frac{d\psi(t,x)}{dt} = \left(-\frac{\partial^2}{\partial x^2} - V(x) + \tilde{\beta}|\psi(t,x)|^2\right)\psi(t,x)$$



✓ The optimal control law is slightly modified

✓ Speed up of the control process with the non-linearity

Conclusion

Optimal control theory: A versatile and efficient tool for quantum technologies

- ✓ Open-loop control process: Towards closed-loop protocols
- ✓ Derivation of non-intuitive solutions
- ✓ Reaching the physical limits of experimental setups
- ✓ Optimal control is not a magic tool, it is constrained by the dynamics

- ✓ Application in quantum computing: Generation of quantum gates
- ✓ Application in quantum sensing and atom interferometry : Estimation of Hamiltonian parameters
- ✓ Application in quantum simulation: Preparation of the initial state

Numerical codes

In a recent tutorial paper, we share the original version of our numerical codes (Python codes):

- ✓ Control of a BEC in a one-dimensional optical lattice with GRAPE
- ✓ Control in a two-level quantum system with GRAPE
- ✓ Control in a two-level quantum system with a shooting algorithm
- ✓ Control in a two level quantum system with control constraints.

Q. Ansel et al., *Introduction to theoretical and experimental aspects of quantum optimal control*, J. Phys B, 57, 133001 (2024)

Thank you !