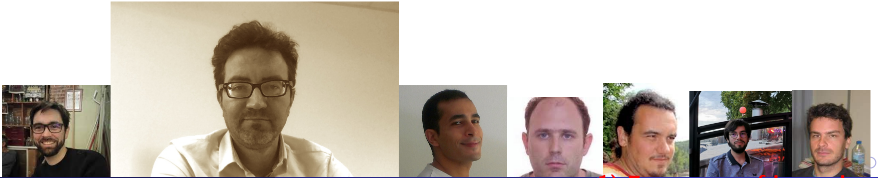
Quantum Technologies Workshop : Homogeneization of Markovian process and application to limit of strong quantum Measurments

1) **Emergence of jumps in quantum trajectories via homogeneization. T Benoist, C Bernardin, R Chétrite, R Chhaibi, J Najnudel, C Pellegrini, Comm Math Phys 2021.**

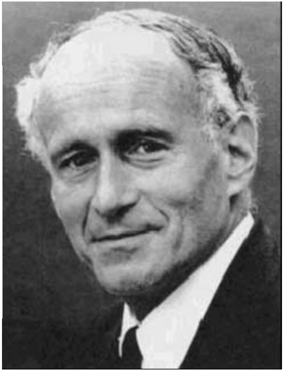
2) **Spiking and collapsing in large noise limits of SDEs, C Bernardin, R Chetrite, R Chhaibi, J Najnudel, C Pellegrini. Ann App Prob 2022.**



Caveat....Mathematical Physics and....Probability

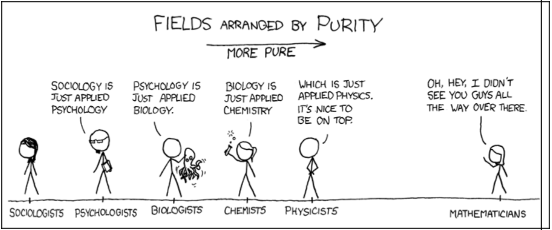
"When you speak in front of an audience... you can assume that everybody knows about Stein's manifold or Betti's numbers of topological space ; but if you need a stochastic integral, you need to define filtrations, previsible processes, martingales... from scratch. There is something abnormal in this. There are of course many reasons for this - probabilists' esoteric vocabulary, to start with.'

Laurent Schwartz



Probabilités ∈ Mathématiques

Math : traite d'idées abstraites qui existent indépendamment de notre monde.



Physique Théorique ∈ Physique

Physique : traite les propriétés fondamentales de notre monde.

Essentialization of a Markovian process

Markovian process on male characters

< 2020 :  > 2020 :

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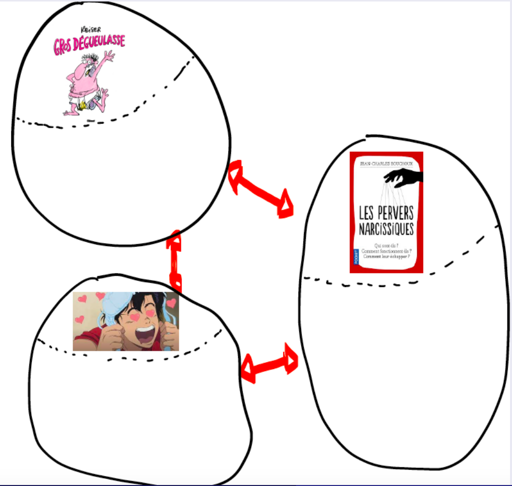

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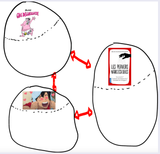
> 2020 : 

Decimation of states : Markov property lost in general

Coarse-graining : Dynkin criterium for conserve Markov property

Slow-Fast Quantum Trajectories

- Field of open quantum system: quantum system which can exchange energy, particles with the surrounding world **AND** which can be subjected to measurements.

Unread measure : Lindblad Eqn with 3 time scale : ode on $E \equiv \{\rho \in M_d(\mathbb{C}) | \rho \geq 0, Tr\rho = 1\}$

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- **Basics : One shot projective measurment**

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Mathematicaly trivial : For a state ρ and a measure "of projector P_i" such that $\sum_i P_i = Id$, we have

- Born rule : the probability to obtain i is $Tr(P_i \rho)$
- Collapse : just after the measure, the new state is $\frac{P_i \rho P_i}{Tr(P_i \rho)}$
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Max Born 1926

John von Neumann 1932

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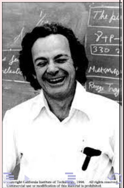
Physically subtil : Random, Non linear, Discontinuous ! Instantaneous !
Two major fact :

1. theory works in an extraordinarly accurate way
2. No one fully understand quantum theory. *Richard Feynman : "I think I can safely say that nobody really understands quantum mechanics..."*



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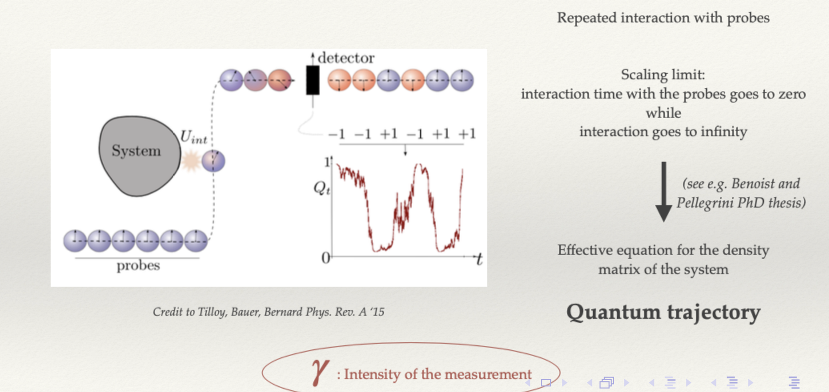
- **Generalization : Continuous Measurments**



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Vocabulary, Biblio....

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Example without quantum complication : unidimensional sde with strong noise

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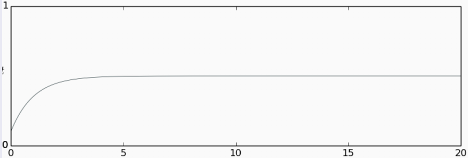

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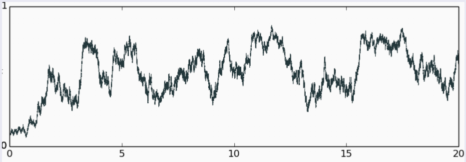

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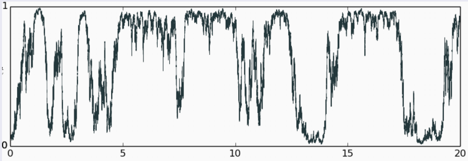

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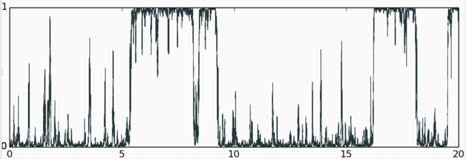

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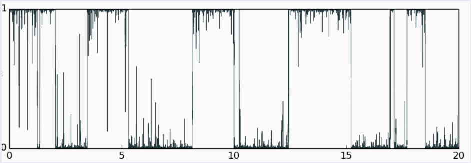

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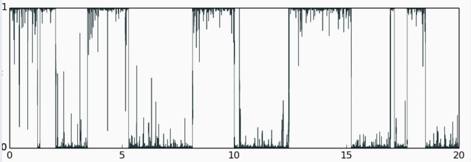

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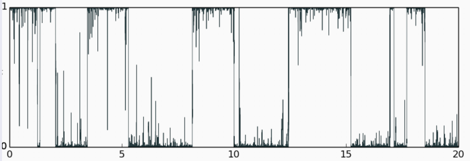

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Pense bête

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(QND) Quantum Non Demolition : H_2, L_2, L_1 ortho-codiagonalis in $(e_i)_{i=1}^d$ BON of $\mathbb{C}^d$ **(pointer basis)**

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Ex : Competition between 3 phenomenons :

- 1 : **Measurement** of the basis $(e_i)_{i=1}^{d}$: Strong (L_2) and moderate (L_1)

Slow-Fast Quantum Trajectories

Pense bête

$$\begin{cases} d\rho_{t,\gamma} = \left(\gamma^2\mathcal{L}_2 + \gamma\mathcal{L}_1 + \mathcal{L}_0\right)\left(\rho_{t,\gamma}\right) dt + \gamma\sigma_2\left(\rho_{t,\gamma}\right).dW_{t,2} + \sqrt{\gamma}\sigma_1\left(\rho_{t,\gamma}\right).dW_{t,1} + \sigma_0\left(\rho_{t,\gamma}\right) dW_{t,0} \\ \mathcal{L}_\alpha = -i\left(l_{H_\alpha} - r_{H_\alpha}\right) + \left[l_{L_\alpha} r_{[L_\alpha]^\dagger} - \frac{1}{2}\left(l_{[L_\alpha]^\dagger L_\alpha} + r_{[L_\alpha]^\dagger L_\alpha}\right)\right]. \\ \sigma_\alpha(\rho) \equiv \eta_\alpha\left(L_\alpha\rho + \rho\left(L_\alpha\right)^\dagger - Tr\left(\left(L_\alpha + \left(L_\alpha\right)^\dagger\right)\rho\right)\rho\right) \end{cases}$$

(QND) Quantum Non Demolition : H_2, L_2, L_1 ortho-codiagonalis in $(e_i)_{i=1}^d$ BON of $\mathbb{C}^d$ **(pointer basis)**

Ex : Competition between 3 phenomenons :

- 1 : **Measurement** of the basis $(e_i)_{i=1}^d$: Strong (L_2) and moderate (L_1)
- 2 : **Unitary evolution** strong (H_2) diagonale in $(e_i)_{i=1}^d$, general moderate H_1 and weak H_0

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$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = \gamma^2 \boxed{0X_{t,\gamma} + 0Y_{t,\gamma}} - \gamma\left(B_1^\dagger Y_{t,\gamma} + \boxed{0X_{t,\gamma}}\right) + \left(A_0 X_{t,\gamma} + C_0 Y_{t,\gamma}\right) + \gamma\sigma_2^X(X)\frac{dW_{2,X,t}}{dt} + \sqrt{\gamma}\sigma_1^X(X)\frac{dW_{1,X,t}}{dt} + \sigma_0^X \frac{dW}{} \\ \frac{dY_{t,\gamma}}{dt} = \gamma^2\left(D_2 Y_{t,\gamma} + \boxed{0X_{t,\gamma}}\right) + \gamma\left(B_1 X_{t,\gamma} + D_1 Y_{t,\gamma}\right) + \left(B_0 X_{t,\gamma} + D_0 Y_{t,\gamma}\right) + \gamma\sigma_2^Y \frac{dW_{2,Y,t}}{dt} + \sqrt{\gamma}\sigma_1^Y \frac{dW_{1,Y,t}}{dt} + \sigma_0^Y \frac{dW}{} \end{cases}$$

- We have finaly that :
  - No γ^2 −drift on X. The manifold $Y = 0$ is stable by the strong dominant (red) dynamics (but not by the intermediate (blue) and weak (black) dynamics) , it is also absorbing (i.e. decoherence) thanks to $D_2 < 0$.
  - On this manifold, the strong (red) and intermediate (blue) dynamics are driftess (Martingales) and live on the symplexe with the pointer state $(e_j, 0)$ as absorbing points and extremal points, i.e. on the edges theses dynamics are driftelss (as everywhere on this manifold) and the noise flow are along the edge : i.e.

$$\forall t \in [0,1] : \begin{cases} \sigma_2^X\left(te_i + (1-t)e_j\right) \sim \left(e_i - e_j\right) \\ \sigma_1^X\left(te_i + (1-t)e_j\right) \sim \left(e_i - e_j\right) \end{cases}$$

Theorem : Emergence of Quantum Jump when $\gamma \to \infty$ (strong measure)

Assumptions : 1 : QND and 2: technical assumption

Weak convergence of $\rho_{t,\gamma}$: $\lim_{\gamma \to \infty} \rho_{t,\gamma} = \left| X_t^{eff} \right\rangle \left\langle X_t^{eff} \right|$ with X_t^{eff} jump process on $(e_i)_{i=1}^{d}$

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- 1 : Independant of L_1 .
- 2 : Independant of H_0 (zeno effect) : explain the renormalisation $\gamma H_1 + \gamma^2 H_2$.
- 3. Independant of efficiency η_0 and η_1 (to have access to the weak and intermediate measurment change nothing).
- 4. Depend of $(L_2)_{ii}$: eigenvalue of the mesured operator have an impact on the effective dynamics.

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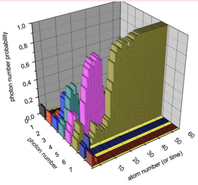
- 2 : "Quantum jump see exp (Guerlin-Haroche et al, Nature 2007) :



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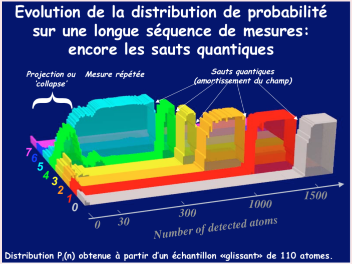


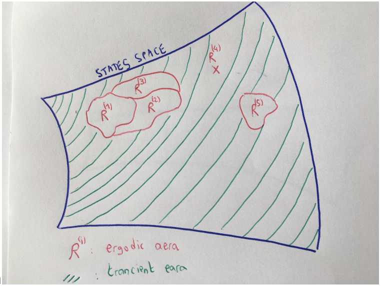
Serge Haroche college de France Cours 2007-2008 :

"Si le système est couplé à un réservoir qui le fait relaxer, son évolution sous observation continue se manifeste par des sauts discrets, à des instants aléatoires entre valeurs propres différentes. Ce sont les sauts quantiques. Ils ont été observés sur des atomes et des ions piégés et récemment sur un champ électro-magnétique"

Homogeneization of Markov Process : Mother-result

- Set-up :
  1. $X_{t,\gamma}$ is a markov process on E with gen $\gamma^2 L_2 + \gamma L_1 + L_0$ (op on V ≡Scalar field on E)
  2. $X_{t,2}$ fictious process with L_2 and decomposition $E = T \cup_{a \in F_2} R^{(a)}$ with $R^{(a)}$ ergodic region

     and T set of transcient region



  3.
  4.

- **Then** $\lim_{\gamma \to \infty} X_{t,\gamma} = X_t^{eff}$, **markov process on** E^{eff} **with generator** L^{eff}**, i.e.**

$$\lim_{\gamma \to \infty} \mathbb{E}_{\rho_0} \left[f\left(X_{t,\gamma}\right) \right] = \mathbb{E}_{\rho_0^{eff}} \left[f^{eff}\left(X_t^{eff}\right) \right],$$

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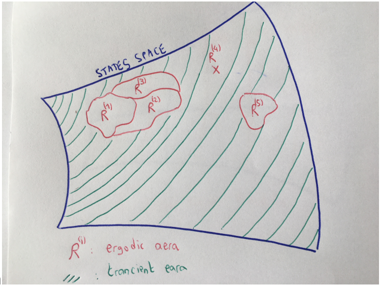

  3. We define $\begin{cases} r^{(a)}(x) = \mathbb{P}_x \left(X_{2,t} \text{ absorbé dans } R^{(a)} \right) \\ \rho_{inv}^{(a)} \text{ unique invariant measure when restricted to be in } R^{(a)} \end{cases}$
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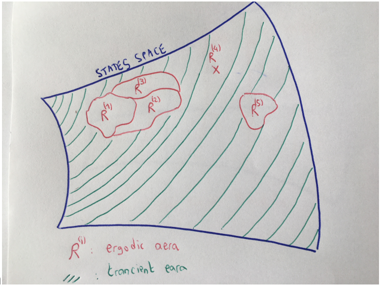

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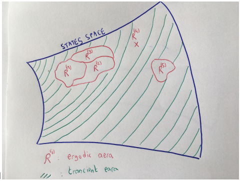

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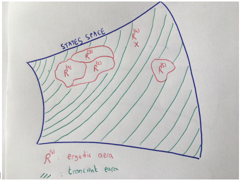

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  3. We define $\begin{cases} r^{(a)}(x) = \mathbb{P}_x \left(X_{2,t} \text{ absorbé dans } R^{(a)} \right) \\ \rho^{(a)}_{inv} \text{ unique invariant measure when restricted to be in } R^{(a)} \end{cases}$
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- **Then** $\boxed{\lim_{\gamma \to \infty} X_{t,\gamma} = X^{eff}_t}$**, markov process on** E^{eff} **with generator** L^{eff}**, i.e.**

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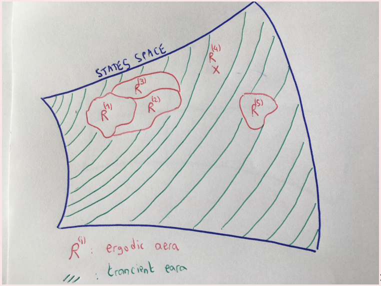
with $\begin{cases} E^{eff} \equiv \left\{ R^{(a)}, a \in F_2 \right\} \\ L^{eff} \left(R^{(a)}, R^{(b)} \right) \equiv \left\langle \rho^{(a)}_{inv}, \left(L_0 - L_1 L_2^{-1} L_1 \right) r^{(b)} \right\rangle_V \end{cases}$ **and** $\begin{cases} f^{eff} \left(R^{(a)} \right) \equiv \left\langle \rho^{(a)}_{inv}, f \right\rangle_V \\ \rho^{eff}_0 \left(R^{(a)} \right) \equiv \left\langle \rho_0, r^{(a)} \right\rangle_V \end{cases}$.

End-Ouverture

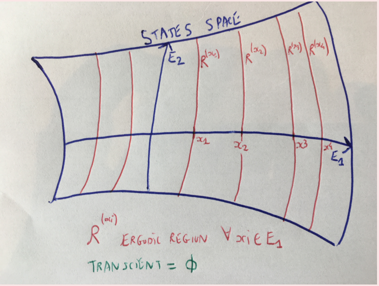
1. **TCL :** $\lim_{\gamma\to\infty} \frac{\rho_{t,\gamma} - |X_t\rangle\langle X_t|}{\gamma^a}$ ...
2. **Ldev :** $\mathbb{P}\left(\left(\rho_\gamma - |X\rangle\langle X|\right)_{[0,t]} = q_{[0,t]}\right) \asymp \exp\left(-\gamma \int_0^t du I(q_u)\right)$....
3. Reversibility properties of the effective process
4. Limit of functional $\int_0^T f\left(\rho_{t,\gamma}\right) dt + \langle g\left(\rho_{t,\gamma}\right), d\rho_{t,\gamma}\rangle$, when limit not just in term of X_t ...**anomaly** (Yash Chopra PHD in the case of the entropy production).
5. **The jew joke of the year : the wedding of the Lion**

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$\Longrightarrow$



Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68

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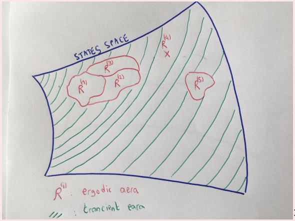

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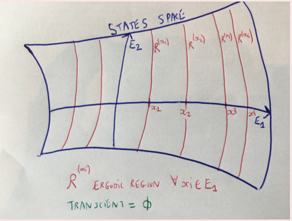

$$\begin{cases} E = {\color{green}T} \cup_{a \in F_2} R^{(a)} \qquad E_{eff} \equiv \left\{ R^{(a)}, a \in F_2 \right\} \\ Ker\left({\color{red}L_2^{\dagger}}\right) = Span\left\{ \rho_{inv}^{(a)}, {\color{red}a \in F_2} \right\} \\ Ker\left({\color{red}L_2}\right) = Span\left\{ \left| {\color{red}r^{(a)}} \right\rangle, {\color{red}a \in F_2} \right\} \\ \lim_{\gamma \to \infty} \mathbb{E}_{\rho_0}\left[f\left(X_{t,\gamma}\right)\right] = \mathbb{E}_{\rho_0^{eff}}\left[f^{eff}\left(X_t^{eff}\right)\right] \\ f^{eff}\left(R^{(a)}\right) \equiv \left\langle \rho_{inv}^{(a)}, f \right\rangle_V \end{cases} \Rightarrow$$

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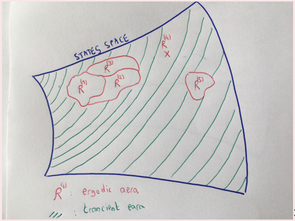

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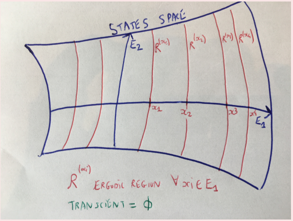

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T = \emptyset \qquad R^{(a)} = \{a\} \times E_2 \qquad E_{eff} \sim E_1 \\
Ker\left(L_2^{\dagger}\right) = Span\left\{ l^a(x,y) = \delta_a(x) \rho_{inv}^{(a)}(y), a \in E_1 \right\} \\
Ker\left(L_2\right) = \begin{cases} \{ f / f(x,y) = f(x) \} \\ Span\left\{ r^a(x,y) = \delta_a(x); a \in E_1 \right\} \end{cases} \\
\lim_{\gamma \to \infty} \mathbb{E}_{\rho_0}\left[f\left(X_{t,\gamma}, Y_{t,\gamma}\right) \right] = \mathbb{E}_{\rho_0^{eff}}\left[f^{eff}\left(X_t^{eff}\right) \right] \\
f^{eff}(x) = \int_{E_2} dy' \rho_{inv}^{(x)}(y') f(x,y') \equiv \bar{f}(x)
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$$

Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

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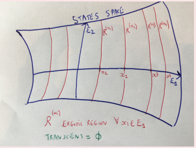

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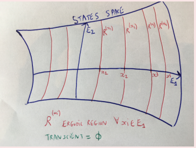

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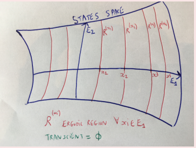
\end{cases}
\Rightarrow
$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



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Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

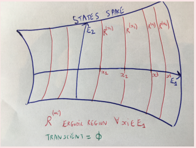
- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68

Sde :

$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = F_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{X,0,t}}{dt} \\ \frac{dY_{t,\gamma}}{dt} = \left(\begin{array}{c} \gamma^2 F_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) \\ +\gamma\sigma_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{Y,2,t}}{dt} \\ + F_0^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^Y \frac{dW_{Y,0,t}}{dt} \end{array} \right) \end{cases} \Rightarrow \begin{cases} L_2 = \left\langle F_2^Y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_2^Y\left(\sigma_2^Y\right)^{\dagger}\nabla_y, \nabla_y \right\rangle \\ L_1 = 0 \\ L_0 = \left(\begin{array}{c} \left\langle F_0^Y, \nabla_y \right\rangle + \left\langle F_0^X, \nabla_x \right\rangle + \\ +\frac{1}{2}\left\langle \sigma_0^Y\left(\sigma_0^Y\right)^{\dagger}\nabla_y, \nabla_y \right\rangle \\ +\frac{1}{2}\left\langle \sigma_0^X\left(\sigma_0^X\right)^{\dagger}\nabla_x, \nabla_x \right\rangle + (.)\nabla_x\nabla_y \end{array} \right) \end{cases}$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$$\begin{cases} T = \emptyset \qquad R^{(a)} = \{a\} \times E_2 \qquad E_{eff} \sim E_1 \\ Ker\left(L_2^\dagger\right) = Span\left\{ l^a(x,y) = \delta_a(x)\rho_{inv}^{(a)}(y), a \in E_1 \right\} \\ Ker\left(L_2\right) = \{f / f(x,y) = f(x)\} = Span\left\{ r^a(x,y) = \delta_a(x); a \in E_1 \right\} \\ \lim_{\gamma\to\infty} \mathbb{E}_{\rho_0}\left[f\left(X_{t,\gamma}, Y_{t,\gamma}\right)\right] = \mathbb{E}_{\rho_0^{eff}}\left[f^{eff}\left(X_t^{eff}\right)\right] \\ f^{eff}(x) = \int_{E_2} dy' \rho_{inv}^{(x)}(y') f(x,y') \equiv \overline{f}(x) \end{cases}$$

Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

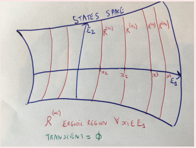
- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68

Sde :
$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = F_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{X,0,t}}{dt} \\ \frac{dY_{t,\gamma}}{dt} = \begin{pmatrix} \gamma^2 F_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) \\ +\gamma\sigma_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{Y,2,t}}{dt} \\ + F_0^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^Y \frac{dW_{Y,0,t}}{dt} \end{pmatrix} \end{cases} \Rightarrow \begin{cases} L_2 = \left\langle F_2^Y, \nabla_y\right\rangle + \frac{1}{2}\left\langle \sigma_2^Y\left(\sigma_2^Y\right)^\dagger \nabla_y, \nabla_y\right\rangle \\ L_1 = 0 \\ L_0 = \begin{pmatrix} \left\langle F_0^Y, \nabla_y\right\rangle + \left\langle F_0^X, \nabla_x\right\rangle + \\ +\frac{1}{2}\left\langle \sigma_0^Y\left(\sigma_0^Y\right)^\dagger \nabla_y, \nabla_y\right\rangle \\ +\frac{1}{2}\left\langle \sigma_0^X\left(\sigma_0^X\right)^\dagger \nabla_x, \nabla_x\right\rangle + (.)\nabla_x\nabla_y \end{pmatrix} \end{cases}$$

Effective process : $L^{eff} = \left\langle \overline{F_0^X}, \nabla_x\right\rangle + \frac{1}{2}\left\langle \overline{\sigma_0^X\left(\sigma_0^X\right)^\dagger}\nabla_x, \nabla_x\right\rangle \Rightarrow$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$$\begin{cases} T = \emptyset \qquad R^{(a)} = \{a\} \times E_2 \qquad E_{eff} \sim E_1 \\ Ker\left(L_2^{\dagger}\right) = Span\left\{ l^a(x,y) = \delta_a(x)\rho_{inv}^{(a)}(y),\, a \in E_1 \right\} \\ Ker\left(L_2\right) = \{f / f(x,y) = f(x)\} = Span\left\{ r^a(x,y) = \delta_a(x);\, a \in E_1 \right\} \\ \lim_{\gamma\to\infty} \mathbb{E}_{\rho_0}\left[f\left(X_{t,\gamma}, Y_{t,\gamma}\right)\right] = \mathbb{E}_{\rho_0^{eff}}\left[f^{eff}\left(X_t^{eff}\right)\right] \\ f^{eff}(x) = \int_{E_2} dy'\, \rho_{inv}^{(x)}(y') f(x,y') \equiv \overline{f}(x) \end{cases}$$

Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

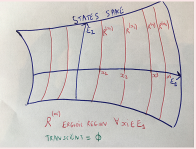
- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68

Sde :
$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = F_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{X,0,t}}{dt} \\ \frac{dY_{t,\gamma}}{dt} = \begin{pmatrix} \gamma^2 F_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) \\ +\gamma\sigma_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{Y,2,t}}{dt} \\ + F_0^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^Y \frac{dW_{Y,0,t}}{dt} \end{pmatrix} \end{cases} \Rightarrow \begin{cases} L_2 = \left\langle F_2^Y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_2^Y\left(\sigma_2^Y\right)^{\dagger}\nabla_y, \nabla_y\right\rangle \\ L_1 = 0 \\ L_0 = \begin{pmatrix} \left\langle F_0^Y, \nabla_y\right\rangle + \left\langle F_0^X, \nabla_x\right\rangle + \\ +\frac{1}{2}\left\langle \sigma_0^Y\left(\sigma_0^Y\right)^{\dagger}\nabla_y, \nabla_y\right\rangle \\ +\frac{1}{2}\left\langle \sigma_0^X\left(\sigma_0^X\right)^{\dagger}\nabla_x, \nabla_x\right\rangle + (.)\nabla_x\nabla_y \end{pmatrix} \end{cases}$$

Effective process : $L^{eff} = \left\langle \overline{F_0^X}, \nabla_x \right\rangle + \frac{1}{2}\left\langle \overline{\sigma_0^X\left(\sigma_0^X\right)^{\dagger}}\nabla_x, \nabla_x\right\rangle \Rightarrow \boxed{\frac{dX_t^{eff}}{dt} = \overline{F_0^X} + \sqrt{\overline{\sigma_0^X\left(\sigma_0^X\right)^{\dagger}}}\frac{dW_t}{dt}}$.

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$$\begin{cases} T = \emptyset \qquad R^{(a)} = \{a\} \times E_2 \qquad E_{eff} \sim E_1 \\ Ker\left(L_2^{\dagger}\right) = Span\left\{ l^a(x,y) = \delta_a(x)\rho_{inv}^{(a)}(y),\, a \in E_1 \right\} \\ Ker\left(L_2\right) = \{f / f(x,y) = f(x)\} = Span\left\{ r^a(x,y) = \delta_a(x);\, a \in E_1 \right\} \\ \lim_{\gamma\to\infty} \mathbb{E}_{\rho_0}\left[f\left(X_{t,\gamma}, Y_{t,\gamma}\right)\right] = \mathbb{E}_{\rho_0^{eff}}\left[f^{eff}\left(X_t^{eff}\right)\right] \\ f^{eff}(x) = \int_{E_2} dy' \rho_{inv}^{(x)}(y') f(x,y') \equiv \overline{f}(x) \end{cases}$$

Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68

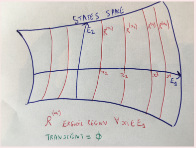
Sde :

$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = F_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{X,0,t}}{dt} \\ \frac{dY_{t,\gamma}}{dt} = \begin{pmatrix} \gamma^2 F_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) \\ +\gamma\sigma_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{Y,2,t}}{dt} \\ +F_0^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^Y \frac{dW_{Y,0,t}}{dt} \end{pmatrix} \end{cases} \Rightarrow$$

Effective process : $L^{eff} = \left\langle \overline{F_0^X}, \nabla_x \right\rangle + \frac{1}{2}\left\langle \overline{\sigma_0^X\left(\sigma_0^X\right)^{\dagger}}\nabla_x, \nabla_x \right\rangle \Rightarrow \boxed{\frac{dX_t^{eff}}{dt} = \overline{F_0^X} + \sqrt{\overline{\sigma_0^X\left(\sigma_0^X\right)^{\dagger}}}\frac{dW_t}{dt}}$.

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$$\begin{cases} T = \emptyset \qquad R^{(a)} = \{a\} \times E_{\mathbf{2}} \qquad E_{eff} \sim E_{\mathbf{1}} \\ Ker\left(L_{\mathbf{2}}^{\dagger}\right) = Span\left\{ l^{a}(x,y) = \delta_a(x)\rho_{inv}^{(a)}(y), a \in E_{\mathbf{1}} \right\} \\ Ker\left(L_{\mathbf{2}}\right) = \{ f / f(x,y) = f(x) \} = Span\left\{ r^{a}(x,y) = \delta_a(x); a \in E_{\mathbf{1}} \right\} \\ \lim_{\gamma\to\infty} \mathbb{E}_{\rho_{\mathbf{0}}}\left[f\left(X_{t,\gamma}, Y_{t,\gamma}\right)\right] = \mathbb{E}_{\rho_{\mathbf{0}}^{eff}}\left[f^{eff}\left(X_t^{eff}\right)\right] \\ f^{eff}(x) = \int_{E_{\mathbf{2}}} dy' \rho_{inv}^{(x)}(y') f(x,y') \equiv \overline{f}(x) \end{cases}$$

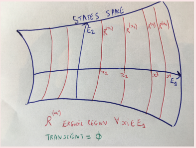
Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68

Sde :
$$\begin{cases} \dfrac{dX_{t,\gamma}}{dt} = F_{\mathbf{0}}^{X}\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_{\mathbf{0}}^{X}\left(X_{t,\gamma}, Y_{t,\gamma}\right)\dfrac{dW_{X,\mathbf{0},t}}{dt} \\ \dfrac{dY_{t,\gamma}}{dt} = \left(\begin{array}{c} \gamma^{\mathbf{2}} F_{\mathbf{2}}^{Y}\left(X_{t,\gamma}, Y_{t,\gamma}\right) \\ +\gamma\sigma_{\mathbf{2}}^{Y}\left(X_{t,\gamma}, Y_{t,\gamma}\right)\dfrac{dW_{Y,\mathbf{2},t}}{dt} \\ + F_{\mathbf{0}}^{Y}\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_{\mathbf{0}}^{Y}\dfrac{dW_{Y,\mathbf{0},t}}{dt} \end{array} \right) \end{cases} \Rightarrow \boxed{\frac{dX_t^{eff}}{dt} = \overline{F_{\mathbf{0}}^{X}} + \sqrt{\overline{\sigma_{\mathbf{0}}^{X}\left(\sigma_{\mathbf{0}}^{X}\right)^{\dagger}}}\frac{dW_t}{dt}}$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$$
\begin{cases}
T = \emptyset \qquad R^{(a)} = \{a\} \times E_2 \qquad E_{eff} \sim E_1 \\
Ker\left(L_2^\dagger\right) = Span\left\{l^a(x,y) = \delta_a(x)\rho_{inv}^{(a)}(y),\, a \in E_1\right\} \\
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\lim_{\gamma\to\infty} \mathbb{E}_{\rho_0}\left[f\left(X_{t,\gamma}, Y_{t,\gamma}\right)\right] = \mathbb{E}_{\rho_0^{eff}}\left[f^{eff}\left(X_t^{eff}\right)\right] \\
f^{eff}(x) = \int_{E_2} dy' \rho_{inv}^{(x)}(y') f(x,y') \equiv \overline{f}(x)
\end{cases}
$$

Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68

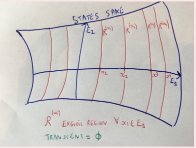
Sde :
$$
\begin{cases}
\frac{dX_{t,\gamma}}{dt} = F_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{X,0,t}}{dt} \\
\frac{dY_{t,\gamma}}{dt} = \begin{pmatrix} \gamma^2 F_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) \\ +\gamma\sigma_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{Y,2,t}}{dt} \\ + F_0^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^Y \frac{dW_{Y,0,t}}{dt} \end{pmatrix}
\end{cases}
\Rightarrow \boxed{\frac{dX_t^{eff}}{dt} = \overline{F_0^X} + \sqrt{\overline{\sigma_0^X\left(\sigma_0^X\right)^\dagger}}\frac{dW_t}{dt}}
$$

- **2 : Raffinement** : Phys : overdamp limit of Langevin, Math : Papanicolau-Stroock-Varadhan 76

$$
\begin{cases}
\frac{dX_{t,\gamma}}{dt} = \begin{pmatrix} \gamma \boxed{F_1^X\left(X_{t,\gamma}, Y_{t,\gamma}\right)} \\ +\ldots \end{pmatrix} \\
\ldots
\end{cases}
$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



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Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68

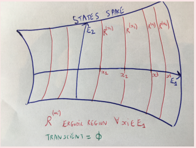
Sde :
$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = F_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^X\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{X,0,t}}{dt} \\ \frac{dY_{t,\gamma}}{dt} = \begin{pmatrix} \gamma^2 F_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) \\ +\gamma\sigma_2^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right)\frac{dW_{Y,2,t}}{dt} \\ + F_0^Y\left(X_{t,\gamma}, Y_{t,\gamma}\right) + \sigma_0^Y \frac{dW_{Y,0,t}}{dt} \end{pmatrix} \end{cases} \Rightarrow \begin{cases} L_2 = \left\langle F_2^Y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_2^Y\left(\sigma_2^Y\right)^{\dagger}\nabla_y, \nabla_y\right\rangle \\ L_1 = 0 \\ L_0 = \begin{pmatrix} \left\langle F_0^Y, \nabla_y\right\rangle + \left\langle F_0^X, \nabla_x\right\rangle + \\ +\frac{1}{2}\left\langle \sigma_0^Y\left(\sigma_0^Y\right)^{\dagger}\nabla_y, \nabla_y\right\rangle \\ +\frac{1}{2}\left\langle \sigma_0^X\left(\sigma_0^X\right)^{\dagger}\nabla_x, \nabla_x\right\rangle + (.)\nabla_x\nabla_y \end{pmatrix} \end{cases}$$

- **2 : Raffinement** : Phys : overdamp limit of Langevin, Math : Papanicolau-Stroock-Varadhan 76

$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = \begin{pmatrix} \gamma \boxed{F_1^X\left(X_{t,\gamma}, Y_{t,\gamma}\right)} \\ +\dots \end{pmatrix} \end{cases} \qquad \begin{cases} L_2 \\ L_1 = \begin{pmatrix} \left\langle F_1^Y, \nabla_y\right\rangle + \left\langle \boxed{F_1^X}, \nabla_x\right\rangle \end{pmatrix} \end{cases}$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



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Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

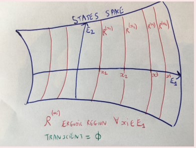
- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68
- **2 : Raffinement :** Phys : overdamp limit of Langevin, Math : Papanicolau-Stroock-Varadhan 76

Sde : $$\begin{cases} \frac{dX_{t,\gamma}}{dt} = \gamma \boxed{F_1^X} + F_0^X + \sigma_0^X \frac{dW_{X,0,t}}{dt} \\ \frac{dY_{t,\gamma}}{dt} = \gamma^2 F_2^Y + \gamma F_1^Y + F_0^Y + \gamma\sigma_2^Y \frac{dW_{Y,2,t}}{dt} + \sqrt{\gamma}\sigma_1^Y \frac{dW_{Y,1,t}}{dt} + \sigma_0^Y \frac{dW_{Y,0,t}}{dt} \end{cases}$$

$$\begin{cases} L_2 = \left\langle F_2^Y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_2^Y \left(\sigma_2^Y\right)^{\dagger} \nabla_y, \nabla_y \right\rangle \\ L_1 = \left\langle F_1^Y, \nabla_y \right\rangle + \left\langle \boxed{F_1^X}, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_1^Y \left(\sigma_1^Y\right)^{\dagger} \nabla_y, \nabla_y \right\rangle + (\ldots)\nabla_x\nabla_y \\ L_0 = \left\langle F_0^Y, \nabla_y \right\rangle + \left\langle F_0^X, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_0^Y \left(\sigma_0^Y\right)^{\dagger} \nabla_y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_0^X \left(\sigma_0^X\right)^{\dagger} \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y \end{cases}$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



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f^{eff}(x) = \int_{E_2} dy' \rho_{inv}^{(x)}(y') f(x,y') \equiv \overline{f}(x)
\end{cases}
$$

Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

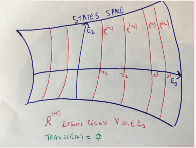
- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68
- **2 : Raffinement** : Phys : overdamp limit of Langevin, Math : Papanicolau-Stroock-Varadhan 76

Sde :
$$
\begin{cases}
\frac{dX_{t,\gamma}}{dt} = \gamma \boxed{F_1^X} + F_0^X + \sigma_0^X \frac{dW_{X,0,t}}{dt} \\
\frac{dY_{t,\gamma}}{dt} = \gamma^2 F_2^Y + \gamma F_1^Y + F_0^Y + \gamma\sigma_2^Y \frac{dW_{Y,2,t}}{dt} + \sqrt{\gamma}\sigma_1^Y \frac{dW_{Y,1,t}}{dt} + \sigma_0^Y \frac{dW_{Y,0,t}}{dt}
\end{cases}
$$

$$
\begin{cases}
L_2 = \left\langle F_2^Y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_2^Y \left(\sigma_2^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle \\
L_1 = \left\langle F_1^Y, \nabla_y \right\rangle + \left\langle \boxed{F_1^X}, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_1^Y \left(\sigma_1^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + (\ldots)\nabla_x\nabla_y \\
L_0 = \left\langle F_0^Y, \nabla_y \right\rangle + \left\langle F_0^X, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_0^Y \left(\sigma_0^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_0^X \left(\sigma_0^X\right)^\dagger \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y
\end{cases}
$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$$\begin{cases} T = \emptyset \qquad R^{(a)} = \{a\} \times E_2 \qquad E_{eff} \sim E_1 \\ Ker\left(L_2^\dagger\right) = Span\left\{ l^a(x,y) = \delta_a(x)\rho_{inv}^{(a)}(y), \, a \in E_1 \right\} \\ Ker\left(L_2\right) = \{f / f(x,y) = f(x)\} = Span\left\{ r^a(x,y) = \delta_a(x); \, a \in E_1 \right\} \\ \lim_{\gamma\to\infty} \mathbb{E}_{\rho_0}\left[f\left(X_{t,\gamma}, Y_{t,\gamma}\right)\right] = \mathbb{E}_{\rho_0^{eff}}\left[f^{eff}\left(X_t^{eff}\right)\right] \\ f^{eff}(x) = \int_{E_2} dy' \rho_{inv}^{(x)}(y') f(x,y') \equiv \overline{f}(x) \end{cases}$$

Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

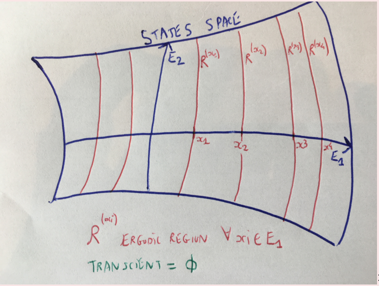
- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68
- **2 : Raffinement** : Phys : overdamp limit of Langevin, Math : Papanicolau-Stroock-Varadhan 76

$$\begin{cases} L_2 = \left\langle F_2^Y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_2^Y \left(\sigma_2^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle \\ L_1 = \quad \left\langle F_1^Y, \nabla_y \right\rangle + \left\langle \boxed{F_1^X}, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_1^Y \left(\sigma_1^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle \quad + (\ldots)\nabla_x\nabla_y \\ L_0 = \quad \left\langle F_0^Y, \nabla_y \right\rangle + \left\langle F_0^X, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_0^Y \left(\sigma_0^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_0^X \left(\sigma_0^X\right)^\dagger \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y \end{cases}$$

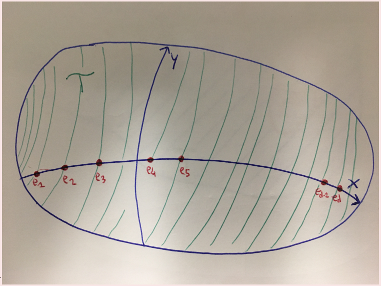
$$\Rightarrow L^{eff} = \left(\overline{\left(\begin{matrix} \left\langle F_1^Y, \nabla_y \right\rangle + \left\langle \boxed{F_1^X}, \nabla_x \right\rangle + \\ + \frac{1}{2}\left\langle \sigma_1^Y \left(\sigma_1^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + (\ldots)\nabla_x\nabla_y \end{matrix} \right) \left[\left\langle \left(L_2^{-1}\right) \left(\boxed{F_1^X}\right), \nabla_x \right\rangle \right]} \right.$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case

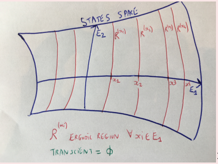


$\Longrightarrow$

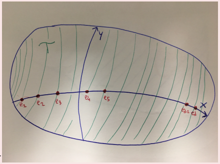


APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$\Longrightarrow$

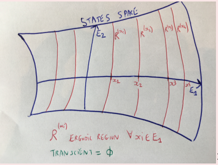


$$\begin{cases} T = \emptyset \quad R^{(a)} = \{a\} \times E_2 \quad E_{eff} \sim E_1 \\ Ker\left(L_2^{\dagger}\right) = Span\left\{ I^a(x,y) = \delta_a(x)\rho_{inv}^{(a)}(y);\, a \in E_1 \right\} \end{cases}$$

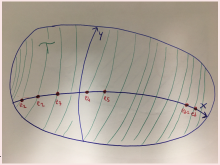
$\Rightarrow$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$\Longrightarrow$

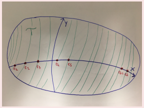


$$\begin{cases} T = \emptyset \qquad R^{(a)} = \{a\} \times E_{\mathbf{2}} \qquad E_{eff} \sim E_{\mathbf{1}} \\ Ker\left(L_{\mathbf{2}}^{\dagger}\right) = Span\left\{ I^{a}(x,y) = \delta_a(x)\rho_{inv}^{(a)}(y);\, a \in E_{\mathbf{1}} \right\} \end{cases}$$

$$\Rightarrow \begin{cases} T = E - \{\cup_i (e_i, 0)\} \qquad R^{(i)} = \{(e_i, 0)\} \;\; E_{eff} = \{(e_i, 0),\, i = [\![1, d]\!]\} \\ Ker\left(L_{\mathbf{2}}^{\dagger}\right) = Span\left\{ \;\; I^{(i)}(x,y) = \delta_{e_i}(x)\delta_{\mathbf{0}}(y),\, i \in [\![1, d]\!] \,, \;\; \right\} \end{cases}$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$$\begin{cases} T = E - \{\cup_i (e_i, 0)\} \qquad R^{(i)} = \{(e_i, 0)\} \;\; E_{eff} = \{(e_i, 0), i = [\![1, d]\!]\} \\ Ker\left(L_2^\dagger\right) = Span\left\{ I^{(i)}(x, y) = \delta_{e_i}(x)\delta_0(y) \forall i \in [\![1, d]\!] \right\} \end{cases}$$

Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68
- **2 : Raffinement :** Phys : overdamp limit of Langevin, Math : Papanicolau-Stroock-Varadhan 76

Sde :

$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = \gamma \boxed{F_1^X} + F_0^X + \sigma_0^X \frac{dW_{X,0,t}}{dt} \\ \frac{dY_{t,\gamma}}{dt} = \gamma^2 F_2^Y + \gamma F_1^Y + F_0^Y + \gamma \sigma_2^Y \frac{dW_{Y,2,t}}{dt} + \sqrt{\gamma}\sigma_1^Y \frac{dW_{Y,1,t}}{dt} + \sigma_0^Y \frac{dW_{Y,0,t}}{dt} \end{cases}$$

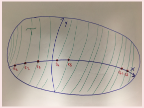
"Dominant absorbing" case : discrete number of ergodic singleton

Rappel Quantum traj with QND :

$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = \gamma^2 \boxed{0} - \gamma\left(B_1^\dagger Y_{t,\gamma} + \boxed{0 X_{t,\gamma}}\right) + \left(A_0 X_{t,\gamma} + C_0 Y_{t,\gamma}\right) + \sigma_0^X \frac{dW_{0X,,t}}{dt} + \\ \frac{dY_{t,\gamma}}{dt} = \gamma^2 \left(D_2 Y_{t,\gamma} + \boxed{0 X_{t,\gamma}}\right) + \gamma\left(B_1 X_{t,\gamma} + D_1 Y_{t,\gamma}\right) + \left(B_0 X_{t,\gamma} + D_0 Y_{t,\gamma}\right) dt + \gamma\sigma_2^Y \frac{dW_{2,Y,t}}{dt} + \sqrt{\gamma}\sigma_1^Y \frac{dW_{1,Y,t}}{dt} + \end{cases}$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$$\begin{cases} T = E - \{\cup_i (e_i, 0)\} \quad R^{(i)} = \{(e_i, 0)\} \quad E_{eff} = \{(e_i, 0), i = [\![1, d]\!]\} \\ Ker\left(L_2^{\dagger}\right) = Span\left\{ I^{(i)}(x, y) = \delta_{e_i}(x)\delta_{\mathbf{0}}(y) \forall i \in [\![1, d]\!] \right\} \end{cases}$$

Usual case : $E = E_1 \times E_2$ and continuous number of ergodic set

- **1 : Historique** : Phys ...., ,Math : Stratonovich 63-Khasminskii 68
- **2 : Raffinement :** Phys : overdamp limit of Langevin, Math : Papanicolau-Stroock-Varadhan 76

Sde :
$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = \gamma \boxed{F_{\mathbf{1}}^X} + F_{\mathbf{0}}^X + \sigma_{\mathbf{0}}^X \frac{dW_{X,\mathbf{0},t}}{dt} \\ \frac{dY_{t,\gamma}}{dt} = \gamma^{\mathbf{2}} F_{\mathbf{2}}^Y + \gamma F_{\mathbf{1}}^Y + F_{\mathbf{0}}^Y + \gamma \sigma_{\mathbf{2}}^Y \frac{dW_{Y,\mathbf{2},t}}{dt} + \sqrt{\gamma}\sigma_{\mathbf{1}}^Y \frac{dW_{Y,\mathbf{1},t}}{dt} + \sigma_{\mathbf{0}}^Y \frac{dW_{Y,\mathbf{0},t}}{dt} \end{cases}$$

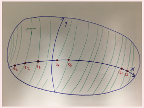
"Dominant absorbing" case : discrete number of ergodic singleton

Rappel Quantum traj with QND :

$$\frac{dX_{t,\gamma}}{dt} = \gamma^{\mathbf{2}} \boxed{\mathbf{0}} - \gamma \left(B_{\mathbf{1}}^{\dagger} Y_{t,\gamma} + \boxed{\mathbf{0} X_{t,\gamma}} \right) + \left(A_{\mathbf{0}} X_{t,\gamma} + C_{\mathbf{0}} Y_{t,\gamma} \right) + \sigma_{\mathbf{0}}^X \frac{dW_{\mathbf{0}X,,t}}{dt} + \boxed{\gamma \sigma_{\mathbf{2}}^X \frac{dW_{\mathbf{2},X,t}}{dt} + \sqrt{\gamma}\sigma_{\mathbf{1}}^X \frac{dW_{\mathbf{1},X,t}}{dt}}$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$$\begin{cases} T = E - \{\cup_i (e_i, 0)\} \quad R^{(i)} = \{(e_i, 0)\} \;\; E_{eff} = \{(e_i, 0), i = [\![1, d]\!]\} \\ Ker\left(L_2^\dagger\right) = Span\left\{ I^{(i)}(x, y) = \delta_{e_i}(x)\delta_0(y) \forall i \in [\![1, d]\!] \right\} \end{cases}$$

''Dominant absorbing'' case : discrete number of ergodic singleton

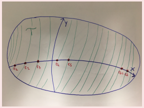
Rappel Quantum traj with QND :

$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = \gamma^2 \boxed{0} - \gamma \left(B_1^\dagger Y_{t,\gamma} + \boxed{0 X_{t,\gamma}} \right) + \left(A_0 X_{t,\gamma} + C_0 Y_{t,\gamma} \right) + \sigma_0^X \frac{dW_{0X,,t}}{dt} + \boxed{\gamma \sigma_2^X \frac{dW_{2,X,t}}{dt} + \sqrt{\gamma}\sigma_1^X \frac{dW_{1,X,t}}{dt}} \\ \frac{dY_{t,\gamma}}{dt} = \gamma^2 \left(D_2 Y_{t,\gamma} + \boxed{0 X_{t,\gamma}} \right) + \gamma \left(B_1 X_{t,\gamma} + D_1 Y_{t,\gamma} \right) + \left(B_0 X_{t,\gamma} + D_0 Y_{t,\gamma} \right) dt + \gamma \sigma_2^Y \frac{dW_{2,Y,t}}{dt} + \sqrt{\gamma}\sigma_1^Y \frac{dW_{1,Y,t}}{dt} + \dots \end{cases}$$

- **Generalisation :** $$\begin{cases} \frac{dX_{t,\gamma}}{dt} = \boxed{0\gamma^2} + \left(\gamma F_1^X + F_0^X \right) + \sigma_0^X \frac{dW_{0,t}}{dt} + \boxed{\gamma \sigma_2^X \frac{dW_{X,2,t}}{dt} + \sqrt{\gamma}\sigma_1^X \frac{dW_{X,1,t}}{dt}} \\ \frac{dY_{t,\gamma}}{dt} = \gamma^2 F_2^Y + \gamma F_1^Y + F_0^Y + \gamma \sigma_2^Y \frac{dW_{Y,2,t}}{dt} + \sqrt{\gamma}\sigma_1^Y \frac{dW_{,Y,1,t}}{dt} + \sigma_0^X \frac{dW_{Y,0,t}}{dt} \end{cases}$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$$\begin{cases} T = E - \{\cup_i (e_i, 0)\} \qquad R^{(i)} = \{(e_i,0)\} \quad E_{eff} = \{(e_i, 0), i = [\![1, d]\!]\} \\ Ker\left(L_2^{\dagger}\right) = Span\left\{ I^{(i)}(x,y) = \delta_{e_i}(x)\delta_0(y) \forall i \in [\![1,d]\!] \right\} \end{cases}$$

"Dominant absorbing" case : discrete number of ergodic singleton

Rappel Quantum traj with QND :

$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = \gamma^2 \boxed{0} - \gamma\left(B_1^{\dagger} Y_{t,\gamma} + \boxed{0 X_{t,\gamma}}\right) + \left(A_0 X_{t,\gamma} + C_0 Y_{t,\gamma}\right) + \sigma_0^X \frac{dW_{0X,,t}}{dt} + \boxed{\gamma\sigma_2^X \frac{dW_{2,X,t}}{dt} + \sqrt{\gamma}\sigma_1^X \frac{dW_{1,X,t}}{dt}} \\ \frac{dY_{t,\gamma}}{dt} = \gamma^2\left(D_2 Y_{t,\gamma} + \boxed{0 X_{t,\gamma}}\right) + \gamma\left(B_1 X_{t,\gamma} + D_1 Y_{t,\gamma}\right) + \left(B_0 X_{t,\gamma} + D_0 Y_{t,\gamma}\right) dt + \gamma\sigma_2^Y \frac{dW_{2,Y,t}}{dt} + \sqrt{\gamma}\sigma_1^Y \frac{dW_{1,Y,t}}{dt} + \dots \end{cases}$$

- **Generalisation :**

$$\begin{cases} \frac{dX_{t,\gamma}}{dt} = \boxed{0\gamma^2} + \left(\gamma F_1^X + F_0^X\right) + \sigma_0^X \frac{dW_{0,t}}{dt} + \boxed{\gamma\sigma_2^X \frac{dW_{X,2,t}}{dt} + \sqrt{\gamma}\sigma_1^X \frac{dW_{X,1,t}}{dt}} \\ \frac{dY_{t,\gamma}}{dt} = \gamma^2 F_2^Y + \gamma F_1^Y + F_0^Y + \gamma\sigma_2^Y \frac{dW_{Y,2,t}}{dt} + \sqrt{\gamma}\sigma_1^Y \frac{dW_{,Y,1,t}}{dt} + \sigma_0^X \frac{dW_{Y,0,t}}{dt} \end{cases}$$

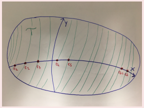
avec la varieté $Y = 0$ stable by the strong dominant red dynamics $\boxed{\begin{cases} \sigma_2^Y(x, 0) = 0 \\ F_2^Y(x,0) = 0 \end{cases}}$ **and on this manifold the strong (red)**

and intermediate (blue) dynamics live on the simplex with the pointer state $(e_i, 0)$ as stable

$\left\{ \sigma^X(x, 0) = 0 \quad \sigma^X(x, 0) = 0 \right.$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case



$$\begin{cases} T = E - \{\cup_i (e_i, 0)\} \qquad R^{(i)} = \{(e_i, 0)\} \;\; E_{eff} = \{(e_i, 0), i = [\![1, d]\!]\} \\ Ker\left(L_2^{\dagger}\right) = Span\left\{ I^{(i)}(x, y) = \delta_{e_i}(x)\delta_{\mathbf{0}}(y) \forall i \in [\![1, d]\!] \right\} \end{cases}$$

"Dominant absorbing" case : discrete number of ergodic singleton

- Hyp $\begin{cases} \sigma_2^Y(x, 0) = 0 \\ F_2^Y(x, 0) = 0 \\ \sigma_2^X(e_i, 0) = 0 \\ \boxed{F_1^X(e_i, 0) = 0} \end{cases}$

$$\begin{cases} L_2 = \left\langle F_2^Y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_2^Y \left(\sigma_2^Y\right)^{\dagger} \nabla_y, \nabla_y \right\rangle + \boxed{\frac{1}{2}\left\langle \sigma_2^X \left(\sigma_2^X\right)^{\dagger} \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y} + \boxed{0\nabla_x} \\ L_1 = \left\langle F_1^Y, \nabla_y \right\rangle + \left\langle \boxed{F_1^X}, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_1^Y \left(\sigma_1^Y\right)^{\dagger} \nabla_y, \nabla_y \right\rangle + \boxed{\frac{1}{2}\left\langle \sigma_1^X \left(\sigma_1^X\right)^{\dagger} \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y} \\ L_0 = \left\langle F_0^Y, \nabla_y \right\rangle + \left\langle F_0^X, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_0^Y \left(\sigma_0^Y\right)^{\dagger} \nabla_y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_0^X \left(\sigma_0^X\right)^{\dagger} \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y \end{cases}$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case

$$\begin{cases} \sigma_2^Y(x,0) = 0 \\ F_2^Y(x,0) = 0 \\ \sigma_2^X(e_i,0) = 0, \\ \sigma_1^X(e_i,0) = 0 \\ \boxed{F_1^X(e_i,0) = 0} \end{cases}$$

et

$$\begin{cases} L_2 = \left\langle F_2^Y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_2^Y \left(\sigma_2^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + \boxed{\boxed{\frac{1}{2}\left\langle \sigma_2^X \left(\sigma_2^X\right)^\dagger \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y}} + \boxed{0\nabla_x} \\ Ker\left(L_2^\dagger\right) = Span\left\{ l^{(i)}(x,y) = \delta_{e_i}(x)\delta_0(y) \forall i \in [\![1,d]\!] \right\} \\ Ker\left(L_2\right) = Span\left\{ r^{(i)}(x,y) = x_i \forall i \in [\![1,d]\!] \right\} \\ L_1 = \left(\begin{array}{c} \left\langle F_1^Y, \nabla_y \right\rangle + \left\langle \boxed{F_1^X}, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_1^Y \left(\sigma_1^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + \\ + \boxed{\boxed{\frac{1}{2}\left\langle \sigma_1^X \left(\sigma_1^X\right)^\dagger \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y}} \end{array} \right) \\ L_0 = \left\langle F_0^Y, \nabla_y \right\rangle + \left\langle F_0^X, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_0^Y \left(\sigma_0^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_0^X \left(\sigma_0^X\right)^\dagger \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y \end{cases}$$

"Dominant absorbing" case : discrete number of ergodic singleton

$\Longrightarrow$ centering hypothesis : $\left\langle l^{(i)}, L_1 r^{(j)} \right\rangle_V = \left(\boxed{F_1^X(e_i,0)} \right)_j = 0$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case

$$\begin{cases} \sigma_2^Y(x,0) = 0 \\ F_2^Y(x,0) = 0 \\ \sigma_2^X(e_i,0) = 0, \\ \sigma_1^X(e_i,0) = 0 \\ \boxed{F_1^X(e_i,0) = 0} \end{cases}$$

et

$$\begin{cases} L_2 = \left\langle F_2^Y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_2^Y \left(\sigma_2^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + \boxed{\boxed{\frac{1}{2}\left\langle \sigma_2^X \left(\sigma_2^X\right)^\dagger \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y}} + \boxed{0\nabla_x} \\ Ker\left(L_2^\dagger\right) = Span\left\{ l^{(i)}(x,y) = \delta_{e_i}(x)\delta_0(y) \forall i \in [\![1,d]\!] \right\} \\ Ker\left(L_2\right) = Span\left\{ r^{(i)}(x,y) = x_i \forall i \in [\![1,d]\!] \right\} \\ L_1 = \left(\begin{array}{l} \left\langle F_1^Y, \nabla_y \right\rangle + \left\langle \boxed{F_1^X}, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_1^Y \left(\sigma_1^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + \\ + \boxed{\boxed{\frac{1}{2}\left\langle \sigma_1^X \left(\sigma_1^X\right)^\dagger \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y}} \end{array} \right) \\ L_0 = \left\langle F_0^Y, \nabla_y \right\rangle + \left\langle F_0^X, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_0^Y \left(\sigma_0^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_0^X \left(\sigma_0^X\right)^\dagger \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y \end{cases}$$

"Dominant absorbing" case : discrete number of ergodic singleton

$\Longrightarrow$ centering hypothesis : $\left\langle l^{(i)}, L_1 r^{(j)} \right\rangle_V = \left(\boxed{F_1^X(e_i,0)} \right)_j = 0$

- (X_t, Y_t) "converg" toward $(X_t, 0)$, where X_t is a jump process on $E^{eff} = \{(e_i, 0)\}$ with

$$\boxed{L^{eff}\left((e_i,0),(e_j,0)\right) = \left\langle l^{(i)}, \left(L_0^{ab} - L_1^{ab}\left(L_2^{ab}\right)^{-1} L_1^{ab} \right) r^{(j)} \right\rangle_V}$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case

$$\begin{cases} \sigma_2^Y(x,0) = 0 \\ F_2^Y(x,0) = 0 \\ \sigma_2^X(e_i,0) = 0, \\ \sigma_1^X(e_i,0) = 0 \\ \boxed{F_1^X(e_i,0) = 0} \end{cases}$$

et

$$\begin{cases} L_2 = \left\langle F_2^Y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_2^Y \left(\sigma_2^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + \boxed{\frac{1}{2}\left\langle \sigma_2^X \left(\sigma_2^X\right)^\dagger \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y} + \boxed{0\nabla_x} \\ Ker\left(L_2^\dagger\right) = Span\left\{ l^{(i)}(x,y) = \delta_{e_i}(x)\delta_0(y) \forall i \in [\![1,d]\!] \right\} \\ Ker\left(L_2\right) = Span\left\{ r^{(i)}(x,y) = x_i \forall i \in [\![1,d]\!] \right\} \\ L_1 = \left(\left\langle F_1^Y, \nabla_y \right\rangle + \left\langle \boxed{F_1^X}, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_1^Y \left(\sigma_1^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + \boxed{\frac{1}{2}\left\langle \sigma_1^X \left(\sigma_1^X\right)^\dagger \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y} \right) \\ L_0 = \left\langle F_0^Y, \nabla_y \right\rangle + \left\langle F_0^X, \nabla_x \right\rangle + \frac{1}{2}\left\langle \sigma_0^Y \left(\sigma_0^Y\right)^\dagger \nabla_y, \nabla_y \right\rangle + \frac{1}{2}\left\langle \sigma_0^X \left(\sigma_0^X\right)^\dagger \nabla_x, \nabla_x \right\rangle + (\ldots)\nabla_x\nabla_y \end{cases}$$

''Dominant absorbing'' case : discrete number of ergodic singleton

- (X_t, Y_t) ''converg'' toward $(X_t, 0)$, where X_t is a jump process on $E^{eff} = \{(e_i, 0)\}$ with

$$\boxed{L^{eff}\left((e_i,0),(e_j,0)\right) = \left\langle l^{(i)}, \left(L_0^{ab} - L_1^{ab}\left(L_2^{ab}\right)^{-1} L_1^{ab}\right) r^{(j)} \right\rangle_V}$$

APENDIX : Example : Diffusion process with non irreductible fast part

Famous case and new case

"Dominant absorbing" case : discrete number of ergodic singleton

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Explcit calculation of L_{eff} : Easy and difficult part