Course on quantum control engineering: dynamics, estimation and feedback.

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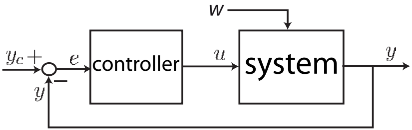
Underlying issues

Quantum Error Correction (QEC) is based on a discrete-time feedback loop

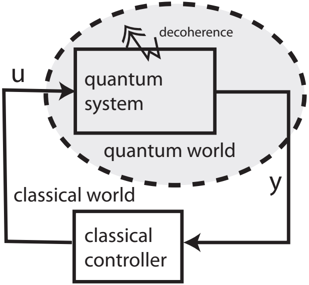
- A typical stabilizing feedback-loop for a classical system



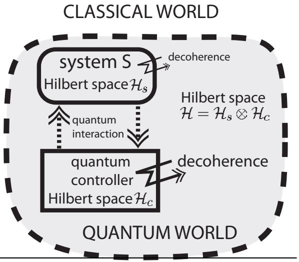
- Current experiments: 10^{-3} is the typical error probability during elementary gates (manipulations) involving few physical qubits.
- Need for QEC, a feedback loop involving: many physical qubit encoding a logical qubit (redundancy), local-error syndrome measurement (quantum non-demolitions measure), control-input pulses correcting detected errors on the physical qubits.
- High-order error-correcting codes with an important overhead;
- Today, no such controllable logical qubit has been built.
- Key issue: reduction by several magnitude orders of such error rates, far below the threshold required by actual QEC, to build a controllable logical qubit encoded in a reasonable number of physical qubits and protected by QEC.

Control engineering can play a crucial role to build a controllable logical qubit protected by **adapted feedback schemes increasing precision and stability.**

Two kinds of quantum feedback[1]



Measurement-based feedback: **controller is classical**; measurement back-action on the quantum system of Hilbert space $\mathcal{H}$ is stochastic (collapse of the wave-packet); the measured output y is a classical signal; the control input u is a classical variable appearing in some controlled Schrödinger equation; $u(t)$ depends on the past measurements $y(\tau)$, $\tau \leq t$.



Coherent/autonomous feedback and reservoir/dissipation engineering: the **system of Hilbert space $\mathcal{H}_s$** is coupled to **the controller, another quantum system**; the composite system of Hilbert space $\boldsymbol{\mathcal{H}_s} \otimes \boldsymbol{\mathcal{H}_c}$, is an open-quantum system relaxing to some target (separable) state. Relaxation behaviors in open quantum systems can be exploited: optical pumping of Alfred Kastler.

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[1] Wiseman/Milburn: Quantum Measurement and Control, 2009, Cambridge University Press.

Outline

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- Feedback stabilization of the photon box
  - Discrete-time dynamics
  - Stabilization of Fock-state with a classical controller
  - Stabilization of Schrödinger cats with a quantum controller
- Dynamics governed by Stochastic Master Equations (SME)
  - Discrete-time SME
  - Continuous-time SME
  - Estimation based on SME
  - Two kinds of feedback schemes
- Stabilization with quantum controller
  - Quantum dissipation engineering
  - Convergence analysis based on Lindblad master equations
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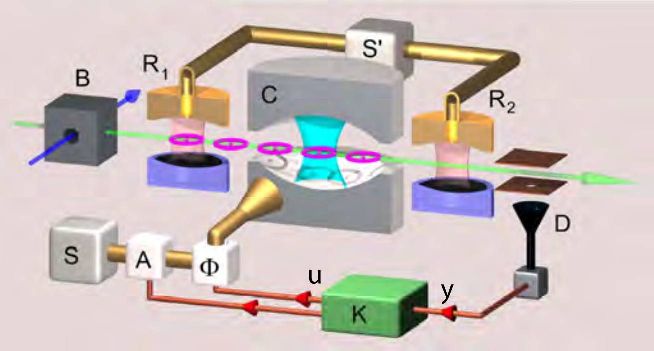
  - Cat-qubit and autonomous correction of bit-flips
  - GKP-qubit and autonomous correction of bit/phase-flips

The first experimental realization of a **quantum state feedback**

The photon box of the Laboratoire Kastler-Brossel (LKB):

S.Haroche, J.M.Raimond and M. Brune.[2]



Stabilization of a quantum state with exactly $n = 0, 1, 2, 3, \ldots$ photon(s).

Experiment: C. Sayrin et. al., Nature 477, 73-77, September 2011.

Theory: I. Dotsenko et al., Physical Review A, 80: 013805-013813, 2009.

R. Somaraju et al., Rev. Math. Phys., 25, 1350001, 2013.

H. Amini et. al., Automatica, 49 (9): 2683-2692, 2013.

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[2] Courtesy of Igor Dotsenko. **Sampling period** $80\ \mu s$.

Three quantum features emphasized by the LKB photon box[3]

1. Schrödinger: wave funct. $|\psi\rangle \in \mathcal{H}$ or density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -\tfrac{i}{\hbar}\widehat{H}|\psi\rangle, \quad \frac{d}{dt}\rho = -\tfrac{i}{\hbar}[\widehat{H},\rho], \quad \widehat{H} = \widehat{H}_0 + u\widehat{H}_1$$

2. Origin of dissipation: collapse of the wave packet induced by the measurement of observable $\widehat{O}$ with spectral decomp. $\sum_y \lambda_y \widehat{P}_y$:
    - measurement outcome y with proba. $\mathbb{P}_= \langle\psi|\widehat{P}_y|\psi\rangle = \text{Tr}\left(\rho\widehat{P}_y\right)$ depending on $|\psi\rangle$, ρ just before the measurementy
    - measurement back-action if outcome y:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\widehat{P}_y|\psi\rangle}{\sqrt{\langle\psi|\widehat{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\widehat{P}_y\rho\widehat{P}_y}{\text{Tr}\left(\rho\widehat{P}_y\right)}$$

3. Tensor product for the description of composite systems (S, M):
    - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
    - Hamiltonian $\widehat{H} = \widehat{H}_S \otimes \widehat{I}_M + \widehat{H}_{int} + \widehat{I}_S \otimes \widehat{H}_M$
    - observable on sub-system M only: $\widehat{O} = \widehat{I}_S \otimes \boldsymbol{O}_M$.

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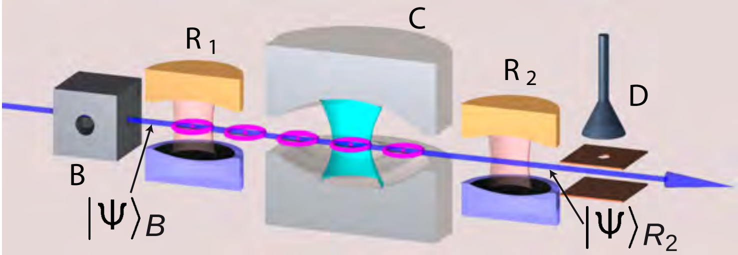
[3] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

The Markov model (1)

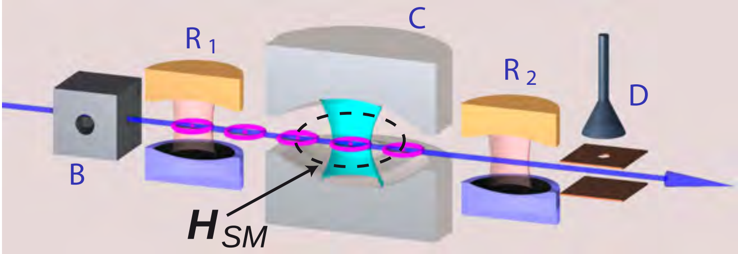


- When atom comes out B, $|\Psi\rangle_B$ of the full system is separable $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = \widehat{U}_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(\widehat{M}_g|\psi\rangle\big) \otimes |g\rangle + \big(\widehat{M}_e|\psi\rangle\big) \otimes |e\rangle$$

where $\widehat{U}_{SM}$ is a unitary transformation (Schrödinger propagator) defining the linear measurement operators $\widehat{M}_g$ and $\widehat{M}_e$ on $\mathcal{H}_S$. Since $\widehat{U}_{SM}$ is unitary, $\widehat{M}_g^\dagger \widehat{M}_g + \widehat{M}_e^\dagger \widehat{M}_e = \widehat{I}$.

The Markov model (2)



The unitary propagator $\widehat{U}_{SM}$ is derived from Jaynes-Cummings Hamiltonian $\widehat{H}_{SM}$ in the interaction frame.

Two kind of qubit/cavity Halmitonians:

resonant, $\widehat{H}_{SM}/\hbar = i\big(\Omega(vt)/2\big)\ \big(\widehat{a}^\dagger \otimes \widehat{\sigma}_- - \widehat{a} \otimes \widehat{\sigma}_+\big)$,

dispersive, $\widehat{H}_{SM}/\hbar = \big(\Omega^2(vt)/(2\delta)\big)\ \widehat{N} \otimes \widehat{\sigma}_z$,

where $\Omega(x) = \Omega_0 e^{-\frac{x^2}{w^2}}$, $x = vt$ with v atom velocity, Ω_0 vacuum Rabi pulsation, w radial mode-width and where $\delta = \omega_q - \omega_c$ is the detuning between qubit pulsation ω_q and cavity pulsation ω_c ($|\delta| \ll \Omega_0$).

The Markov model (3)

Just before D, the field/atom state is **entangled**:

$$\widehat{M}_g|\psi\rangle \otimes |g\rangle + \widehat{M}_e|\psi\rangle \otimes |e\rangle$$

Denote by $y \in \{g, e\}$ the measurement outcome in detector D: with probability $\mathbb{P}_y = \langle\psi|\widehat{M}_y^\dagger \widehat{M}_y|\psi\rangle$ we get y. Just after the measurement outcome y, the state becomes separable:

$$|\Psi\rangle_D = \tfrac{1}{\sqrt{\mathbb{P}_y}} \left(\widehat{M}_y|\psi\rangle\right) \otimes |y\rangle = \underbrace{\left(\frac{\widehat{M}_y}{\sqrt{\langle\psi|\widehat{M}_y^\dagger \widehat{M}_y|\psi\rangle}}|\psi\rangle\right)}_{\triangleq|\psi_+\rangle} \otimes|y\rangle.$$

Markov process (density matrix formulation $\rho \sim |\psi\rangle\langle\psi|$)

$$\rho_+ = \begin{cases} \frac{\widehat{M}_g\rho\widehat{M}_g^\dagger}{\mathrm{Tr}\left(\widehat{M}_g\rho\widehat{M}_g^\dagger\right)}, & \text{with probability } \mathbb{P}_g = \mathrm{Tr}\left(\widehat{M}_g\rho\widehat{M}_g^\dagger\right); \\ \frac{\widehat{M}_e\rho\widehat{M}_e^\dagger}{\mathrm{Tr}\left(\widehat{M}_e\rho\widehat{M}_e^\dagger\right)}, & \text{with probability } \mathbb{P}_e = \mathrm{Tr}\left(\widehat{M}_e\rho\widehat{M}_e^\dagger\right). \end{cases}$$

Kraus map: $\mathbb{E}\left(\rho_+/\rho\right) = \widehat{K}(\rho) = \widehat{M}_g\rho\widehat{M}_g^\dagger + \widehat{M}_e\rho\widehat{M}_e^\dagger$.

Markov model with detection inefficiency

- With pure state $\rho = |\psi\rangle\langle\psi|$, we have

  $$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{\widehat{M}_y \rho \widehat{M}_y^\dagger}{\mathrm{Tr}\left(\widehat{M}_y \rho \widehat{M}_y^\dagger\right)}$$

  when the atom collapses in $y = g, e$ with probabilty $\mathrm{Tr}\left(\widehat{M}_y \rho \widehat{M}_y^\dagger\right)$.
- Detection efficiency: the probability to detect the atom is $\eta \in [0, 1]$. Three possible outcomes for y: $y = g$ if detection in g, $y = e$ if detection in e and $y = \emptyset$ if no detection.

The only possible update is based on ρ: expectation ρ_+ of $|\psi_+\rangle\langle\psi_+|$ knowing ρ and the outcome $y \in \{g, e, \emptyset\}$ (**Bayesian inference**).

$$\rho_+ = \begin{cases} \frac{\eta \widehat{M}_g \rho \widehat{M}_g^\dagger}{\mathrm{Tr}\left(\eta \widehat{M}_g \rho \widehat{M}_g\right)} & \text{if } y = g, \text{ probability } \eta \ \mathrm{Tr}\left(\widehat{M}_g \rho \widehat{M}_g\right) \\ \frac{\eta \widehat{M}_e \rho \widehat{M}_e^\dagger}{\mathrm{Tr}\left(\eta \widehat{M}_e \rho \widehat{M}_e\right)} & \text{if } y = e, \text{ probability } \eta \ \mathrm{Tr}\left(\widehat{M}_e \rho \widehat{M}_e\right) \\ \frac{(1-\eta)\left(\widehat{M}_g \rho \widehat{M}_g^\dagger + \widehat{M}_e \rho \widehat{M}_e^\dagger\right)}{\mathrm{Tr}\left((1-\eta)\left(\widehat{M}_g \rho \widehat{M}_g^\dagger + \widehat{M}_e \rho \widehat{M}_e^\dagger\right)\right)} & \text{if } y = \emptyset, \text{ probability } 1 - \eta \end{cases}$$

For $\eta = 0$: $\rho_+ = \widehat{M}_g \rho \widehat{M}_g^\dagger + \widehat{M}_e \rho \widehat{M}_e^\dagger = \mathcal{K}(\rho) = \mathbb{E}\left(\rho_+ \mid \rho\right)$ defines a quantum channel with Kraus operators $\widehat{M}_g$ and $\widehat{M}_e$.

Markov model with detection errors

- With pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\mathrm{Tr}\left(\widehat{M}_\mu \rho \widehat{M}_\mu^\dagger\right)} \widehat{M}_\mu \rho \widehat{M}_\mu^\dagger$$

  when the atom collapses in $\mu = g, e$ with proba. $\mathrm{Tr}\left(\widehat{M}_\mu \rho \widehat{M}_\mu^\dagger\right)$.

- Detection error rates: $\mathbb{P}(y = e/\mu = g) = \eta_g \in [0,1]$ the probability of erroneous assignation to e when the atom collapses in g; $\mathbb{P}(y = g/\mu = e) = \eta_e \in [0,1]$ (given by the contrast of the Ramsey fringes).

Bayesian inference: expectation ρ_+ of $|\psi_+\rangle\langle\psi_+|$ knowing ρ and the imperfect detection y.

$$\rho_+ = \begin{cases} \frac{(1-\eta_g)\widehat{M}_g \rho \widehat{M}_g^\dagger + \eta_e \widehat{M}_e \rho \widehat{M}_e^\dagger}{\mathrm{Tr}\left((1-\eta_g)\widehat{M}_g \rho \widehat{M}_g^\dagger + \eta_e \widehat{M}_e \rho \widehat{M}_e^\dagger\right)} \text{if } y = g, \text{ prob. } \mathrm{Tr}\left((1-\eta_g)\widehat{M}_g \rho \widehat{M}_g^\dagger + \eta_e \widehat{M}_e \rho \widehat{M}_e^\dagger\right); \\ \frac{\eta_g \widehat{M}_g \rho \widehat{M}_g^\dagger + (1-\eta_e)\widehat{M}_e \rho \widehat{M}_e^\dagger}{\mathrm{Tr}\left(\eta_g \widehat{M}_g \rho \widehat{M}_g^\dagger + (1-\eta_e)\widehat{M}_e \rho \widehat{M}_e^\dagger\right)} \text{if } y = e, \text{ prob. } \mathrm{Tr}\left(\eta_g \widehat{M}_g \rho \widehat{M}_g^\dagger + (1-\eta_e)\widehat{M}_e \rho \widehat{M}_e^\dagger\right). \end{cases}$$

ρ_+ does not remain pure: the quantum state ρ_+ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant.

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A controlled Markov process (input u, hidden state ρ, output y)

Input u: classical amplitude of a coherent micro-wave pulse.
State ρ: the density operator of the photon(s) trapped in the cavity.
Output y: quantum projective measurement of the probe atom.
The ideal model reads

$$\rho_{k+1} = \begin{cases} \dfrac{\widehat{D}_{u_k} \widehat{M}_g \rho_k \widehat{M}_g^\dagger \widehat{D}_{u_k}^\dagger}{\mathrm{Tr}\left(\widehat{M}_g \rho_k \widehat{M}_g^\dagger\right)} & y_k = g \text{ with probability } \mathbb{P}_{g,k} = \mathrm{Tr}\left(\widehat{M}_g \rho_k \widehat{M}_g^\dagger\right) \\ \dfrac{\widehat{D}_{u_k} \widehat{M}_e \rho_k \widehat{M}_e^\dagger \widehat{D}_{u_k}^\dagger}{\mathrm{Tr}\left(\widehat{M}_e \rho_k \widehat{M}_e^\dagger\right)} & y_k = e \text{ with probability } \mathbb{P}_{e,k} = \mathrm{Tr}\left(\widehat{M}_e \rho_k \widehat{M}_e^\dagger\right) \end{cases}$$

- Displacement unitary operator ($u \in \mathbb{R}$): $\widehat{D}_u = e^{u\widehat{a}^\dagger - u\widehat{a}}$ with $\widehat{a} =$ upper diag$(\sqrt{1}, \sqrt{2}, \ldots)$ the photon annihilation operator.
- Measurement Kraus operators in the linear dispersive case $\widehat{M}_g = \cos\left(\frac{\phi_0 \widehat{N} + \phi_R}{2}\right)$ and $\widehat{M}_e = \sin\left(\frac{\phi_0 \widehat{N} + \phi_R}{2}\right)$: $\widehat{M}_g^\dagger \widehat{M}_g + \widehat{M}_e^\dagger \widehat{M}_e = \widehat{I}$ with $\widehat{N} = \widehat{a}^\dagger \widehat{a} = \mathrm{diag}(0, 1, 2, \ldots)$ the photon number operator.

Structure of the stabilizing quantum-state feedback scheme

With a sampling time of 80 μs, the controller is classical

- Goal: stabilization of the steady-state $|\bar{n}\rangle\langle\bar{n}|$ (controller set-point).
- At each time step k:
  1. read y_k the measurement outcome for probe atom k.
  2. update the quantum state estimation $\rho_{\mathrm{k}-1}$ to ρ_{k} from y_k
  3. compute u_k as a function of ρ_{k} (state feedback).
  4. apply the micro-wave pulse of amplitude u_k.

Observer/controller exploiting the **quantum separation principle**[4]:

1. real-time state estimation based on asymptotic observer: here quantum filtering techniques;
2. state feedback stabilization towards a stationary regime: here control Lyapunov techniques constructed with open-loop martingales $\operatorname{Tr}\left(g(\widehat{N})\rho\right)$ and inversion of a Laplacian matrix.

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[4] L. Bouten and R. van Handel: On the separation principle of quantum control. In *Quantum Stochastics and Information: Statistics, Filtering and Control*, V. P Belavkin and M. I. Guta (Eds.) World Scientific, 2008.

Experimental closed-loop data

C. Sayrin et. al., Nature 477, 73-77, Sept. 2011.

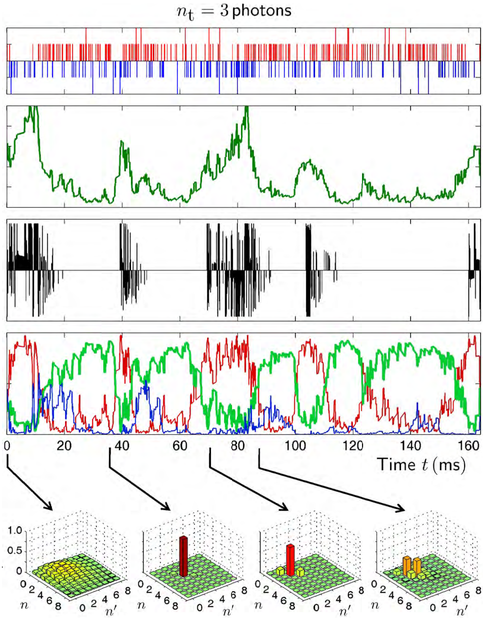
Decoherence due to finite photon life time around 70 ms)

Detection efficiency 40%
Detection error rate 10%
Delay 4 sampling periods

The quantum filter takes into account cavity decoherence, measure imperfections and delays (Bayesian inference).

Truncation to 9 photons

Stabilization around 3-photon state



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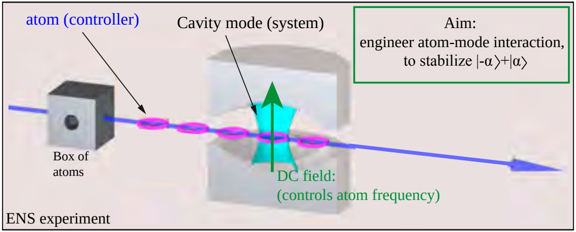
Stabilizing "Schrödinger cats" $(|\alpha\rangle + i|\text{-}\alpha\rangle)/\sqrt{2}$.[5]



Jaynes-Cumming Hamiltionian

$$\widehat{H}(t)/\hbar = \omega_c \widehat{a}^\dagger \widehat{a} \otimes \widehat{I}_M + \omega_{\mathrm{q}}(\mathrm{t}) \widehat{I}_S \otimes \widehat{\sigma}_z/2 + i\Omega(t)\left(\widehat{a}^\dagger \otimes \widehat{\sigma}_- - \widehat{a} \otimes \widehat{\sigma}_+\right)/2$$

with the open-loop control $t \mapsto \omega_{\mathrm{q}}(\mathrm{t})$ combining dispersive $\omega_q \neq \omega_c$ and resonant $\omega_q = \omega_c$ interactions.

Key issues: **convergence** of $\rho_{k+1} = \mathcal{K}(\rho_k) = \widehat{M}_g \rho_k \widehat{M}_g^\dagger + \widehat{M}_e \rho_k \widehat{M}_e^\dagger$

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[5] A. Sarlette et al: Stabilization of Nonclassical States of the Radiation Field in a Cavity by Reservoir Engineering. Physical Review Letters, Volume 107, Issue 1, 2011.

Convergence of $\widehat{K}$ iterates towards $(|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle)/\sqrt{2}$

Iterations $\rho_{k+1} = \widehat{K}(\rho_k) = \widehat{M}_g \rho_k \widehat{M}_g^\dagger + \widehat{M}_e \rho_k \widehat{M}_e^\dagger$ in the Kerr frame $\rho = \mathrm{e}^{-i\widehat{\mathrm{H}}^{\mathrm{Kerr}}} \rho^{\mathrm{Kerr}} \mathrm{e}^{i\widehat{\mathrm{H}}^{\mathrm{Kerr}}}$ yields

$$\rho_{k+1}^{\mathrm{Kerr}} = \widehat{K}^{\mathrm{Kerr}}(\rho_k^{\mathrm{Kerr}}) = \widehat{M}_g^{\mathrm{Kerr}} \rho_k^{\mathrm{Kerr}} (\widehat{M}_g^{\mathrm{Kerr}})^\dagger + \widehat{M}_e^{\mathrm{Kerr}} \rho_k^{\mathrm{Kerr}} (\widehat{M}_e^{\mathrm{Kerr}})^\dagger.$$

with $\widehat{M}_g^{\mathrm{Kerr}} = \cos(\frac{u}{2}) \cos(\theta_{\widehat{N}}/2) + \sin(\frac{u}{2}) \frac{\sin(\theta_{\widehat{N}}/2)}{\sqrt{\widehat{N}}}\, \widehat{a}^\dagger$ and

$\widehat{M}_e^{\mathrm{Kerr}} = \sin(\frac{u}{2}) \cos(\theta_{\widehat{N}+1}/2) - \cos(\frac{u}{2})\, \widehat{a} \frac{\sin(\theta_{\widehat{N}}/2)}{\sqrt{\widehat{N}}}$.

Assume $|u| \leq \pi/2$, $\theta_0 = 0$, $\theta_n \in]0,\pi[$ for $n > 0$ and $\lim_{n \mapsto +\infty} \theta_n = \pi/2$, then (Zaki Leghtas, PhD thesis (2012))

- exists a unique common eigen-state $|\psi^{\mathrm{Kerr}}\rangle$ of $\widehat{M}_g^{\mathrm{Kerr}}$ and $\widehat{M}_e^{\mathrm{Kerr}}$: $\rho_\infty^{\mathrm{Kerr}} = |\psi^{\mathrm{Kerr}}\rangle\langle\psi^{\mathrm{Kerr}}|$ fixed point of $\widehat{K}^{\mathrm{Kerr}}$.
- if, moreover $n \mapsto \theta_n$ is increasing, $\lim_{k \mapsto +\infty} \rho_k^{\mathrm{Kerr}} = \rho_\infty^{\mathrm{Kerr}}$.

For well chosen experimental parameters, $\rho_\infty^{\mathrm{Kerr}} \approx |\alpha_\infty\rangle\langle\alpha_\infty|$ and $\widehat{H}^{\mathrm{Kerr}} \approx \pi \widehat{N}^2/2$. Since $e^{-i\frac{\pi}{2}\widehat{N}^2}|\alpha_\infty\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}}\big(|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle\big)$:

$$\lim_{k \mapsto +\infty} \rho_k = \tfrac{1}{2}\Big(|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle\Big)\Big(\langle\alpha_\infty| + i\langle\text{-}\alpha_\infty|\Big)$$

$$\neq \tfrac{1}{2}|\alpha_\infty\rangle\langle\alpha_\infty| + \tfrac{1}{2}|\text{-}\alpha_\infty\rangle\langle\text{-}\alpha_\infty|.$$

Outline

Structure of dynamical models

Four modeling features[6]:

1. Schrödinger equations defining unitary transformations.
2. **Randomness**, irreversibility and dissipation induced by the measurement of observables with degenerate spectra.
3. Entanglement and tensor product for composite systems.
4. Classical probability (Bayesian inference) to include classical noises, measurement errors and uncertainties.

$\Rrightarrow$ **Hidden-state controlled Markov system**

Control input u, state ρ (density op.), measured output y:

$$\rho_{\mathbf{t+1}} = \frac{\mathcal{K}_{\mathbf{u_t},\mathbf{y_t}}(\rho_{\mathbf{t}})}{\mathrm{Tr}\Big(\mathcal{K}_{\mathbf{u_t},\mathbf{y_t}}(\rho_{\mathbf{t}})\Big)}, \text{ with proba. } \mathbb{P}\Big(\mathsf{y_t}\ /\rho_t, \mathsf{u_t}\Big) = \ \mathrm{Tr}\left(\mathcal{K}_{\mathsf{u_t},\mathsf{y_t}}(\rho_t)\right)$$

where $\mathcal{K}_{\mathsf{u},\mathsf{y}}(\rho) = \sum_{\mu=1}^{m} \eta_{\mathsf{y},\mu} \widehat{M}_{\mathsf{u},\mu} \rho \widehat{M}_{\mathsf{u},\mu}^{\dagger}$ with **left stochastic matrix** $(\eta_{\mathsf{y},\mu})$ and Kraus operators $\widehat{M}_{\mathsf{u},\mu}$ satisfying $\sum_{\mu} \widehat{M}_{\mathsf{u},\mu}^{\dagger} \widehat{M}_{\mathsf{u},\mu} = \widehat{I}$.

Kraus map $\mathcal{K}_{\mathsf{u}}$ (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{t+1} | \rho_t\right) = \mathcal{K}_{\mathsf{u}}(\rho_t) = \sum_{\mathsf{y}} \mathcal{K}_{\mathsf{u},\mathsf{y}}(\rho_t) = \sum_{\mu} \widehat{M}_{\mathsf{u},\mu} \rho_t \widehat{M}_{\mathsf{u},\mu}^{\dagger}.$$

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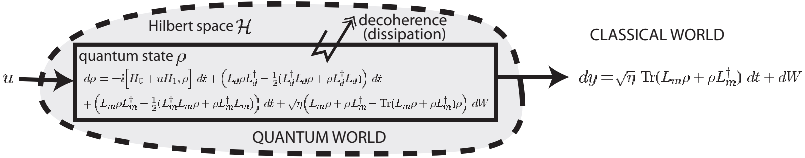
[6] See, e.g., books: E.B Davies in 1976; S. Haroche with J.M. Raimond in 2006; C. Gardiner with P. Zoller in 2014/2015.

Outline

Continuous dynamical models relying on Stochastic Master Equation (SME) [7]



Continuous-time models: stochastic differential systems (Itō formulation)
Control input $\boldsymbol{u}$, state ρ (density op.), measured output $\boldsymbol{y}$:

$$d\rho_t = \Big(-i[\widehat{H}_0 + \boldsymbol{u_t}\widehat{H}_1, \rho_t] + \sum_{\nu=d,m} \widehat{L}_\nu \rho_t \widehat{L}_\nu^\dagger - \frac{1}{2}(\widehat{L}_\nu^\dagger \widehat{L}_\nu \rho_t + \rho_t \widehat{L}_\nu^\dagger \widehat{L}_\nu) \Big) dt$$

$$+ \sqrt{\eta_m} \Big(\widehat{L}_m \rho_t + \rho_t \widehat{L}_m^\dagger - \ \mathrm{Tr}\left((\widehat{L}_m + \widehat{L}_m^\dagger) \rho_t \right) \rho_t \Big) \boldsymbol{dW_t}$$

driven by the Wiener process $\boldsymbol{W_t}$, with measurement $\boldsymbol{y_t}$,

$$\boldsymbol{dy_t} = \sqrt{\eta_m}\ \mathrm{Tr}\left((\widehat{L}_m + \widehat{L}_m^\dagger)\, \rho_t \right)\ dt + \boldsymbol{dW_t} \quad \text{detection efficiencies } \eta_m \in [0,1].$$

Measurement backaction: $d\rho_t$ and $\boldsymbol{dy_t}$ share the same noises $\boldsymbol{dW_t}$. Very different from Kalman I/O state-space description used in control engineering.

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[7] A. Barchielli, M. Gregoratti (2009): Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case. Springer Verlag.

Kraus maps and diffusive SME[8]

Linearity/positivity/trace preserving numerical integration scheme for

$$d\rho_t = \left(-i[\widehat{H}, \rho_t] + \sum_\nu \widehat{L}_\nu \rho_t \widehat{L}_\nu^\dagger - \frac{1}{2}(\widehat{L}_\nu^\dagger \widehat{L}_\nu \rho_t + \rho_t \widehat{L}_\nu^\dagger \widehat{L}_\nu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu} \left(\widehat{L}_\nu \rho_t + \rho_t \widehat{L}_\nu^\dagger - \operatorname{Tr}\left((\widehat{L}_\nu + \widehat{L}_\nu^\dagger)\rho_t\right) \rho_t\right) dW_{\nu,t},$$
$$dy_{\nu,t} = \sqrt{\eta_\nu} \operatorname{Tr}\left(\widehat{L}_\nu \rho_t + \rho_t \widehat{L}_\nu^\dagger\right) dt + dW_{\nu,t}$$

With $\widehat{M}_0 = \widehat{I} + \big(-i\widehat{H} - \frac{1}{2}\sum_\nu \widehat{L}_\nu^\dagger \widehat{L}_\nu\big)dt$, $\quad \widehat{S} = \widehat{M}_0^\dagger \widehat{M}_0 + \left(\sum_\nu \widehat{L}_\nu^\dagger \widehat{L}_\nu\right) dt$ set

$$\widetilde{\mathrm{M}}_0 = \widehat{M}_0 \widehat{S}^{-1/2}, \quad \widetilde{\mathrm{L}}_\nu = \widehat{L}_\nu \widehat{S}^{-1/2}.$$

Sampling of $dy_{\nu,t} = s_{\nu,t}\sqrt{dt}$ according to the following probability law:

$$\mathbb{P}\Big((s_{\nu,t} \in [s_\nu, s_\nu + ds_\nu])_\nu \mid \rho_t \Big) = \operatorname{Tr}\left(\widetilde{\mathrm{M}}_{s\sqrt{dt}} \rho_t \widetilde{\mathrm{M}}_{s\sqrt{dt}}^\dagger + \sum_\nu (1-\eta_\nu) \widetilde{\mathrm{L}}_\nu \rho_t \widetilde{\mathrm{L}}_\nu^\dagger dt\right) \prod_\nu \frac{e^{-\frac{s_\nu^2}{2}} ds_\nu}{\sqrt{2\pi}}.$$

where $\widetilde{\mathrm{M}}_{dy_t} = \widetilde{\mathrm{M}}_0 + \sum_\nu \sqrt{\eta_\nu} dy_{\nu,t} \widetilde{\mathrm{L}}_\nu$. Exact Kraus-map formulation:

$$\rho_{t+dt} = \frac{\widetilde{\mathrm{M}}_{dy_t} \rho_t \widetilde{\mathrm{M}}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \widetilde{\mathrm{L}}_\nu \rho_t \widetilde{\mathrm{L}}_\nu^\dagger dt}{\operatorname{Tr}\left(\widetilde{\mathrm{M}}_{dy_t} \rho_t \widetilde{\mathrm{M}}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \widetilde{\mathrm{L}}_\nu \rho_t \widetilde{\mathrm{L}}_\nu^\dagger dt\right)}.$$

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[8] A. Jordan, A. Chantasri, PR, and B.Huard. Anatomy of fluorescence: quantum trajectory statistics from continuously measuring spontaneous emission. *Quantum Studies: Mathematics and Foundations*, 3(3):237–263, 2016.

Jump SME in continuous-time[9] (1)

General structure of a Jump SME in continuous time with counting process N_t and increment expectation value knowing ρ_t given by $\langle dN_t \rangle = \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(V\rho_t V^\dagger\right)\right) dt$, with $\bar{\theta} \geq 0$ (dark-noise rate) and $\bar{\eta} \in [0,1]$ (detection efficiency):

$$d\rho_t = \left(-i[\widehat{H},\rho_t] + \widehat{V}\rho_t\widehat{V}^\dagger - \tfrac{1}{2}(\widehat{V}^\dagger\widehat{V}\rho_t + \rho_t\widehat{V}^\dagger\widehat{V})\right) dt \\ + \left(\tfrac{\bar{\theta}\rho_t + \bar{\eta}\widehat{V}\rho_t\widehat{V}^\dagger}{\bar{\theta}+\bar{\eta}\ \mathrm{Tr}\left(\widehat{V}\rho_t\widehat{V}^\dagger\right)} - \rho_t\right)\left(dN_t - \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\widehat{V}\rho_t\widehat{V}^\dagger\right)\right) dt\right).$$

Here $\widehat{H}$ and $\widehat{V}$ are operators on an underlying Hilbert space $\mathcal{H}$, $\widehat{H}$ being Hermitian. At each time-step between t and $t + dt$, one has the following recipe

- $dN_t = 0$ with probability $1 - \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\widehat{V}\rho_t\widehat{V}^\dagger\right)\right) dt$

  $$\rho_{t+dt} = \frac{\widehat{M}_0\rho_t\widehat{M}_0^\dagger + (1-\bar{\eta})\widehat{V}\rho_t\widehat{V}^\dagger dt}{\mathrm{Tr}\left(\widehat{M}_0\rho_t\widehat{M}_0^\dagger + (1-\bar{\eta})\widehat{V}\rho_t\widehat{V}^\dagger dt\right)}$$

  where $\widehat{M}_0 = I - \left(iH + \frac{1}{2}\widehat{V}^\dagger\widehat{V}\right) dt$.
- $dN_t = 1$ with probability $\left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\widehat{V}\rho_t\widehat{V}^\dagger\right)\right) dt$,

  $$\rho_{t+dt} = \frac{\bar{\theta}\rho_t + \bar{\eta}\widehat{V}\rho_t\widehat{V}^\dagger}{\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\widehat{V}\rho_t\widehat{V}^\dagger\right)}.$$

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[9] J. Dalibard, Y. Castin, and K. Mølmer. Wave-function approach to dissipative processes in quantum optics. *Phys. Rev. Lett.*, 68(5):580–583, 1992.

General mixed diffusive/jump SME (1)

Combine in a single SME Wiener and Poisson noises induced by diffusive and counting measurements:

$$
\begin{aligned}
d\rho_t = \Big(-i[\widehat{H},\rho_t] + \widehat{L}\rho_t\widehat{L}^\dagger - \tfrac{1}{2}(\widehat{L}^\dagger\widehat{L}\rho_t + \rho_t\widehat{L}^\dagger\widehat{L}) + \widehat{V}\rho_t\widehat{V}^\dagger - \tfrac{1}{2}(\widehat{V}^\dagger\widehat{V}\rho_t + \rho_t\widehat{V}^\dagger\widehat{V})\Big)\, dt \\
+ \sqrt{\eta}\Big(\widehat{L}\rho_t + \rho_t\widehat{L}^\dagger - \operatorname{Tr}\left((\widehat{L}+\widehat{L}^\dagger)\rho_t\right)\rho_t\Big)dW_t \\
+ \left(\tfrac{\bar{\theta}\rho_t + \bar{\eta}\widehat{V}\rho_t\widehat{V}^\dagger}{\bar{\theta}+\bar{\eta}\operatorname{Tr}\left(\widehat{V}\rho_t\widehat{V}^\dagger\right)} - \rho_t\right)\left(dN_t - \left(\bar{\theta} + \bar{\eta}\operatorname{Tr}\left(\widehat{V}\rho_t\widehat{V}^\dagger\right)\right)dt\right)
\end{aligned}
$$

With $dy_t = \sqrt{\eta}\operatorname{Tr}\left((\widehat{L}+\widehat{L}^\dagger)\,\rho_t\right) dt + dW_t$ and $dN_t = 0$ with probability $1 - \left(\bar{\theta} + \bar{\eta}\operatorname{Tr}\left(\widehat{V}\rho_t\widehat{V}^\dagger\right)\right) dt$. Kraus-map equivalent formulation:

- for $dN_t = 0$ of probability $1 - \left(\bar{\theta} + \bar{\eta}\operatorname{Tr}\left(\widehat{V}\rho_t\widehat{V}^\dagger\right)\right) dt$

$$
\rho_{t+dt} = \frac{\widehat{M}_{dy_t}\rho_t\widehat{M}_{dy_t}^\dagger + (1-\eta)\widehat{L}\rho_t\widehat{L}^\dagger dt + (1-\bar{\eta})\widehat{V}\rho_t\widehat{V}^\dagger dt}{\operatorname{Tr}\left(\widehat{M}_{dy_t}\rho_t\widehat{M}_{dy_t}^\dagger + (1-\eta)\widehat{L}\rho_t\widehat{L}^\dagger dt + (1-\bar{\eta})\widehat{V}\rho_t\widehat{V}^\dagger dt\right)}
$$

  with $\widehat{M}_{dy_t} = I - \left(i\widehat{H} + \frac{1}{2}\widehat{L}^\dagger\widehat{L} + \frac{1}{2}\widehat{V}^\dagger\widehat{V}\right) dt + \sqrt{\eta}dy_t\widehat{L}$.

- for $dN_t = 1$ of probability $\left(\bar{\theta} + \bar{\eta}\operatorname{Tr}\left(\widehat{V}\rho_t\widehat{V}^\dagger\right)\right) dt$:

$$
\rho_{t+dt} = \frac{\widehat{M}_{dy_t}\tilde{\rho}_t\widehat{M}_{dy_t}^\dagger + (1-\eta)\widehat{L}\tilde{\rho}_t\widehat{L}^\dagger dt + (1-\bar{\eta})\widehat{V}\tilde{\rho}_t\widehat{V}^\dagger dt}{\operatorname{Tr}\left(\widehat{M}_{dy_t}\tilde{\rho}_t\widehat{M}_{dy_t}^\dagger + (1-\eta)\widehat{L}\tilde{\rho}_t\widehat{L}^\dagger dt + (1-\bar{\eta})\widehat{V}\tilde{\rho}_t\widehat{V}^\dagger dt\right)} \text{ with } \tilde{\rho}_t = \frac{\bar{\theta}\rho_t + \bar{\eta}\widehat{V}\rho_t\widehat{V}^\dagger}{\bar{\theta} + \bar{\eta}\operatorname{Tr}\left(\widehat{V}\rho_t\widehat{V}^\dagger\right)}
$$

General mixed diffusive/jump SME (2)[10]

$$d\rho_t = \left(-i[\widehat{H},\rho_t] + \sum_\nu \widehat{L}_\nu \rho_t \widehat{L}_\nu^\dagger - \tfrac{1}{2}(\widehat{L}_\nu^\dagger \widehat{L}_\nu \rho_t + \rho_t \widehat{L}_\nu^\dagger \widehat{L}_\nu) + \sum_\mu \widehat{V}_\mu \rho_t \widehat{V}_\mu^\dagger - \tfrac{1}{2}(\widehat{V}_\mu^\dagger \widehat{V}_\mu \rho_t + \rho_t \widehat{V}_\mu^\dagger \widehat{V}_\mu)\right) dt$$

$$+ \sum_\nu \sqrt{\eta_\nu}\left(\widehat{L}_\nu\rho_t + \rho_t\widehat{L}_\nu^\dagger - \ \mathrm{Tr}\left((\widehat{L}_\nu+\widehat{L}_\nu^\dagger)\rho_t\right)\rho_t\right) dW_{\nu,t}$$

$$+\sum_\mu \left(\frac{\bar\theta_\mu\rho_t + \sum_{\mu'}\bar\eta_{\mu,\mu'}\widehat{V}_{\mu'}\rho_t\widehat{V}_{\mu'}^\dagger}{\bar\theta_\mu + \sum_{\mu'}\bar\eta_{\mu,\mu'}\ \mathrm{Tr}\left(\widehat{V}_{\mu'}\rho_t\widehat{V}_{\mu'}^\dagger\right)} - \rho_t\right)\left(dN_{\mu,t} - \left(\bar\theta_\mu + \sum_{\mu'}\bar\eta_{\mu,\mu'}\ \mathrm{Tr}\left(\widehat{V}_{\mu'}\rho_t\widehat{V}_{\mu'}^\dagger\right)\right)dt\right)$$

where $\eta_\nu \in [0,1]$, $\bar\theta_\mu, \bar\eta_{\mu,\mu'} \geq 0$ with $\bar\eta_{\mu'} = \sum_\mu \bar\eta_{\mu,\mu'} \leq 1$. The equivalent Kraus-map formulation

- When $\forall \mu,\ dN_{\mu,t} = 0$ (probability $1 - \sum_\mu \left(\bar\theta_\mu + \bar\eta_\mu\ \mathrm{Tr}\left(\widehat{V}_\mu\rho_t\widehat{V}_\mu^\dagger\right)\right) dt$) we have

$$\rho_{t+dt} = \frac{\widehat{M}_{dy_t}\rho_t\widehat{M}_{dy_t}^\dagger + \sum_\nu(1-\eta_\nu)\widehat{L}_\nu\rho_t\widehat{L}_\nu^\dagger dt + \sum_\mu(1-\bar\eta_\mu)\widehat{V}_\mu\rho_t\widehat{V}_\mu^\dagger dt}{\mathrm{Tr}\left(\widehat{M}_{dy_t}\rho_t\widehat{M}_{dy_t}^\dagger + \sum_\nu(1-\eta_\nu)\widehat{L}_\nu\rho_t\widehat{L}_\nu^\dagger dt + \sum_\mu(1-\bar\eta_\mu)\widehat{V}_\mu\rho_t\widehat{V}_\mu^\dagger dt\right)}$$

with $\widehat{M}_{dy_t} = I - \left(iH + \frac{1}{2}\sum_\nu \widehat{L}_\nu^\dagger\widehat{L}_\nu + \frac{1}{2}\sum_\mu\ \widehat{V}_\mu^\dagger\widehat{V}_\mu\right)dt + \sum_\nu \sqrt{\eta_\nu}dy_{\nu t}\widehat{L}_\nu$ and where $dy_{\nu,t} = \sqrt{\eta_\nu}\ \mathrm{Tr}\left((\widehat{L}_\nu + \widehat{L}_\nu^\dagger)\,\rho_t\right)\ dt + dW_{\nu,t}$.

- If, for some μ, $dN_{\mu,t} = 1$ (probability $\left(\bar\theta_\mu + \sum_{\mu'}\bar\eta_{\mu,\mu'}\ \mathrm{Tr}\left(\widehat{V}_{\mu'}\rho_t\widehat{V}_{\mu'}^\dagger\right)\right) dt$) we have

$$\rho_{t+dt} = \frac{\widehat{M}_{dy_t}\tilde\rho_t\widehat{M}_{dy_t}^\dagger + \sum_\nu(1-\eta_\nu)\widehat{L}_\nu\tilde\rho_t\widehat{L}_\nu^\dagger dt + \sum_{\mu'}(1-\bar\eta_{\mu'})\widehat{V}_{\mu'}\tilde\rho_t\widehat{V}_{\mu'}^\dagger dt}{\mathrm{Tr}\left(\widehat{M}_{dy_t}\tilde\rho_t\widehat{M}_{dy_t}^\dagger + \sum_\nu(1-\eta_\nu)\widehat{L}_\nu\tilde\rho_t\widehat{L}_\nu^\dagger dt + \sum_{\mu'}(1-\bar\eta_{\mu'})\widehat{V}_{\mu'}\tilde\rho_t\widehat{V}_{\mu'}^\dagger dt\right)}$$

with $\tilde\rho_t = \frac{\bar\theta_\mu\rho_t + \sum_{\mu'}\bar\eta_{\mu,\mu'}\widehat{V}_{\mu'}\rho_t\widehat{V}_{\mu'}^\dagger}{\bar\theta_\mu + \sum_{\mu'}\bar\eta_{\mu,\mu'}\ \mathrm{Tr}\left(\widehat{V}_{\mu'}\rho_t\widehat{V}_{\mu'}^\dagger\right)}$.

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[10] H. Amini, C. Pellegrini, and PR. Stability of continuous-time quantum filters with measurement imperfections. *Russian Journal of Mathematical Physics*, 21(3):297–315–, 2014.

Outline

- Feedback stabilization of the photon box
  - Discrete-time dynamics
  - Stabilization of Fock-state with a classical controller
  - Stabilization of Schrödinger cats with a quantum controller
- Dynamics governed by Stochastic Master Equations (SME)
  - Discrete-time SME
  - Continuous-time SME
  - Estimation based on SME
  - Two kinds of feedback schemes
- Stabilization with quantum controller
  - Quantum dissipation engineering
  - Convergence analysis based on Lindblad master equations
- Autonomous Quantum Error Correction for bosonic codes
  - Assumption of local errors in the phase-space (q, p)
  - Cat-qubit and autonomous correction of bit-flips
  - GKP-qubit and autonomous correction of bit/phase-flips

Discrete-time models of open quantum systems

Four features[11]:

1. Bayes inference: $\mathbb{P}(\mu/y) = \frac{\mathbb{P}(y/\mu)\mathbb{P}(\mu)}{\sum_{\mu'} \mathbb{P}(y/\mu')\mathbb{P}(\mu')}$,
2. Schrödinger equations defining unitary transformations.
3. Randomness, irreversibility and dissipation induced by the measurement of observables with degenerate spectra.
4. Entanglement and tensor product for composite systems.

$\Rrightarrow$ **Control input u, constant parameter p and measured output y**
Set of operators $\widehat{M}^{\mathsf{p}}_{\mathsf{u},\mu}$ satisfying $\sum_\mu (\widehat{M}^{\mathsf{p}}_{\mathsf{u},\mu})^\dagger \widehat{M}^{\mathsf{p}}_{\mathsf{u},\mu} = \widehat{I}$, left stochastic matrix $(\eta^{\mathsf{p}}_{\mathsf{y},\mu})$ (i.e. $\eta^{\mathsf{p}}_{\mathsf{y},\mu} \geq 0$ and $\sum_{\mathsf{y}} \eta^{\mathsf{p}}_{\mathsf{y},\mu} = 1$), underling the SME:

$$\rho_{\mathsf{t}+1} = \frac{\mathcal{K}^{\mathsf{p}}_{\mathsf{u_t},\mathsf{y_t}}(\rho_{\mathsf{t}})}{\mathrm{Tr}\left(\mathcal{K}^{\mathsf{p}}_{\mathsf{u_t},\mathsf{y_t}}(\rho_{\mathsf{t}})\right)}, \text{ with proba. } \mathbb{P}_{\mathsf{u_t},\mathsf{y_t}}(\rho_t) = \mathrm{Tr}\left(\mathcal{K}^{\mathsf{p}}_{\mathsf{u_t},\mathsf{y_t}}(\rho_t)\right)$$

with $\mathcal{K}^{\mathsf{p}}_{\mathsf{u},\mathsf{y}}(\rho) = \sum_{\mu=1}^{m} \eta^{\mathsf{p}}_{\mathsf{y},\mu} \widehat{M}^{\mathsf{p}}_{\mathsf{u},\mu} \rho (\widehat{M}^{\mathsf{p}}_{\mathsf{u},\mu})^\dagger$. It is associated to the Kraus map (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{t+1} | \rho_t, \mathsf{u_t}\right) = \mathcal{K}^{\mathsf{p}}_{\mathsf{u_t}}(\rho_t) = \sum_{\mathsf{y}} \mathcal{K}^{\mathsf{p}}_{\mathsf{u_t},\mathsf{y}}(\rho_t) = \sum_{\mu} \widehat{M}^{\mathsf{p}}_{\mathsf{u_t},\mu} \rho_t (\widehat{M}^{\mathsf{p}}_{\mathsf{u_t},\mu})^\dagger.$$

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[11] See, e.g., books: E.B Davies in 1976; S. Haroche with J.M. Raimond in 2006; C. Gardiner with P. Zoller in 2014/2015.

Likelihood function and adjoint state (1)

Consider N independent realizations with Input/Output trajectory data $\mathsf{U}^{(\mathsf{n})}$/ $\mathsf{Y}^{(\mathsf{n})}$ indexed by $n \in \{1, \ldots, N\}$.

- Denote by $\mathbb{P}_n(\mathsf{Y}^{(\mathsf{n})} \mid \mathsf{U}^{(\mathsf{n})}, \rho_0, \mathsf{p})$ the probability of getting measurement trajectory number n, $\mathsf{Y}^{(\mathsf{n})} = (\mathsf{y}_\mathsf{t}^{(\mathsf{n})})_{t=0,\ldots,T}$, knowing the initial state $\rho_0^{(n)} = \rho_0$, input control $\mathsf{U}^{(\mathsf{n})} = (\mathsf{u}_\mathsf{t}^{(\mathsf{n})})_{t=0,\ldots,T}$ and constant parameter p.
- Since $\rho_{t+1}^{(n)} = \dfrac{\mathcal{K}^{\mathbf{p}}_{\mathbf{u}_\mathbf{t}^{(\mathbf{n})},\mathbf{y}_\mathbf{t}^{(\mathbf{n})}}\left(\rho_t^{(n)}\right)}{\mathrm{Tr}\left(\mathcal{K}^{\mathbf{p}}_{\mathbf{u}_\mathbf{t}^{(\mathbf{n})},\mathbf{y}_\mathbf{t}^{(\mathbf{n})}}\left(\rho_t^{(n)}\right)\right)}$ with $\mathrm{Tr}\left(\mathcal{K}^{\mathbf{p}}_{\mathbf{u}_\mathbf{t}^{(\mathbf{n})},\mathbf{y}_\mathbf{t}^{(\mathbf{n})}}\left(\rho_t^{(n)}\right)\right)$ the probability of having detected $\mathsf{y}_\mathsf{t}^{(\mathsf{n})}$ knowing $\rho_t^{(n)}$, $\mathsf{u}_\mathsf{t}^{(\mathsf{n})}$ and p, one has

$$\mathbb{P}_n(\mathsf{Y}^{(\mathsf{n})} \mid \mathsf{U}^{(\mathsf{n})}, \rho_0, \mathsf{p}) = \prod_{t=0}^{T} \mathrm{Tr}\left(\mathcal{K}^{\mathbf{p}}_{\mathbf{u}_\mathbf{t}^{(\mathbf{n})},\mathbf{y}_\mathbf{t}^{(\mathbf{n})}}\left(\rho_t^{(n)}\right)\right) = \mathrm{Tr}\left(\mathcal{K}^{\mathbf{p}}_{\mathbf{u}_\mathbf{T}^{(\mathbf{n})},\mathbf{y}_\mathbf{T}^{(\mathbf{n})}} \circ \ldots \circ \mathcal{K}^{\mathbf{p}}_{\mathbf{u}_\mathbf{0}^{(\mathbf{n})},\mathbf{y}_\mathbf{0}^{(\mathbf{n})}}(\rho_0)\right).$$

Computation of the likelihood function via the adjoint state (2)

- With adjoint map $\mathcal{K}^{\mathsf{p}*}_{\mathbf{u},\mathbf{y}}$ ($\forall A, B,$ $\mathrm{Tr}\left(\mathcal{K}^{\mathsf{p}}_{\mathbf{u},\mathbf{y}}(A)\ B\right) \equiv \mathrm{Tr}\left(A\ \mathcal{K}^{\mathsf{p}*}_{\mathbf{u},\mathbf{y}}(B)\right)$):

$$\mathrm{Tr}\left(\mathcal{K}^{\mathsf{p}}_{\mathbf{u}^{(\mathbf{n})}_{\mathbf{T}},\mathbf{y}^{(\mathbf{n})}_{\mathbf{T}}}\circ\ldots\circ\mathcal{K}^{\mathsf{p}}_{\mathbf{u}^{(\mathbf{n})}_{\mathbf{0}},\mathbf{y}^{(\mathbf{n})}_{\mathbf{0}}}(\rho_0)\quad \widehat{I}\right) = \mathrm{Tr}\left(\rho_0\quad \mathcal{K}^{\mathsf{p}*}_{\mathbf{u}^{(\mathbf{n})}_{\mathbf{0}},\mathbf{y}^{(\mathbf{n})}_{\mathbf{0}}}\circ\ldots\circ\mathcal{K}^{\mathsf{p}*}_{\mathbf{u}^{(\mathbf{n})}_{\mathbf{T}},\mathbf{y}^{(\mathbf{n})}_{\mathbf{T}}}(\widehat{I})\right).$$

- With the normalized adjoint quantum filter[12]: $E_t^{(n)} = \dfrac{\mathcal{K}^{\mathsf{p}*}_{\mathbf{u}^{(\mathbf{n})}_{\mathbf{t}},\mathbf{y}^{(\mathbf{n})}_{\mathbf{t}}}\left(E^{(n)}_{t+1}\right)}{\mathrm{Tr}\left(\mathcal{K}^{\mathsf{p}*}_{\mathbf{u}^{(\mathbf{n})}_{\mathbf{t}},\mathbf{y}^{(\mathbf{n})}_{\mathbf{t}}}\left(E^{(n)}_{t+1}\right)\right)}$

  with $E^{(n)}_{T+1} = \widehat{I}/\ \mathrm{Tr}\left(\widehat{I}\right)$, the likelihood probability $\mathbb{P}_n(\mathsf{Y}^{(\mathsf{n})} \mid \mathsf{U}^{(\mathsf{n})}, \rho_0, \mathsf{p})$ is equal to

$$\prod_{t=T}^{0}\ \mathrm{Tr}\left(\mathcal{K}^{\mathsf{p}*}_{\mathbf{u}^{(\mathbf{n})}_{\mathbf{t}},\mathbf{y}^{(\mathbf{n})}_{\mathbf{t}}}\left(E^{(n)}_{t+1}\right)\right)\ \mathrm{Tr}\left(\rho_0 E_0^{(n)}\right) \triangleq g_n(\mathsf{U},\mathsf{Y},\mathsf{p})\ \mathrm{Tr}\left(\rho_0 E_0^{(n)}\right).$$

- A simple expression of its gradients versus ρ_0 and p:

$$\nabla_{\rho_0}\log\mathbb{P}_n = \frac{E_0^{(n)}}{\mathrm{Tr}\left(\rho_0 E_0^{(n)}\right)},\quad \nabla_{\mathsf{p}} \log\mathbb{P}_n\cdot\delta\mathsf{p} = \sum_{t=0}^{T} \frac{\mathrm{Tr}\left(E^{(n)}_{t+1}\ \left(\nabla_{\mathsf{p}}\mathcal{K}^{\mathsf{p}}_{\mathbf{u}^{(\mathbf{n})}_{\mathbf{t}},\mathbf{y}^{(\mathbf{n})}_{\mathbf{t}}}\left(\rho_t^{(n)}\right)\cdot\delta\mathsf{p}\right)\right)}{\mathrm{Tr}\left(E^{(n)}_{t+1}\ \mathcal{K}^{\mathsf{p}}_{\mathbf{u}^{(\mathbf{n})}_{\mathbf{t}},\mathbf{y}^{(\mathbf{n})}_{\mathbf{t}}}\left(\rho_t^{(n)}\right)\right)},$$

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[12] M. Tsang. Time-symmetric quantum theory of smoothing. PRL 2009. S. Gammelmark, B. Julsgaard, and K. Mølmer. Past quantum states of a monitored system. PRL 2013.

MaxLike tomography from I/O data $\mathsf{U} = \left\{\mathsf{u}_{\mathsf{t}=0,\ldots,\mathsf{T}}^{(\mathsf{n}=1,\ldots,\mathsf{N})}\right\}$ and $\mathsf{Y} = \left\{\mathsf{y}_{\mathsf{t}=0,\ldots,\mathsf{T}}^{(\mathsf{n}=1,\ldots,\mathsf{N})}\right\}$

From $\mathbb{P}_n(\mathsf{Y}^{(\mathsf{n})} \mid \mathsf{U}^{(\mathsf{n})}, \rho_0, \mathsf{p}) = g_n(\mathsf{U}, \mathsf{Y}, \mathsf{p})\ \mathrm{Tr}\left(\rho_0 E_0^{(n)}\right)$ we have

$$\mathbb{P}(\mathsf{Y} \mid \mathsf{U}, \rho_0, \mathsf{p}) \triangleq \prod_{\mathbf{n}=\mathbf{1}}^{N} \mathbb{P}_n(\mathsf{Y}^{(\mathsf{n})} \mid \mathsf{U}^{(\mathsf{n})}, \rho_0, \mathsf{p}) = \left(\prod_{n=\mathbf{1}}^{N} g_n(\mathsf{U}, \mathsf{Y}, \mathsf{p})\right) \left(\prod_{n=\mathbf{1}}^{N}\ \mathrm{Tr}\left(\rho_0 E_0^{(n)}\right)\right).$$

- MaxLike **state tomography:** p is known and ${\rho_0}^{ML}$ maximizes

$$\rho_0 \mapsto \sum_{n=1}^{N} \log\left(\ \mathrm{Tr}\left(\rho_0 E_0^{(n)}\right)\right)$$

  a concave function on the convex set of density operators ρ:
  a well structured convex optimization problem.

- MaxLike **process tomography:** ρ_0 is known and p^{ML} maximizes $\mathsf{p} \mapsto f(\mathsf{p}) = \log \mathbb{P}_n(\mathsf{Y}^{(\mathsf{n})} \mid \mathsf{U}^{(\mathsf{n})}, \rho_0, \mathsf{p})$ those gradient is given by

$$\nabla_{\mathsf{p}} f(\mathsf{p}) \cdot \delta\mathsf{p} = \sum_{n=1}^{N} \sum_{t=0}^{T} \frac{\mathrm{Tr}\left(E_{t+\mathbf{1}}^{(n)}\ \left(\nabla_{\mathsf{p}} \mathcal{K}^{\mathbf{p}}_{\mathbf{u}_{\mathbf{t}}^{(\mathbf{n})}, \mathbf{y}_{\mathbf{t}}^{(\mathbf{n})}}\left(\rho_t^{(n)}\right) \cdot \delta\mathsf{p}\right)\right)}{\mathrm{Tr}\left(E_{t+\mathbf{1}}^{(n)}\ \mathcal{K}^{\mathbf{p}}_{\mathbf{u}_{\mathbf{t}}^{(\mathbf{n})}, \mathbf{y}_{\mathbf{t}}^{(\mathbf{n})}}\left(\rho_t^{(n)}\right)\right)},$$

  The Hessian $\nabla_{\mathsf{p}}^2 f$ can be computed similarly (Fisher information).

Stability issues

- For $\rho_{k+1} = \mathcal{K}(\rho_k)$, contraction for many distances[13] (nuclear norm, fidelity,...)
- Adjoint map (unital map) $A_{k+1} = \mathcal{K}^*(A_k)$ contracts spectrum[14]:
$$\lambda_{min}(A_k) \leq \lambda_{min}(A_{k+1}) \leq \lambda_{max}(A_{k+1}) \leq \lambda_{max}(A_k).$$
- Quantum filter $\rho^{\textbf{est}}_{k+1} = \frac{\mathcal{K}_{y_k}(\rho^{\textbf{est}}_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho^{\textbf{est}}_k)\right)}$ where y_k is governed by $\rho_{k+1} = \frac{\mathcal{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right)}$ with $\rho^{\textbf{est}}_0 \neq \rho_0$: fidelity $\mathrm{Tr}^2\left(\sqrt{\sqrt{\rho^{\textbf{est}}_k}\rho_k\sqrt{\rho^{\textbf{est}}_k}}\right)$ is always a sub-martingale[15]
- Convergence issues around filtering and parameter estimation along quantum trajectories: seminal works of Belavkin in continuous-time, Van-Handel thesis at Caltech 2007. See also recent works of Nina Amini, Maël Bompais, Tristan Benoit and Clément Pellegrini.

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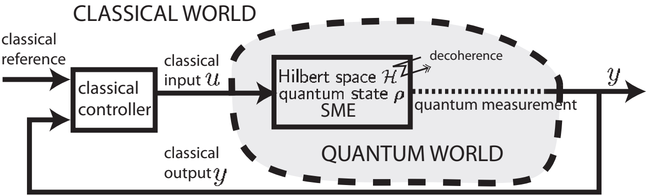
[13] D. Petz. Monotone metrics on matrix spaces. *Linear Algebra and its Applications*, 244:81–96, 1996.

[14] R. Sepulchre, A. Sarlette, and PR.. Consensus in non-commutative spaces. In *Decision and Control (CDC), 2010 49th IEEE Conference on*, pages 6596–6601, 2010.

[15] PR. Fidelity is a sub-martingale for discrete-time quantum filters. *IEEE Transactions on Automatic Control*, 56(11):2743–2747, 2011.

Outline

Measurement-based feedback



- P-controller (Markovian feedback[16]) for $u_t\ dt = k\ dy_t$, the ensemble average closed-loop dynamics of ρ remains governed by a linear Lindblad master equation.
- PID controller: no Lindblad master equation in closed-loop for dynamics output feedback
- Nonlinear hidden-state stochastic systems: Lyapunov state-feedback[17]; many open issues on convergence rates, delays, robustness, ...
- **Short sampling times limit feedback complexity**

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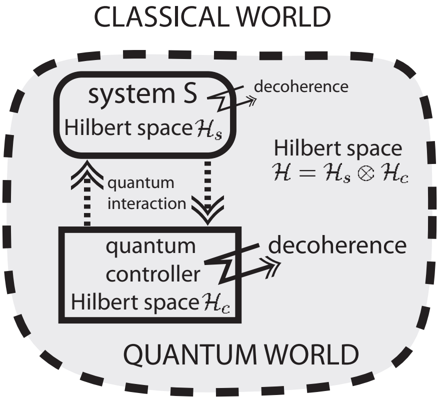
[16] H. Wiseman, G. Milburn (2009). Quantum Measurement and Control. Cambridge University Press.

[17] See e.g.: C. Ahn et. al (2002): Continuous quantum error correction via quantum feedback control. Phys. Rev. A 65;
M. Mirrahimi, R. Handel (2007): Stabilizing feedback controls for quantum systems. SIAM Journal on Control and Optimization, 46(2), 445-467;
G. Cardona, A. Sarlette, PR (2019): Continuous-time quantum error correction with noise-assisted quantum feedback. IFAC Mechatronics & Nolcos Conf.

Coherent (autonomous) feedback (dissipation engineering)

Quantum analogue of Watt speed governor: a **dissipative** mechanical system controls another mechanical system [18]



Optical pumping (Kastler 1950), coherent population trapping (Arimondo 1996)

Dissipation engineering, autonomous feedback: (Zoller, Cirac, Wolf, Verstraete, Devoret, Schoelkopf, Siddiqi, Martinis, Mølmer, Raimond, Brune,. . . , Lloyd, Viola, Ticozzi, Leghtas, Mirrahimi, Sarlette, PR, ...)

(S,L,H) theory and linear quantum systems: quantum feedback networks based on stochastic Schrödinger equation, Heisenberg picture (Gardiner, Yurke, Mabuchi, Genoni, Serafini, Milburn, Wiseman, Doherty, . . . , Gough, James, Petersen, Nurdin, Yamamoto, Zhang, Dong, . . .)

Stability analysis: Kraus maps and Lindblad propagators are always contractions (non commutative diffusion and consensus).

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[18] J.C. Maxwell (1868): On governors. Proc. of the Royal Society, No.100.

Coherent feedback involves tensor products and many time-scales

The closed-loop Lindblad master equation on $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$:

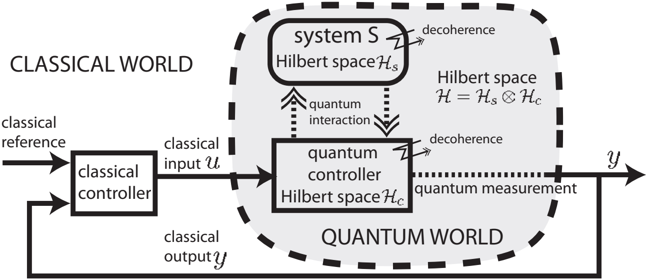
$$\frac{d}{dt}\rho = -i\left[\widehat{H}_s \otimes \widehat{I}_c + \widehat{I}_s \otimes \widehat{H}_c + \widehat{H}_{sc}\ ,\ \rho\right] + \sum_{\nu} \mathbb{D}_{\widehat{L}_{s,\nu}\otimes\widehat{I}_c}(\rho) + \sum_{\nu'} \mathbb{D}_{\widehat{I}_s\otimes\widehat{L}_{c,\nu'}}(\rho)$$

with $\mathbb{D}_{\widehat{L}}(\rho) = \widehat{L}\rho\widehat{L}^\dagger - \frac{1}{2}\left(\widehat{L}^\dagger\widehat{L}\rho + \rho\widehat{L}^\dagger\widehat{L}\right)$ and operators made of tensor products.

- Consider a convex subset $\overline{\mathcal{D}}_s$ of steady-states for original system S: each density operator $\overline{\rho}_s$ on $\mathcal{H}_s$ belonging to $\overline{\mathcal{D}}_s$ satisfy $i[\widehat{H}_s, \overline{\rho}_s] = \sum_\nu \mathbb{D}_{\widehat{L}_{s,\nu}}(\overline{\rho}_s)$.

- Designing a **realistic** quantum controller C ($\widehat{H}_c, \widehat{L}_{c,\nu'}$) and coupling Hamiltonian $\widehat{H}_{sc}$ stabilizing $\overline{\mathcal{D}}_s$ is non trivial. **Realistic** means in particular relying on physical time-scales and constraints:
  - Fastest time-scales attached to $\widehat{H}_s$ and $\widehat{H}_c$ (Bohr frequencies) and averaging approximations: $\|\widehat{H}_s\|, \|\widehat{H}_c\| \gg \|\widehat{H}_{sc}\|$,
  - High-quality oscillations: $\|\widehat{H}_s\| \gg \|\widehat{L}_{s,\nu}^\dagger\widehat{L}_{s,\nu}\|$ and $\|\widehat{H}_c\| \gg \|\widehat{L}_{c,\nu'}^\dagger\widehat{L}_{c,\nu'}\|$.
  - Decoherence rates of S much slower than those of C: $\|\widehat{L}_{s,\nu}^\dagger\widehat{L}_{s,\nu}\| \ll \|\widehat{L}_{c,\nu'}^\dagger\widehat{L}_{c,\nu'}\|$: model reduction by quasi-static approximations (adiabatic elimination, singular perturbations).

Quantum feedback engineering for robust quantum information processing



To protect quantum information stored in system S (alternative to usual QEC):

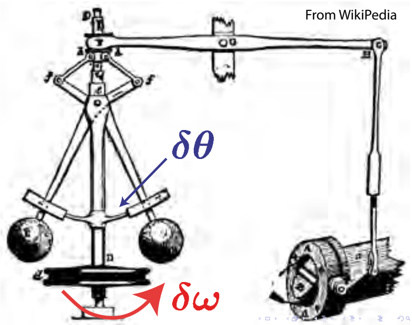
- fast stabilization and protection mainly achieved by a **quantum controller** (coherent feedback stabilizing decoherence-free sub-spaces);
- slow decoherence and perturbations mainly tackled by a **classical controller** (measurement-based feedback "finishing the job")

Underlying mathematical methods for high-precision dynamical modeling and control based on stochastic master equations (SME):

- High-order averaging methods and geometric singular perturbations for coherent feedback.
- Stochastic control Lyapunov methods for exponential stabilization via measurement-based feedback.

Outline

Watt regulator: classical analogue of a quantum controller. [19]



Third order system

The first variations of speed $\delta\omega$ and governor angle $\delta\theta$ obey to

$$\frac{d}{dt}\delta\omega = -a\delta\theta$$

$$\frac{d^2}{dt^2}\delta\theta = -\Lambda\frac{d}{dt}\delta\theta - \Omega^2(\delta\theta - b\delta\omega)$$

with (a, b, Λ, Ω) positive parameters.

$$\frac{d^3}{dt^3}\delta\omega + \Lambda\frac{d^2}{dt^2}\delta\omega + \Omega^2\frac{d}{dt}\delta\omega + ab\Omega^2\delta\omega = 0.$$

Characteristic polynomial $P(s) = s^3 + \Lambda s^2 + \Omega^2 s + ab\Omega^2$ with roots having negative real parts iff $\Lambda > ab$: governor damping must be strong enough to ensure asymptotic stability.

Key issues: asymptotic stability and convergence rates.

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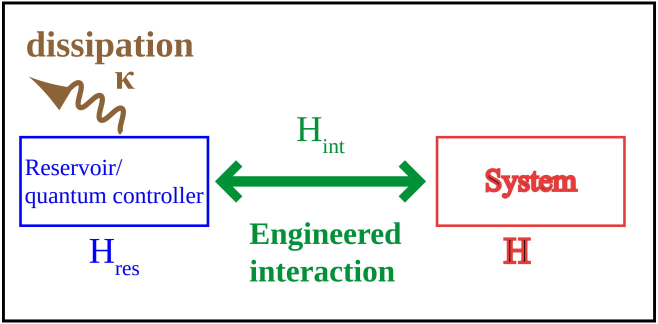
[19] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.

Reservoir/dissipation engineering and quantum controller (1) [20]



$$\widehat{H} = \widehat{H}_{res} + \widehat{H}_{int} + \widehat{H}$$

If $\rho \underset{t\to\infty}{\rightarrow} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}|$ exponentially with rate $1/\tau > 0$ then ......

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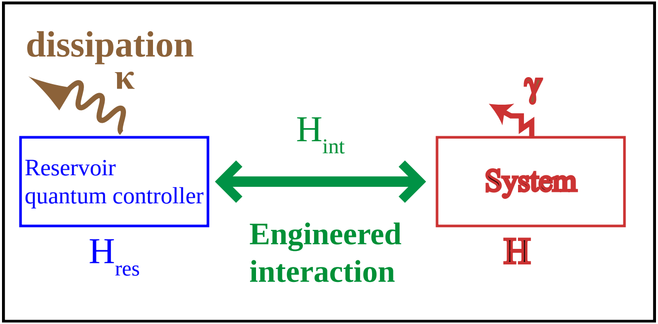
[20] See, e.g., the lectures of H. Mabuchi delivered at the "Ecole de physique des Houches", July 2011.

Reservoir/dissipation engineering and quantum controller (2)



$$\widehat{H} = \widehat{H}_{\text{res}} + \widehat{H}_{\text{int}} + \widehat{H}$$

…… $\rho \underset{t\to\infty}{\rightarrow} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}| + \overline{\delta\rho}$, if $\tau\gamma \ll 1$ then $|\overline{\delta\rho}| \ll 1$

Outline

- Feedback stabilization of the photon box
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  - Stabilization of Fock-state with a classical controller
  - Stabilization of Schrödinger cats with a quantum controller
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  - Discrete-time SME
  - Continuous-time SME
  - Estimation based on SME
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Quantum dynamics with dissipation (decoherence)

Gorini–Kossakowski –Sudarshan–Lindblad (GKSL) master equation:

$$\frac{d}{dt}\rho = -i[\widehat{H}_0 + u\widehat{H}_1, \rho] + \sum_\nu \left(\widehat{L}_\nu \rho \widehat{L}_\nu^\dagger - \frac{1}{2}(\widehat{L}_\nu^\dagger \widehat{L}_\nu \rho + \rho \widehat{L}_\nu^\dagger \widehat{L}_\nu) \right)$$

- Preservation of trace, hermiticity and positivity: ρ lies in the set of Hermitian and trace-class operators that are non-negative and of trace one.
- Invariance under unitary transformations.
  A time-varying change of frame $\rho \mapsto \widehat{U}_t^\dagger \rho \widehat{U}_t$ with $\widehat{U}_t$ unitary.
  The new density operator obeys to a similar master equation where $\widehat{H}_0 + u\widehat{H}_1 \mapsto \widehat{U}_t^\dagger(\widehat{H}_0 + u\widehat{H}_0)\widehat{U}_t + i\widehat{U}_t^\dagger \left(\frac{d}{dt}\widehat{U}_t\right)$ and $\widehat{L}_\nu \mapsto \widehat{U}_t^\dagger \widehat{L}_\nu \widehat{U}_t$.
- "L^1-contraction" properties. Such master equations generate contraction semi-groups for many distances (nuclear distance[21], Hilbert metric on the cone of non negative operators[22]).
- If the Hermitian operator $\widehat{A}$ satisfies the operator inequality
  $$i[\widehat{H}_0 + u\widehat{H}_1, \widehat{A}] + \sum_\nu \left(\widehat{L}_\nu^\dagger \widehat{A}\widehat{L}_\nu - \frac{1}{2}(\widehat{L}_\nu^\dagger \widehat{L}_\nu \widehat{A} + \widehat{A}\widehat{L}_\nu^\dagger \widehat{L}_\nu) \right) \leq 0$$
  then $V(\rho) = \text{Tr}\left(\widehat{A}\rho\right)$ is a Lyapunov function when $\widehat{A} \geq 0$.

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[21] D.Petz (1996). Monotone metrics on matrix spaces. Linear Algebra and its Applications

[22] R. Sepulchre, A. Sarlette, PR (2010). Consensus in non-commutative spaces. IEEE-CDC.

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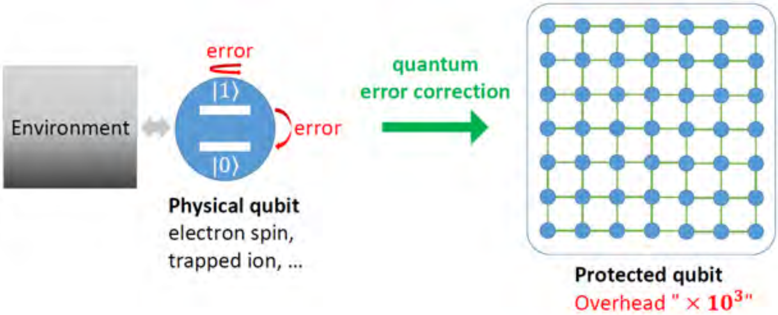
QEC: 2D redundancy to correct bit-flip and phase-flip errors

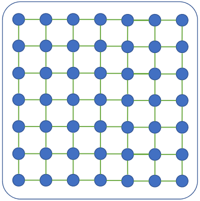


Physical qubit
electron spin,
trapped ion, ...

Protected qubit
Overhead "$\times 10^3$"

Local noise assumption (1)

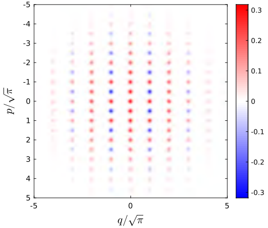
DV-QEC



$$\mathcal{H} = \left(\mathbb{C}^2\right)^{\otimes N}$$

Dimension: 2^N

CV-QEC



$$\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$$

Dimension: $+\infty$

Wave function $|\psi\rangle : \mathbb{R} \ni q \mapsto \psi(q) \in \mathbb{C}$, and Wigner function

$$\mathbb{R}^2 \ni (q,p) \mapsto W^{|\psi\rangle\langle\psi|}(q,p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \psi^*\left(q - \frac{u}{2}\right)\psi\left(q + \frac{u}{2}\right) e^{-2ipu} du.$$

Local error operators $\hat{q}$ and $\hat{p}$ ($[\hat{q}, \hat{p}] = i$) on $|\psi\rangle$: small random shifts along q ($e^{i\pm\epsilon\hat{p}} \equiv e^{\pm\epsilon d/dq}$) and p ($e^{i\pm\epsilon\hat{q}} \equiv e^{\mp\epsilon d/dp}$) similar to diffusion along q and p axis for $W^{|\psi\rangle\langle\psi|}$.

Local noise assumption (2)

For a density operator ρ, its Wigner function

$$\mathbb{R}^2 \ni (q,p) \mapsto W^\rho(q,p) \in \mathbb{R}$$

reads ($\widehat{a} = \frac{\widehat{q}+i\widehat{p}}{\sqrt{2}}$)

$$W^\rho(q,p) = \tfrac{1}{\pi}\ \mathrm{Tr}\left(e^{i\pi \widehat{a}^\dagger \widehat{a}}\ e^{i(p\widehat{q}-q\widehat{p})}\ \rho\ e^{-i(p\widehat{q}-q\widehat{p})}\right)$$

Since

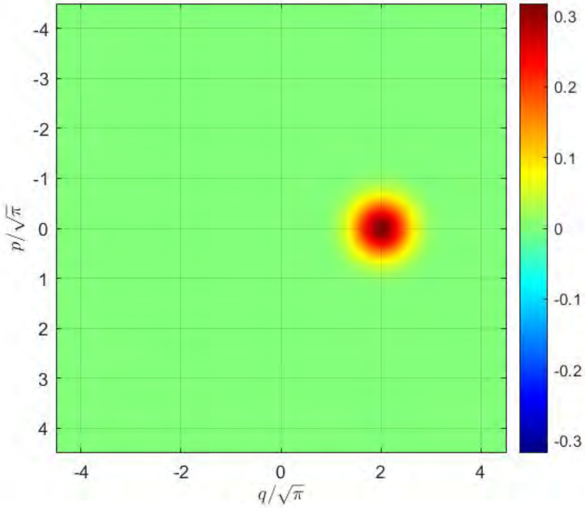
$$W^{\mathcal{D}_{\widehat{q}}(\rho)} = \frac{1}{2}\frac{\partial^2}{\partial p^2} W^\rho, \quad W^{\mathcal{D}_{\widehat{p}}(\rho)} = \frac{1}{2}\frac{\partial^2}{\partial q^2} W^\rho$$

and

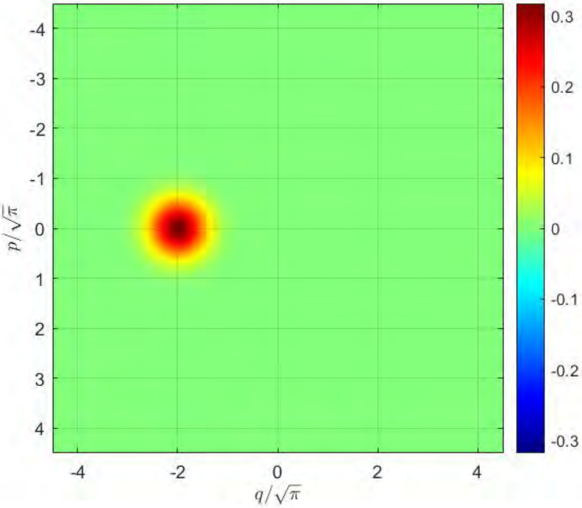
$$W^{\mathcal{D}_{\widehat{a}}(\rho)} = \frac{1}{2}\frac{\partial}{\partial q}(qW^\rho) + \frac{1}{2}\frac{\partial}{\partial p}(pW^\rho) + \frac{1}{2}\frac{\partial^2}{\partial q^2} W^\rho + \frac{1}{2}\frac{\partial^2}{\partial p^2} W^\rho$$

dominant errors on ρ correspond to local differential operators in the phase-space (q,p).

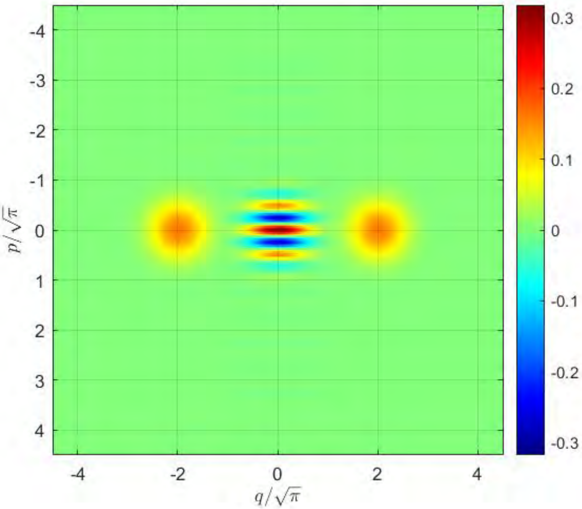
Wigner function of coherent state $|\sqrt{2\pi}\rangle \equiv \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(q-2\sqrt{\pi})^2}{2}\right) \approx |0_L\rangle$



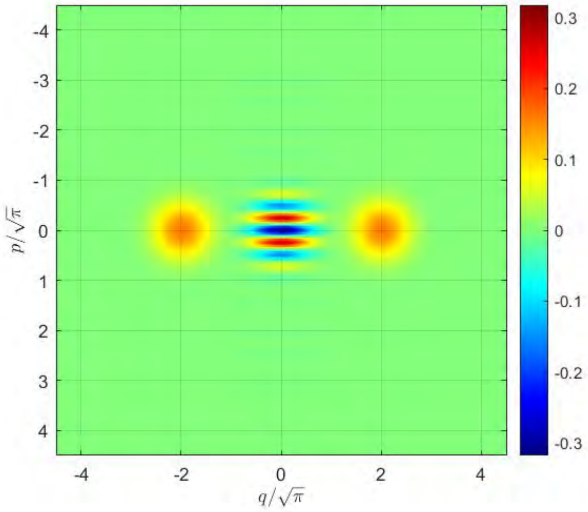
Wigner function of coherent state $|\text{-}\sqrt{2\pi}\rangle \equiv \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(q-2\sqrt{\pi})^2}{2}\right) \approx |1_L\rangle$



Wigner function of $|+_L\rangle \propto \frac{|\sqrt{2\pi}\rangle + |\text{-}\sqrt{2\pi}\rangle}{\sqrt{2}}$ ("Schrödinger phase cat")



Wigner function of $|-_L\rangle \propto \frac{|\sqrt{2\pi}\rangle - |\text{-}\sqrt{2\pi}\rangle}{\sqrt{2}}$ ("Schrödinger phase cat")



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Bosonic code with cat-qubits

- Quantum error corrrection requires redundancy.
- Bosonic code: instead of encoding a logical qubit in N physical qubits living in $\mathbb{C}^{2^N}$, encode a logical qubit in an harmonic oscillator living in Fock space $\text{span}\{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\} \sim L^2(\mathbb{R}, \mathbb{C})$ of infinite dimension.
- Cat-qubit [23]: $|\psi_L\rangle \in \text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ where $|\alpha\rangle$ is the coherent state of real amplitude α: $\widehat{a}|\alpha\rangle = \alpha|\alpha\rangle$ with $\widehat{a} = (\widehat{q} + i\widehat{p})/\sqrt{2}$ and $[\widehat{q}, \widehat{p}] = i$:

$$|\psi\rangle \sim \psi(q) \in L^2(\mathbb{R}, \mathbb{C}),\ \widehat{q}|\psi\rangle \sim q\psi(q),\ \widehat{p}|\psi\rangle \sim -i\frac{d\psi}{dq}(q),\ |\alpha\rangle \sim \frac{\exp\left(-\frac{(q-\alpha\sqrt{2})^2}{2}\right)}{\sqrt{2\pi}}.$$

- Stabilisation of cat-qubit via a single Lindblad dissipator $\widehat{L} = \widehat{a}^2 - \alpha^2$. For any initial density operator $\rho(0)$, the solution $\rho(t)$ of

$$\frac{d}{dt}\rho = \widehat{L}\rho\widehat{L}^\dagger - \frac{1}{2}(\widehat{L}^\dagger\widehat{L}\rho + \rho\widehat{L}^\dagger\widehat{L})$$

  converges exponentially towards a steady-state density operator since

$$\frac{d}{dt}\ \text{Tr}\left(\widehat{L}^\dagger\widehat{L}\rho\right) \leq -2\ \text{Tr}\left(\widehat{L}^\dagger\widehat{L}\rho\right), \quad \ker\widehat{L} = \text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}.$$

  Any density operator with support in $\text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ is a steady-state.

---

[23] M. Mirrahimi, Z. Leghtas, . . . , M. Devoret: Dynamically protected cat-qubits: a new paradigm for universal quantum computation. 2014, New Journal of Physics.

Asymmetrically Threaded SQUID (ATS) stabilizing a cat-qubit [24]

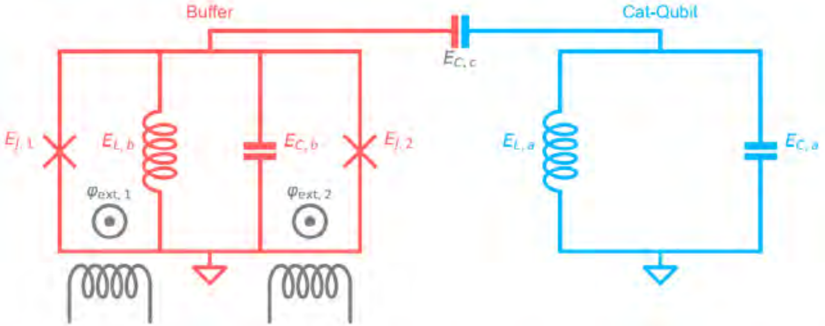

Figure S3. Equivalent circuit diagram. The cat-qubit (blue), a linear resonator, is capacitively coupled to the buffer (red). One recovers the circuit of Fig. 2 by replacing the buffer inductance with a 5-junction array and by setting $\varphi_\Sigma = (\varphi_{\text{ext},1} + \varphi_{\text{ext},2})/2$ and $\varphi_\Delta = (\varphi_{\text{ext},1} - \varphi_{\text{ext},2})/2$. Not shown here: the buffer is capacitively coupled to a transmission line, the cat-qubit resonator is coupled to a transmon qubit

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[24] R. Lescanne, . . . , Z. Leghtas: Exponential suppression of bit-flips in a qubit encoded in an oscillator. Nature Physics (2020)
Startup Alice&Bob (2020)
U. Reglade, . . . , Z. Leghtas: Quantum control of a cat-qubit with bit-flip times exceeding ten seconds. Nature (2024)

Master equations of the ATS super-conducting circuit

Oscillator $\widehat{a}$ with **quantum controller based on a damped oscillator** $\widehat{b}$:

$$\frac{d}{dt}\rho = g_2\Big[(\widehat{a}^2 - \alpha^2)\widehat{b}^\dagger - ((\widehat{a}^\dagger)^2 - \alpha^2)\widehat{b}\,,\ \rho\Big] + \kappa_b\Big(\widehat{b}\rho\widehat{b}^\dagger - (\widehat{b}^\dagger\widehat{b}\rho + \rho\widehat{b}^\dagger\widehat{b})/2\Big)$$

with $\alpha \in \mathbb{R}$ such that $\alpha^2 = u/g_2$, the drive amplitude $u \in \mathbb{R}$ applied to mode $\widehat{b}$ and $1/\kappa_b > 0$ the life-time of photon in mode $\widehat{b}$.
Any density operators $\bar{\rho} = \bar{\rho}_a \otimes |0\rangle\langle 0|_b$ **is a steady-state as soon as the support of** $\bar{\rho}_a$ **belongs to the two dimensional vector space spanned by the quasi-classical wave functions** $|\alpha\rangle$ **and** $|\text{-}\alpha\rangle$ (range($\bar{\rho}_a$) $\subset$ span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$)

Usually $\kappa_b \gg |g_2|$, mode $\widehat{b}$ relaxes rapidly to vaccuum $|0\rangle\langle 0|_b$, can be eliminated adiabatically (**singular perturbations**, second order corrections) to provides the slow evolution of mode $\widehat{a}$

$$\frac{d}{dt}\rho_a = \tfrac{4|g_2|^2}{\kappa_b}\Big(\widehat{L}\rho\widehat{L}^\dagger - \frac{1}{2}(\widehat{L}^\dagger\widehat{L}\rho + \rho\widehat{L}^\dagger\widehat{L})\Big) \text{ with } \widehat{L} = \widehat{a}^2 - \alpha^2.$$

Convergence via the exponential Lyapunov function $V(\rho) = \operatorname{Tr}\Big(\widehat{L}^\dagger\widehat{L}\rho\Big)$ [25]

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[25] For a mathematical proof of convergence analysis in an adapted Banach space, see :R. Azouit, A. Sarlette, PR: Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping. 2016, ESAIM: COCV.

Cat-qubit: exponential suppression of bit-flip for large α.

Since $\langle\alpha|\text{-}\alpha\rangle = e^{-2\alpha^2} \approx 0$:

$$|0_L\rangle \approx |\alpha\rangle,\ |1_L\rangle \approx |\text{-}\alpha\rangle,\ |+_L\rangle \propto \tfrac{|\alpha\rangle+|\text{-}\alpha\rangle}{\sqrt{2}},\ |-_L\rangle \propto \tfrac{|\alpha\rangle-|\text{-}\alpha\rangle}{\sqrt{2}}.$$

Photon loss as dominant error channel (dissipator $\widehat{a}$ with $0 < \kappa_1 \ll 1$):

$$\frac{d}{dt}\rho_a = \mathcal{D}_{\widehat{a}^2-\alpha^2}(\rho) + \kappa_1\mathcal{D}_{\widehat{a}}(\rho)$$

with $\mathcal{D}_{\widehat{L}}(\rho) = \widehat{L}\rho\widehat{L}^\dagger - \frac{1}{2}(\widehat{L}^\dagger\widehat{L}\rho + \rho\widehat{L}^\dagger\widehat{L})$.

- if $\rho(0) = |0_L\rangle\langle 0_L|$ or $|1_L\rangle\langle 1_L|$, $\rho(t)$ converges to a statistical mixture of quasi-classical states close to $\frac{1}{2}|\alpha\rangle\langle\alpha| + \frac{1}{2}|\text{-}\alpha\rangle\langle\text{-}\alpha|$ in a time

  $$T_{bit-flip} \sim \frac{e^{2\alpha^2}}{\kappa_1}$$

  since $\widehat{a}|0_L\rangle \approx \alpha|0_L\rangle$ and $\widehat{a}|1_L\rangle \approx -\alpha|1_L\rangle$.
- if $\rho(0) = |+_L\rangle\langle +_L|$ or $|-_L\rangle\langle -_L|$, $\rho(t)$ converges also to the same statistical mixture in a time

  $$T_{phase-flip} \sim \frac{1}{\kappa_1\alpha^2}$$

  since $\widehat{a}|+_L\rangle = \alpha|-L\rangle$ and $\widehat{a}|-_L\rangle = \alpha|+L\rangle$.

Take α large to ignore bit-flip and to correct only the phase-flip with $1D$ code: important overhead reduction.

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Grid-states and GKP-qubits

- Poisson summation formula: the Fourier transform of Dirac comb $f(q)$ of period T is a Dirac comb $g(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(q) e^{-iqp} dq$ of period $2\pi/T$.

infinite energy grid-states	q representation	p representation
$\|0_L\rangle$	$\sum_k \delta(q - 2k\sqrt{\pi})$	$\sum_k \delta(p - k\sqrt{\pi})$
$\|1_L\rangle$	$\sum_k \delta(q - 2(k+1)\sqrt{\pi})$	$\sum_k (-1)^k \delta(p - k\sqrt{\pi})$
$\|+_L\rangle \sim \|0_L\rangle + \|1_L\rangle$	$\sum_k \delta(q - k)\sqrt{\pi})$	$\sum_k \delta(p - 2k\sqrt{\pi})$
$\|-_L\rangle \sim \|0_L\rangle - \|1_L\rangle$	$\sum_k (-1)^k \delta(q - k)\sqrt{\pi})$	$\sum_k \delta(p - 2(k+1)\sqrt{\pi})$

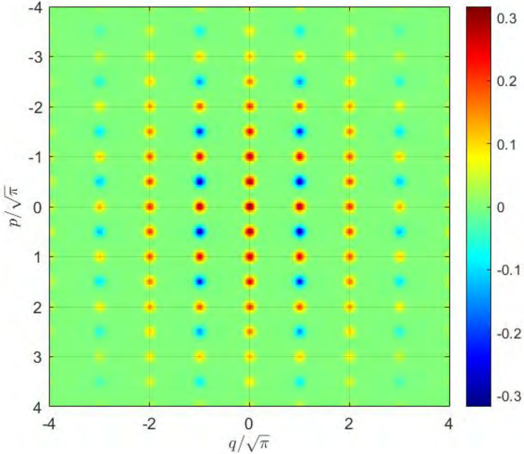
- Pauli operators of a GKP-qubit[26] with Bloch coordinates $(x, y, z) \in \mathbb{R}^3$:
$$\widehat{Z} = \text{sign}(\cos(\sqrt{\pi}\widehat{q})),\ \widehat{X} = \text{sign}(\cos(\sqrt{\pi}\widehat{p})) \text{ and } \widehat{Y} = -i\widehat{Z}\widehat{X}.$$
- 4 stabilizer operators $\widehat{S}$ relying on commuting modular operators in $\widehat{q}$ and $\widehat{p}$:
$$\forall \widehat{S} \in \{e^{i2\sqrt{\pi}\widehat{q}}, e^{i2\sqrt{\pi}\widehat{p}}, e^{-i2\sqrt{\pi}\widehat{q}}, e^{-i2\sqrt{\pi}\widehat{p}}\} \text{ and } \forall |\psi_L\rangle \in \text{span}\{|0_L\rangle, |1_L\rangle\}\text{: } \widehat{S}|\psi_L\rangle = |\psi_L\rangle$$
- Finite energy regularization with $0 < \epsilon \ll 1$,
$$|0_\epsilon\rangle \approx e^{-\epsilon \widehat{a}^\dagger \widehat{a}} |0_L\rangle, \quad |1_\epsilon\rangle \approx e^{-\epsilon \widehat{a}^\dagger \widehat{a}} |0_L\rangle,$$
where $\widehat{a}^\dagger \widehat{a} = \frac{1}{2}(\widehat{q}^2 + \widehat{p}^2) \sim \frac{1}{2}(q^2 + \partial^2/\partial q^2)$, provides a finite-energy code space where **any small local error can be corrected**[27].

---

[26] D. Gottesman, A. Kitaev and J. Preskill: Encoding a qubit in an oscillator. Physical Review A, 2001.

[27] 3 recent experiments stabilizing GKP-qubits via classical controllers: Ph. Campagne-Ibarcq et al. “Quantum error correction of a qubit encoded in grid states of an oscillator” Nature (2020); B. de Neeve et al. “Error correction of a logical grid state qubit by dissipative pumping” Nature (2022); V. Sivak et al. “Real-Time Quantum Error Correction beyond Break-Even” Nature (2023).

Wigner function of the GKP finite energy grid-state $|0_\epsilon\rangle$ [28]



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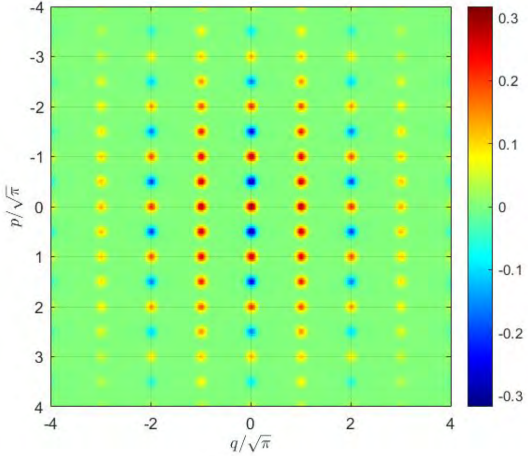
[28] $|0_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k \exp\left(-\frac{(q-2k\sqrt{\pi})^2}{\epsilon}\right)$ with $\epsilon = \frac{1}{30}$.

Wigner function of the GKP finite energy grid-state $|1_\epsilon\rangle$ [29]



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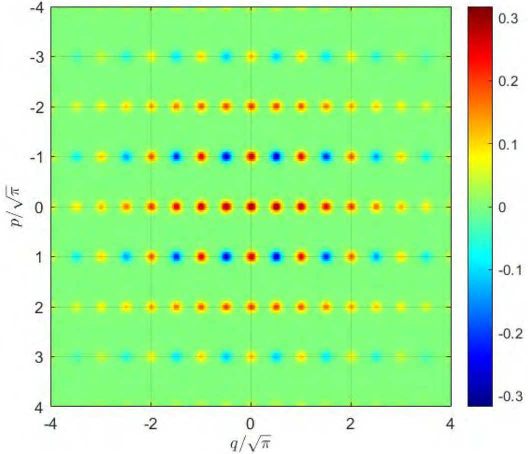
[29] $|1_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k \exp\left(-\frac{(q-(2k+1)\sqrt{\pi})^2}{\epsilon}\right)$ with $\epsilon = \frac{1}{30}$.

Wigner function of the GKPfinite energy grid-state $|+_\epsilon\rangle$ [30]



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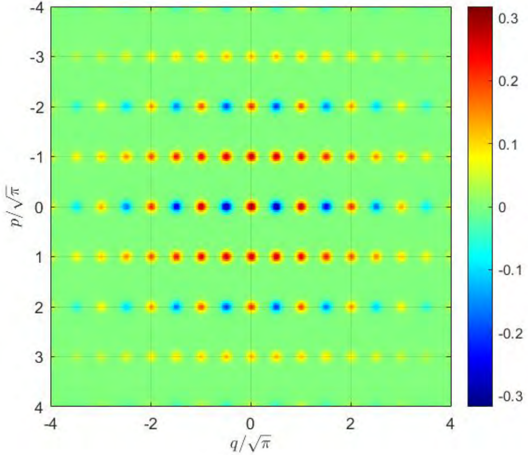
[30] $|+_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k \exp\left(\frac{(q-k\sqrt{\pi})^2}{\epsilon}\right) \equiv e^{-\epsilon p^2} \sum_k \exp\left(\frac{(p-2k\sqrt{\pi})^2}{\epsilon}\right)$.

Wigner function of the GKP finite energy grid-state $|-_\epsilon\rangle$ [31]



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[31] $|-_\epsilon\rangle \approx e^{-\epsilon q^2} \sum_k (-1)^k \exp\left(-\frac{(q-k\sqrt{\pi})^2}{\epsilon}\right) \equiv e^{-\epsilon p^2} \sum_k \exp\left(-\frac{(p-(2k+1)\sqrt{\pi})^2}{\epsilon}\right)$.

Exponential stabilisation of finite energy GKP-qubits[32]

- 4 regularized stabilizers:

$$\widehat{S}_{\epsilon,k} \triangleq e^{-(\epsilon - i\frac{k\pi}{2})\widehat{a}^\dagger \widehat{a}}\ e^{i2\sqrt{\pi}\widehat{q}}\ e^{(\epsilon - i\frac{k\pi}{2})\widehat{a}^\dagger \widehat{a}}, \quad k = 0, 1, 2, 3.$$

- Master equation with 4 dissipators $\widehat{M}_{\epsilon,k} = \widehat{S}_{\epsilon,k} - \widehat{I}$

$$\frac{d}{dt}\rho = \sum_{k=0}^{3} \mathcal{D}_{\widehat{M}_{\epsilon,k}}(\rho)$$

- Lyapunov function:

$$V(\rho) = \sum_k \operatorname{Tr}\left(\widehat{M}_{\epsilon,k}^\dagger \widehat{M}_{\epsilon,k}\rho\right) \text{ with } \frac{d}{dt}V \leq -\left(32\pi^2\epsilon^2 + O(\epsilon^3)\right)V$$

ensuring exponential convergence towards the finite-energy code space

$$\operatorname{span}\left\{e^{-\epsilon\widehat{a}^\dagger \widehat{a}}|0_L\rangle, e^{-\epsilon\widehat{a}^\dagger \widehat{a}}|1_L\rangle\right\}$$

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[32] L.A. Sellem, Ph. Campagne-Ibarcq, M. Mirrahimi, A. Sarlette, PR: Exponential convergence of a dissipative quantum system towards finite-energy grid states of an oscillator: IEEE CDC 2022 (arXiv:2203.16836).

Approximated Lindblad dissipators with exponentially small decoherence rates [33]

Replace the ideal dissipators $\widehat{M}_{\epsilon,k}$ by more realistic dissipators $\widehat{L}_{\epsilon,k}$ derived from a first-order approximation in ϵ:

$$\widehat{L}_{\epsilon,k} \triangleq e^{i\frac{k\pi}{2}\widehat{a}^\dagger\widehat{a}} \left(e^{-2\pi\epsilon} e^{i2\sqrt{\pi}\widehat{q}}(\widehat{I} - 2\epsilon\sqrt{\pi}\widehat{p}) - \widehat{I}\right) e^{-i\frac{k\pi}{2}\widehat{a}^\dagger\widehat{a}}$$

For ρ governed by master equation $\frac{d}{dt}\rho = \sum_{k=0}^{3} \mathcal{D}_{\widehat{L}_{\epsilon,k}}(\rho)$:

- Energy $\operatorname{Tr}\left(\widehat{a}^\dagger\widehat{a}\rho\right)$ remains finite and for t large is less than $\frac{1}{2\epsilon} + 0(1)$.
- For any 2π periodic function $f(\theta)$, one has

$$\frac{d}{dt}\operatorname{Tr}\left(f(\sqrt{\pi}\widehat{q})\rho\right) = -4\epsilon\pi e^{-2\pi\epsilon}\operatorname{Tr}\left(\left(\sin(2\sqrt{\pi}\widehat{q})f'(\sqrt{\pi}\hat{q}) - \epsilon\pi e^{-2\pi\epsilon}f''(\sqrt{\pi}\widehat{q})\right)\rho\right).$$

- Spect. $(\lambda_n)_{n\geq 0}$ of Witten Laplacian$\mathcal{L}_\sigma(f(\theta)) = \sin(2\theta)f'(\theta) - \sigma f''(\theta)$ with 2π-periodic function $f(\theta)$ and $0 < \sigma \ll 1$:
  $$\lambda_0 = 0 < \boldsymbol{\lambda_1} \sim \boldsymbol{\tfrac{4}{\pi} e^{-1/\sigma}} < 1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq \ldots$$
  with eigenfunction $f_1(\theta) \approx \operatorname{sign}(\cos\theta)$ corresponding to λ_1. Thus $z \approx \operatorname{Tr}\left(f_1(\sqrt{\pi}\widehat{q})\rho\right)$ is almost constant: $\frac{d}{dt}z \approx \boldsymbol{-16\epsilon\exp\left(-\frac{1}{\epsilon\pi}\right)}\, z$.
  Similar exponentially small decays for (x, y, z) with quadrature noises, i.e. when $\frac{d}{dt}\rho = \sum_{k=0}^{3} \mathcal{D}_{\widehat{L}_{\epsilon,k}}(\rho) + \kappa_q\mathcal{D}_{\widehat{q}}(\rho) + \kappa_p\mathcal{D}_{\widehat{p}}(\rho)$ $(\kappa_q, \kappa_q \ll 1)$

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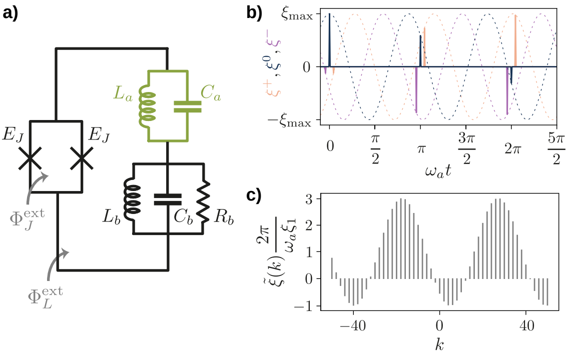
[33] L.A. Sellem, R. Robin, Ph. Campagne-Ibarcq, PR: Stability and decoherence rates of a GKP qubit protected by dissipation. IFAC WC 2023 (arXiv:2304.03806).

Engineering modular dissipation with super-conducting Josephson circuits [34]



High impedance $\sqrt{L_a/C_a}$ and low pulsation $1/\sqrt{L_a C_a}$ for storage mode $\widehat{a}$.

High pulsation $1/\sqrt{L_b C_b}$ of damped mode $\widehat{b}$ (quantum controller $R_b > 0$).

Josephson energy E_J between $\hbar/\sqrt{L_a C_a}$ and $\hbar/\sqrt{L_b C_b}$.

Classical open-loop control signals $\Phi_J^{ext}(t)$ and $\Phi_L^{ext}(t)$ made of short pulses.

Mathematical analysis to recover master equation with dissipators $\widehat{L}_k$.

Numerical simulations to test robustness versus experimental imperfections.

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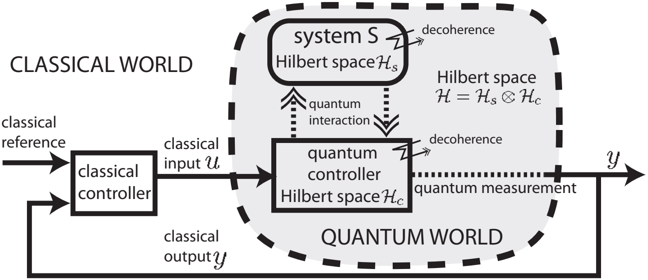
[34] L.A. Sellem, A. Salette, Z. Leghtas, M. Mirrahimi, PR, Ph. Campagne-Ibarcq: A GKP qubit protected by dissipation in a high-impedance superconducting circuit driven by a microwave frequency comb. PRX 2024 in press (arXiv:2304.01425).

Quantum feedback engineering for robust quantum information processing



To protect quantum information stored in system S:

- fast stabilization and protection mainly achieved by **quantum controllers** (autonomous feedback stabilizing decoherence-free sub-spaces);
- slow decoherence and perturbations, parameter estimation mainly tackled by **classical controllers and estimation algorithms** (measurement-based feedback and estimation "finishing the job")

Need of adapted mathematical and numerical methods for high-precision dynamical modeling and control with (stochastic) master equations.

Quantic research group ENS/Inria/Mines/CNRS/PSL, June 2023