On Richards' equation and its numerical solution

Konstantin Brenner

Laboratoire J.A. Dieudonné Inria & Univ. Côte d'Azur

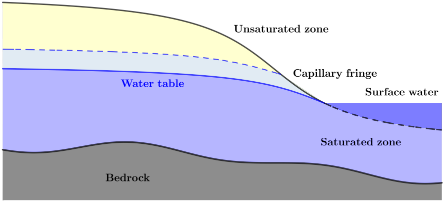
Séminaire d'analyse numérique, IRMAR

June 9, 2022

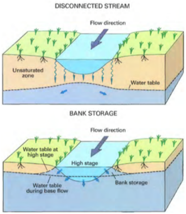
Applications of Richards' equation

Unsaturated/saturated groundwater flow

- pores occupied by water + air



Michel et al. '19



Reilly '01

Applications:

- Water resource estimation
- Irrigation
- Contaminant transport
- Interaction with surface water

Applications of Richards' equation

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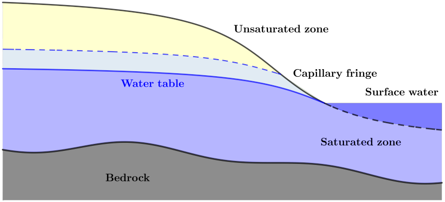

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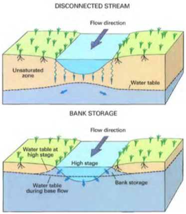

Reilly '01

Alternatives:

- Saturated flow
- Richards' (air at equilibrium)
- Two-phase flow

Why Richards' equation?

French Wikipedia: La résolution numérique de l'équation de Richards demeure l'un des problèmes d'analyse numérique les plus difficiles pour les sciences naturelles[1].

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Major numerical challenge: Robustness and efficiency of the nonlinear solvers.

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Outline

Introduction to Richards' equation

- From saturated to unsaturated flow

Monotone Newton Theorem

- Convergence proof $\neq$ performance

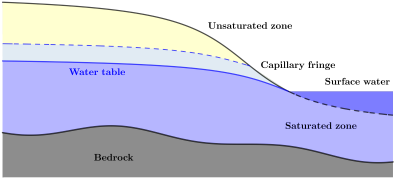
Nonlinear preconditioning

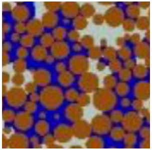
- Convergence proof $+$ performance

Conclusion

Outline

Saturated vs. unsaturated incompressible groundwater flow





$$s = \frac{|\text{water}|}{|\text{void}|} \text{ in a REV}$$

Saturated flow

Continuity equation:

$$\operatorname{div} \boldsymbol{v} = 0$$

Darcy law:

$$\boldsymbol{v} = -\frac{\mathbb{K}}{\mu}(\nabla p - \rho \boldsymbol{g})$$

Find p satisfying

$$-\operatorname{div}\left(\frac{\mathbb{K}}{\mu}(\nabla p - \rho \boldsymbol{g})\right) = 0$$

Unsaturated flow

Continuity equation:

$$\phi \partial_t s + \operatorname{div} \boldsymbol{v} = 0$$

Darcy-Buckingham law:

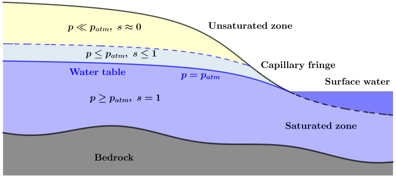
$$\boldsymbol{v} = -\frac{\mathbb{K} k(s)}{\mu}(\nabla p - \rho \boldsymbol{g})$$

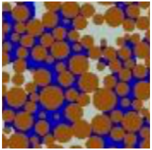
Find p and s satisfying

$$\phi \partial_t s - \operatorname{div}\left(\frac{\mathbb{K} k(s)}{\mu}(\nabla p - \rho \boldsymbol{g})\right) = 0$$

together with $s = S(p)$

Saturated vs. unsaturated incompressible groundwater flow





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Hydrodynamical properties

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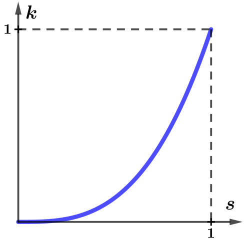
Parameters:

- Porosity $\phi \approx 0.01 - 1$
- Permeability $\mathbb{K} \approx 10^{-7} - 10^{-20} m^2$, possibly a full tensor

Closure laws:

- Relative permeability $k(s) \approx s^m$, $m > 1$
- Retention curve S: increasing, $S(-\infty) = 0$ and $S(p \geq p_e) = 1$



Relative permeability k as a function of saturation

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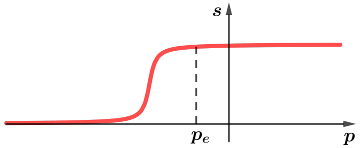

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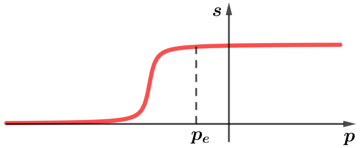

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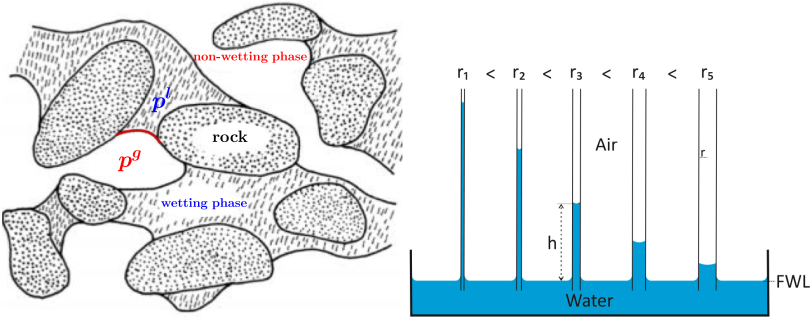
Capillary pressure at the pore-scale



Nemes '16

Assumptions:

- Two immiscible phases: sharp interfaces at pore scale
- Wetting (liquid) and non-wetting (gas) phases

Pressure jump across the **interface**

- Young–Laplace law

$$p^g - p^l = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

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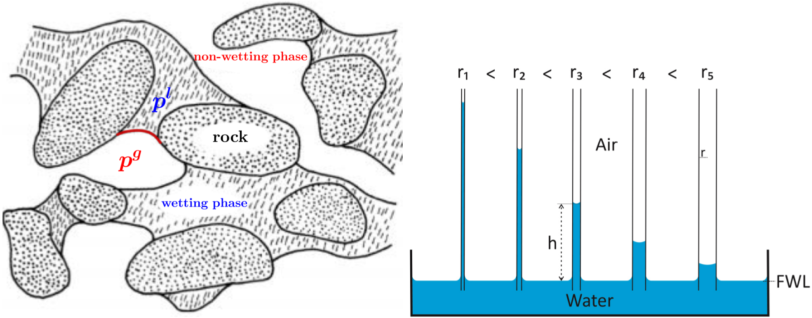

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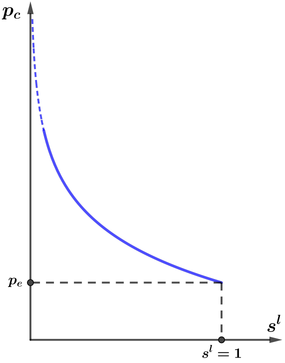
Observation: Wetting phase "prefers" small pores

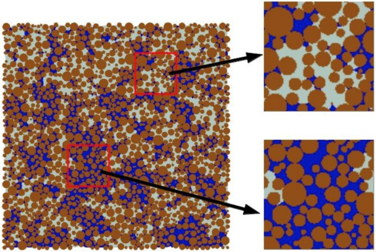
Macroscopic capillary pressure

Capillary pressure at Darcy (macroscopic) scale

$$p^g - p^l = p_c(s^l)$$



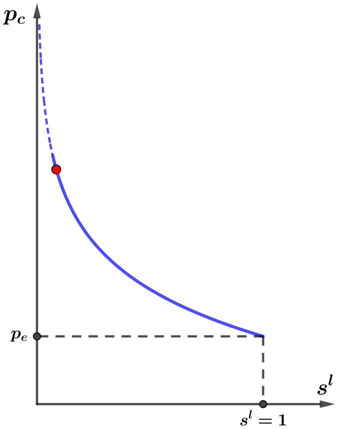
- Entry pressure p_e

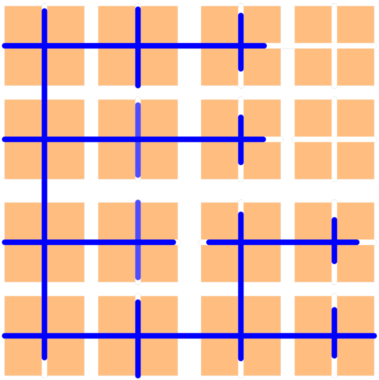


Yuan et al. '16

- Capillary pressure law depends on pore-size distribution

Macroscopic capillary pressure





Wetting of the rock:

- Interface between the phases moves to larger pores $\Rightarrow$ pressure jump decreases

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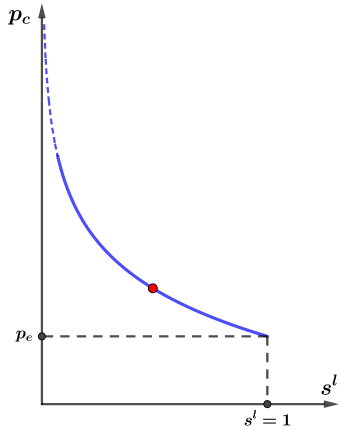

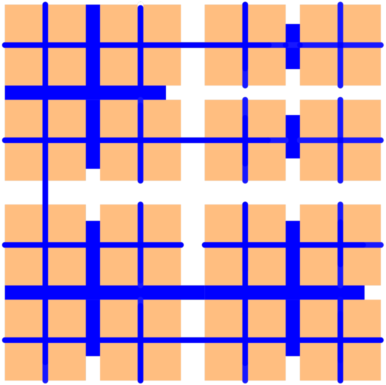

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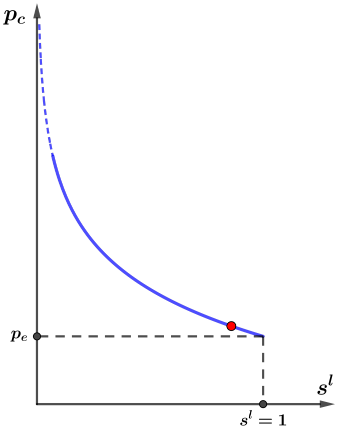

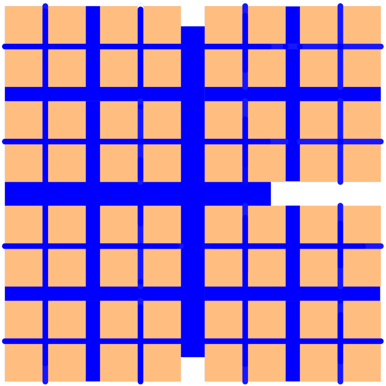

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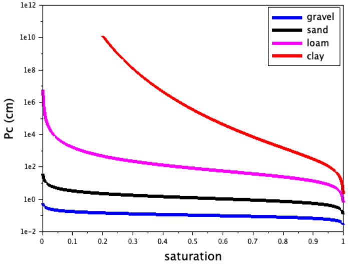
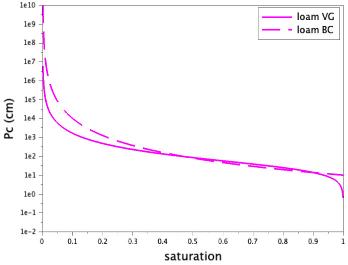
Capillary pressure curves





Macroscopic capillary pressure:

- p_c law depends on pore-size distribution;
- pressure jump may be neglected/assumed constant for some soils.

Van Genuchten model

$$p_c(s) = p_\alpha \left(s^{-\frac{1}{m}} - 1 \right)^{\frac{1}{n}}$$

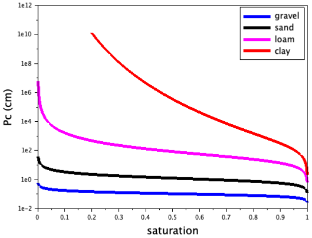
Brooks-Corey model

$$p_c(s) = p_e s^{-\lambda}$$

Retention curve

Set $p^g = p_{atm}$ from capillary pressure law:

$$p^l = p_{atm} - p_c(s) \qquad \text{for} \quad 0 < s < 1$$



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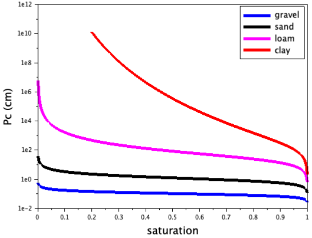

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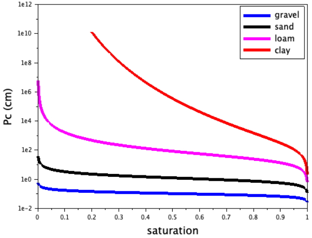

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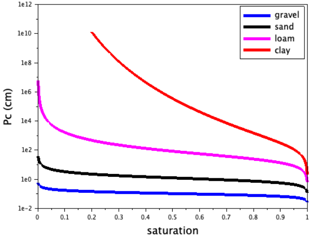

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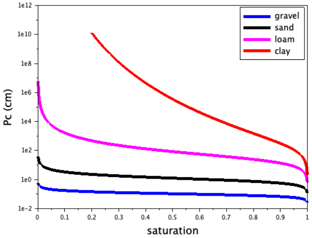
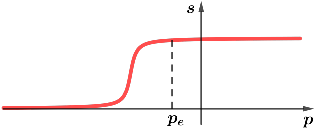


Functional closure $s = S(p)$



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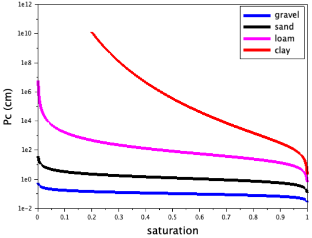

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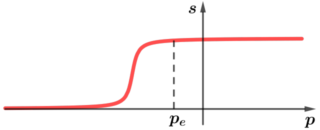
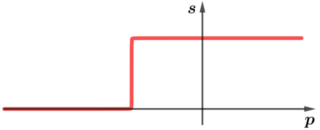


Graphical closure $s \in S(p)$



Formulation using Kirchhoff transform

Richards' equation

$$\partial_t s - \mathrm{div}\left(k(s)\left(\nabla p - \boldsymbol{g}\right)\right) = 0$$

Natural energy estimate

$$\sup_{t\leq T}\int_\Omega \Psi(p)\,\mathrm{d}\mathbf{x} + \int_0^T\int_\Omega k(s)|\nabla p|^2\,\mathrm{d}\mathbf{x} < +\infty, \qquad \Psi(p) = S(p)p - \int^p S(\pi)\,\mathrm{d}\pi$$

does not provide control on $\|\nabla p\|_{L^2}$

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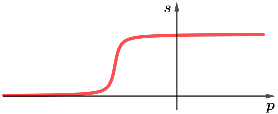
Kirchhoff transform

$$U(p) = \int^p k(S(\pi))\,\mathrm{d}\pi$$

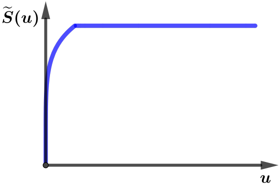


Reformulated equation

$$\partial_t s - \operatorname{div}\left(\nabla u - k(s)\boldsymbol{g}\right) = 0$$

with $s = \widetilde{S}(u) := S(U^{-1}(u))$

Equation is linear w.r.t. "generalized pressure" u.



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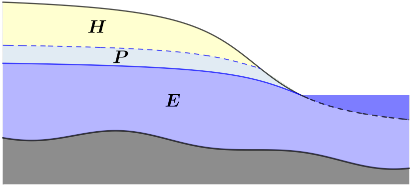
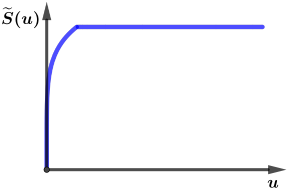
Coexistence of multiple "regimes"





Typically $\widetilde{S}(u) \sim u^{1/m}$, $m > 1$ near $u = 0$,

- connections to porous medium equation: $\partial_t u^{1/m} = \Delta u$
- almost hyperbolic behavior near $u = 0$: $\partial_t s + \operatorname{div} k(s)\boldsymbol{g} = 0$.
  Set of $1d$ problems!

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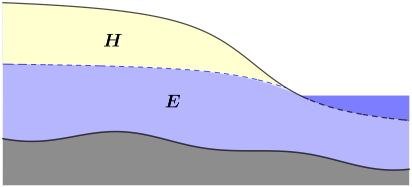

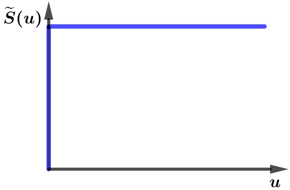

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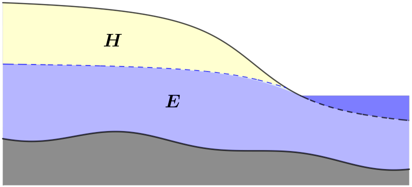

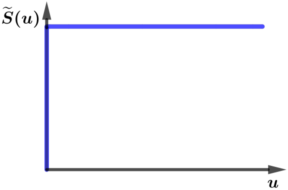

Existence and uniqueness

- Parabolic-elliptic: Van Duijn & Peletier '82, Alt & Luckhaus '83
- Hyperbolic-elliptic (Dam problem): Visintin '80, Carrillo '94
- Hyperbolic-parabolic-elliptic: Carrillo '99

Formulation using Kirchhoff transform

Richards' equation using Kirchhoff transform

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Pros & cons of Kirchhoff formulation

Makes mathematicians 😀

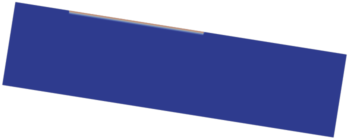
- Analysis and discretization friendly
- Easier to solve the discrete problem

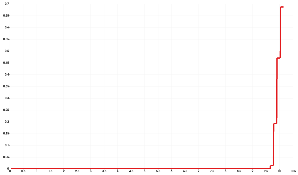
Makes engineers 😕

- No analytical expression of $U(p)$ for some closure laws
- Harder to incorporate additional physics

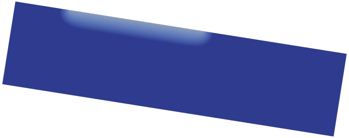
Rainfall over a dry soil

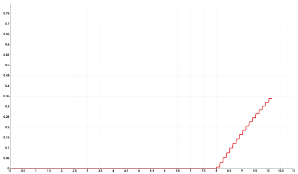
Sand



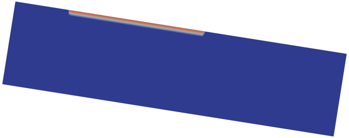
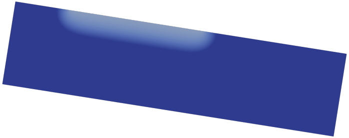
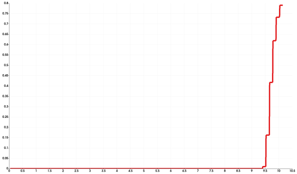
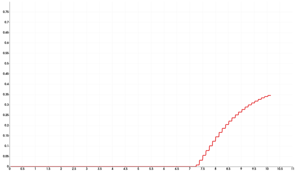


Clay



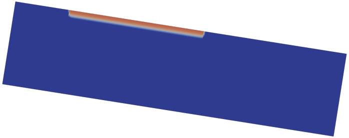


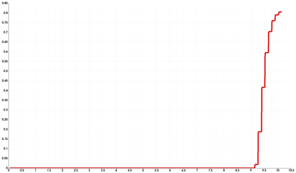
Rainfall over a dry soil

Sand	Clay
	
	

Rainfall over a dry soil

Sand





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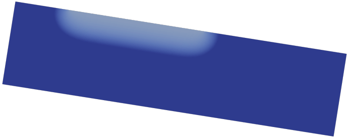
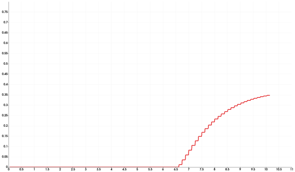
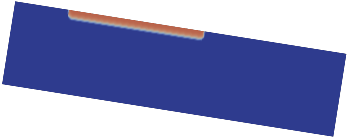
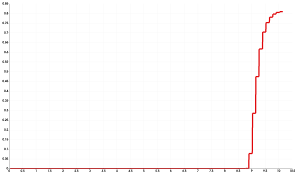




Rainfall over a dry soil

Sand





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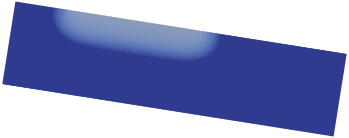
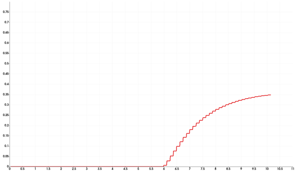
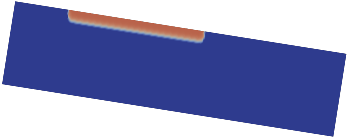




Rainfall over a dry soil

Sand



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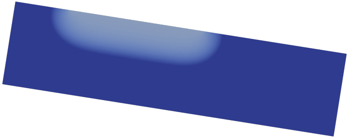
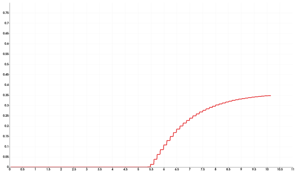
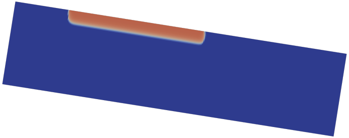
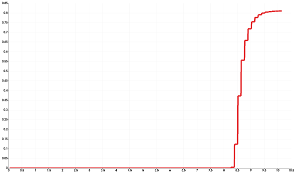




Rainfall over a dry soil

Sand





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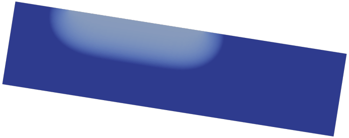
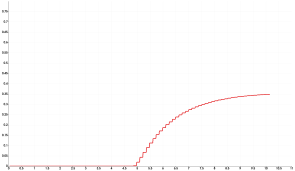
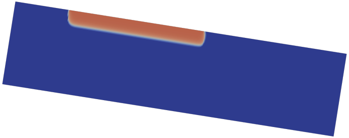
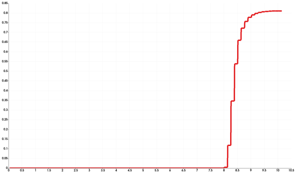




Rainfall over a dry soil

Sand





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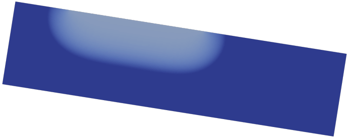
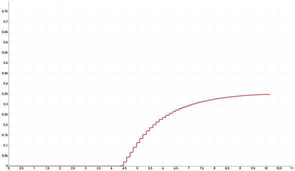
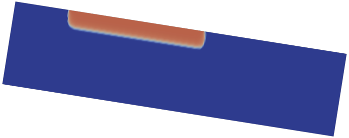


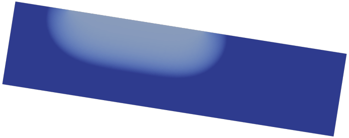


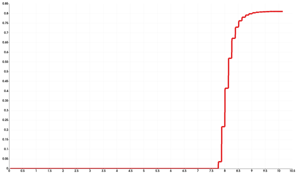
Rainfall over a dry soil

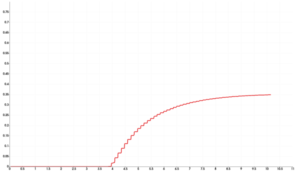
Sand

Clay



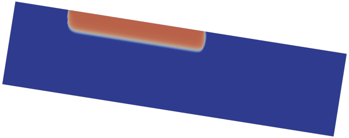


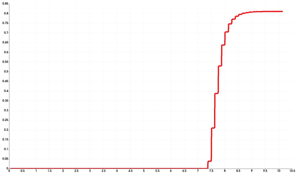




Rainfall over a dry soil

Sand





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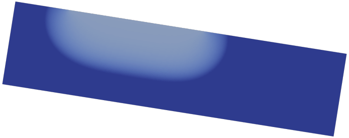
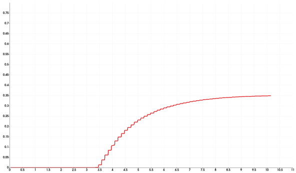
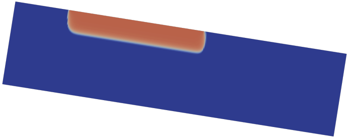
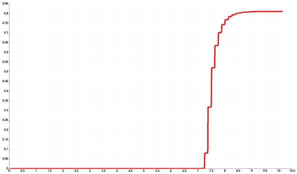




Rainfall over a dry soil

Sand





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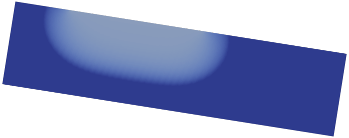
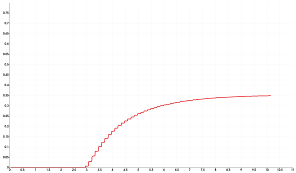
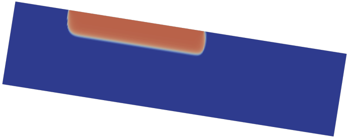
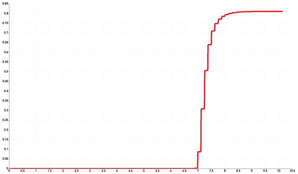




Rainfall over a dry soil

Sand





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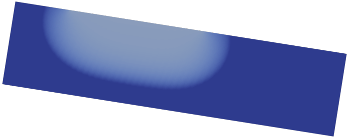

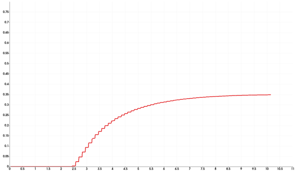

Rainfall over a dry soil

Sand

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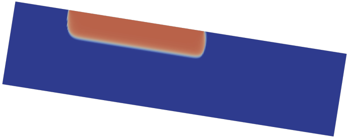
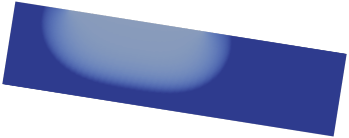
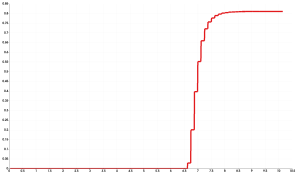
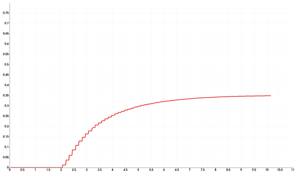


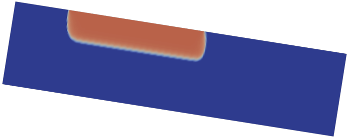


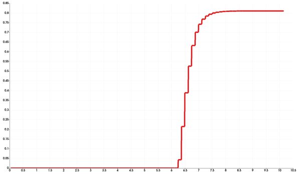




Rainfall over a dry soil

Sand





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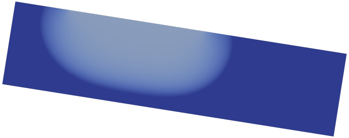
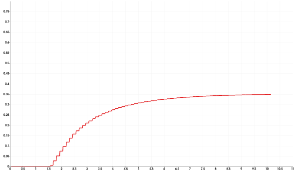
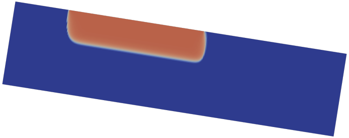
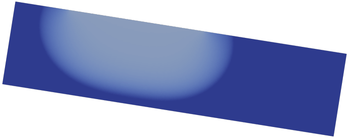
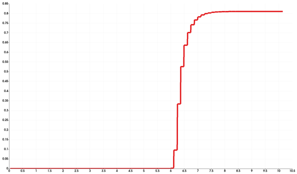
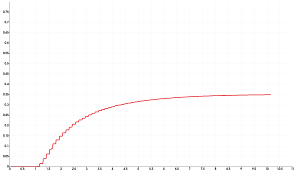


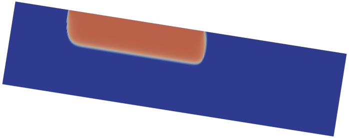


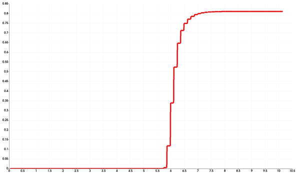
Rainfall over a dry soil

Sand	Clay
	
	

Rainfall over a dry soil

Sand





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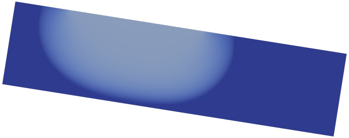
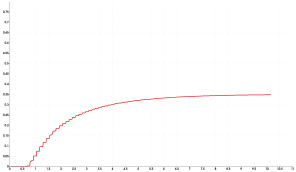
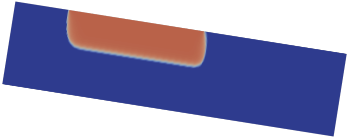
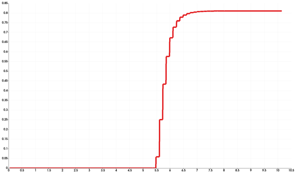




Rainfall over a dry soil

Sand





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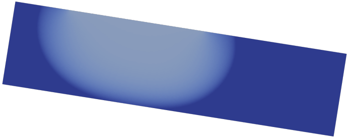
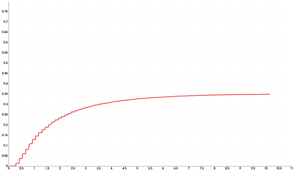
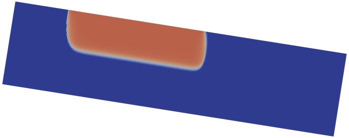
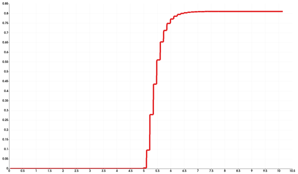




Rainfall over a dry soil

Sand





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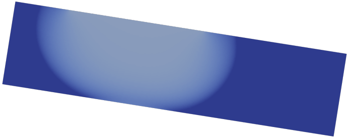
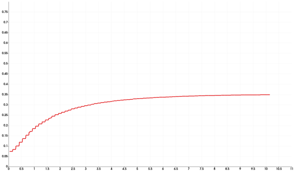
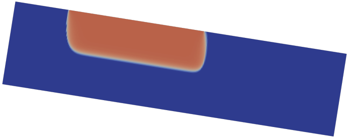
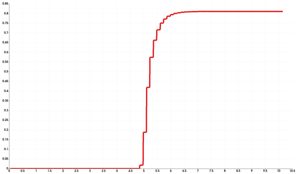




Rainfall over a dry soil

Sand





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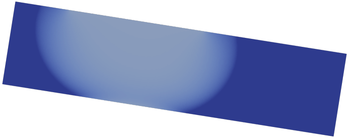
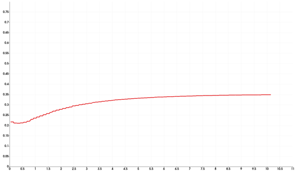
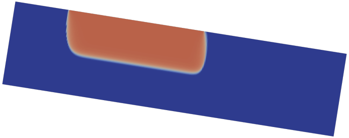
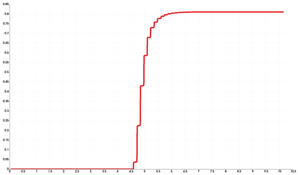




Rainfall over a dry soil

Sand





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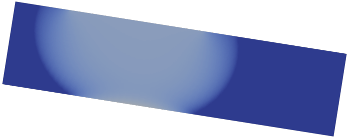
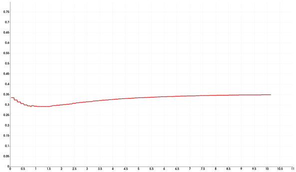
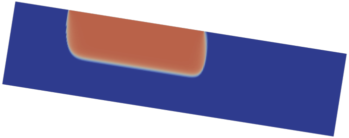
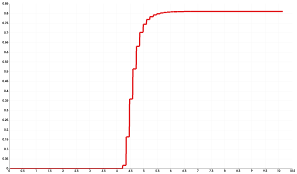




Rainfall over a dry soil

Sand





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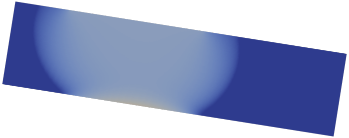
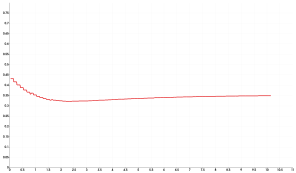
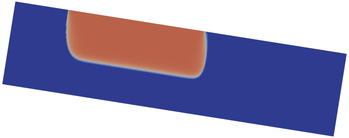
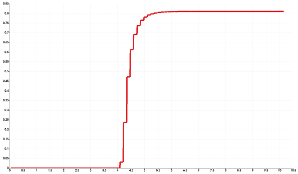




Rainfall over a dry soil

Sand





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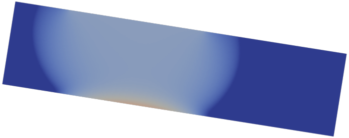
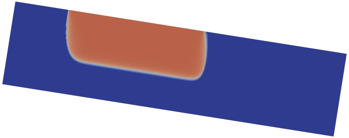
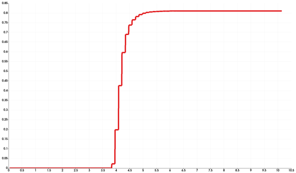


Rainfall over a dry soil

Sand





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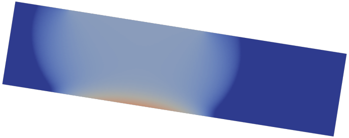
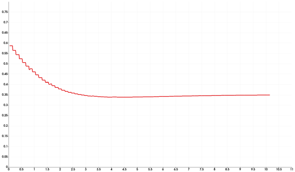
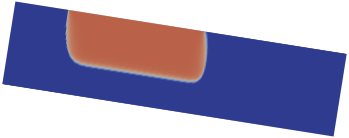
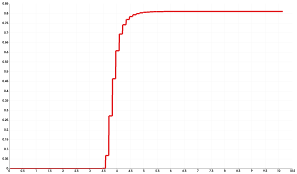




Rainfall over a dry soil

Sand





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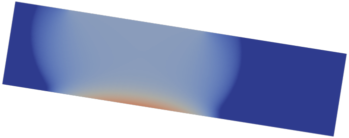
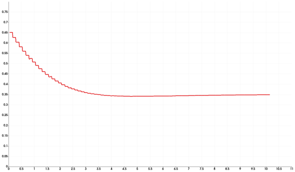
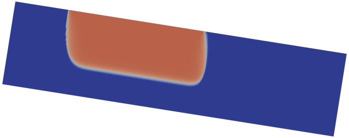
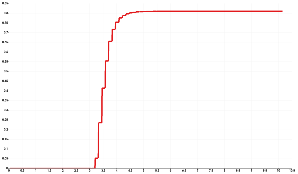




Rainfall over a dry soil

Sand





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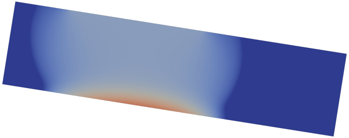
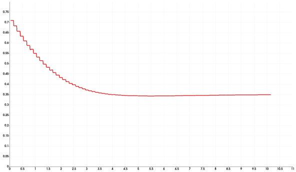
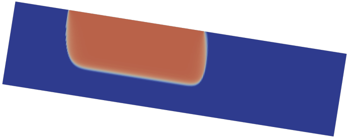
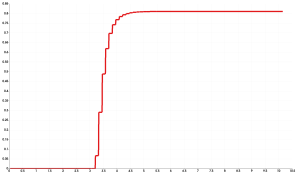




Rainfall over a dry soil

Sand





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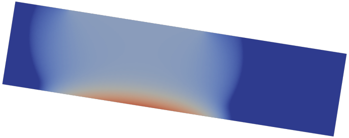
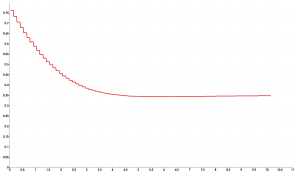
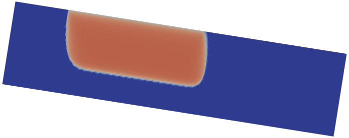
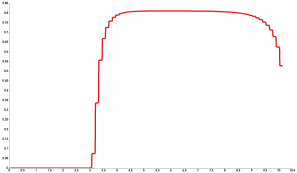




Rainfall over a dry soil

Sand





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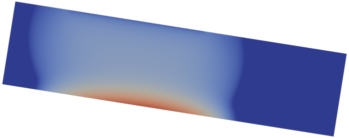
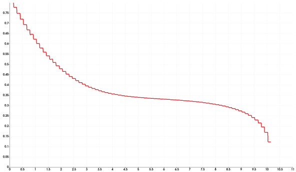
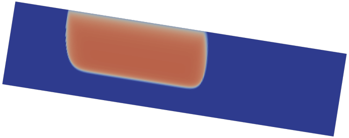
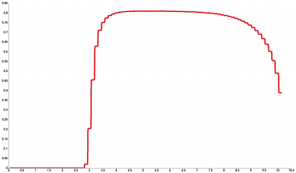




Rainfall over a dry soil

Sand





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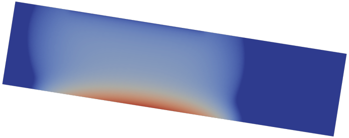
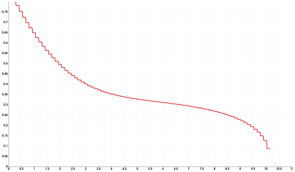
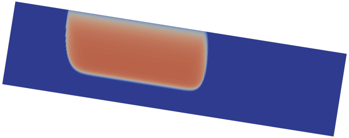
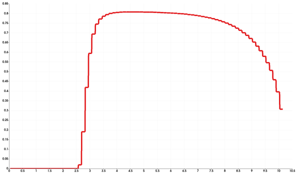




Rainfall over a dry soil

Sand





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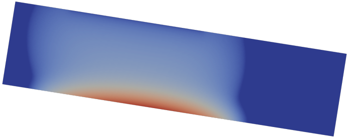
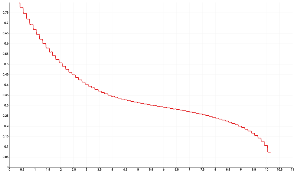
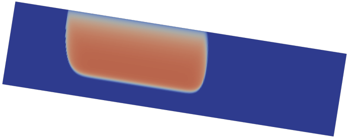


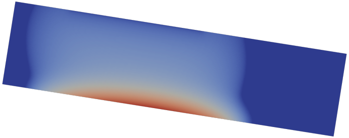


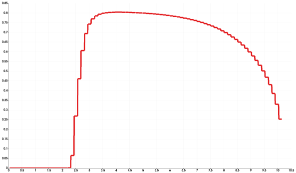
Rainfall over a dry soil

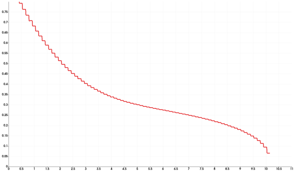
Sand

Clay





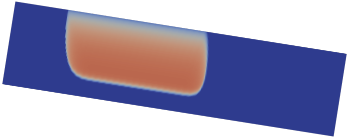


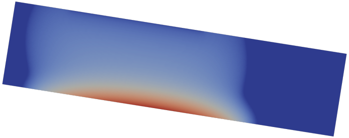


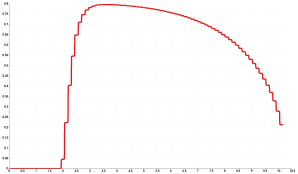
Rainfall over a dry soil

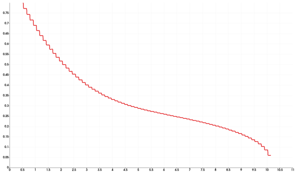
Sand

Clay



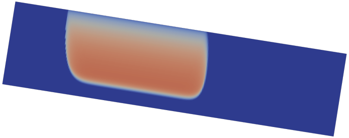


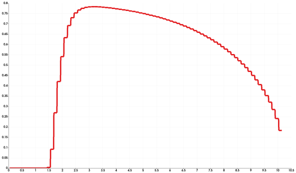




Rainfall over a dry soil

Sand





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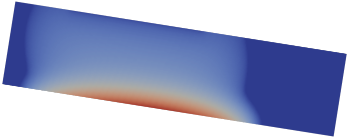
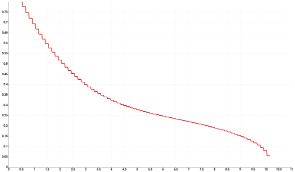
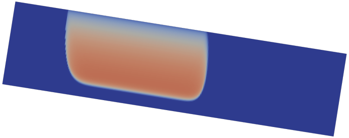
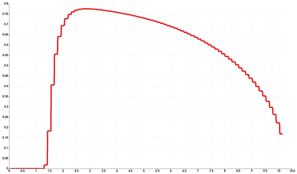




Rainfall over a dry soil

Sand





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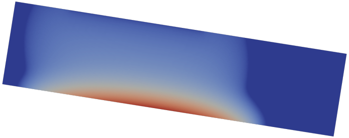
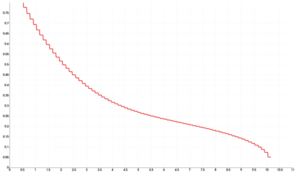
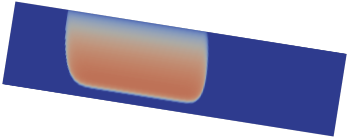
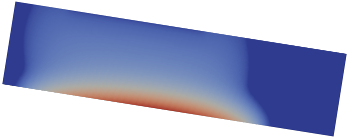
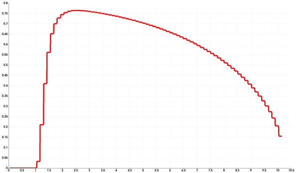
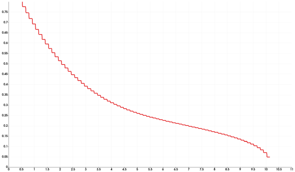


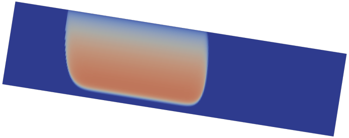


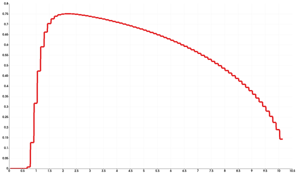
Rainfall over a dry soil

Sand	Clay
	
	

Rainfall over a dry soil

Sand





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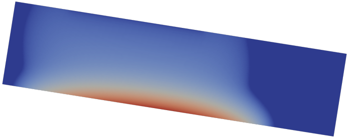
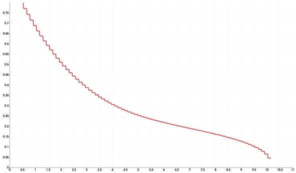
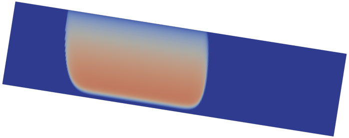
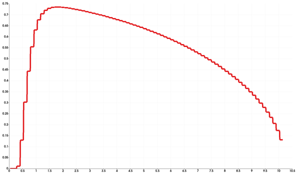




Rainfall over a dry soil

Sand





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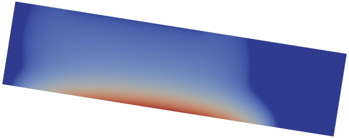
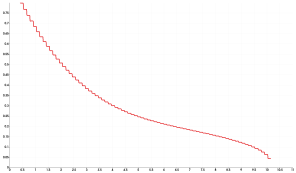
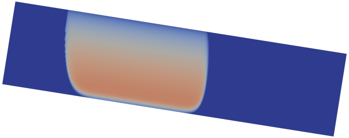
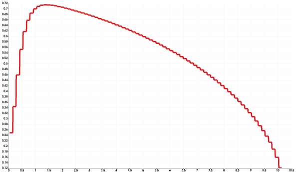




Rainfall over a dry soil

Sand





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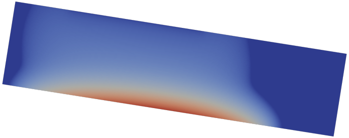
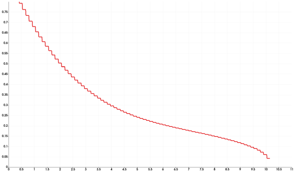
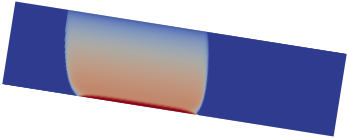
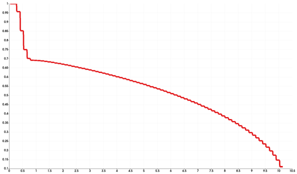




Rainfall over a dry soil

Sand





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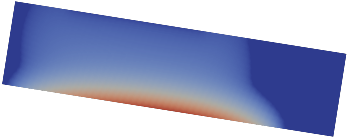
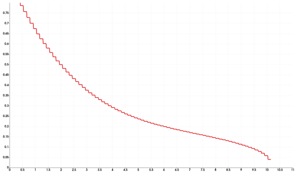
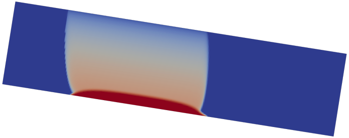
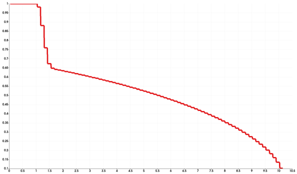




Rainfall over a dry soil

Sand





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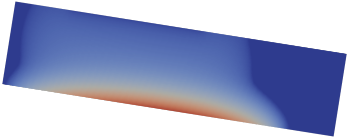
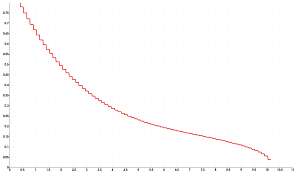
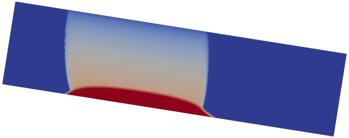
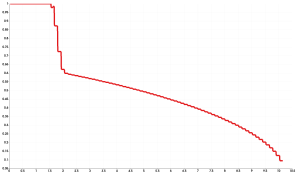




Rainfall over a dry soil

Sand





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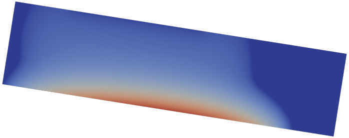
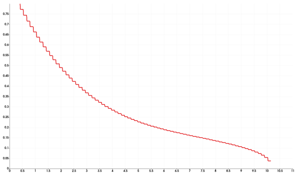
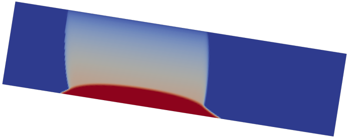




Rainfall over a dry soil

Sand



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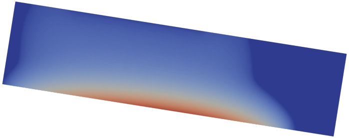

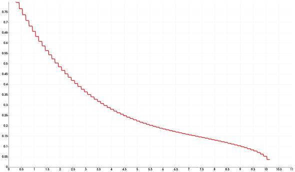

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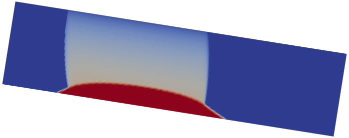

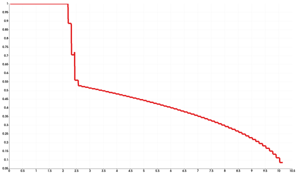

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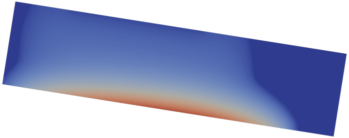

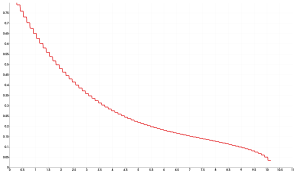

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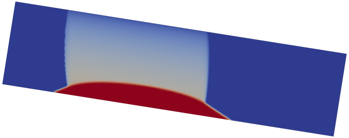

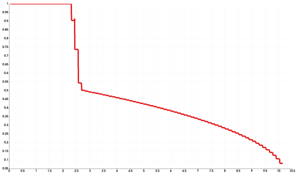

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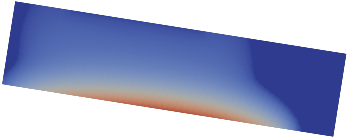

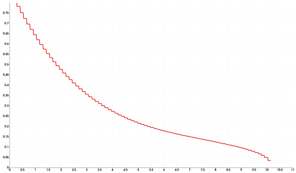

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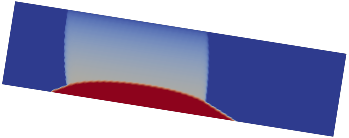

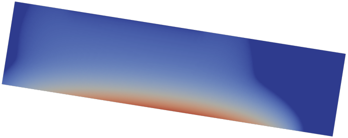

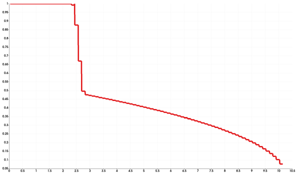

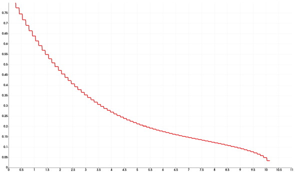

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Sand

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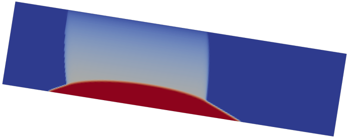

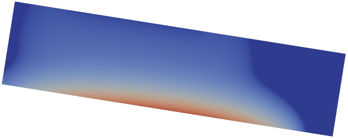

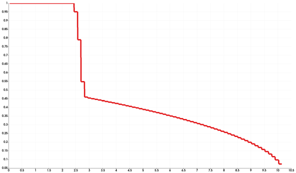

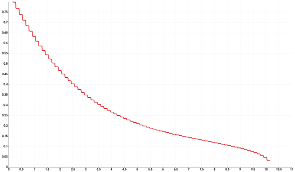

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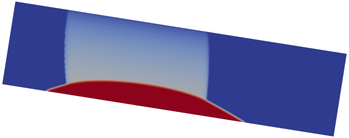

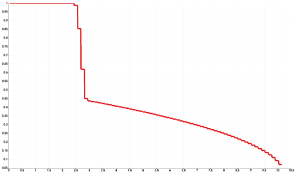

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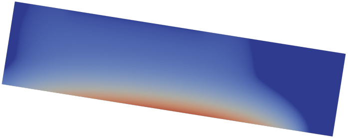

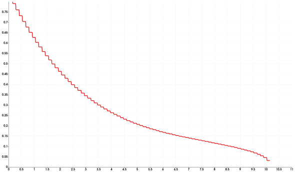

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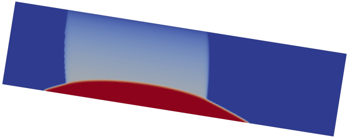

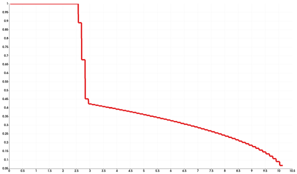

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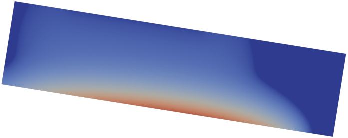

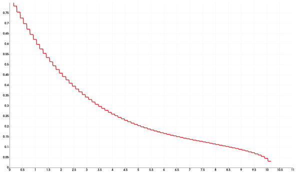

Rainfall over a dry soil

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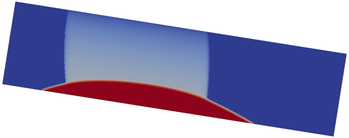

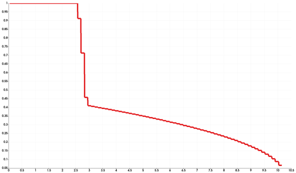

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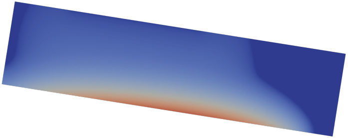

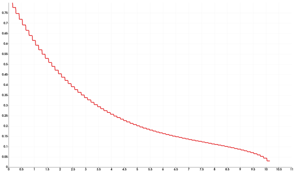

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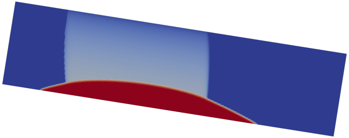

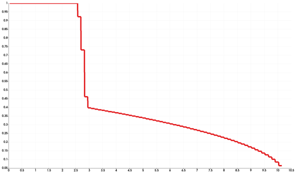

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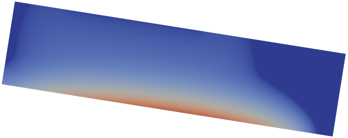

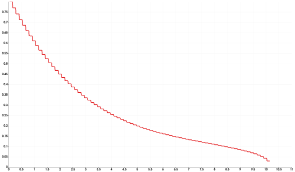

Rainfall over a dry soil

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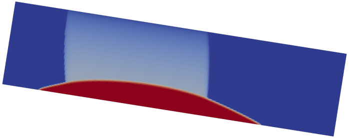

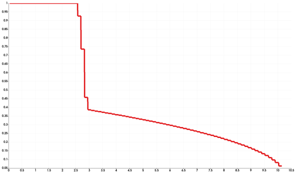

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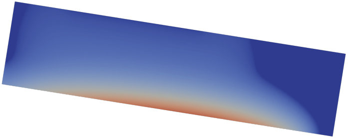
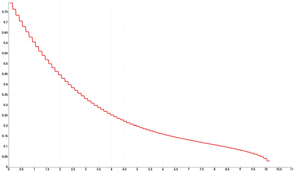
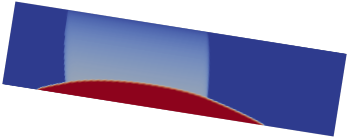
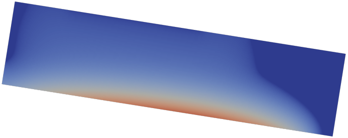
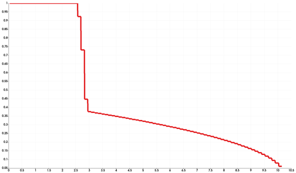
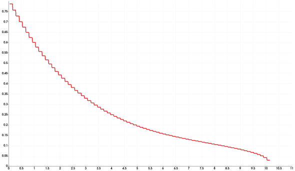




Rainfall over a dry soil

Sand	Clay
	
	

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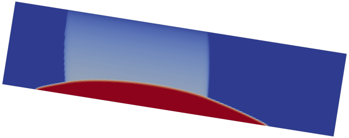

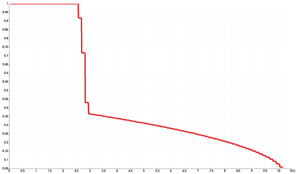

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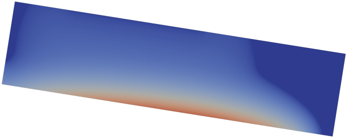
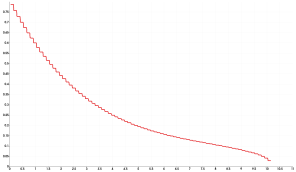




Classical discretizations: Kirchhoff pressure

Implicit finite volume scheme

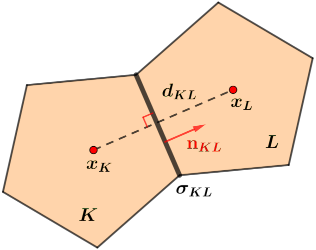
$$|K|\frac{s_K^n - s_K^{n-1}}{\Delta t_n} + \sum_L q_{KL}^n = 0$$

Flux discretiaztion

$$\frac{1}{|\sigma_{KL}|}\int_{\sigma_{KL}} \nabla u - k(s)\boldsymbol{g}\mathrm{d}\sigma \approx \frac{u_K - u_L}{|x_K - x_L|} - k(s_{KL})\,\boldsymbol{g}\cdot\mathbf{n}_{KL}$$

with upwinding

$$s_{KL} = \begin{cases} s_K, & -\boldsymbol{g}\cdot\mathbf{n}_{KL} \geq 0 \\ s_L, & \text{else} \end{cases}$$



Admissible mesh

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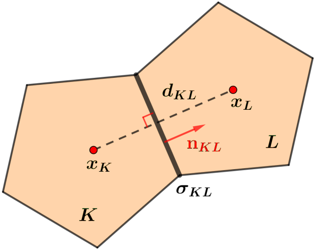

Admissible mesh

Implicit discretization:

$$\boldsymbol{s} + A\boldsymbol{u} + Bk(\boldsymbol{s}) = \boldsymbol{s}^{n-1}, \quad \boldsymbol{s} = \widetilde{S}(\boldsymbol{u})$$

Structural properties:

- A and $Bk'(\boldsymbol{s})$ are M-matrices

Classical discretizations: Kirchhoff pressure

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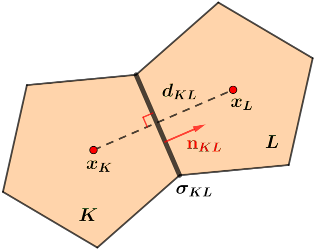

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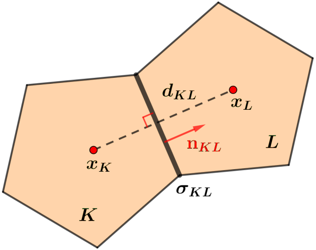


Admissible mesh

Semi-implicit discretization:

$$\boldsymbol{s} + A\boldsymbol{u} + Bk(\boldsymbol{s}^{n-1}) = \boldsymbol{s}^{n-1}, \quad \boldsymbol{s} = \widetilde{S}(\boldsymbol{u})$$

Structural properties:

- A and $Bk'(\boldsymbol{s})$ are M-matrices $\Rightarrow$ discrete maximum principle under CFL cond.

Outline

Introduction to Richards' equation

- From saturated to unsaturated flow

Monotone Newton Theorem

- Convergence proof $\neq$ performance

Nonlinear preconditioning

- Convergence proof $+$ performance

Conclusion

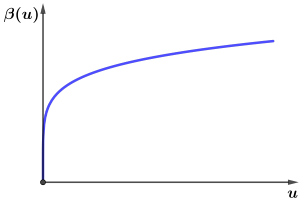
Model algebraic system

Find $\boldsymbol{u} \in \mathbb{R}^N$

$$\beta(\mathbf{u}) + A\mathbf{u} = \boldsymbol{b}, \qquad \boldsymbol{b} \geq 0$$

Objective: Newton-like iterative method

- efficient and robust w.r.t. to the shape of β
- with guarantied (semi-)global convergence



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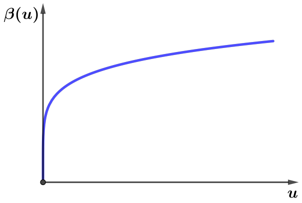


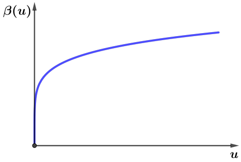
Assumptions:

- $J(\boldsymbol{u}) = \beta'(\boldsymbol{u}) + A$ is M-matrix:
  $J(\boldsymbol{u})^{-1} \geq 0$ and $(J(\boldsymbol{u}))_{ij} \leq 0, i \neq j$
- $\beta_i : \mathbb{R}^+ \to \mathbb{R}^+$ diagonal, increasing and concave, $\beta_i'(0) \leq +\infty$

Other applications in geosciences

Porous medium equation

$$\partial_t u^{1/m} - \Delta u = 0, \qquad m > 1$$

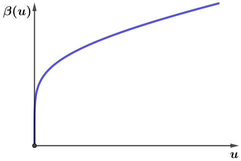


Contaminant transport with adsorption

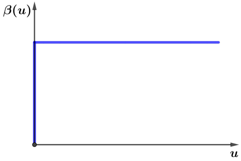
$$\partial_t \underbrace{(u + a(u))}_{\text{dissolved + adsorbed conc.}} - \operatorname{div}(\nabla u + u\mathbf{V}) = 0$$

Freundlich isotherm

$$a(u) = cu^{1/m}, \quad c > 0, \quad m > 1$$



Other applications in geosciences



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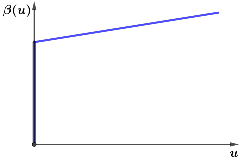

Freundlich isotherm

$$a(u) = cu^{1/m}, \quad c > 0, \quad m > 1$$

Limit case $m \to +\infty$

$$\partial_t v + L(u) = 0, \qquad v \in \beta(u)$$

Other applications in geosciences

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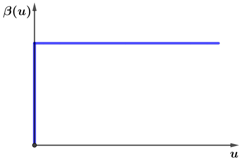
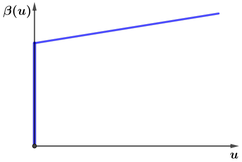


Contaminant transport with adsorption

$$\partial_t \underbrace{(u + a(u))}_{\text{dissolved + adsorbed conc.}} - \mathrm{div}\,(\nabla u + u\mathbf{V}) = 0$$

Freundlich isotherm

$$a(u) = cu^{1/m}, \quad c > 0, \quad m > 1$$

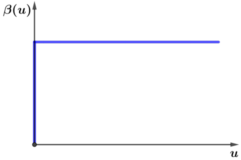


Limit case $m \to +\infty$

$$\partial_t v + L(u) = 0, \qquad v \in \beta(u)$$

- β is maximal monotone
- Connections to obstacle problems (Brugnano & Sestini '09)

Other applications in geosciences



Porous medium equation

$$\partial_t u^{1/m} - \Delta u = 0, \qquad m > 1$$

Contaminant transport with adsorption

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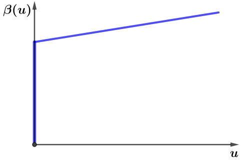

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Limit case $m \to +\infty$

$$\partial_t v + L(u) = 0, \qquad v \in \beta(u)$$

Nonlinear solver must be robust w.r.t. to the shape of β

Monotone Newton's method

Notations

$$F(\boldsymbol{u}) = \beta(\boldsymbol{u}) + A\boldsymbol{u} - \boldsymbol{b}$$

Newton's method

$$F'(\boldsymbol{u}_k)\,(\boldsymbol{u}_{k+1} - \boldsymbol{u}_k) + F(\boldsymbol{u}_k) = 0$$

Monotone Newton Theorem (Baluev '52; Ortega & Rheinboldt '70)
Let $\boldsymbol{u}_0$ satisfy $F(\boldsymbol{u}_0) \leq 0$, then

- $\boldsymbol{u}_k$ converges to the unique solution $\boldsymbol{u}_\star$
- $F(\boldsymbol{u}_k) \leq 0$ and $\boldsymbol{u}_k \leq \boldsymbol{u}_{k+1} \leq \boldsymbol{u}_\star$ for all $k \geq 0$

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Main ingredients:

- F is concave (or convex)
- $F'(\boldsymbol{u})^{-1} \geq 0$

Illustration (N = 1)

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Illustration (N = 1)

The method is semi-globally convergent. Is it efficient?

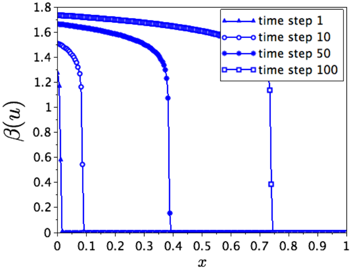
1D numerical experiment

Porous medium equation on $(0, 1) \times (0, T)$

$$\partial_t \beta(u) - \partial^2_{xx} u = 0, \qquad \beta(u) = u^{1/m}$$

with Neumann boundary conditions

- Inflow at $x = 0$: $-\partial_x u(0, t) = q > 0$
- No-flow at $x = 1$
- Almost "dry" initial condition: $\beta(u(x, 0)) = 10^{-10}$

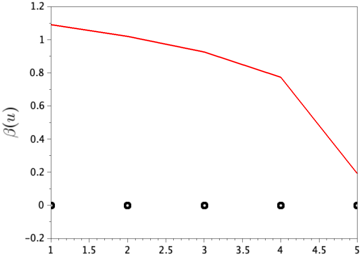


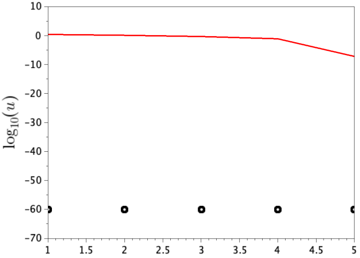
Solution profile at different time steps

Newton's iterates over a single time step

Newton's method applied to

$$\beta(\boldsymbol{u}) + A\boldsymbol{u} = \boldsymbol{b}, \qquad \beta(\boldsymbol{u}) = \boldsymbol{u}^{1/10}$$





first last jacobi

- Slow convergence
- Iteration count grows with m and N

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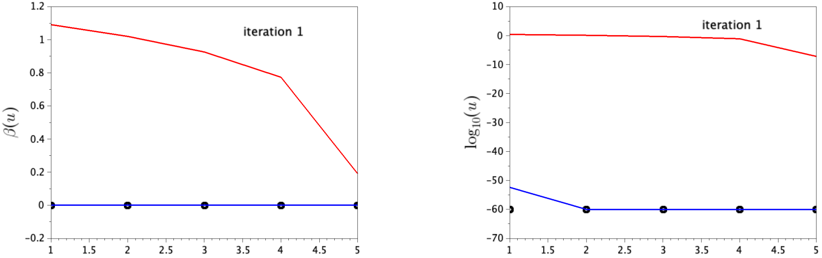

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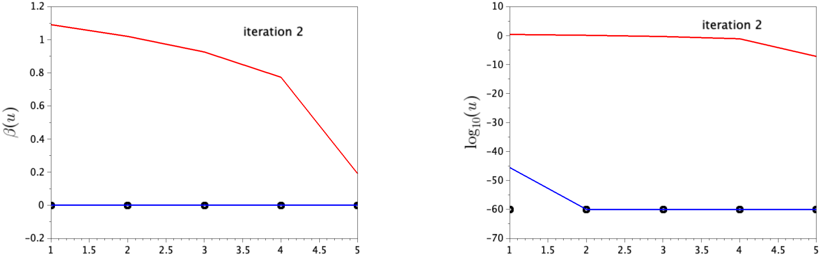

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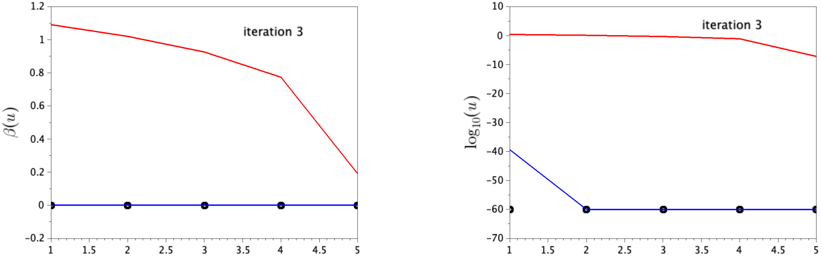

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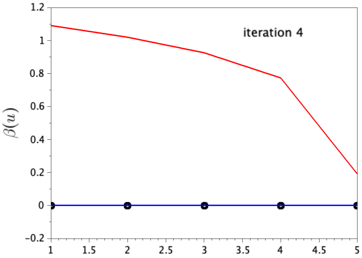

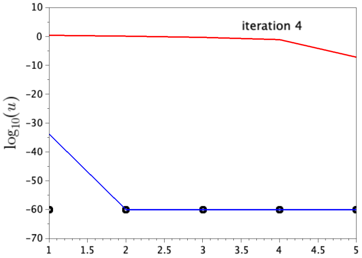

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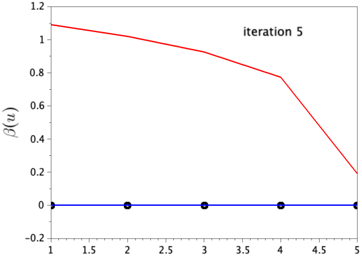

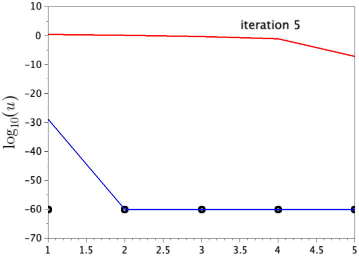

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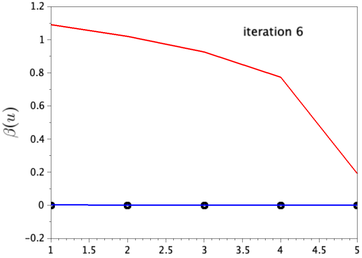

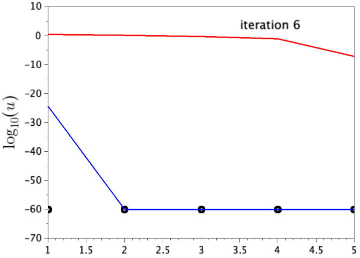

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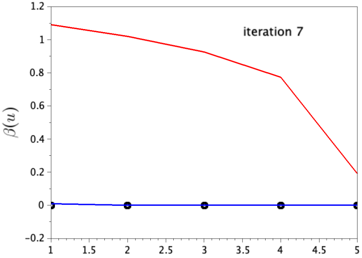

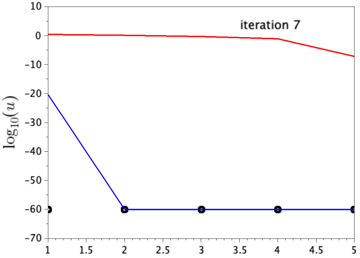

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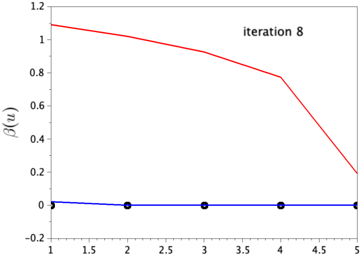

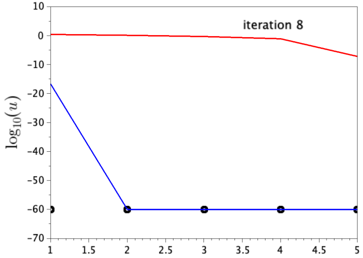

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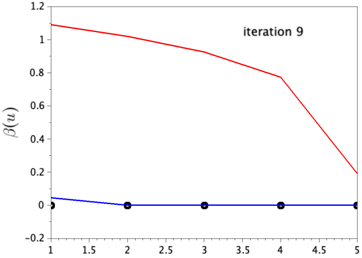

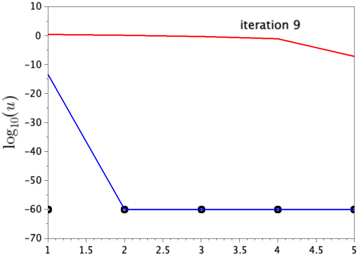

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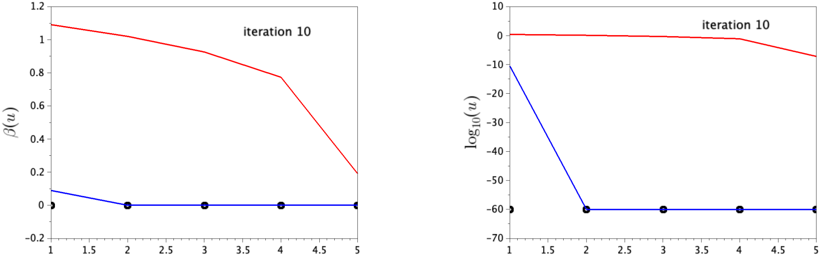

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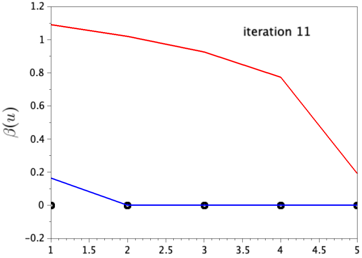

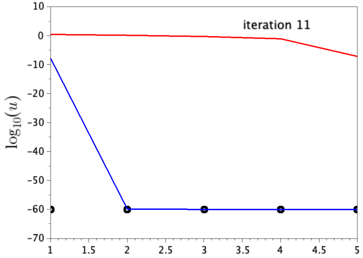

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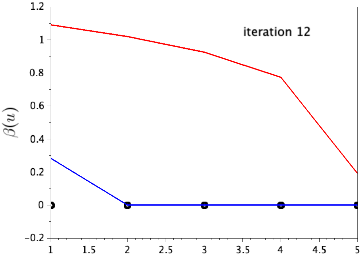

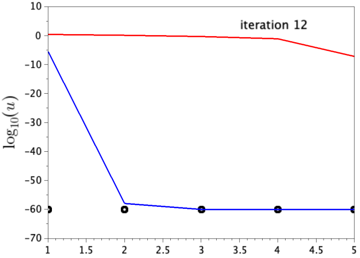

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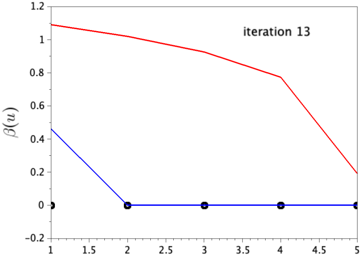

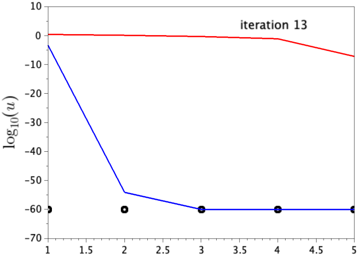

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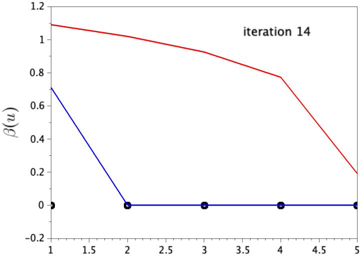

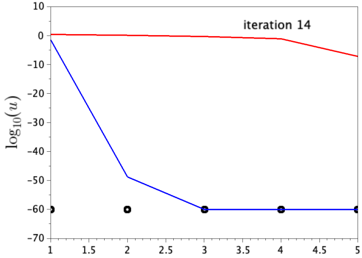

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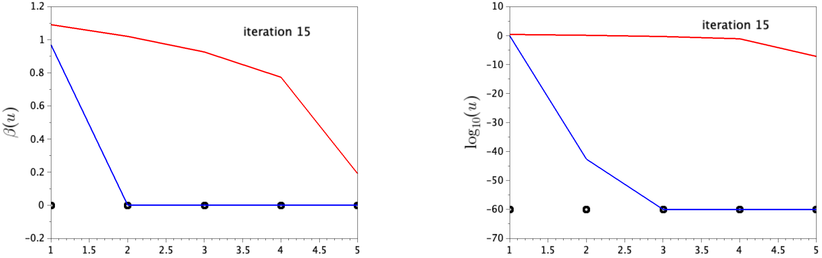

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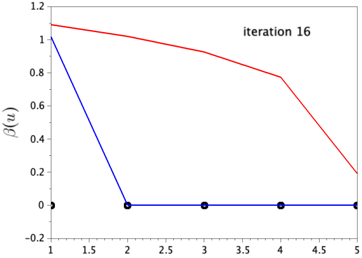

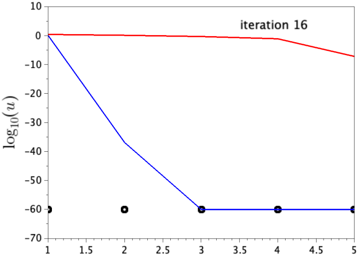

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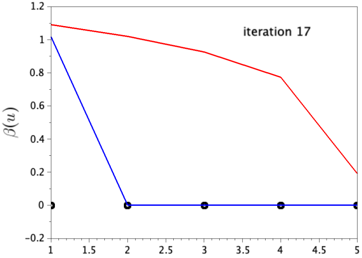

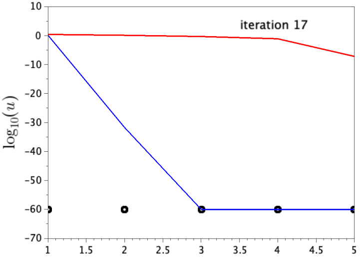

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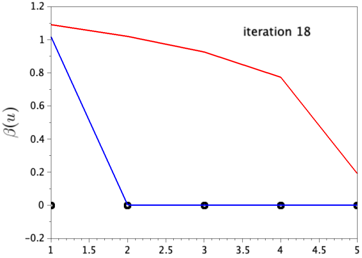

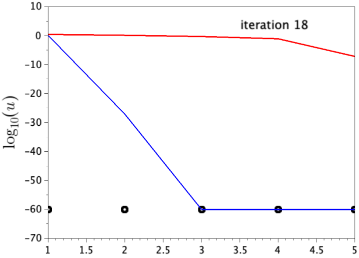

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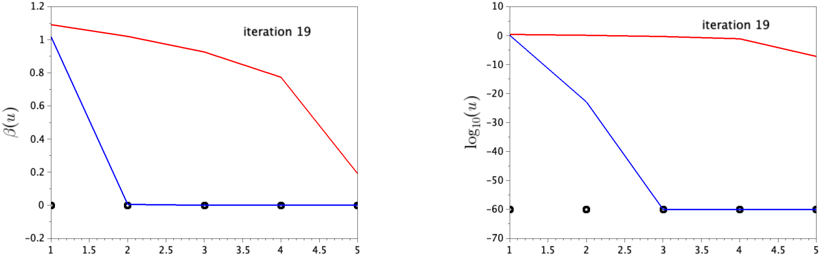

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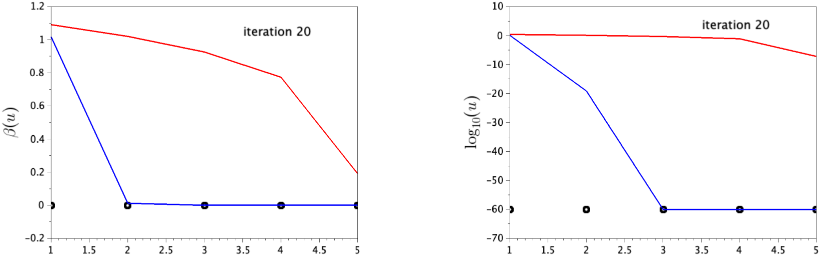

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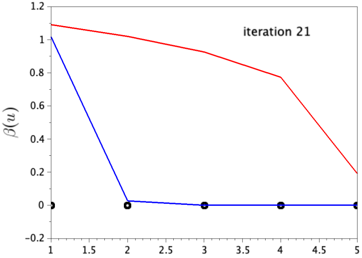

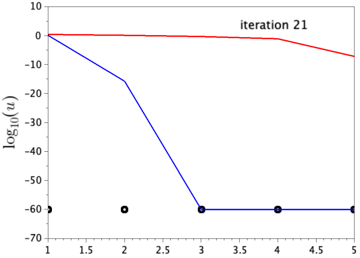

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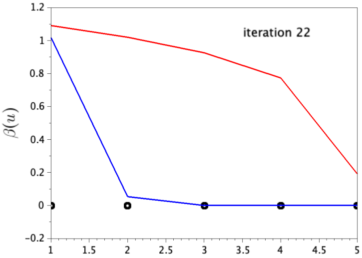

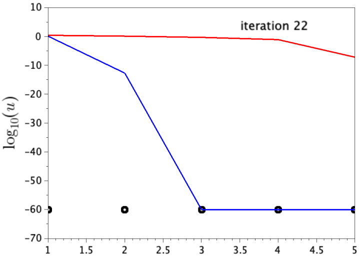

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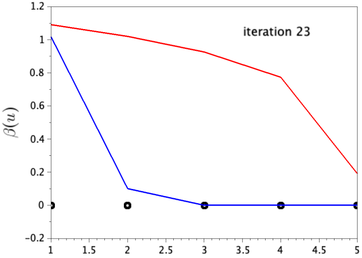

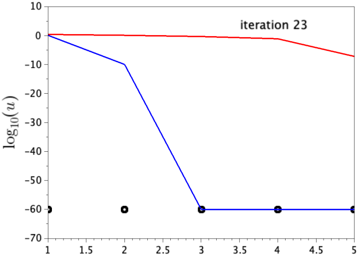

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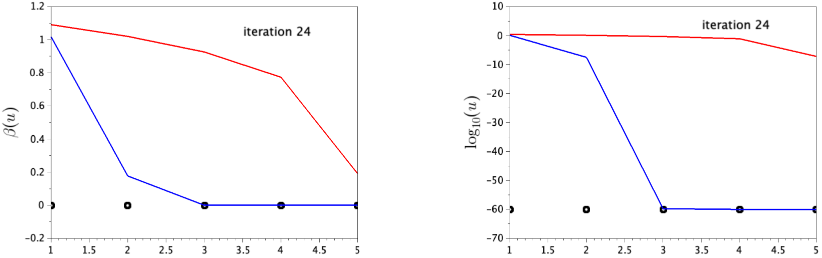

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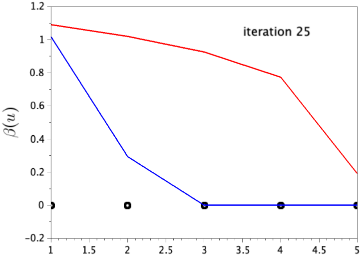

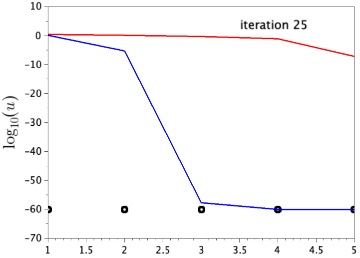

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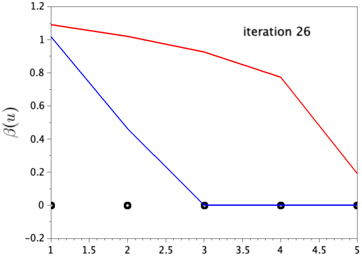

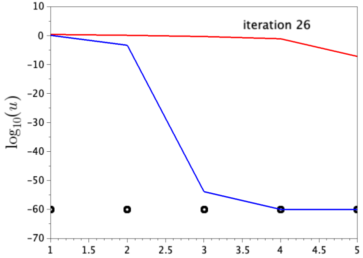

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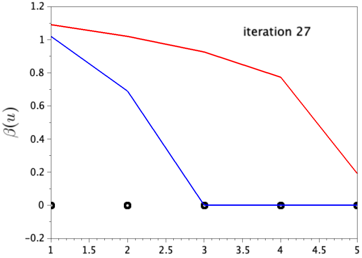

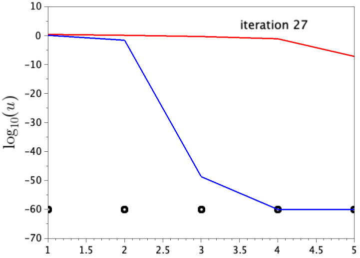

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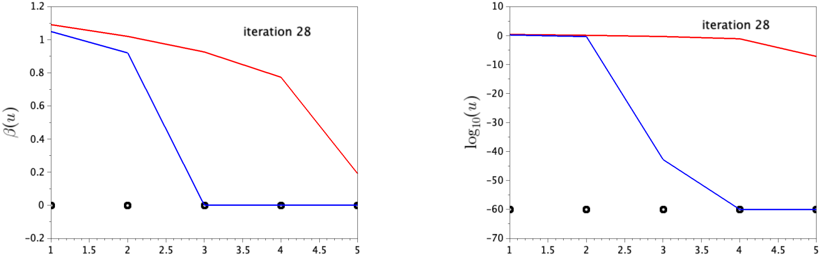

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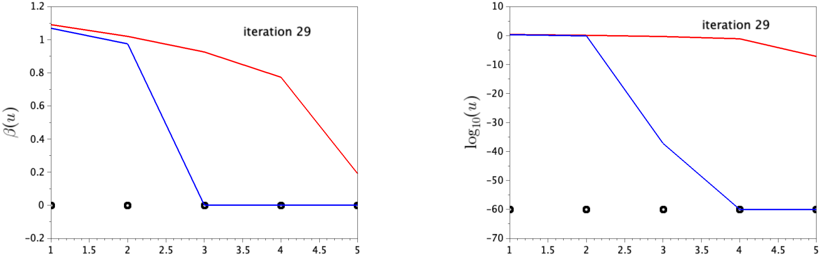

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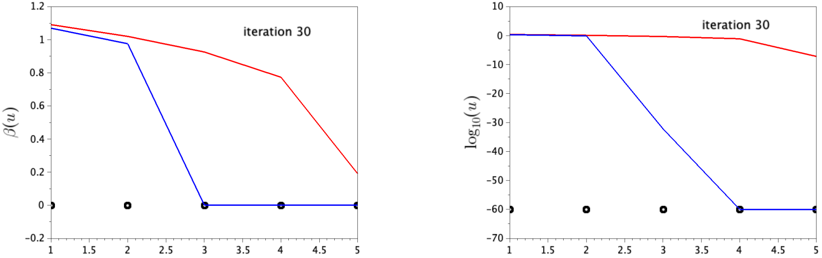

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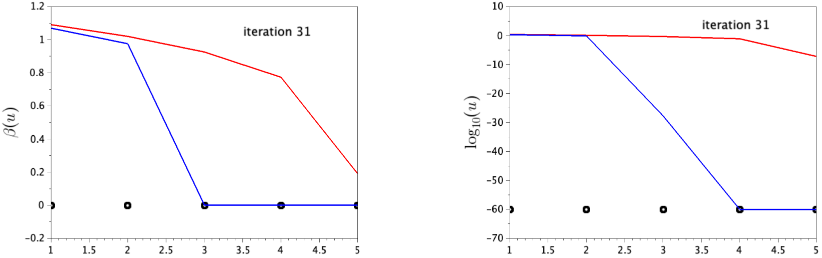

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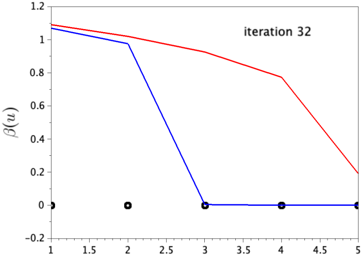

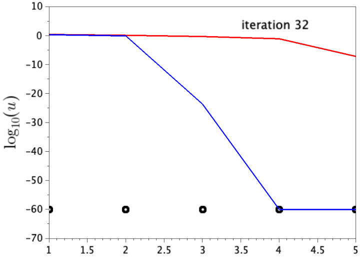

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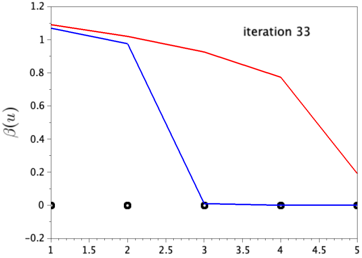

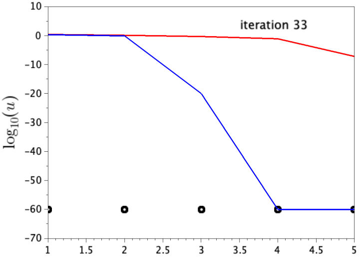

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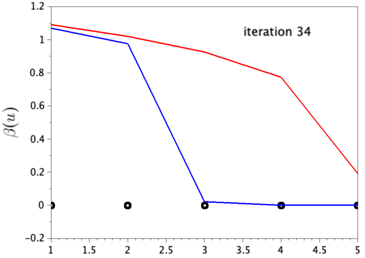

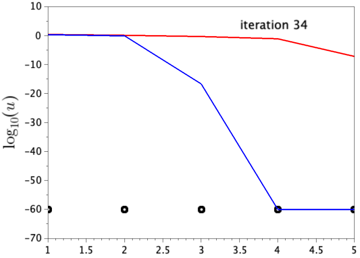

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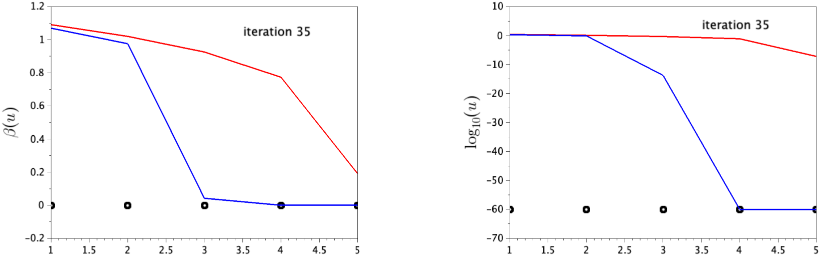

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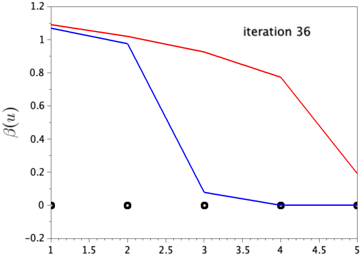

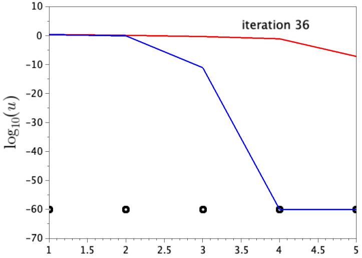

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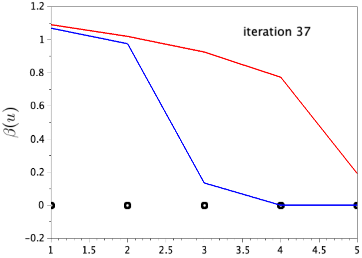

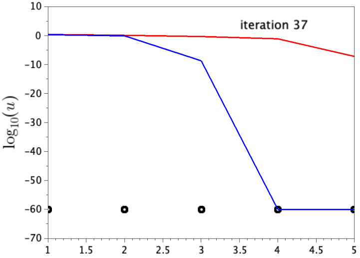

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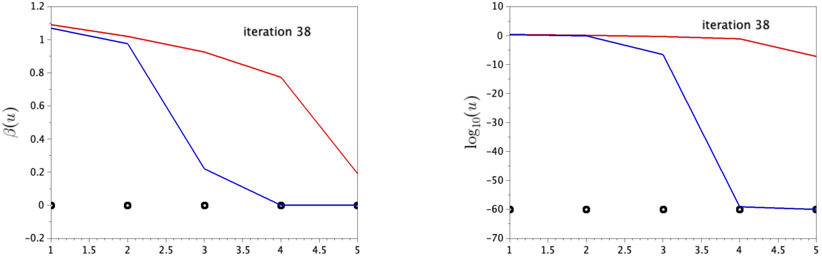

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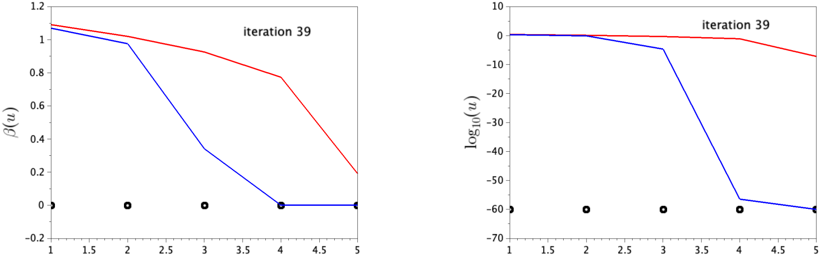

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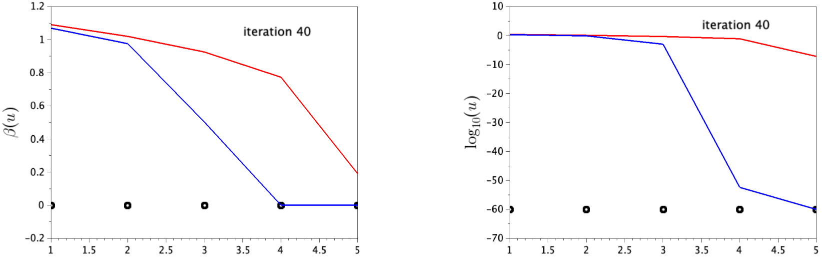

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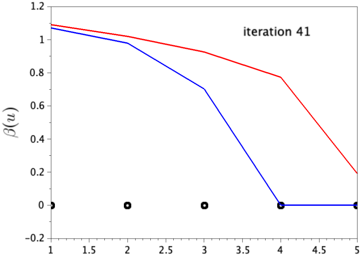

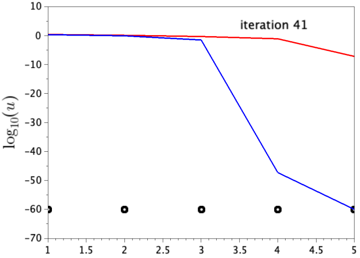

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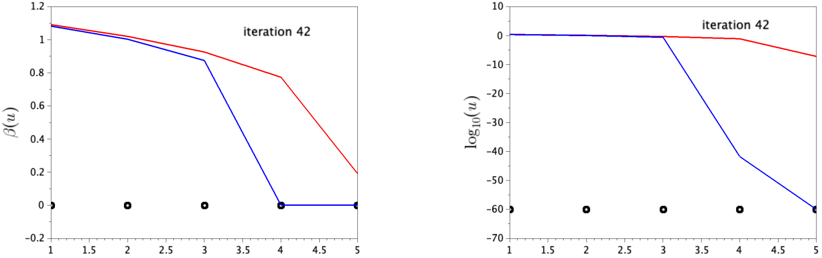

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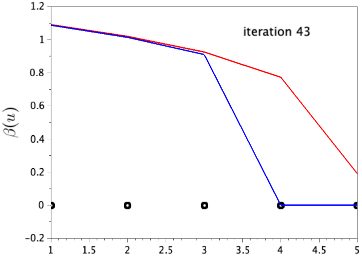

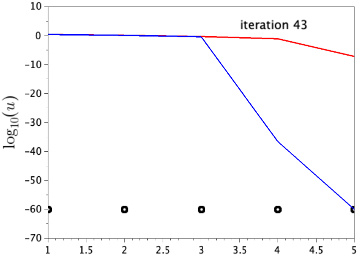

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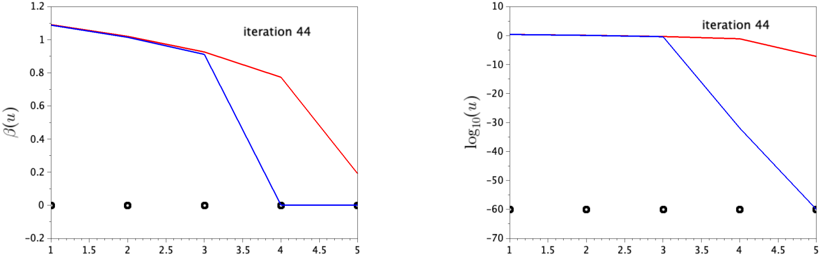

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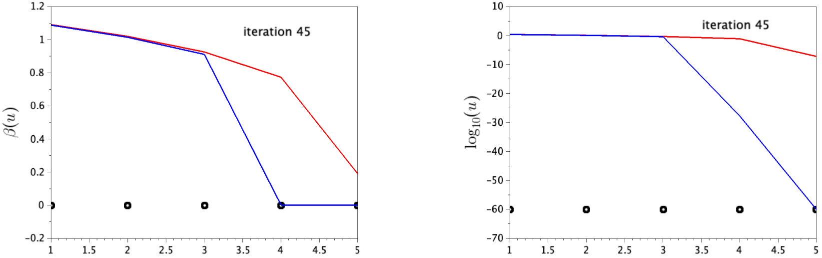

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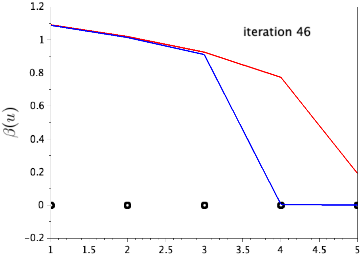

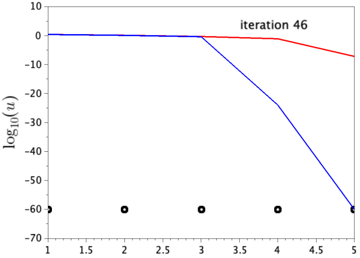

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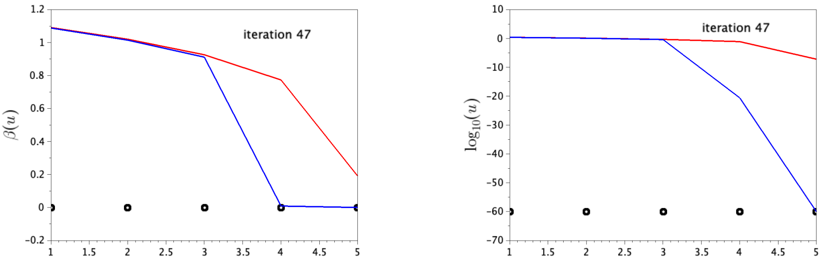

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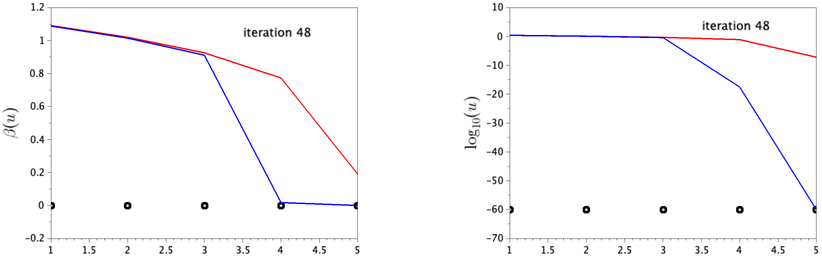

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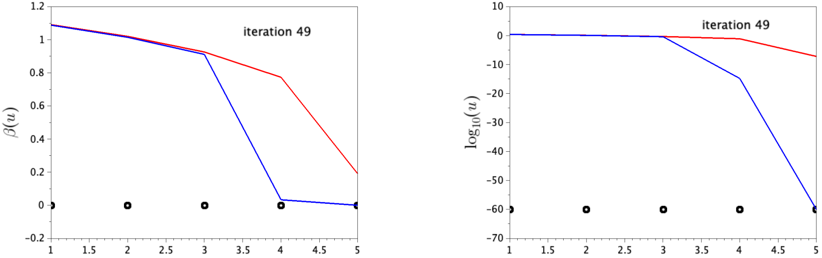

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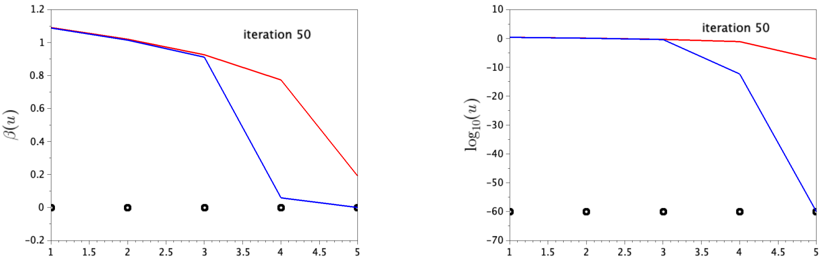

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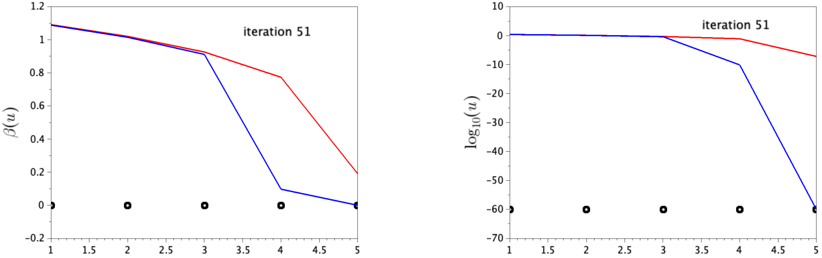

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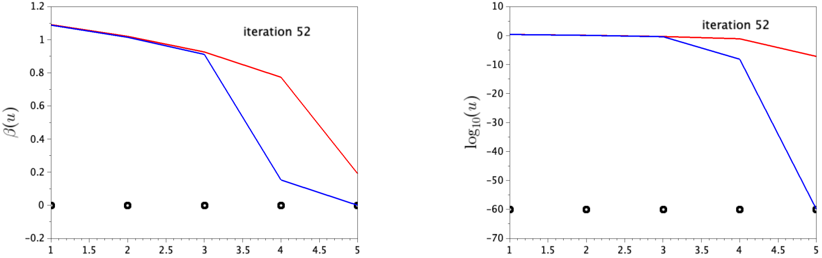

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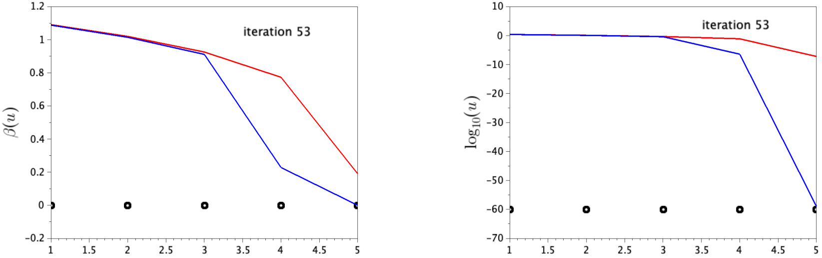

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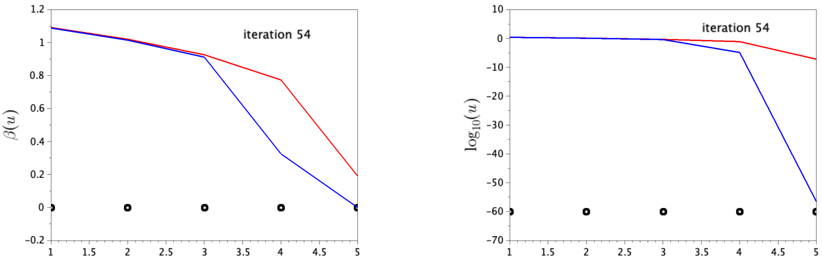

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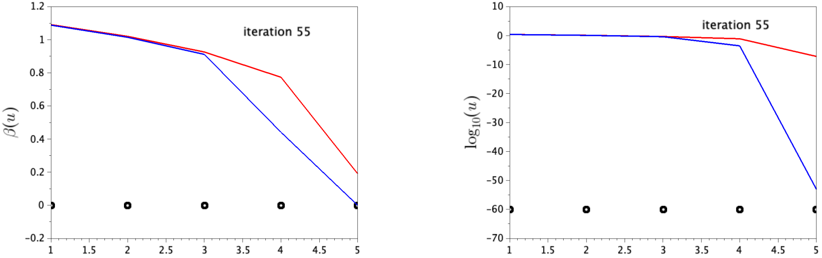

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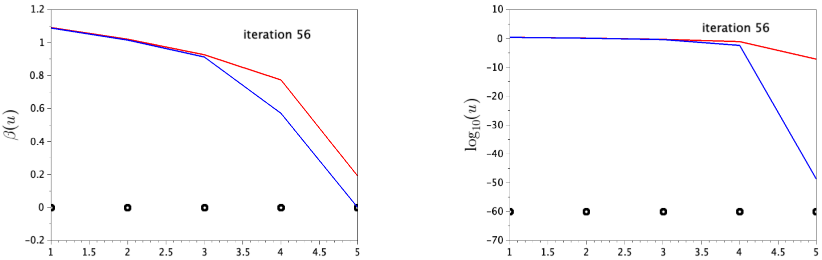

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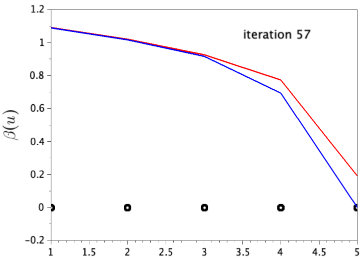

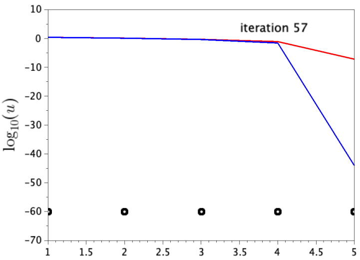

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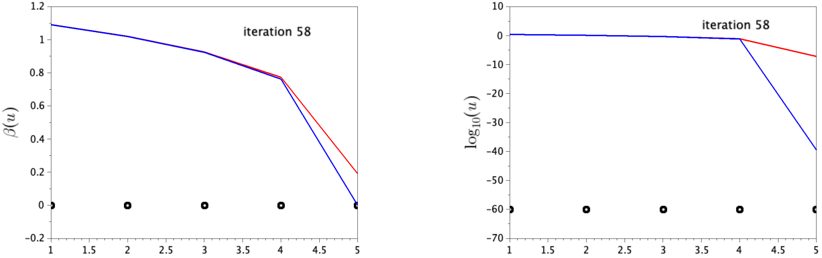

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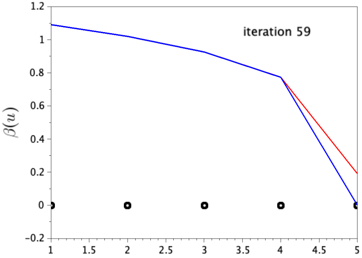

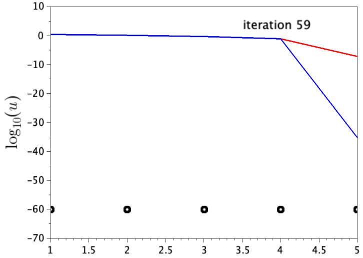

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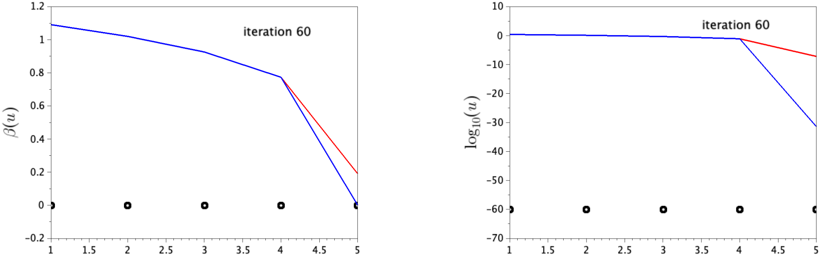

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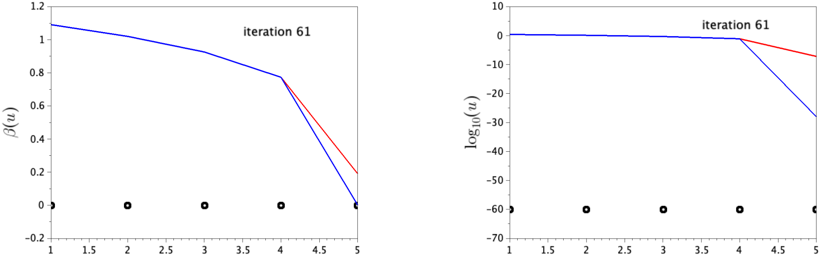

first last jacobi

- Slow convergence
- Iteration count grows with m and N

Newton's iterates over a single time step

Newton's method applied to

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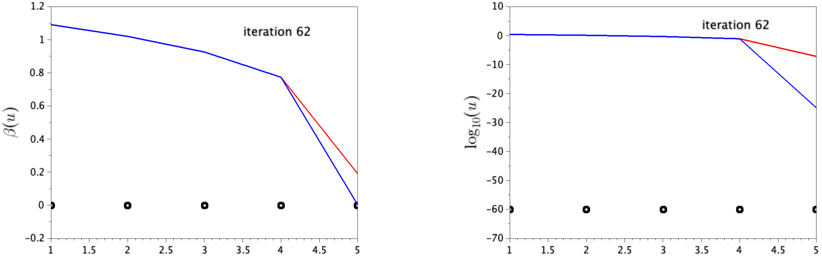

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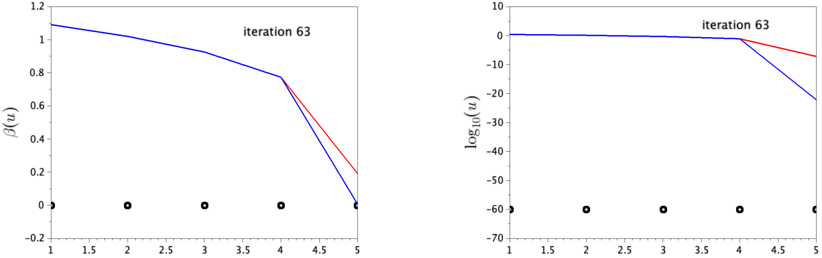

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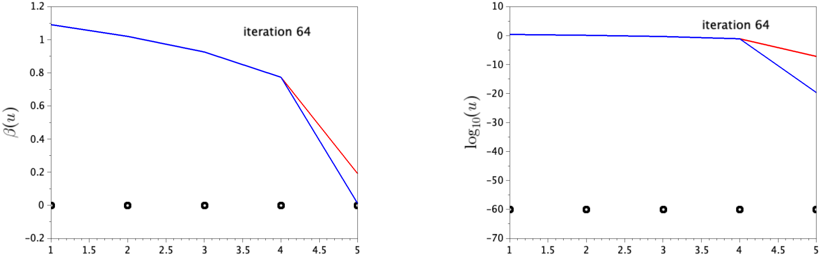

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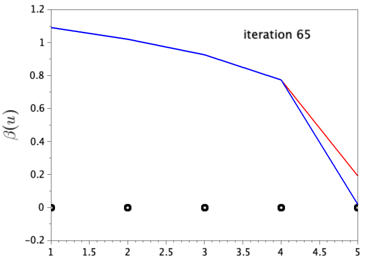

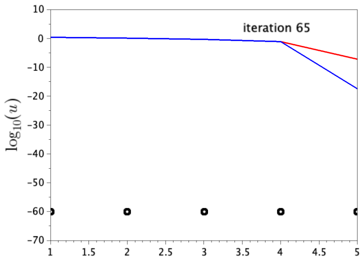

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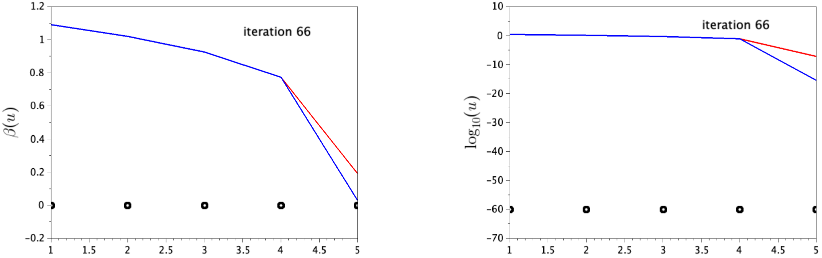

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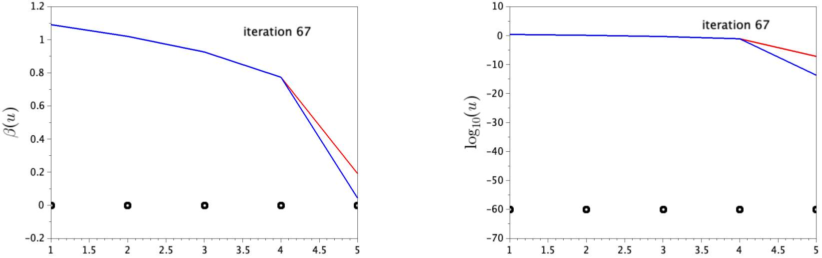

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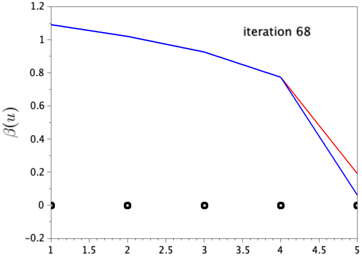

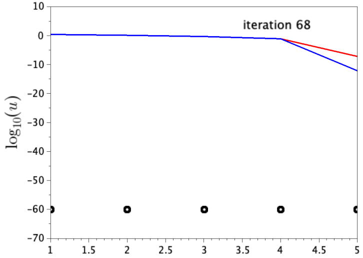

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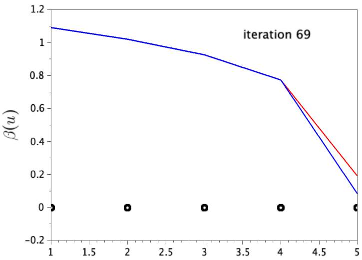

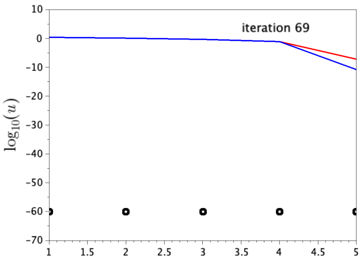

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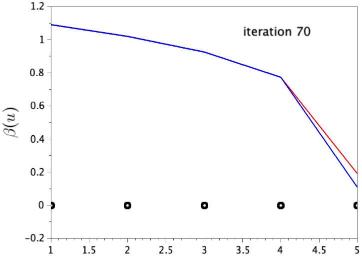

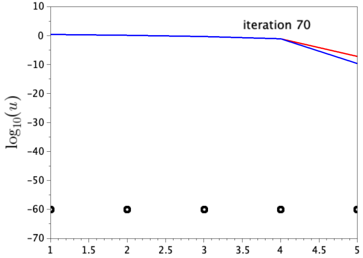

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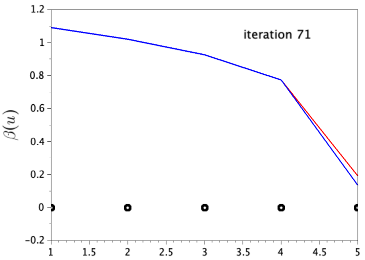

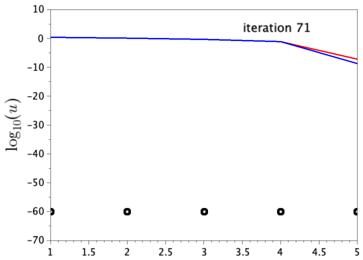

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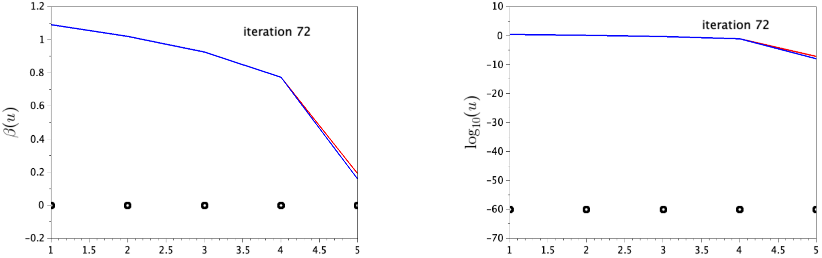

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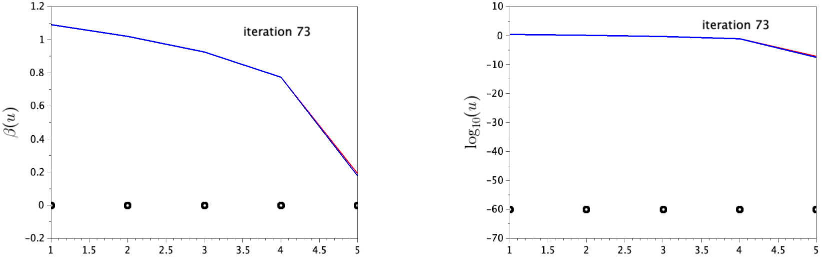

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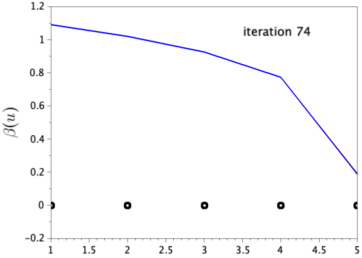

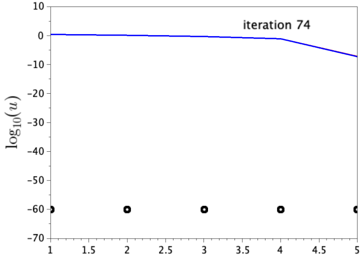

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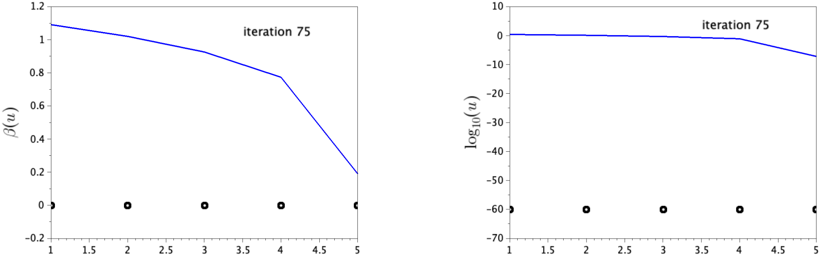

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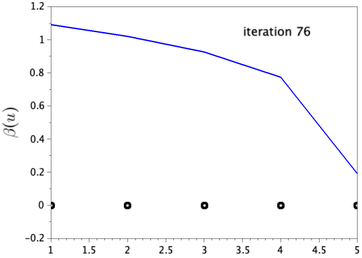

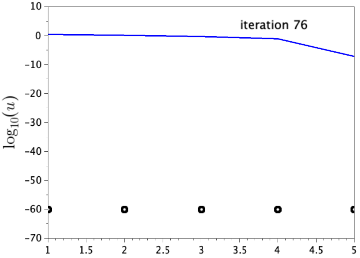

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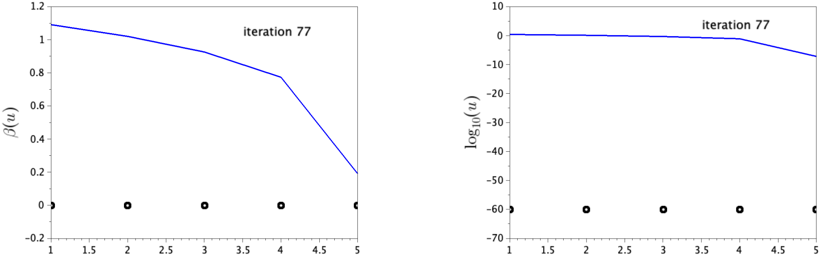

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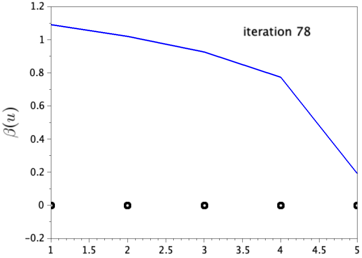

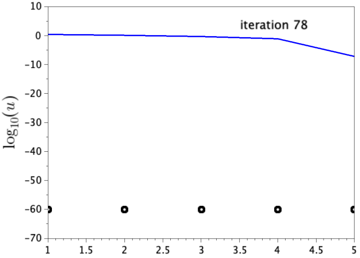

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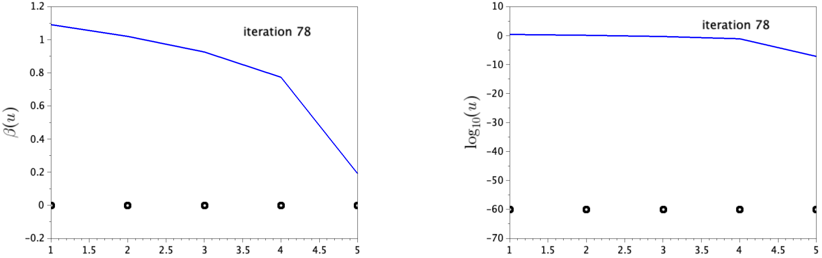

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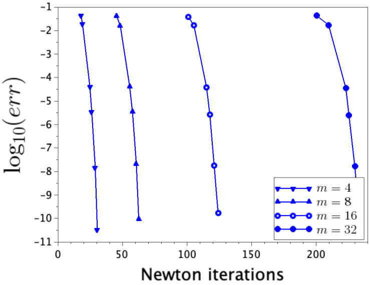
Alternative formulations

Original u-formulation:

$$\beta(\boldsymbol{u}) + A\boldsymbol{u} - \boldsymbol{b} = 0$$



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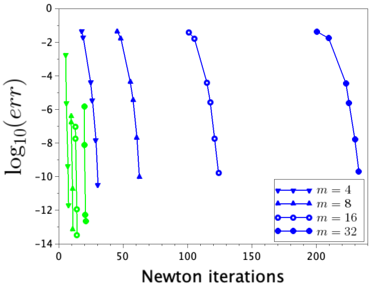

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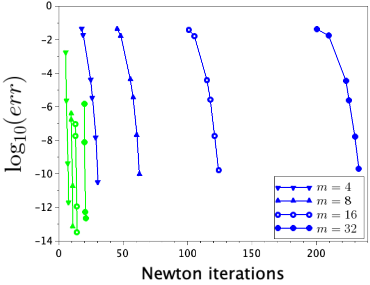

- Blue: Original formulation is inefficient, manly because $\beta'(0) = +\infty$.
- Green: Alternative formulation is more efficient, but concavity is lost: note that $(A)_{ii}(A)_{ij} \leq 0, i \neq j$

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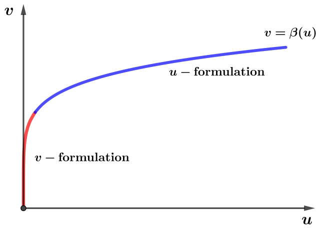

Parameterized τ-formulation:

$$\overline{v}(\boldsymbol{\tau}) + A\overline{u}(\boldsymbol{\tau}) - \boldsymbol{b} = 0$$

- Generic parameterization: $\overline{v}(\tau) = \beta(\overline{u}(\tau))$ for all τ
- Variable switching: $\max(\overline{v}'(\tau), \overline{u}'(\tau)) = 1$

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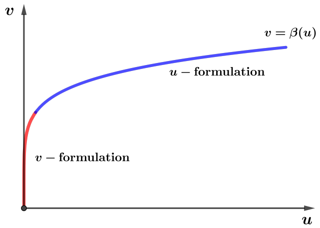

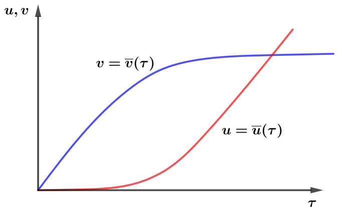

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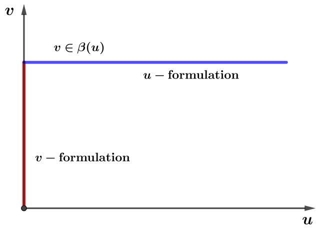

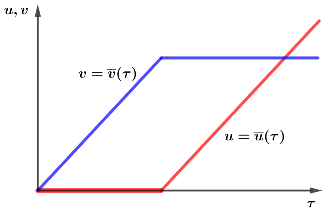

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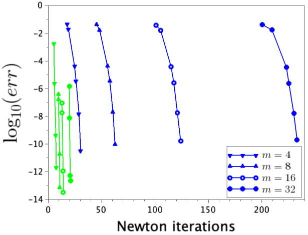

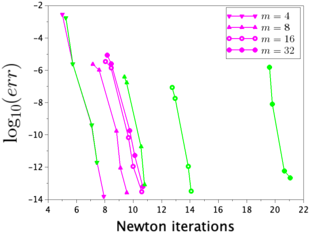

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Outline

Introduction to Richards' equation

- From saturated to unsaturated flow

Monotone Newton Theorem

- Convergence proof $\neq$ performance

Nonlinear preconditioning

- Convergence proof $+$ performance

Conclusion

Splitting methods: linear case

Model problem

$$A\boldsymbol{u} = \boldsymbol{b}$$

(Weak) regular splitting

$$A = M - N$$

such that

$$M^{-1} \geq 0, \quad N \geq 0 \quad \left(M^{-1}N,\, NM^{-1} \geq 0\right)$$

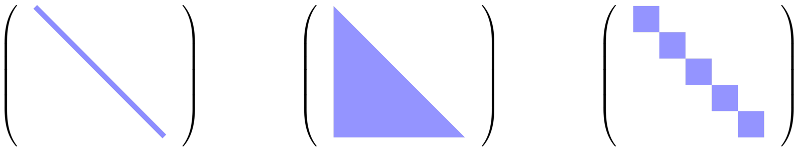
Stationary iterations

$$\boldsymbol{u}_{k+1} = M^{-1}\left(N\boldsymbol{u}_k + \boldsymbol{b}\right)$$

Preconditioned Krylov applied to

$$\left(I - M^{-1}N\right)\boldsymbol{u} = M^{-1}\boldsymbol{b}$$

Jacobi | Gauss-Seidel | Block Jacobi



Splitting methods: linear case

Model problem

$$Au = b$$

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$$M^{-1} \geq 0, \quad N \geq 0 \quad (M^{-1}N,\, NM^{-1} \geq 0)$$

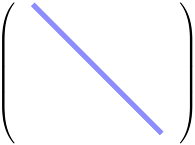
Stationary iterations

$$u_{k+1} = M^{-1}(Nu_k + b)$$

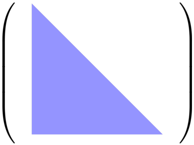
Preconditioned Krylov applied to

$$\left(I - M^{-1}N\right) u = M^{-1}b$$

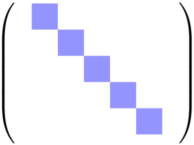
Jacobi



Gauss-Seidel



Block Jacobi



If $A = M - N$ is a weak regular splitting, then

$$\rho\left(M^{-1}N\right) < 1 \quad \Leftrightarrow \quad A^{-1} \geq 0.$$

Splitting methods: mildly nonlinear case

Model problem

$$\beta(\boldsymbol{u}) + A\boldsymbol{u} = \boldsymbol{b}, \qquad A = P - Q$$

Splitting

$$\beta(\boldsymbol{u}) + P\boldsymbol{u} = Q\boldsymbol{u} + \boldsymbol{b}$$

Splitting methods: mildly nonlinear case

Model problem

$$\beta(\boldsymbol{u}) + A\boldsymbol{u} = \boldsymbol{b}, \qquad A = P - Q$$

Splitting

$$\underbrace{\beta(\boldsymbol{u}) + P\boldsymbol{u}}_{M(u)} = \underbrace{Q\boldsymbol{u} + \boldsymbol{b}}_{N(u)}$$

Splitting methods: mildly nonlinear case

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$$\boldsymbol{u}_{k+1} = M^{-1}\left(N(\boldsymbol{u}_k)\right)$$

Newton's method applied to

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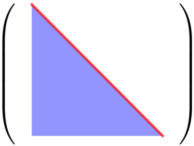

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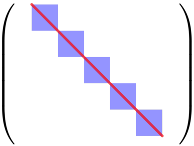

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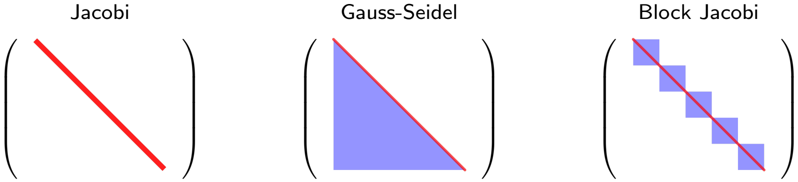


If $\beta'(\boldsymbol{u}) + A = M'(\boldsymbol{u}) - N'(u)$ is a w.r.s., then

$$\mathcal{F}(\boldsymbol{u}) = \boldsymbol{u} - M^{-1}(N(\boldsymbol{u}))$$

satisfies the MNT.

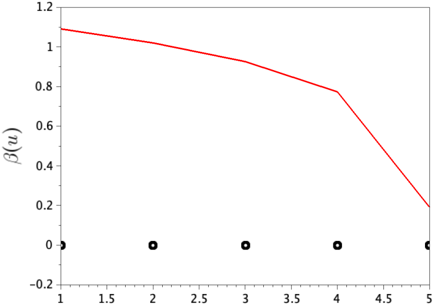
Jacobi-Newton's iterates

Preconditioned Newton's method

$$\boldsymbol{u} - M^{-1}(M(\boldsymbol{u}) - F(\boldsymbol{u})) = 0$$

with

$$M(\boldsymbol{u}) = \beta(\boldsymbol{u}) + \mathrm{diag}(A)\boldsymbol{u}$$



$\#\text{unk} = 5$

baseline

Jacobi-Newton's iterates

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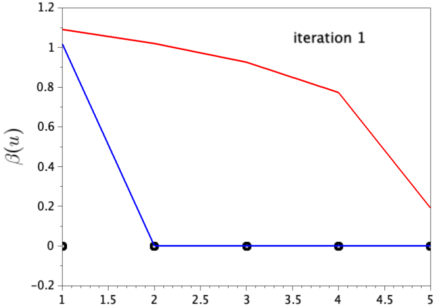

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baseline

Jacobi-Newton's iterates

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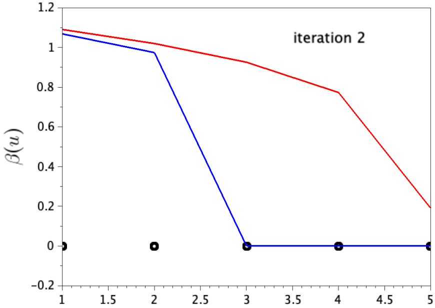

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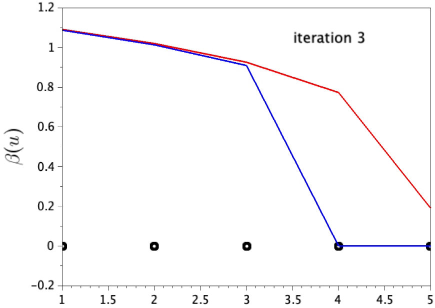

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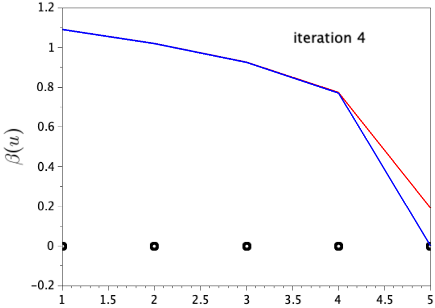

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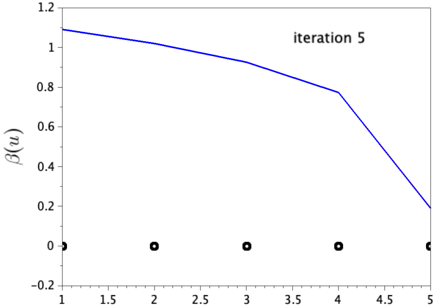

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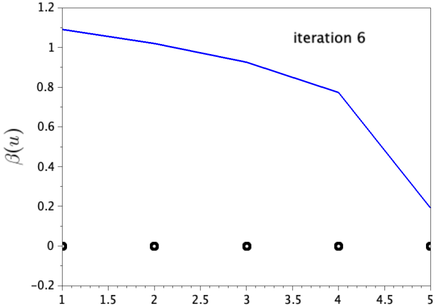

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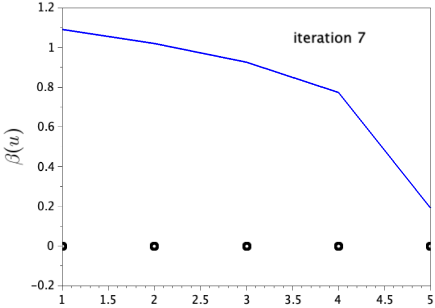

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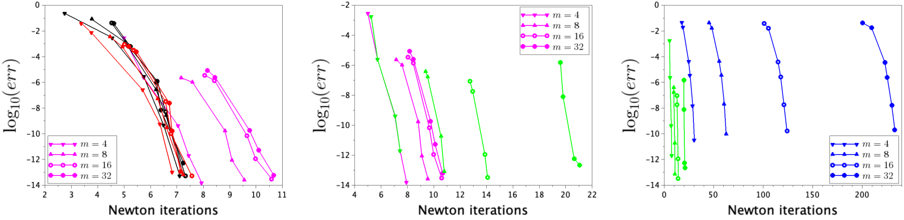
baseline

Jacobi-Newton versus variable substitution methods

Nonlinear Jacobi preconditioning

- accelerates convergence of Newton's method,
- while preserving monotone convergence.



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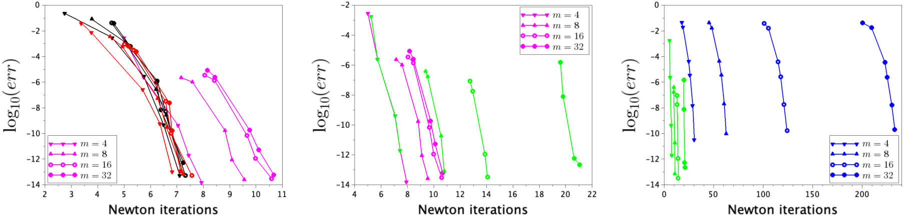

$$
\begin{aligned}
u - \text{formulation} : &\quad \beta(\boldsymbol{u}) + A\boldsymbol{u} - \boldsymbol{b} &= 0\\
v - \text{formulation} : &\quad \boldsymbol{v} + A\beta^{-1}(\boldsymbol{v}) - \boldsymbol{b} &= 0\\
\tau - \text{formulation} : &\quad \overline{v}(\boldsymbol{\tau}) + A\overline{u}(\boldsymbol{\tau}) - \boldsymbol{b} &= 0\\
\text{Left-preconditioned} : &\quad \boldsymbol{u} - M^{-1}(Q\boldsymbol{u} + \boldsymbol{b}) &= 0\\
\text{Right-preconditioned} : &\quad \boldsymbol{\xi} + QM^{-1}(\boldsymbol{\xi}) - \boldsymbol{b} &= 0
\end{aligned}
$$

Jacobi-Newton's iterates

Preconditioned Newton's method

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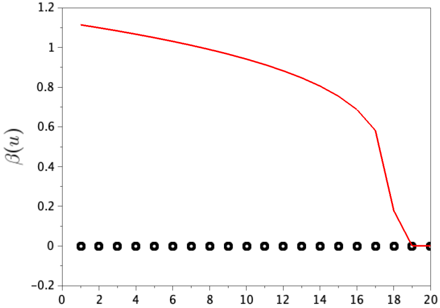

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Jacobi-Newton's iterates

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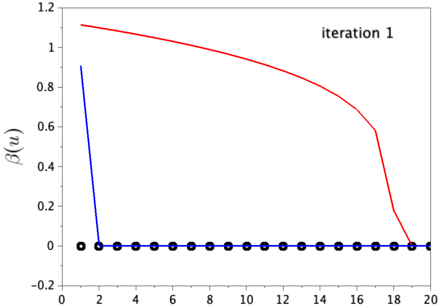

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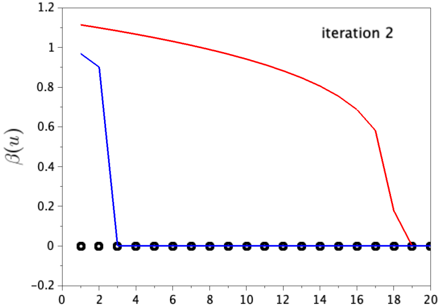

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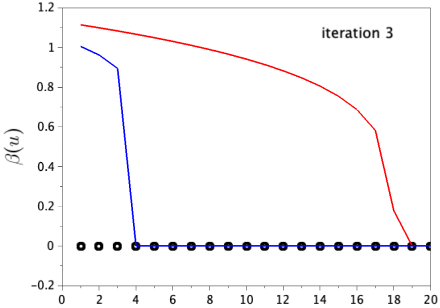

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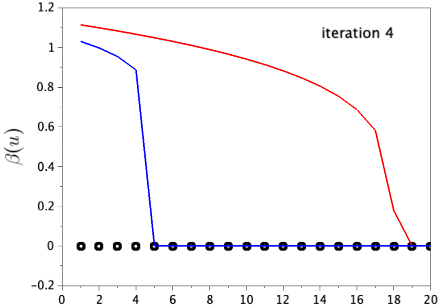

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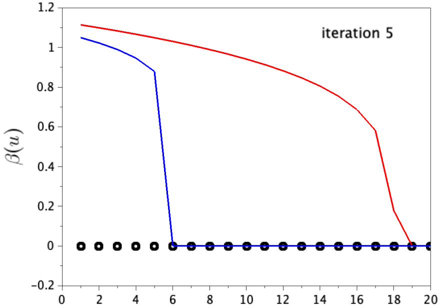

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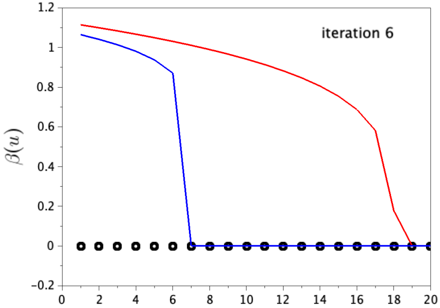

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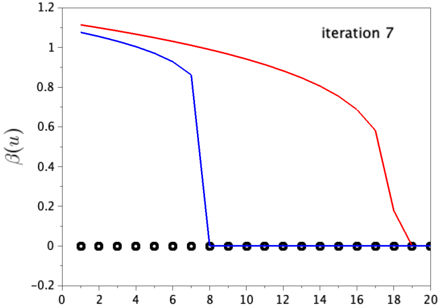

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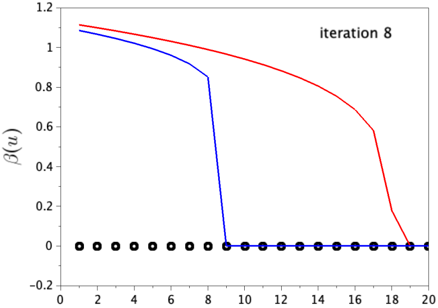

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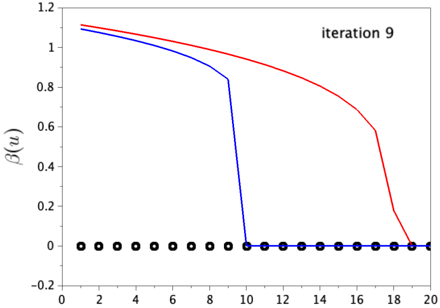

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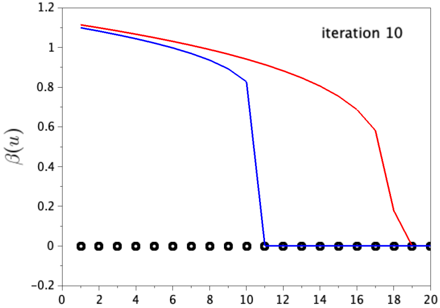

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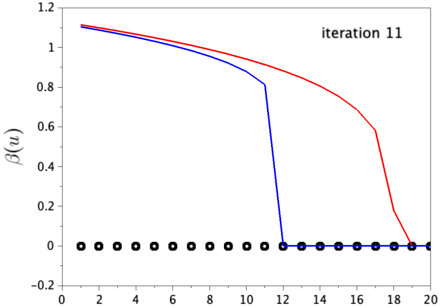

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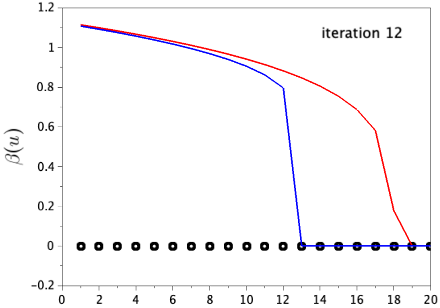

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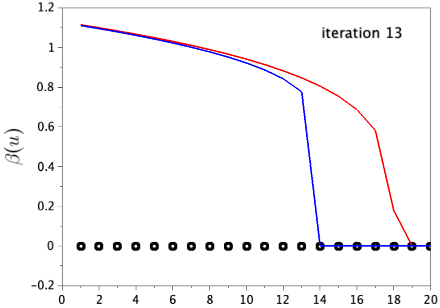

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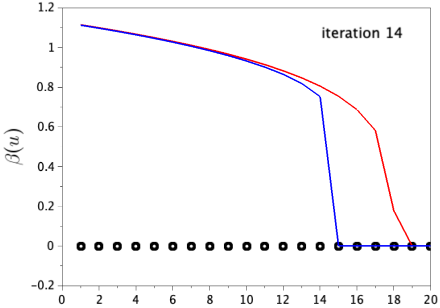

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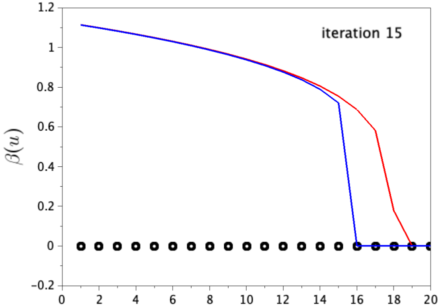

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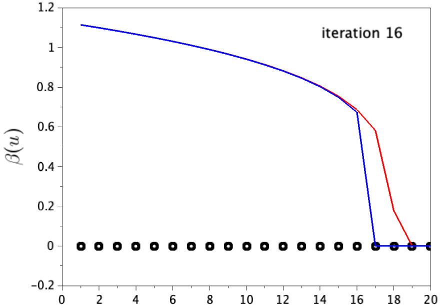

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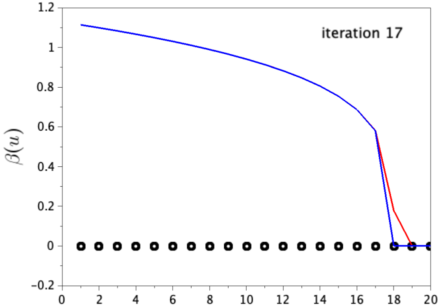

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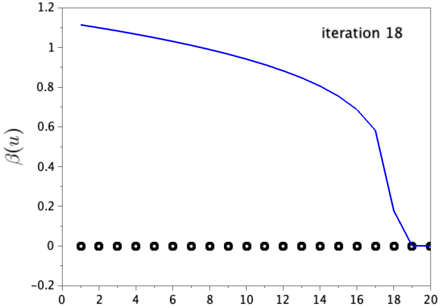

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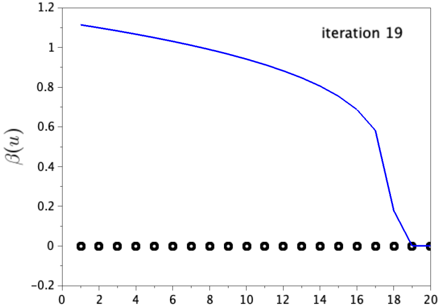

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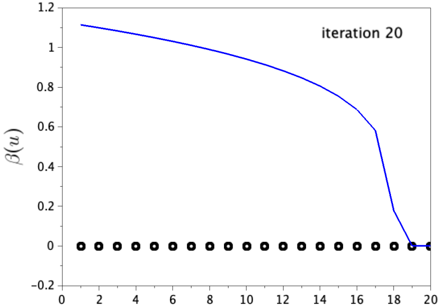

#unk = 20

baseline

Block Jacobi-Newton's iterates

Preconditioned Newton's method

$$\boldsymbol{u} - M^{-1}(M(\boldsymbol{u}) - F(\boldsymbol{u})) = 0$$

with

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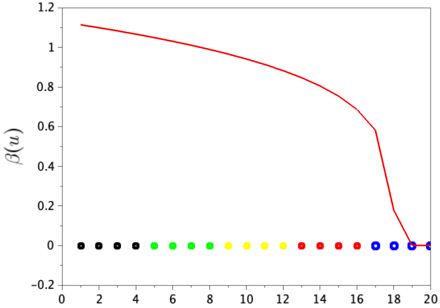

#blocks = 5, #unk = 20

baseline jacobi

Block Jacobi-Newton's iterates

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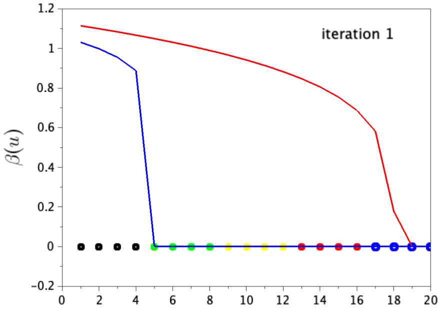

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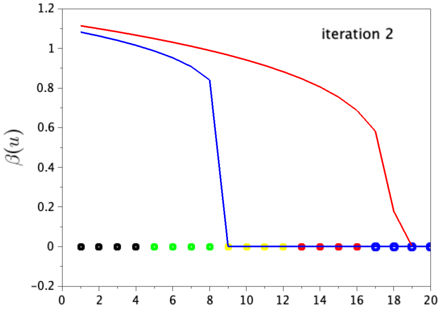

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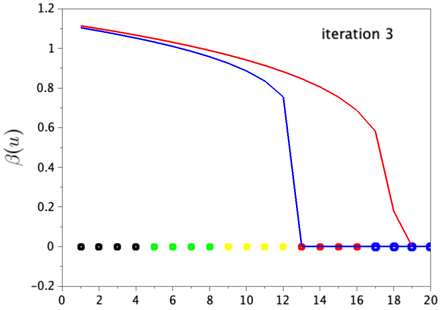

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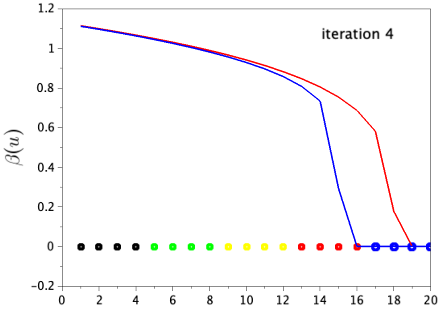

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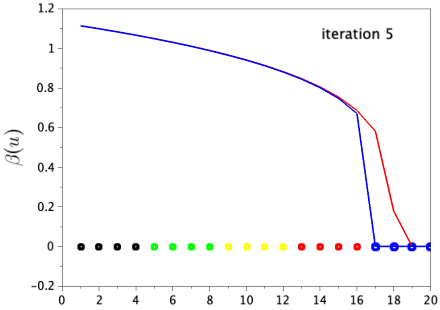

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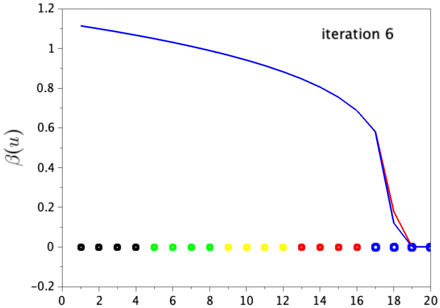

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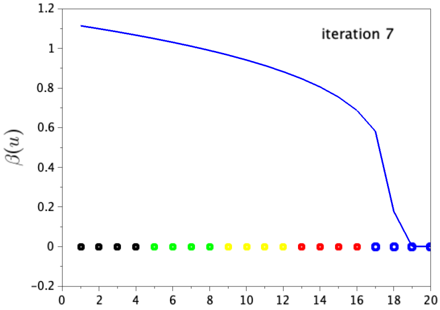

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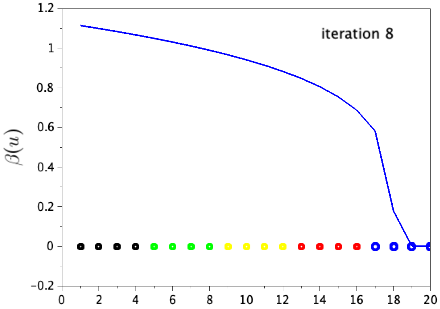

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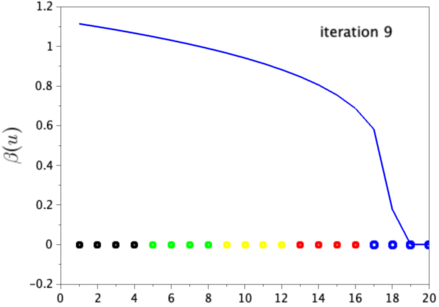

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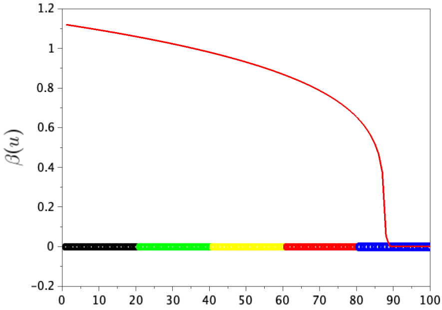

#blocks = 5, #unk = 100

baseline jacobi

Block Jacobi-Newton's iterates

Preconditioned Newton's method

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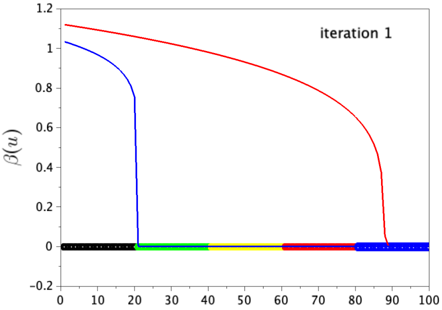

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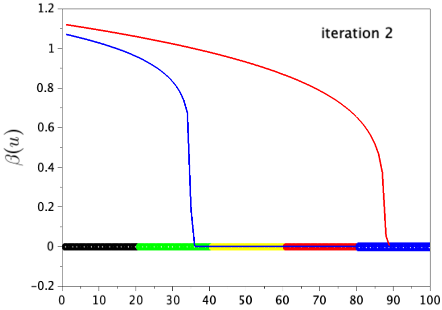

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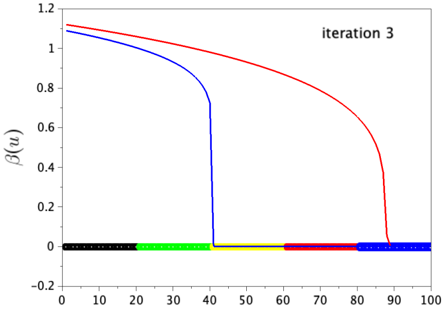

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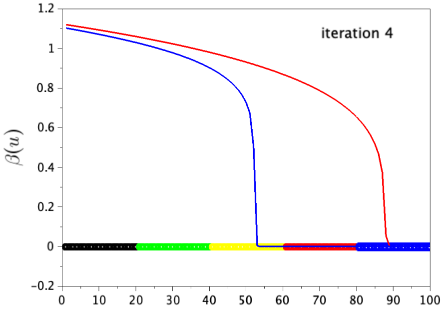

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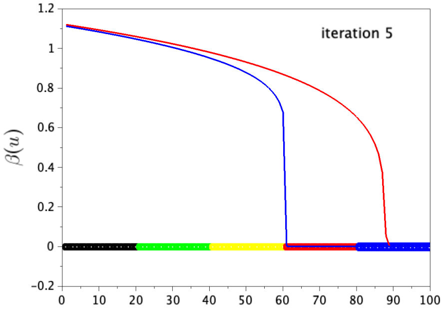

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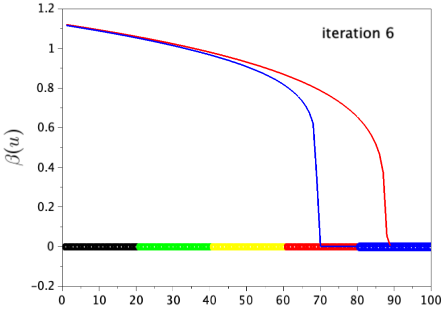

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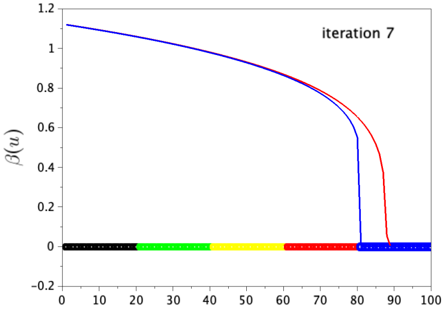

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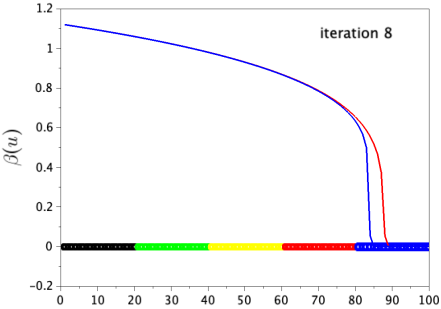

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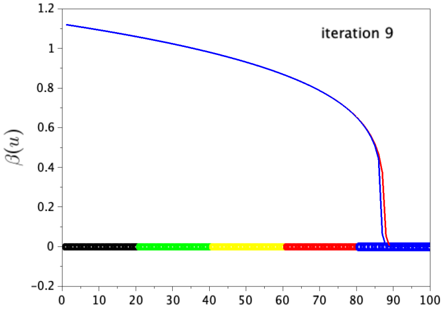

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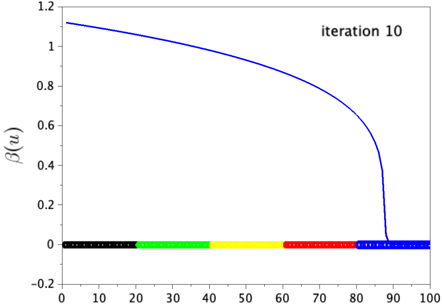

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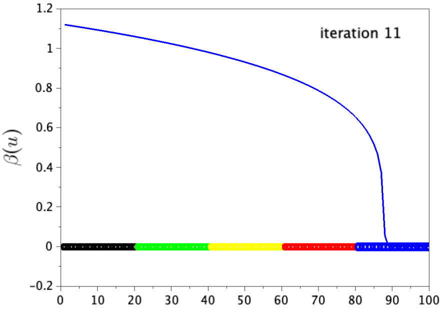

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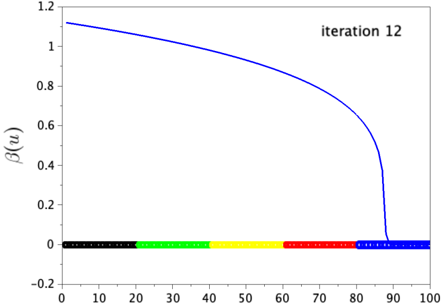

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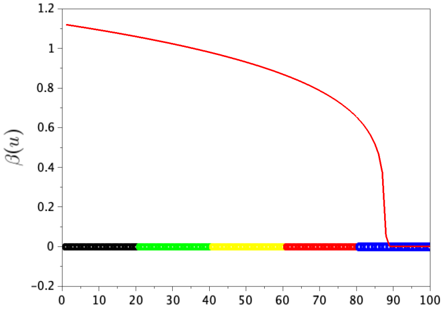

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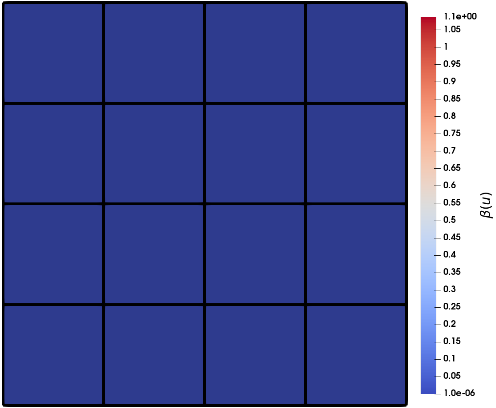
Porous medium equation in 2D case

$$\partial_t \beta(u) - \Delta u = \delta_{\mathbf{x}=0}, \qquad \beta(u) = u^{1/10}$$



	$Nt=1$			$Nt=5$			$Nt=10$		
$\sqrt{\#\text{unk}}$	20	40	80	20	40	80	20	40	80
Jac.	28	54	106	43	66	114	66	86	136
B.Jac.	10	11	13	30	33	38	50	58	62

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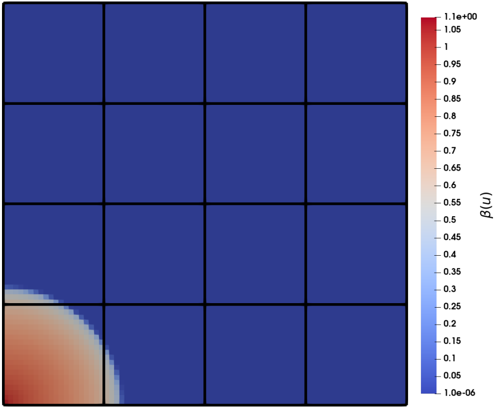

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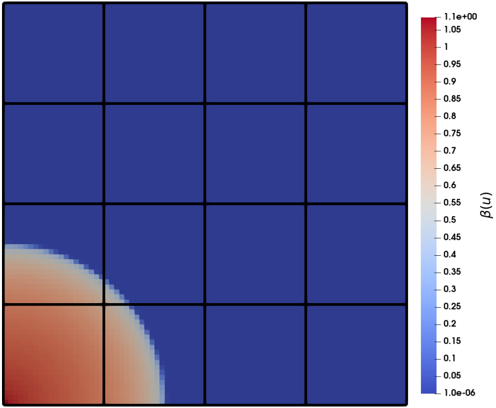

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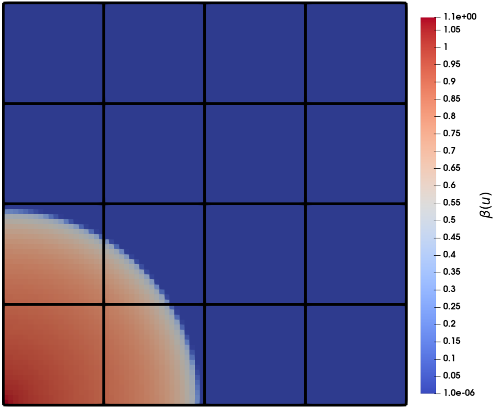

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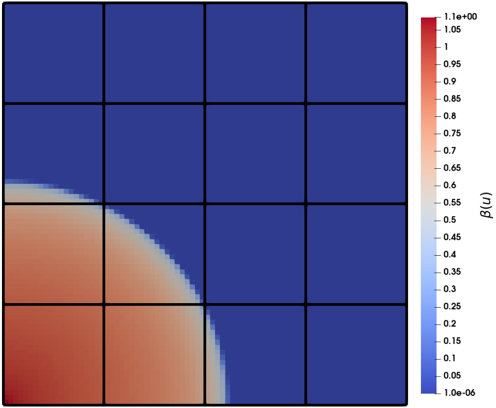

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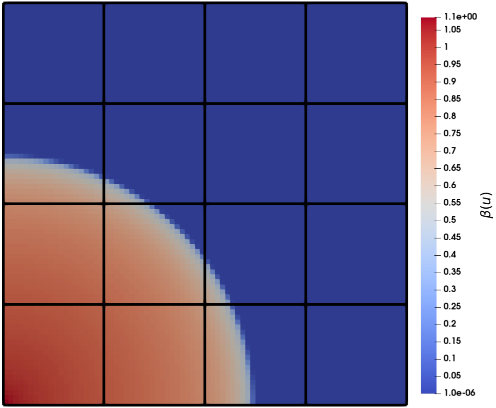

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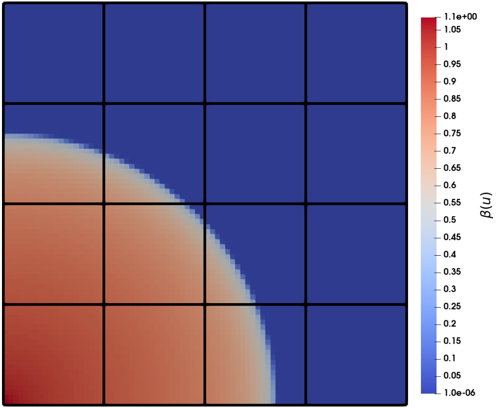

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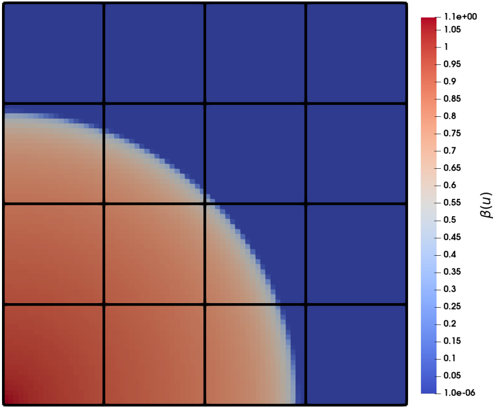

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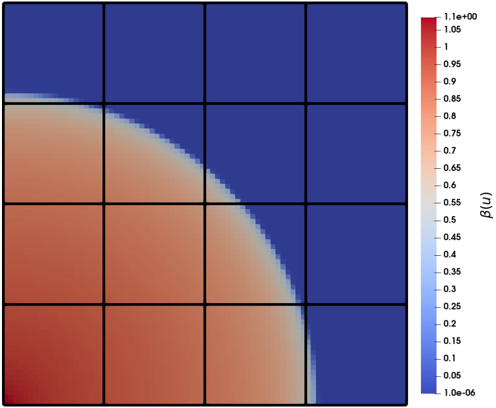

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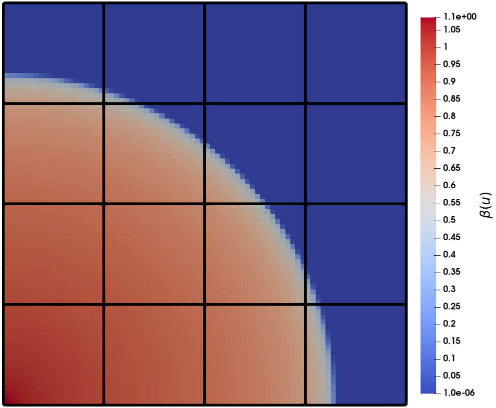

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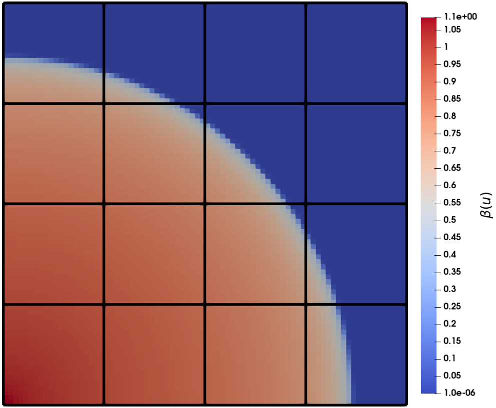

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GS-Newton's iterates

Preconditioned Newton's method

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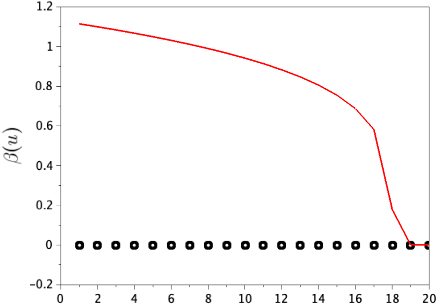

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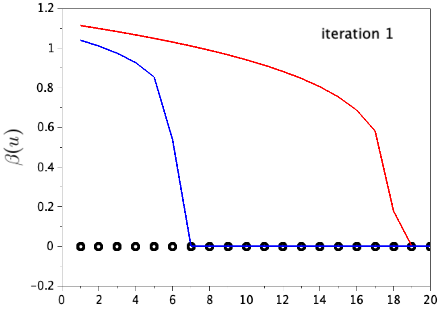

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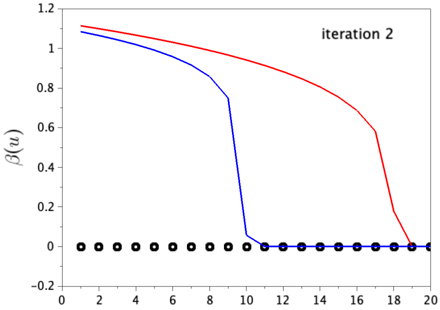

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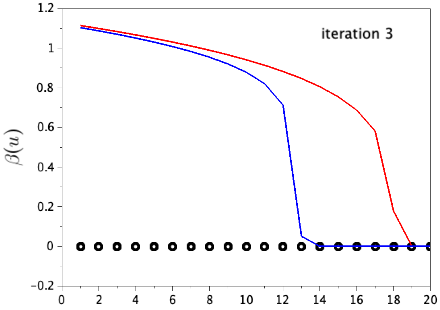

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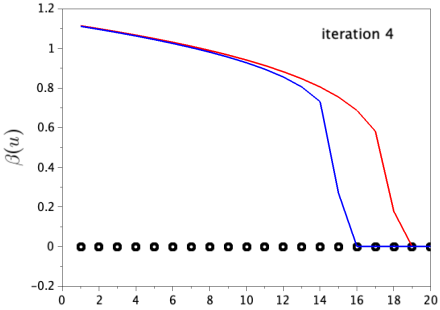

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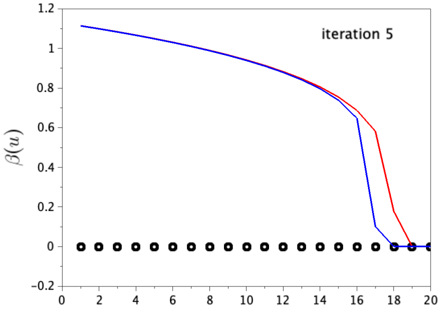

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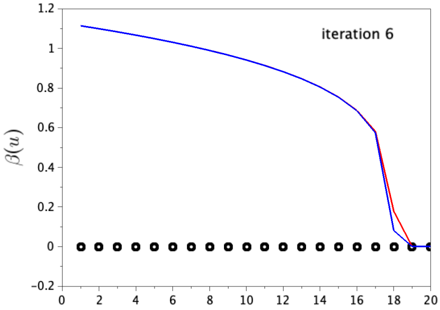

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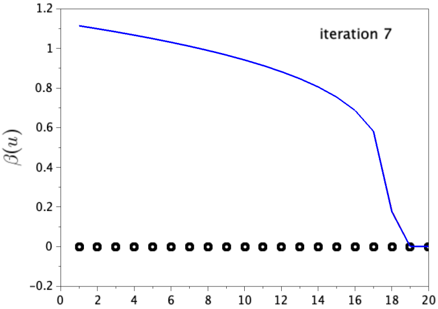

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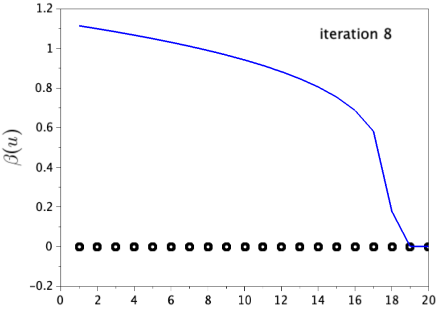

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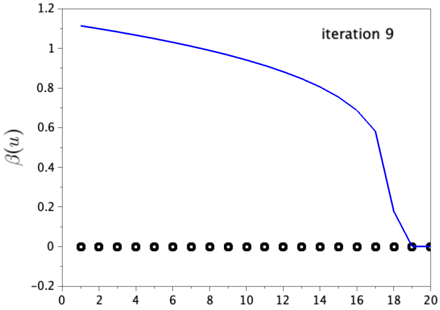

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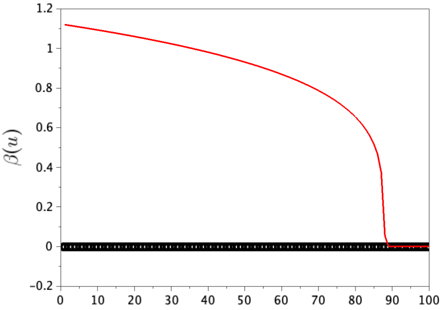

$\#$unk $= 100$

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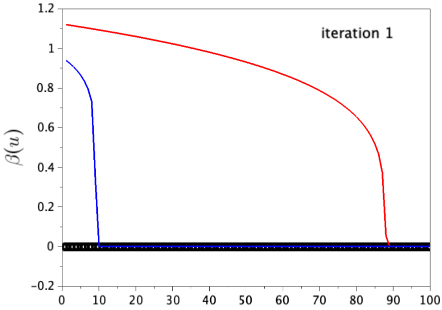

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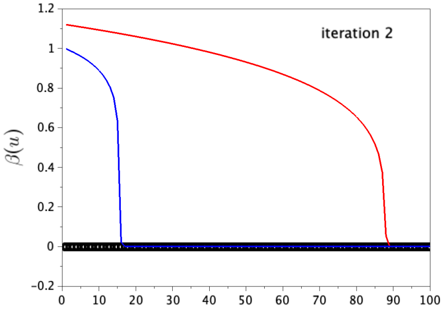

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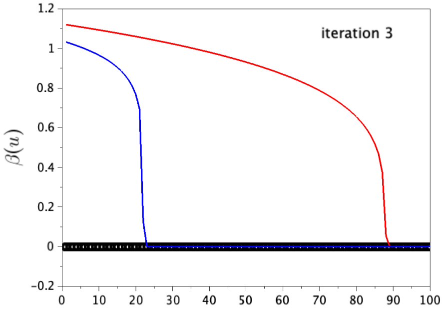

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GS-Newton's iterates

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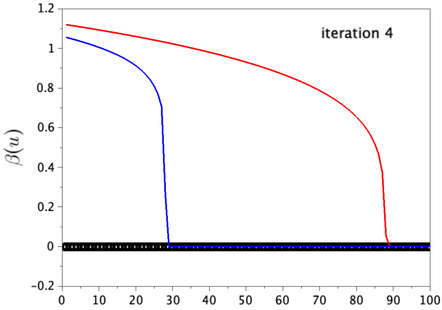

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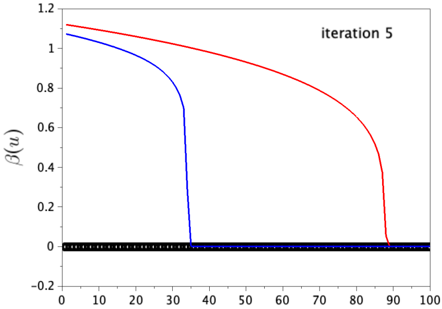

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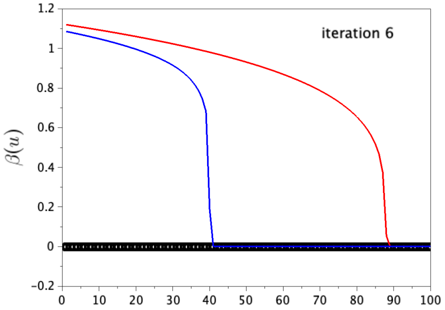

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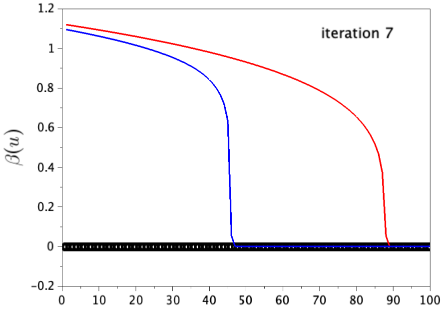

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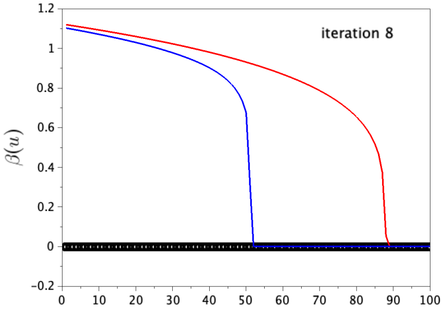

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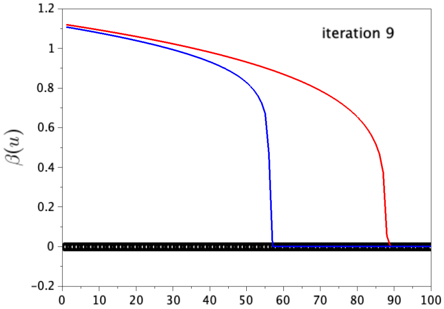

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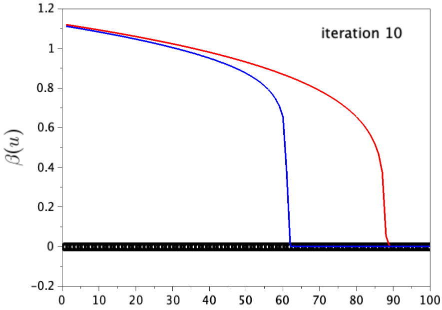

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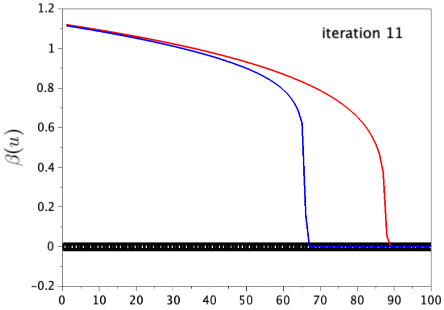

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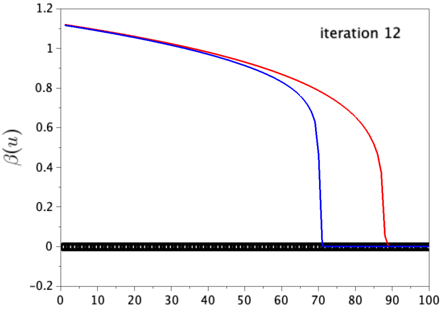

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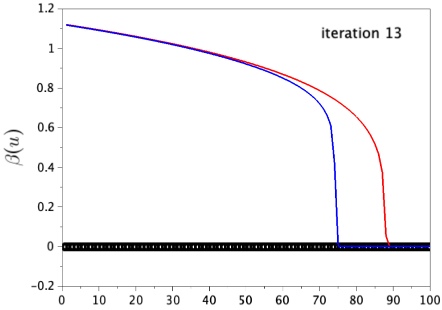

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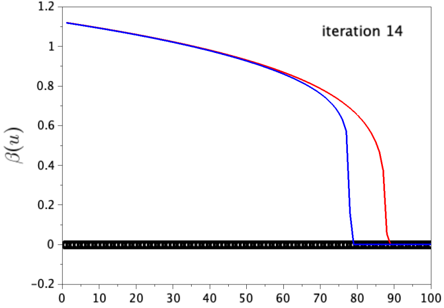

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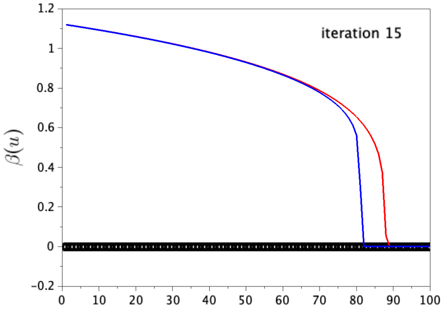

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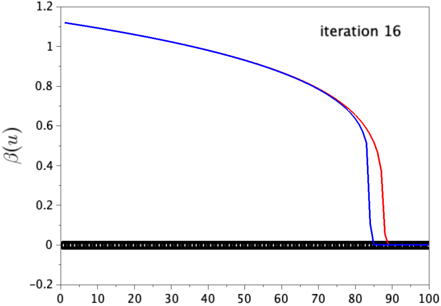

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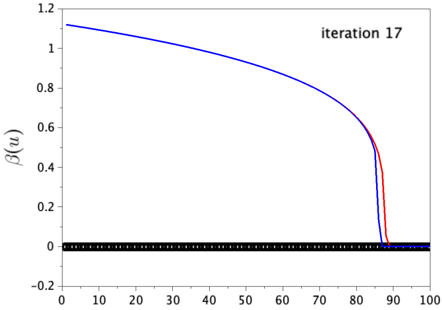

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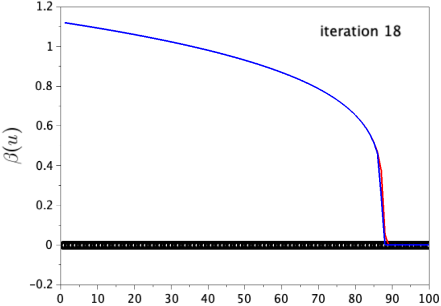

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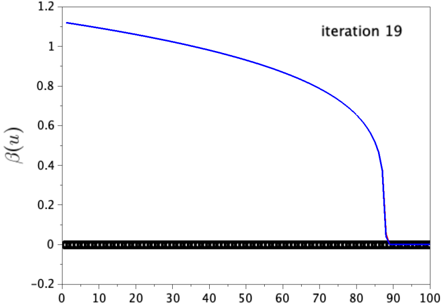

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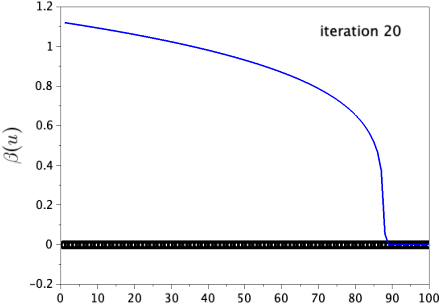

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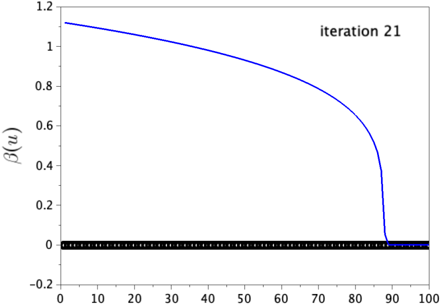

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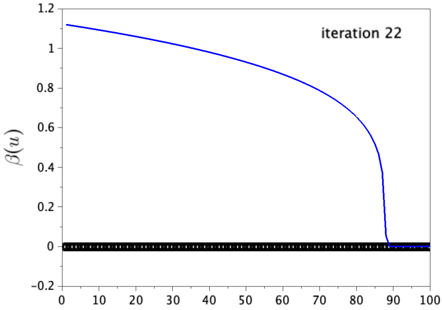

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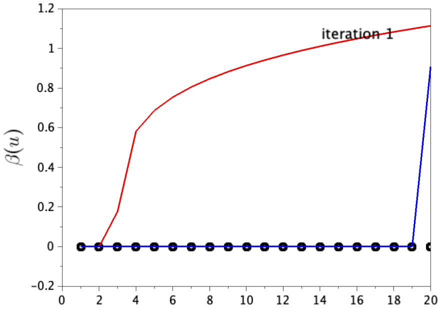

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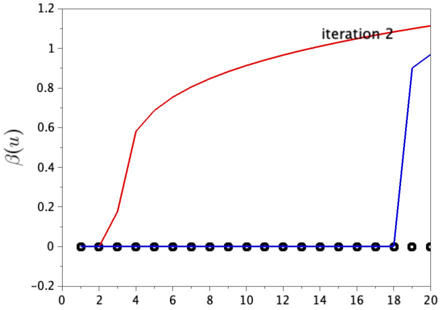

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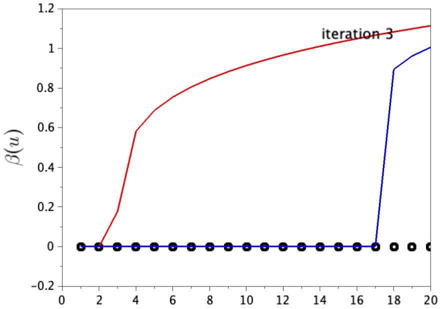

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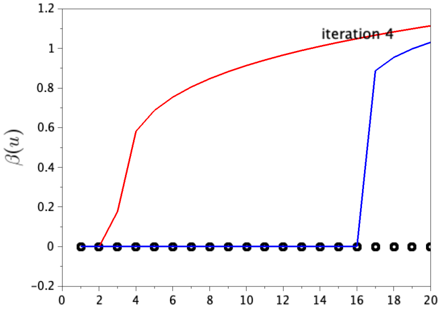

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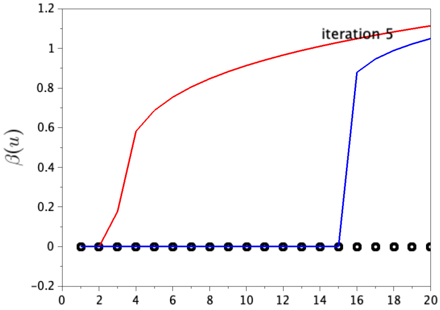

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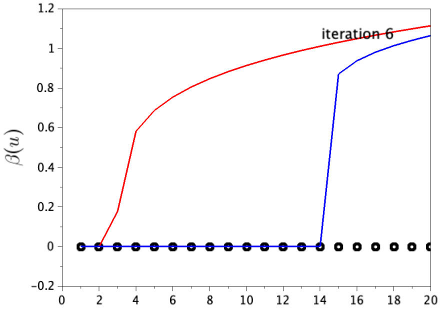

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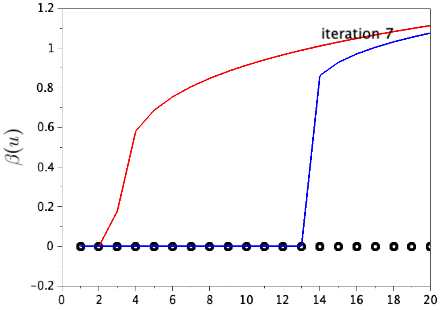

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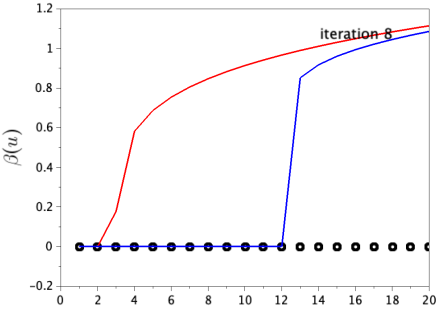

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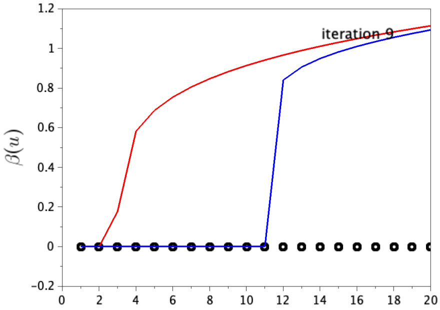

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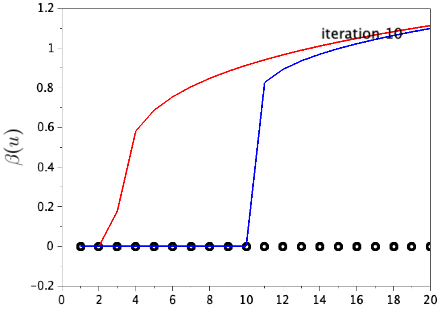

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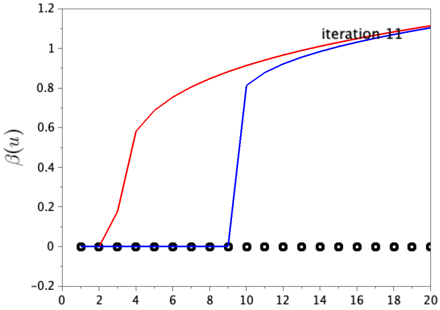

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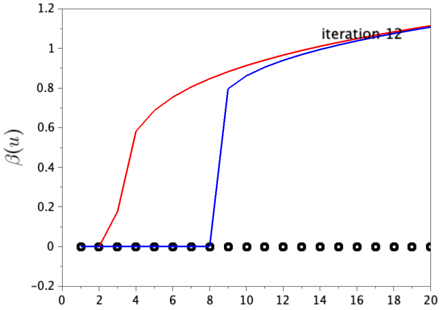

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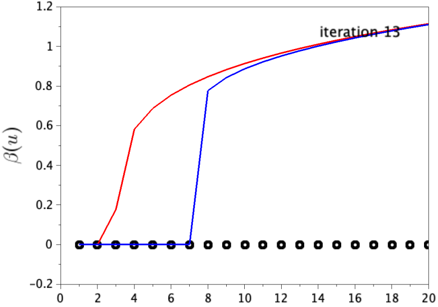

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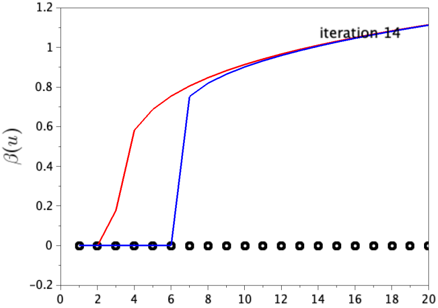

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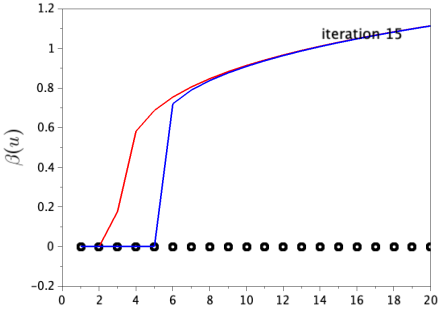

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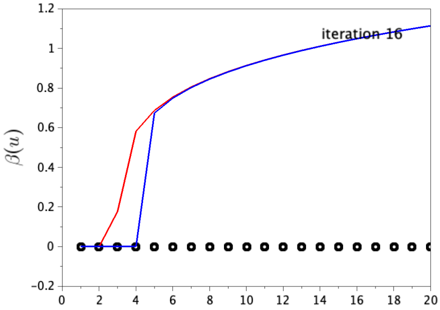

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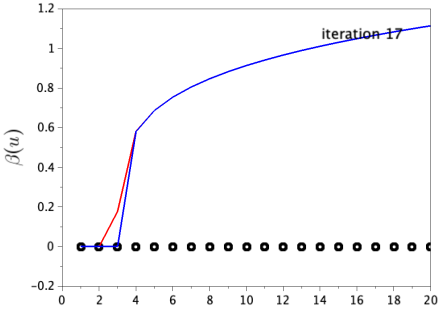

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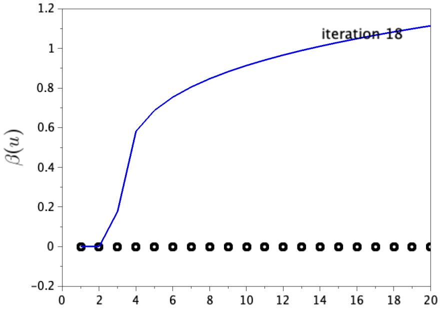

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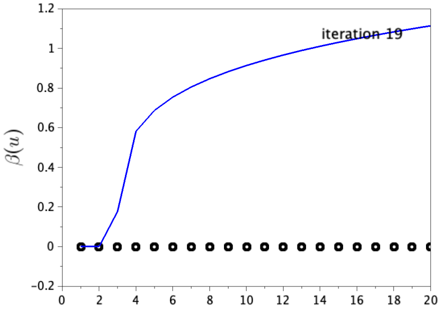

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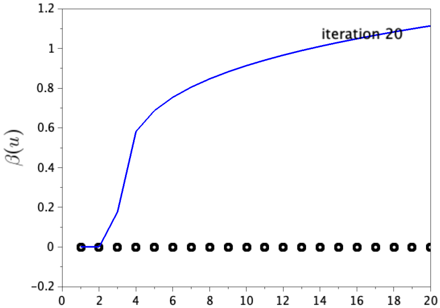

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baseline jacobi

Outline

Introduction to Richards' equation

- From saturated to unsaturated flow

Monotone Newton Theorem

- Convergence proof $\neq$ performance

Nonlinear preconditioning

- Convergence proof $+$ performance

Conclusion

Conclusions

Improved Newton's method

- Nonlinear preconditioning can greatly improve performance and robustness
- Splitting methods with provable global convergence

Perspectives & ongoing work

- Non-convex diagonal nonlinearities
- More complex systems: *true* Richards', heterogeneous problems
- Nonlinear two-level AS/RAS methods (urban flood modeling)

Nonlinear preconditioning via Domain Decomposition

Observation: Small % of dofs is often responsible for the nonlinear convergence issues

- Material interfaces
- Sharp fronts
- Small/ill-shaped elements

Opportunity: Intensity of the computations can be localized in DD approach

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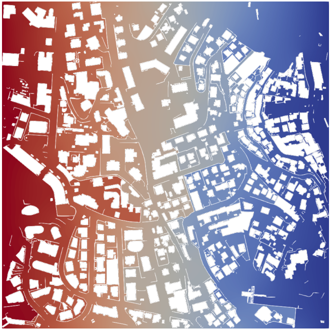

Nonlinear preconditioning via Domain Decomposition

Observation: Small % of dofs is often responsible for the nonlinear convergence issues

- Material interfaces
- Sharp fronts
- **Small/ill-shaped elements**

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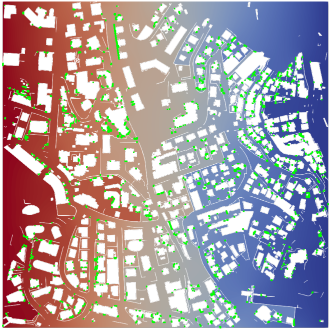

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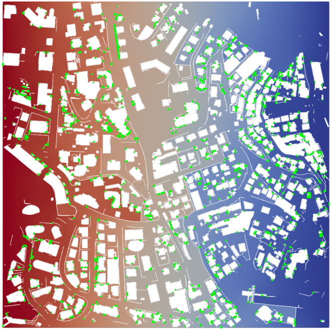
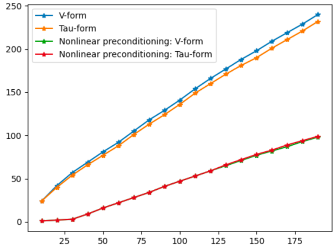


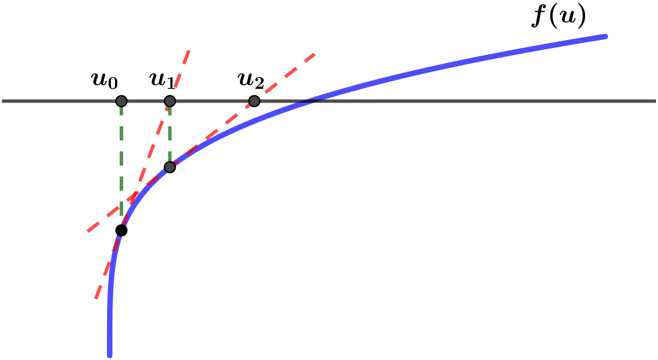


Appendix: Newton's method for a scalar concave problem

Newton's method for

$$f(u) = 0, \qquad u \in \mathbb{R}$$

- f concave and increasing



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