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Par

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RELAXATION HYPERBOLIQUE ET MODÈLES CINÉTIQUES EN DYNAMIQUE DES GAZ

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Résumé

L'objet de cette thèse est l'étude d'équations d'évolution en mécanique des fluides modélisées par des équations aux dérivées partielles. Nous nous intéressons plus spécifiquement au système d'Euler de la dynamique des gaz. Ce système est hyperbolique et constitué de lois de conservation. Nous considérons deux lois de pression : le cas d'une loi isentropique et le cas d'une pression nulle. Nous employons deux approches mathématiques pour étudier ce problème : l'utilisation de modèles cinétiques et celui de modèles continus.

Dans un premier temps, nous étudions un modèle cinétique BGK vectoriel dont la relaxation conduit au système d'Euler avec une pression isentropique. Nous obtenons en particulier l'existence globale de solutions faibles avec des relations d'entropie pour le modèle BGK, puis nous étudions la relaxation du modèle. Ce travail permet d'étendre au cas de la dynamique des gaz isentropiques le fait connu pour les lois de conservation scalaire de l'existence d'un modèle BGK conduisant à toutes les inégalités d'entropie à la limite de relaxation. Nous obtenons également des correspondances entre domaines cinétiques invariants, domaines macroscopiques invariants et entropies cinétiques pour ce modèle.

Dans un deuxième temps, nous considérons le problème mixte constitué d'une donnée initiale et d'une condition de bord pour la dynamique des gaz isentropiques. Nous nous placons dans un cadre L^∞, ce qui est une amélioration par rapport à ce qui se fait habituellement où le contexte d'un problème mixte est plutôt BV. Nous formulons la condition de bord par des inégalités d'entropies avec l'utilisation de traces faibles. Notre méthode est très générale et peut être appliquée à tout système possédant un modèle BGK compatible avec les entropies et ayant compacité forte de ses grandeurs macroscopiques.

Finalement, dans un troisième temps, nous étudions deux modèles avec contraintes. Dans le cas sans pression, nous utilisons les bouchons collants, généralisation des particules collantes, et un résultat de stabilité pour obtenir l'existence pour toute donnée initiale dans L^1 et L^∞. Une méthode numérique par splitting et utilisant un modèle BGK permet d'obtenir la dynamique que doit vérifier les bouchons. Dans le cas d'une pression isentropique d'exposant γ, nous utilisons des modèles cinétiques. La relaxation de ce modèle se fait pour $1 < \gamma < 3$ à l'aide de la compacité par compensation, et pour $\gamma = 3$, à l'aide d'un lemme de moyenne.

Mots clés

dynamique des gaz isentropiques – modèle cinétique BGK – entropie cinétique – limite de relaxation – gaz sans pression – conditions de bord – domaines invariants cinétiques – lois de conservation avec contraintes – compacité par compensation – bouchons collants – saut en temps – trace faible – produit entropique faible – méthodes numériques – lemmes de moyenne – modèles diphasiques

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Abstract

The aim of this PhD thesis is the study of evolution equations in fluid mechanics modelled by Partial Differential Equations. We are interesting more specifically with the Euler system for gas dynamics. This system is an hyperbolic one and is constituted by conservation laws. We consider two pressure laws: the isentropic case and the pressureless case. We use two mathematical approachs to study this problem : the BGK models and the continuous ones.

First, we study a vector kinetic BGK equation leading to isentropic gas dynamics in the relaxation limit. In particular, we obtain the existence of global weak solutions, for the BGK model, satisfying a kinetic entropy inequality, then we study the relaxation limit towards entropy solutions. This work extends to the isentropic case the well-known fact for scalar conservation laws which is the existence of a BGK model leading to all entropy inequalities at the relaxation limit. We also obtain correspondance between kinetic invariant domains, macroscopic invariant domains and kinetic entropies for this model.

Secondly, we consider the mixed problem constituted with an initial data and a boundary condition for the isentropic gas dynamics. We use the L^∞ context, which is an improvement with respect to usual results which takes place in a BV context. We formulate the boundary condition via entropy inequalities with the use of weak traces. Our method is very general and could be applied to any entropy satisfying BGK model as soon as we have strong compactness of the macroscopic variables inside the domain.

Finally, in a third time, we study two models with constraints. In the pressureless case, we use sticky blocks, a generalization of sticky particles, and a stability result to get the existence for any initial data in L^1 and L^∞. A numerical method with a splitting and the use of BGK model allows to obtain the dynamics the sticky blocks have to satisfy. In the isentropic case with an exponent γ, we use kinetic models. For $1 < \gamma < 3$, the relaxation of the model is obtained via compensated compactness and for $\gamma = 3$, via averaging lemma.

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Première partie

Introduction

Nous nous intéressons à des problèmes de Mécanique des fluides que nous modélisons par des Equations aux Dérivées Partielles sur les quantités qui décrivent l'évolution du fluide. En général, ces équations sont données sous la forme d'équations de conservation sur ces quantités physiques et une approche pour les étudier consiste à considérer le milieu comme étant continu et à résoudre le problème avec des estimations sur ces quantités. Une seconde approche, la théorie cinétique, revient à considérer le milieu comme étant constitué de particules décrites par une fonction de distribution. Nous nous intéresserons à ces deux approches. Les différents aspects que nous allons considérer pour ces problèmes sont liés à l'obtention de solutions faibles globales en temps, à la stabilité de ces solutions, à l'obtention d'un nombre important de relations d'entropie, à l'étude de problèmes avec condition initiale et de bord et également au cas de problèmes avec contraintes.
Cette introduction s'organise de la façon suivante : nous commencons par présenter brièvement les résultats que renferme cette thèse (Section 1), puis nous détaillons ceux-ci dans leur contexte dans les sections suivantes selon le plan suivant. Nous faisons tout d'abord quelques rappels sur les équations de la dynamique des gaz (Section 2), puis nous présentons brièvement ce qu'est un modèle de relaxation (Section 3), un modèle cinétique (Section 4) et nous étudions plus en détail le cas particulier des modèles BGK. La section 4.5 présente le modèle BGK que nous utilisons pour la dynamique des gaz isentropiques ; la section 4.6 présente les résultats obtenus relatifs à la relaxation vers la dynamique des gaz isentropiques satisfaisant toutes les inégalités d'entropie. Nous voyons ensuite qu'il est possible de prendre en compte des conditions de bord pour ces modèles (Section 5). Puis nous expliquons comment il est possible de considérer ces équations avec la prise en compte de contraintes unilatérales (Section 6). Finalement, à la Section 7, nous proposons quelques perspectives de prolongement de ces travaux.

1 L'organisation des résultats

La seconde partie de la thèse est constitué de deux articles, *Solution with finite energy to a BGK system relaxing to isentropic gas dynamics*, [13] (travail publié dans la revue Annales de la Faculté des Sciences de Toulouse), et *Kinetic invariant domains and relaxation limit from a BGK model to isentropic gas dynamics* [14] (travail accepté pour publication dans la revue Asymptotic Analysis). Le premier article traite de l'existence globale en temps de solutions faibles pour le modèle cinétique BGK vectoriel associé à la dynamique des gaz isentropiques avec existence de relations d'entropie. Le second traite de la relaxation de ce modèle vers la dynamique des gaz, après avoir prouvé l'existence de domaines invariants cinétiques, leur correspondance avec les domaines invariants macroscopiques et leur correspondance avec les entropies cinétiques. L'Annexe A est un complément de ces travaux et traite de la correspondance cinétique du système des gaz avec notre modèle BGK. L'Annexe B, quant à elle, prouve une partie du résultat de relaxation pour des solutions non bornées, sans avoir recours aux domaines invariants.
La troisième partie de la thèse est constitué de l'article, *Weak entropy boundary conditions for isentropic gas dynamics via kinetic relaxation* [15] (travail soumis à publication). Cet article a pour objet l'étude de la dynamique des gaz avec condition de bord de type Dirichlet, via des relations d'entropie sur

le bord et un modèle de relaxation cinétique. Il est à noter que l'on étudie ainsi un problème mixte avec condition de bord sans théorie BV ; pour cela on prend en compte des traces faibles.
La quatrième partie de la thèse est constitué de deux articles *Existence and weak stability for a pressureless model with unilateral constraint*, [12] (travail accepté pour publication dans la revue Mathematical Models and Methods in Applied Sciences), et *Weak solutions for a hyperbolic system with unilateral constraint and mass loss* [16]. Le premier article prouve l'existence de solutions faibles globales pour le modèle de la dynamique des gaz sans pression, avec contrainte unilatérale et multiplicateur de type pression. Ce problème est relié à un modèle diphasique. Il prend en compte des données initales qui peuvent présenter des sauts en temps, même initial. Le second article propose un autre modèle de dynamique des gaz avec contraintes et le traite dans le cas des gaz isentropiques, par compacité par compensation avec une approximation par un modèle BGK, et dans le cas des gaz sans pression par une approximation BV par les bouchons collants. Il donne une formulation faible pour la prise en compte des produits qui ne sont pas définis à priori.
Détaillons d'avantage chacun de ces résultats en les inscrivant dans leur contexte.

2 Dynamique des gaz et équations d'Euler

2.1 Les lois de conservation, le système d'Euler

Les différentes équations aux dérivées partielles que nous allons étudier sont des systèmes de loi de conservation, c'est-à-dire qu'ils s'écrivent sous la forme

$$\partial_t U + \sum_{i=1}^{N} \partial_{x_i} F_i(U) = 0, \qquad U(t,x) \in \mathbb{R}^k,\ t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{2.1}$$

où U contient les quantités physiques qui sont conservées par le modèle et où F représente le flux du système.
Les équations de la mécanique des fluides ont été introduites voilà plus de deux siècles par Euler puis par Navier ; néanmoins, beaucoup reste à faire d'un point de vue mathématique. Nous nous intéresserons au cas du système d'Euler de la dynamique des gaz en variables eulériennes. Il est donné par le système constitué de la loi de conservation de la masse et la loi de conservation de la quantité de mouvement. Il forme un système fermé d'EDP :

$$\begin{cases} \partial_t \rho + div_x(\rho u) = 0, \\ \partial_t(\rho u_i) + div_x(\rho u_i u) + \partial_{x_i} p(\rho) = 0,\ 1 \le i \le N, \end{cases}$$

où $\rho \geq 0$ est la densité du gaz, $u \in \mathbb{R}^N$ sa vitesse macroscopique et p sa pression. Ces équations expriment bien des lois de conservation d'un point de vue physique. Voyons-le par exemple pour la conservation de la masse. Pour un volume arbitraire V occupé par le fluide, la variation de masse $\int_V \rho\, dV$ est compensée par le transport de fluide à travers la surface du bord S :

$$\frac{d}{dt} \int_V \rho\, dV + \sum_i \int_S \rho l_i u_i\, dS = 0,$$

où l est la normale sortante de S. Le théorème de Stokes donne $\int_S \rho l_i u_i \, dS = \int_V \frac{\partial}{\partial x_i}(\rho u_i) \, dV$. Ainsi, la conservation de masse pour un volume arbitraire de fluide est donnée par $\partial_t \rho + \sum_i \partial_{x_i}(\rho u_i) = 0$.
Pour plus de détails sur la présentation de ces modèles et sur la dérivation des équations d'Euler et de Navier-Stokes, on peut consulter [69], [70], [9], [63], [92] et [93].
Nous allons nous intéresser à deux types de lois de pression : une pression isentropique et une pression nulle. Les deux approches décrites dans le préambule (théorie cinétique et milieu continu) nous serons alors utiles.

2.2 La dynamique des gaz isentropiques

Pour un gaz parfait, on considère la loi d'état $p = \kappa \rho^\gamma$, $\kappa > 0$, $\gamma > 1$ et le modèle isentropique des équations d'Euler correspondant s'écrit :

$$\begin{cases} \partial_t \rho + div_x(\rho u) = 0, \\ \partial_t(\rho u_i) + div_x(\rho u_i u) + \partial_{x_i}(\kappa \rho^\gamma) = 0, \quad 1 \leq i \leq N. \end{cases} \tag{2.2}$$

Ce système est strictement hyperbolique, sauf en $\rho = 0$. Pour $N = 1$, les invariants de Riemann sont donnés par

$$\omega_1 = u - \frac{2\sqrt{\gamma\kappa}}{\gamma - 1} \rho^{\frac{\gamma-1}{2}}, \qquad \omega_2 = u + \frac{2\sqrt{\gamma\kappa}}{\gamma - 1} \rho^{\frac{\gamma-1}{2}}. \tag{2.3}$$

Si les données initiales ρ_0 et u_0 sont suffisamment régulières, alors il existe un intervalle de temps maximal $[0, T_c[$ sur lequel il existe une unique solution régulière. De plus, si $T_c < \infty$, alors (ρ, u) est discontinue lorsque $t \to T_c$. Il est également connu que l'unicité et la stabilité des solutions ne sont pas valides pour des données $\rho, u \in L^\infty$. Pour les obtenir, il est nécessaire d'imposer des conditions d'entropie de Lax. Pour plus de précisions sur ces propriétés, on peut se référer à [65] et [66].

2.3 Les gaz sans pression

Lorsque l'on considère une pression nulle dans les équations d'Euler, le modèle obtenu décrit un plasma froid, ou bien la dynamique des galaxies [105] et s'écrit

$$\begin{cases} \partial_t \rho + div_x(\rho u) = 0, \\ \partial_t(\rho u_i) + div_x(\rho u_i u) = 0, \quad 1 \leq i \leq N. \end{cases}$$

Ce système est faiblement hyperbolique. Il est connu ([21], [31], [25]) que l'on ne peut pas considérer ce problème dans L^1 mais qu'il est nécessaire de considérer des solutions à valeurs mesures. En dimension un, l'existence de solutions globales à valeurs mesures pour des données initiales mesures a été prouvée dans [55] et [48] à l'aide de la notion de particules collantes. Brenier et Grenier [31] ont prouvé que ce modèle est équivalent aux lois de conservation scalaire avec données initiales monotones. Pour le cas multidimensionnel, on consultera également [90]. On peut se référer également aux travaux de [26] et [27] pour d'autres résultats sur les gaz sans pression.

2.4 Les solutions entropiques

Dans les systèmes précédents, le problème de Cauchy est mal posé : on a l'existence de solutions mais pas leur unicité. Il est nécessaire de faire appel à des critères d'origine physique pour avoir unicité, tout en choisissant la "bonne" solution faible. Il s'agit du concept d'entropie introduit dans [49].
Multipliant (2.1) par $\eta'(U)$, on peut écrire l'équation obtenue sous forme conservative s'il existe des fonctions G_j telles que $\eta'(U)F_j'(U) = G_j'(U)$. On appelle entropie pour le système (2.1) toute fonction convexe pour laquelle il existe de telles fonctions G_j, appelées flux d'entropie associées. Dans le cas d'une loi de conservation scalaire, toute fonction convexe convient. Pour des solutions non régulières, on ne peut faire le calcul formel précédent et obtenir une équation de conservation ; cependant la méthode de viscosité dans le cas d'une équation scalaire permet à la limite du terme visqueux d'obtenir une inégalité. On demande alors à une solution de vérifier les inégalités dites d'entropie, à savoir :

$$\partial_t \eta(U) + \sum_{i=1}^{N} \partial_{x_i} G_i(U) \leq 0, \tag{2.4}$$

pour tout couple (η, G) d'entropie - flux d'entropie.
Kružkov [61] utilisa les fonctions $\eta(U) = |U - k|$ avec $k \in \mathbb{R}$ pour obtenir l'unicité pour une loi de conservation scalaire.
Dans le cas de la dynamique des gaz isentropiques, pour $N \geq 2$, les entropies sont

$$\eta(\rho, u) = c_0(\frac{1}{2}\rho\|u\|^2 + \frac{\kappa}{\gamma - 1}\rho^\gamma) + \sum_{i=1}^{N} c_i \rho u_i + b\rho + d.$$

Une des raisons de la très faible compréhension des équations d'Euler pour $N \geq 2$ provient du fait qu'il n'y a pas beaucoup d'entropies.
Au contraire, pour $N = 1$, un grand nombre d'entropies existent et le point de vue cinétique est très adapté pour leur utilisation (voir la Section 4.5). L'existence de solutions entropiques dans L^∞ en dimension un, avec des données initiales dans L^∞, a été obtenu par DiPerna [43] pour $\gamma = (2k+3)/(2k+1)$ avec $k \in \mathbb{N}^*$, puis par Chen [34] pour $1 < \gamma \leq 5/3$, par Lions, Perthame et Tadmor [74] pour $\gamma > 3$ et finalement par Lions, Perthame et Souganidis [72] pour $1 < \gamma < 3$. De plus, ce dernier article prouve un résultat de stabilité, en dimension un, que l'on peut énoncer de la façon suivante :

Theorem 2.1 (Lions-Perthame-Souganidis) *Soit $((\rho_n, u_n))_{n\geq 0}$ une famille, bornée dans $L^\infty(\mathbb{R}\times(0,T))$, de solutions de (2.2) avec $N = 1$ telle que $\rho_n \geq 0$, $\sup_t \int_{\mathbb{R}} \eta(\rho_n, u_n)\, dx < \infty$ et*

$$\partial_t[\eta_S(\rho_n, u_n)] + \partial_x[G_S(\rho_n, u_n)] \leq R_{S,n}(t,x) \underset{n\to+\infty}{\to} 0 \text{ dans } W^{-1,p}_{loc}, \ \forall 1 < p < \infty. \tag{2.5}$$

Alors, à extraction près d'une sous-suite, (ρ_n, u_n) converge p.p. dans $\mathbb{R}\times(0,T)\cap\{\rho > 0\}$ vers une solution entropique (ρ, u) de (2.2) et $\rho_n u_n$ converge p.p. dans $\mathbb{R}\times(0,T)$ vers ρu.

Dans nos diverses approches de la dynamique des gaz, nous nous placerons toujours dans un contexte soigneusement choisi de manière à obtenir toutes les inégalités d'entropie et nous étudierons le cas $N = 1$.

Pour le système des gaz sans pression, les inégalités d'entropie classiquement utilisées sont
$$\partial_t(\rho S(u)) + \partial_x(\rho u S(u)) \leq 0,$$
pour toute fonction convexe S. En fait, cette notion s'avère nécessaire mais insuffisante pour obtenir l'unicité des solutions. A l'aide de la notion de solution en dualité et de la condition d'Oleinik

$$\partial_x u(t,x) \leq \frac{1}{t}, \qquad \text{p.p. } t,x, \tag{2.6}$$

Bouchut et James [25] ont démontré un résultat d'existence et d'unicité pour le système des gaz sans pression en dimension un.

3 Modèles de relaxation

Nous donnons ici une brève présentation des problèmes de relaxation dans le cadre hyperbolique afin de voir les avantages que présentent une telle approche et les difficultés techniques qu'il est nécessaire d'affronter. Ce type de problèmes intervient dans de nombreuses situations physiques : théorie cinétique, visco-élasticité, chromatographie, transition de phases, écoulements de rivières, de traffics routiers, thermodynamique, ..., c'est-à-dire dès qu'un état d'équilibre stable d'un système physique est perturbé. Ces systèmes sont de la forme :

$$\partial_t U + \sum_{i=1}^{N} A_i(U)\partial_{x_i} U = \frac{1}{\varepsilon} Q(U),\ x \in \mathbb{R}^N,\ U(t,x) \in \mathbb{R}^k. \tag{3.1}$$

Le temps de relaxation ε représente par exemple le libre parcours moyen en théorie cinétique, la durée de la mémoire en élasticité,... et est, en général, petit dans beaucoup de situations physiques. Pour être consistant avec les exemples linéaires physiques bien posés, on suppose généralement que le système est hyperbolique, c'est-à-dire que (3.1) peut s'écrire sous la forme :

$$\partial_t U + \nabla \cdot F(U) = \frac{1}{\varepsilon} Q(U),$$

où le flux $F(U)$ est tel que $\forall y \in \mathbb{R}^k$, $\partial_U F(U) \cdot y$ a des valeurs propres réelles et est diagonalisable.
Une illustration du principe de la méthode de relaxation peut se voir sur le problème de Cauchy pour le système de lois de conservation en dimension un :

$$\partial_t U + \partial_x F(U) = 0, \qquad U(x,0) = U_0(x), \tag{3.2}$$

pour $t \geq 0$, $x \in \mathbb{R}$, $U, F(U) \in \mathbb{R}^k$. On introduit le système de relaxation associé suivant :
$$\left\{ \begin{array}{rcl} \partial_t U + \partial_x V &=& 0, \\ \partial_t V + A\partial_x U &=& \frac{F(U)-V}{\varepsilon}, \end{array} \right.$$
$$U(x,0) = U_0(x), \qquad V(x,0) = F(U_0(x)),$$

avec $V \in \mathbb{R}^n$, $A = diag(a_1, ..., a_n)$ matrice diagonale positive. Sous des conditions sur F et un choix approprié de A du type condition sous-caractéristique

$A - F'(U)^2 \geq \nu I$ avec $\nu > 0$, on obtient, en faisant tendre le paramètre de relaxation ε vers 0, l'égalité $V = F(U)$ et le système (3.2).
La méthode consiste à reporter la non-linéarité au second membre ce qui simplifie le problème d'existence ; la difficulté devient alors de pouvoir justifier le passage à la limite.
Cette méthode est à relier aux développements de Chapman-Enskog [103]. On trouvera également à cette référence la théorie linéaire. Pour cette dernière, on consultera aussi [67]. La théorie non-linéaire a débuté récemment et a été initiée par Liu dans [75] et [38], qui révèle le rôle stabilisant de la condition sous-caractéristique. En général, une condition de type sous-caractéristique permet de trouver une fonction d'entropie qui contrôle la distance à l'équilibre [37]. Néanmoins, la structure dissipative est parfois plus délicate [98], [58]. Pour une théorie générale des modèles de relaxation, on peut voir par exemple [80]. Notre modèle de relaxation (voir Section 4.4.1) va utiliser ce genre d'idées et présenter une structure dissipative complexe. La technique de relaxation va, de plus, combiner des techniques cinétiques via des intégrations des grandeurs microscopiques. Voyons maintenant ce que nous entendons par là.

4 La théorie cinétique

4.1 Le modèle cinétique de Boltzmann

La théorie cinétique a été introduite par Maxwell et Boltzmann pour modéliser des gaz raréfiés. Un gaz, un plasma ou tout autre système comprenant un très grand nombre de particules peut ainsi être décrit par une fonction de distribution $f(t,x,\xi)$, dans l'espace des phases (x,ξ) associé aux atomes, où x est la variable macroscopique de position et ξ la variable microscopique/cinétique. La signification physique est la suivante : $f(t,x,\xi)\,dx\,d\xi$ représente la densité de particules dans l'élément de volume $dx\,d\xi$ centré en (x,ξ) au temps t. En supposant que le système est constitué de particules de même espèce et que la dynamique qui les régit est non relativiste et non quantique, la variable microscopique correspond à la variable de vitesse cinétique. Une des idées de base est que toutes les quantités macroscopiques observables vont pouvoir être restituées grâce à des intégrations de la fonction de distribution par rapport à la variable microscopique ξ, c'est-à-dire des intégrales du type

$$\int f(t,x,\xi)S(\xi)\,d\xi.$$

Par exemple dans le modèle de Boltzmann, on peut définir la densité locale ρ, la vitesse macroscopique locale u, la température locale T et donc l'énergie par $\rho(t,x) = \int f(t,x,\xi)\,d\xi$, $\rho u(t,x) = \int f(t,x,\xi)\xi\,d\xi$ et $(\rho|u|^2 + N\rho T)(t,x) = \int f(t,x,\xi)|\xi|^2\,d\xi$. Une hypothèse du type $f(t,.,.) \in L^1_{loc(x)}$ revient alors à supposer que tout domaine borné de l'espace physique ne contient qu'une quantité finie de matière.
L'équation cinétique de Boltzmann pour la dynamique des gaz raréfiés s'écrit :

$$\partial_t f + \xi \cdot \nabla_x f = \frac{Q(f)}{\varepsilon}, \tag{4.1}$$

où $f(t,x,\xi)$ est la densité de particules dans l'espace des phases $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$, ε le libre parcours moyen et Q l'opérateur de collision de Boltzmann.

L'opérateur Q est intégral, agit sur la variable cinétique ξ uniquement et vérifie les moments

$$\int_{\mathbb{R}^N} \left(1, \xi_i, |\xi|^2\right) Q(f)(\xi)\, d\xi = 0,$$

et l'inégalité d'entropie

$$\int_{\mathbb{R}^N} \ln f(\xi) Q(f)(\xi)\, d\xi \leq 0.$$

Ces propriétés permettent, par intégration par rapport à ξ, d'obtenir la conservation locale de la masse, de la quantité de mouvement, de l'énergie définis plus haut et la décroissance de l'entropie

$$\partial_t \int f \ln f\, d\xi + div_x \int \xi f \ln f\, d\xi \leq 0.$$

Une propriété importante de l'opérateur de Boltzmann est que $Q(f) = 0$ si et seulement si f est une fonction d'équilibre thermodynamique appelée Maxwellienne et de la forme

$$M_{\rho,u,T}(\xi) = \frac{\rho}{(2\pi T)^{N/2}} e^{-|\xi-u|^2/(2T)},$$

pour des valeurs $\rho \geq 0$, $T > 0$ et $u \in \mathbb{R}^N$. La relaxation du modèle lorsque ε tend vers 0 donne, formellement, la convergence de f vers une Maxwellienne de paramètre $\rho(t,x)$, $u(t,x)$ et $T(t,x)$, fonctions qui vont vérifier le système d'Euler pour un gaz parfait. La preuve rigoureuse de ce problème de relaxation est encore ouvert à ce jour. Il est connu que l'équation (4.1) ne possède pas, sous cette forme, de solutions globales en temps. DiPerna et Lions ont obtenu l'existence et la stabilité de solutions renormalisées dans [44].
Pour une présentation générale des modèles cinétiques, le lecteur peut consulter [24], [32], [33], [18], [89] et [45].
L'étude de l'équation cinétique de Boltzmann montre que seules certaines propriétés du modèle sont indispensables pour pouvoir espérer obtenir la limite hydrodynamique ; il n'est donc pas nécessaire de conserver le noyau de collision avec toutes les difficultés qu'il contient. C'est l'idée principale du modèle introduit par Bhatnagar, Gross et Krook [17], appelé modèle BGK : remplacer la cinétique de Boltzmann par un modèle plus simple qui redonne la même configuration d'équilibre hydrodynamique.
Nous présentons maintenant deux modèles BGK (pour l'équation de Boltzmann et pour une loi de conservation scalaire), puis nous définissons le cadre général d'un modèle BGK pour un système de lois de conservation et nous l'appliquons à la dynamique des gaz.

4.2 Modèle BGK pour Boltzmann

Le modèle BGK de [17] satisfait les propriétés du modèle de Boltzmann que sont conservation de la masse, quantité de mouvement, d'énergie et propriété d'entropie et s'écrit

$$Q(f) = M_f - f, \qquad M_f = M_{\rho_f,u_f,T_f},$$

où
$$\left(\rho_f, \rho_f u_f, \frac{1}{2}\rho_f|u_f|^2 + \frac{N}{2}\rho_f T_f\right) = \int \left(1, \xi, \frac{|\xi|^2}{2}\right) f(\xi)\, d\xi.$$
L'existence et la stabilité de solutions globales satisfaisant une relation d'entropie pour ce modèle ont été prouvées par Perthame [86]. On pourra voir également [87].

4.3 Modèle BGK pour une loi de conservation scalaire

Le modèle BGK de Lions, Perthame et Tadmor [73] pour une loi de conservation scalaire ($k = 1$) s'écrit :
$$\partial_t f + F'(\xi) \cdot \nabla_x f = \frac{M[f] - f}{\varepsilon},$$
où la Maxwellienne est
$$M[f](t, x, \xi) = \mathbb{1}_{0 \leq \xi \leq \int_{\mathbb{R}} f(t,x,v)\, dv}, \qquad x \in \mathbb{R}^N, \xi \in \mathbb{R}.$$
Lorsque $\varepsilon \to 0$, on obtient
$$\partial_t f + F'(\xi) \cdot \nabla_x f = \partial_\xi \mu, \tag{4.2}$$
où μ est une mesure positive et f une maxwellienne. En intégrant en ξ, on obtient (2.1). Multipliant par $S'(\xi)$ avec S convexe puis intégrant en ξ, on obtient les inégalités d'entropie. Le lecteur pourra consulter également [28], [88] et [29] pour des résultats sur le cas scalaire.
L'équation cinétique (4.2) englobe donc le système et toutes les inégalités d'entropie. Un des intérêts majeurs de la théorie est la possibilité de représenter une famille entière d'inégalités d'entropie à partir d'une seule équation grâce au rajout de la variable microscopique.

4.4 Modèle BGK pour un système de lois de conservation

Une formulation cinétique pour un système de lois de conservation est basée sur une équation de transport faisant intervenir la variable ξ et telle que les moments en ξ de l'équation de transport redonnent le système de lois de conservation et conduisent aux conditions d'entropie. Le cas des lois de conservation scalaire est traité, avec cette approche, dans [88], où les inégalités d'entropie associées sont formulées à partir d'une seule équation cinétique de transport de type BGK.

Propriétés d'un modèle BGK pour obtenir la limite hydrodynamique et les inégalités d'entropie

Un modèle BGK, pour un système de lois de conservation (2.1) et pour une famille $\mathcal{E}$ d'entropies convexes, consiste en
1) une équation de transport avec terme de relaxation
$$\partial_t f + a(\xi) \cdot \nabla_x f = \frac{M_f - f}{\varepsilon}, \quad t > 0, x \in \mathbb{R}^N, \xi \in \mathbb{R}^l, \tag{4.3}$$

d'inconnue $f(t,x,\xi) \in \mathbb{R}^p$ et de vitesse $a : \mathbb{R}^l \to \mathbb{R}^N$, avec

$$M_f(t,x,\xi) = M(U(t,x),\xi), \qquad U(t,x) = \int f(t,x,\xi)\,d\xi,$$

où l'état d'équilibre $M : \mathcal{U} \times \mathbb{R}^l \to \mathbb{R}^p$ (avec $\mathcal{U} \subset \mathbb{R}^k$) vérifie

$$\int M(U,\xi)\,d\xi = U, \qquad \int a_i(\xi) M(U,\xi)\,d\xi = F_i(U), \tag{4.4}$$

2) un domaine convexe D_ξ de $\mathbb{R}^p$ tel que p.p. ξ, $\forall U \in \mathcal{U}$, $M(U,\xi) \in D_\xi$,
3) l'existence d'entropies cinétiques $H_\eta(f,\xi)$ pour tout $\eta \in \mathcal{E}$ telles que p.p. ξ,

$$H_\eta(.,\xi) : D_\xi \to \mathbb{R} \text{ est convexe}, \tag{4.5}$$

$$\int H_\eta(M(U,\xi),\xi)\,d\xi = \eta(U), \qquad \forall U \in \mathcal{U}, \tag{4.6}$$

et $\forall f : \mathbb{R}^l \to \mathbb{R}^p$ telle que $f(\xi) \in D_\xi$ p.p. ξ et $U_f = \int f(\xi)\,d\xi \in \mathcal{U}$,

$$\int H_\eta(M(U_f,\xi),\xi)\,d\xi \le \int H_\eta(f(\xi),\xi)\,d\xi. \tag{4.7}$$

Remarque 4.1 La borne $U(t,x) \in \mathcal{U}$ est nécessaire. On peut l'obtenir par une estimation a priori, ou plus généralement par un principe du maximum. Par exemple une zone invariante $U(0,x) \in \mathcal{U} \Rightarrow U(t,x) \in \mathcal{U}$ permet d'avoir de telles estimations.

Ces propriétés permettent d'obtenir formellement la limite hydrodynamique et les inégalités d'entropie de Lax, détaillons comment.
Si $U_\varepsilon \to U$, alors une étude du terme de relaxation fait apparaître une maxwellienne à la limite $f_\varepsilon \to M(U,\xi)$. A partir de l'équation (4.3), on a par intégration en ξ,

$$\partial_t \int f_\varepsilon\,d\xi + \sum_{i=1}^N \partial_{x_i} \int a_i(\xi) f_\varepsilon\,d\xi = 0,$$

ce qui donne à la limite

$$\partial_t \int M(U,\xi)\,d\xi + \sum_{i=1}^N \partial_{x_i} \int a_i(\xi) M(U,\xi)\,d\xi = 0,$$

soit le système (2.1) d'après les propriétés (4.4) de la Maxwellienne.
Multipliant maintenant (4.3) par $\partial_f H_\eta(f_\varepsilon)$, on obtient l'équation renormalisée

$$\partial_t H_\eta(f_\varepsilon) + a(\xi)\cdot\nabla_x(H_\eta(f_\varepsilon)) = \partial_f H_\eta(f_\varepsilon)\cdot\frac{M_{f_\varepsilon} - f_\varepsilon}{\varepsilon},$$

dont le second membre est inférieur à $\frac{H_\eta(M_{f_\varepsilon}) - H_\eta(f_\varepsilon)}{\varepsilon}$ par l'hypothèse (4.5) de convexité. En intégrant par rapport à ξ, puis en passant à la limite, on trouve grâce à (4.6)-(4.7), les inégalités d'entropie

$$\partial_t \eta(U) + div_x G(U) \le 0, \qquad G(U) = \int a(\xi) H_\eta(M(U,\xi),\xi)\,d\xi,$$

où G est un flux d'entropie associé à η.

Caractérisation de l'existence d'entropies cinétiques

Le point délicat de cette construction d'un modèle BGK est d'obtenir des entropies cinétiques H_η pour toute fonction η dans la famille $\mathcal{E}$ qui est donnée à priori. Heureusement, il est possible de caractériser simplement l'existence d'entropies cinétiques lorsque la maxwellienne et les entropies sont données. On définit pour cela les espaces de maxwelliennes

$$\mathcal{M}^\varepsilon = \{M : \mathcal{U} \to \mathbb{R}^p \ ; \ \forall \eta \in \mathcal{E},\ (M')^t \eta'' \text{ est symétrique partout dans } \mathcal{U}\},$$

$$\mathcal{M}^\varepsilon_+ = \{M \in \mathcal{M}^\varepsilon \ ; \ \forall \eta \in \mathcal{E},\ (M')^t \eta'' \text{ est positive partout dans } \mathcal{U}\}.$$

On a alors les résultats suivants :

Théorème 4.1 (Bouchut) *Soit $\mathcal{U}$ ouvert et p.p. ξ, $M(.,\xi)$ un C^1 difféomorphisme de $\mathcal{U}$ dans $D_\xi = \underset{U\in\mathcal{U}}{conv}\, M(U,\xi)$. Alors il existe des entropies cinétiques $H_\eta(f,\xi)$ satisfaisant à (4.5)-(4.7) et telles que $G_\eta(U,\xi) = H_\eta(M(U,\xi),\xi)$ est C^1 dans $\mathcal{U}$ si et seulement si $M(.,\xi) \in \mathcal{M}^\varepsilon_+$ p.p. ξ. On a alors*

$$G'_\eta(U,\xi) = \eta'(U)M'(U,\xi), \qquad \forall u \in \mathcal{U},\ p.p.\ \xi. \tag{4.8}$$

Théorème 4.2 (Bouchut) *Considérons le système (2.1) supposé symétrique, avec $p = N+1$, c'est-à-dire $F_j^{(i)} = F_i^{(j)}$ où $F_j^{(i)}$ est la ième composante de F_j et $F_0(u) = u$. Alors, pour toute famille d'entropies convexes $\mathcal{E}$, $\mathcal{M}^\varepsilon$ contient toutes les fonctions $M = (M_0, M_1, \ldots, M_N) : \mathcal{U} \to \mathbb{R}^{N+1}$ où M_0 est une entropie pour (2.1) et $(M_1, \ldots, M_N)$ sont les flux d'entropies associés.*

Pour plus de détails, on peut se référer à [19], [94] et [4].

4.5 Modèle BGK pour la dynamique des gaz isentropiques

Appliquons cette théorie à la dynamique des gaz isentropiques. On se place dans le cadre unidimensionnel, avec $\gamma > 1$ et en variables Eulériennes.
Dans [74], le modèle cinétique pour la dynamique des gaz peut être reformulé sous la forme

$$\partial_t \chi(\rho, \xi - u) + \partial_x[(\theta\xi + (1-\theta)u)\chi(\rho, \xi - u)] = -\partial_{\xi\xi} m,$$

où m est une mesure positive bornée, avec $\theta = (\gamma - 1)/2$. Les deux premiers moments en ξ donnent les lois de conservation et le signe de m reflète les inégalités d'entropie. Ce modèle n'est purement cinétique et de type BGK que pour $\gamma = 3$. Dans les autres cas, en effet, u intervient dans le terme d'advection. Pour le cas très particulier $\gamma = 3$, les solutions entropiques vérifient des bornes supplémentaires et la compacité en résulte. On peut consulter également, pour $\gamma = 3$, les travaux de Vasseur [99], [100].
Il est possible de montrer que $\mathcal{M}^\varepsilon$ coïncide pour ce système avec les couples du théorème 4.2. On a, de plus, une caractérisation complète de l'espace des maxwelliennes à savoir :

Théorème 4.3 (Bouchut) *Le couple entropie-flux d'entropie (M_0, M_1) appartient à $\mathcal{M}^\varepsilon_+$ si et seulement si $\frac{\partial M_0}{\partial \omega_2} \geq 0$ et $\frac{\partial M_0}{\partial \omega_1} \leq 0$.*

Exposons le modèle cinétique BGK vectoriel ainsi obtenu. Pour préserver la positivité de ρ, on se place dans $D_\xi \subset \{(f_0, f_1)\ ;\ f_0 \geq 0\}$, ce qui permet d'avoir $U(t,x) \in \mathcal{U} = \{(\rho, u)\ ;\ \rho \geq 0,\ u \in \mathbb{R}\}$. On utilise les entropies standards

$$\eta_S(\rho, u) = \int_{\mathbb{R}} \chi(\rho, v - u) S(v)\, dv, \tag{4.9}$$

avec

$$\chi(\rho, \xi) = c_{\gamma,\kappa} \left(\frac{4\gamma\kappa}{(\gamma-1)^2} \rho^{\gamma-1} - \xi^2 \right)_+^\lambda. \tag{4.10}$$

Pour $1 < \gamma < 3$, on obtient une maxwellienne qui s'écrit

$$M(\rho, u, \xi) = \Big(\chi(\rho, \xi - u), ((1-\theta)u + \theta\xi)\chi(\rho, \xi - u) \Big). \tag{4.11}$$

Pour $\gamma > 3$, la propriété de monotonie du théorème 4.3 n'est pas vérifiée. Pour $\gamma = 3$, on trouve le modèle de [30]. On s'intéresse maintenant au cas $1 < \gamma < 3$. La relation (4.8) avec (4.9) conduit alors à

$$G_\eta(\rho, u) = \int_{\mathbb{R}} \Phi(\rho, u, \xi, v) S(v)\, dv, \tag{4.12}$$

où

$$\Phi(\rho, u, \xi, v) = \frac{(1-\theta)^2}{\theta} \frac{c_{\gamma,\kappa}}{J_\lambda} \mathbb{1}_{\omega_1 < \xi < \omega_2} \mathbb{1}_{\omega_1 < v < \omega_2} |\xi - v|^{2\lambda - 1} \Upsilon_{\lambda-1}(z), \tag{4.13}$$

avec

$$\Upsilon_{\lambda-1}(z) = \int_1^z (y^2 - 1)^{\lambda-1}\, dy, \qquad z = \frac{(\xi + v)(\omega_1 + \omega_2) - 2(\omega_1\omega_2 + \xi v)}{(\omega_2 - \omega_1)|\xi - v|} \geq 1. \tag{4.14}$$

On obtient finalement les entropies cinétiques grâce à

$$H_\eta(f, \xi) = G_\eta(M^{-1}(f, \xi), \xi).$$

Ayant construit formellement tous les éléments du modèle BGK, passons maintenant à l'étude théorique de celui-ci.

4.6 Relaxation d'un modèle cinétique vers la dynamique des gaz satisfaisant toutes les inégalités d'entropie

Pour les lois de conservation scalaires, il est possible d'avoir un principe de minimisation pour toutes les entropies à partir d'un modèle BGK [88]. Pour les systèmes hyperboliques, une seule entropie est habituellement considérée. Le système de Brenier et Corrias [29] considère toutes les entropies mais le contexte est un peu différent. L'équation étudiée est semblable à (4.3) mais avec $\varepsilon = 0$, c'est-à-dire que f est une maxwellienne et le second membre est remplacé par un terme convenable.
Dans la seconde partie de la thèse, on considère une équation BGK cinétique vectorielle conduisant à la dynamique des gaz isentropiques satisfaisant toutes

les inégalités d'entropie à la limite du paramètre de relaxation.
On considère le système en une dimension d'espace de la dynamique des gaz isentropiques, avec $\rho(t,x) \geq 0$, $u(t,x) \in \mathbb{R}$, $\kappa > 0$, $1 < \gamma < 3$,

$$\left\{ \begin{array}{c} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \kappa \rho^\gamma) = 0. \end{array} \right. \tag{4.15}$$

Le modèle BGK vectoriel peut s'écrire

$$\partial_t f + \xi \partial_x f = \frac{M[f] - f}{\varepsilon} \quad \text{dans }]0, \infty[\times \mathbb{R} \times \mathbb{R}, \tag{4.16}$$

avec $t \geq 0$, $x \in \mathbb{R}$, $\xi \in \mathbb{R}$, $f = f(t,x,\xi) \in D = \{(f_0, f_1) \in \mathbb{R}^2,\ f_0 > 0 \text{ or } f_1 = f_0 = 0\}$, où la maxwellienne est définie par

$$M[f](t,x,\xi) = M\left(\rho(t,x), u(t,x), \xi\right), \ (\rho, \rho u)(t,x) = \int_{\mathbb{R}} f(t,x,\xi)\, d\xi, \tag{4.17}$$

$$M(\rho, u, \xi) = (1, u + \frac{\gamma - 1}{2}(\xi - u))) \cdot c_{\gamma,\kappa} \left(\frac{4\gamma\kappa}{(\gamma-1)^2} \rho^{\gamma-1} - (\xi - u)^2 \right)_+^{1/(\gamma-1)-1/2}.$$

Nous définissons, pour toute fonction convexe S, l'entropie cinétique $H_S(f,\xi)$ à partir d'un noyau. Alors, l'entropie η_S et le flux d'entropie G_S pour (4.15) sont définis à partir de cette fonction cinétique et coïncident avec la définition macroscopique classique de [72]. Nous obtenons le principe de minimisation pour toutes les entropies, à savoir

$$\int_{\mathbb{R}} H_S(M[f](\xi), \xi)\, d\xi \leq \int_{\mathbb{R}} H_S(f(\xi), \xi)\, d\xi.$$

Existence de solutions globales pour le modèle BGK

Pour toute donnée initiale $f^0 \in L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$, $f^0 \in D$ p.p. et d'énergie finie $\int_{\mathbb{R}\times\mathbb{R}} H_{v^2/2}(f^0(x,\xi),\xi)\, dx d\xi < +\infty$, nous prouvons l'existence de solutions faibles globales f satisfaisant une inégalité de dissipation d'entropie

$$\frac{1}{\varepsilon} \left(H'_{v^2/2}(f,\xi) - (\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u) \right) \cdot (M[f] - f) \leq 0,$$

cette dernière quantité étant bornée dans $L^1_{t,x,\xi}$. Une difficulté technique provient de la prise en compte du cas $f = 0$, qui est une singularité pour le système ; de plus, en ce point, H_S n'est pas régulier. Les lemmes de compacité en moyenne [53] sont un des outils essentiel du résultat. Ils permettent d'obtenir la compacité forte des quantités macroscopiques à partir de la compacité faible des grandeurs cinétiques.

Domaines invariants et correspondance avec les entropies cinétiques

Pour tout $\omega_{min} < \omega_{max}$, le système (4.16) vérifie la propriété que

$$\tilde{D}_\xi = \{f \in D;\ f = 0 \text{ ou } \omega_{min} \leq \omega_1(f,\xi) \leq \omega_2(f,\xi) \leq \omega_{max}\}$$

est une famille de domaines invariants cinétiques convexes, les $\omega_i(f,\xi)$ étant les invariants cinétiques de Riemann. De plus, l'ensemble $\tilde{D}_\xi$ est en correspondance avec les domaines invariants de (4.15),

$$\tilde{D} = \{(\rho,u) \in \mathbb{R}^+ \times \mathbb{R};\ \rho = 0 \text{ ou } \omega_{min} \leq \omega_1 \leq \omega_2 \leq \omega_{max}\},$$

où les ω_i sont les invariants de Riemann pour ce système. Nous prouvons également une correspondance entre les domaines invariants cinétiques et les entropies cinétiques ; c'est-à-dire, posant $S_M(v) = (v - \omega_{max})_+^2$, $S_m(v) = (\omega_{min} - v)_+^2$, nous avons

$$(H_{S_M}(f,\xi) \leq 0 \text{ et } H_{S_m}(f,\xi) \leq 0) \Leftrightarrow f \in \tilde{D}_\xi.$$

Pour des généralités sur les domaines invariants cinétiques, on peut consulter [94].

Relaxation du modèle

Suivant Lions, Perthame et Souganidis [72], on définit les solutions entropiques de (4.15) comme les solutions d'énergie finie (c'est-à-dire telles que $\sup_t \int \eta_{v^2/2}(\rho(t,x),u(t,x))\,dx < \infty$), qui satisfont les inégalités d'entropie $\partial_t(\eta_S) + \partial_x(G_S) \leq 0$, où η_S et G_S sont paramétrés par les fonctions convexes S à croissance sous-quadratique.
On utilise la convergence vers 0 du terme de dissipation d'entropie

$$\left(H'_{v^2/2}(f_\varepsilon,\xi) - (\frac{\gamma\kappa}{\gamma-1}\rho_\varepsilon^{\gamma-1} - \frac{u_\varepsilon^2}{2}, u_\varepsilon)\right) \cdot (f_\varepsilon - M[f_\varepsilon]),$$

pour obtenir $f_\varepsilon - M[f_\varepsilon] \to 0$ et la relaxation du modèle. Ceci est à rapprocher des inégalités de type Csiszár-Kullback de la forme

$$\textit{Dissipation d'entropie} \geq \|f - M\|_{L^1}^2,$$

(voir [5], [6], [40], [62] et [42]). Ici cependant, on n'a pas une inégalité aussi directe, en particulier du fait de la perte de régularité en 0, et il est nécessaire de faire une étude très fine du terme de dissipation. Grâce aux domaines invariants cinétiques, on obtient des bornes uniformes dans L^∞. Ces estimations permettent d'appliquer les résultats de renormalisation pour une équation de transport selon [20] et d'arriver au résultat

$$\partial_t(H_S(f_\varepsilon,\xi)) + \xi\partial_x(H_S(f_\varepsilon,\xi)) = H'_S(f_\varepsilon,\xi) \cdot \frac{M[f_\varepsilon] - f_\varepsilon}{\varepsilon}.$$

On obtient finalement toutes les inégalités d'entropie à la limite de la relaxation, à savoir :

Théorème 4.4 *Notons f_ε la solution globale obtenue pour le modèle BGK (4.16) avec la même donnée initiale $f^0(x,\xi) \in L^1(\mathbb{R}\times\mathbb{R})$ satisfaisant $f^0(x,\xi) \in \tilde{D}_\xi$ p.p. pour des valeurs de $\omega_{min} < \omega_{max}$, et la borne d'énergie $\int_{\mathbb{R}\times\mathbb{R}} H_{v^2/2}(f^0(x,\xi),\xi)\, dxd\xi < +\infty$. Alors $(\rho_\varepsilon, u_\varepsilon)$, définis par (4.17), sont uniformément bornés dans L^∞, et prenant si nécessaire une sous-suite, $(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)$ converge p.p. dans $]0,\infty[\times\mathbb{R}$ lorsque $\varepsilon \to 0$ vers une solution entropique $(\rho, \rho u)$ de (4.15), de donnée initiale $(\rho^0, \rho^0 u^0) = \int f^0 d\xi$.*

Pour ce résultat, un lemme de moyenne est insuffisant et on utilise la compacité par compensation. Pour d'avantage de détails sur la compacité par compensation, et en particulier sur ses utilisations dans notre cadre, on consultera [95], [96], [97], [43], [78], [72], [35] et [39].

5 Conditions de bord

5.1 Formulations faibles

Dans les différents problèmes précédents, il a toujours été question de problèmes de Cauchy avec uniquement une donnée initiale. D'autres problèmes sont intéressants et utiles, en particulier les problèmes mixtes comportant une donnée initiale et une donnée au bord.
Lorsque l'on considère des problèmes avec des données dans L^1 ou L^∞, la notion de condition de bord devient délicate car on n'a pas clairement l'existence d'une trace. De plus, pour un système hyperbolique linéaire, les valeurs que l'on peut imposer sur le bord sont les composantes de la base de vecteurs propres correspondants aux caractéristiques entrantes. Le problème

$$\begin{cases} \partial_t U + \partial_x F(U) = 0, & x>0, t>0, \\ U(0,x) = U_0(x), & x>0, \\ U(t,0) = U^b(t), & t>0, \end{cases}$$

est mal posé, en général, dès que l'une des vitesses caractéristiques du système est négative [60], [56]. On doit donc retenir une condition de bord moins exigente que $U(t,0) = U^b(t)$.
Dubois et LeFloch proposent dans [47] deux approches pour formuler une condition de bord de type Dirichlet en $x = 0$. La première est basée sur l'idée d'approximer le système hyperbolique de lois de conservation par un modèle parabolique ou visqueux et de voir quelle information sur le bord est obtenue à la limite. La formulation de la condition de bord s'écrit alors selon des inégalités entropiques de bord. On peut l'écrire sous la forme :

$$U(t,0) \in \mathcal{E}(U^b(t)),$$

où

$$\begin{aligned} \mathcal{E}(U^b) = \{w\ ;\ & G(w) - G(U^b) - \eta'(U^b)\cdot(F(w) - F(U^b)) \le 0 \\ & \text{pour tout couple } (\eta, G) \text{ d'entropie - flux d'entropie} \\ & \text{tel que } \eta \text{ est convexe}\}. \end{aligned}$$

Pour un système strictement hyperbolique linéaire, on retrouve que $\mathcal{E}(U^b)$ est le sous-espace affine contenant U^b et engendré par les vecteurs propres associés aux valeurs propres négatives ou nulles. On peut se référer en premier lieu pour ce type de problèmes à [7], [10]. Dans le cas scalaire, Otto [85] a prouvé que cette condition de bord entropique faible donne une unique solution.
Une des difficultés de cette formulation est la nécessité d'avoir l'existence de traces. Avec une théorie BV, on a existence d'une trace forte et il est possible d'étudier de tels systèmes avec une telle approche [7]. Voir par exemple [76], [54], [102], [57], [82], [83] et [81]. Cette formulation avec des inégalités d'entropie présente l'avantage, entre autres, de pouvoir être utilisée à partir d'un modèle de relaxation, par exemple cinétique, cadre dans lequel nous nous plaçons. Pour une approche des lois de conservation scalaire avec un modèle BGK, on renvoie à [84] et [77]. Cependant, il est plus intéressant d'avoir une théorie L^1 ou L^∞ et nous utiliserons ces espaces plutôt que ceux liés à une théorie BV.
La seconde approche repose sur des demi-problèmes de Riemann dans le quadrant $x < 0$, $t > 0$. On peut l'écrire sous la forme :

$$U(t,0) \in \mathcal{R}(U^b(t)),$$

où

$$\begin{aligned}\mathcal{R}(U^b) = \{w \in \mathcal{U}\ ;\ w = w(0^+, U^b, U_R) \text{ où } u_R \in \mathcal{U} \\ \text{et } w(x/t, U_L, U_R) \text{ désigne la solution entropique} \\ \text{du problème de Riemann de l'état } U_L \text{ à } U_R\}.\end{aligned}$$

Cette approche est naturelle pour un problème qui serait implanté numériquement avec un schéma de Godunov. On peut se référer à [51] pour ce point. Il est connu que $\mathcal{R}(U^b) \subset \mathcal{E}(U^b)$ mais que la réciproque n'est pas toujours vraie. Dans le cas scalaire et dans le cas linéaire, il y a équivalence entre les deux approches, voir [11] et [47] ; il en va de même dans le cas des systèmes pour lesquels les courbes de détente sont des droites, et pour lesquels il existe un domaine convexe positivement invariant, borné, dans lequel les valeurs propres restent non nulles. L'article [59] propose également une approche par ces deux techniques et ceci pour des systèmes assez généraux. Pour des conditions de bord de type Riemann dépendant du temps, on peut voir [1].

5.2 Condition de Dirichlet pour la dynamique des gaz

Nous nous intéressons à ce problème mixte pour notre système de la dynamique des gaz isentropiques en dimension un. Dans la troisième partie de la thèse, nous formulons notre condition au bord par les inégalités d'entropie sur le bord. Nous utilisons pour cela notre modèle BGK (Section 4.6) et nous obtenons des solutions faibles globales, des domaines invariants et un principe du maximum, et tout ceci dans un contexte L^∞. Le fait de considérer la singularité $\rho = 0$ nous oblige à donner une formulation plus faible que celle de Dubois-LeFloch aux points où il y a perte de régularité. Elle s'écrit sous la forme

$$\overline{G_S(\rho,u)} - G_S(\rho^b,u^b) - T_S(\rho^b,u^b)\cdot(\overline{F(\rho,u)} - F(\rho^b,u^b)) \leq 0, \quad \text{dans }]0,\infty[_t \times \{0\}_x, \tag{5.1}$$

où $T_S(\rho^b, u^b)$ coïncide avec $\eta'_S(\rho^b, u^b)$ lorsque $\rho^b > 0$ et où $\overline{V}$ désigne la trace faible de V. On obtient, comme dans le cas des données initiales seules, toutes les inégalités d'entropie. Ceci permet d'assurer l'existence des traces faibles considérées : pour le flux du système $F(\rho, u)$ et pour les flux d'entropie $G_S(\rho, u)$. En effet, pour $V = (V_0, V_1) \in L^\infty(]0, \infty[^2)$ et $divV \in \mathcal{M}_{loc}(]0, \infty[^2)$, il existe une trace faible pour V_1. Pour ce type de résultat, on peut également se référer à [2] et [36]. Nous montrons en particulier le résultat suivant :

Théorème 5.1 *Soient ρ^0, $u^0 \in L^\infty(\mathbb{R})$, $\rho^0 \geq 0$, $\rho^0 \in L^1(\mathbb{R})$, ρ_b, $u_b \in L^\infty(\mathbb{R})$, $\rho_b \geq 0$. Alors il existe une solution entropique pour le système de la dynamique des gaz isentropiques, bornée dans L^∞, de données initiales ρ^0, $\rho^0 u^0$ et satisfaisant la condition de bord (5.1).*

Notre méthode est très générale et utilise principalement l'inégalité sous-différentielle

$$H_S(f, \xi) \geq H_S(M(\rho, u, \xi), \xi) + T_S(\rho, u) \cdot (f - M(\rho, u, \xi)), \tag{5.2}$$

qui caractérise la compatibilité du modèle cinétique. Elle peut s'appliquer à tout système possédant un modèle BGK et ayant compacité forte des variables macroscopiques. Elle peut ainsi être utilisée pour améliorer le résultat de [84].

6 Modèles avec contraintes unilatérales

Dans diverses modélisations (modèles diphasiques, équations de Saint-Venant), il est nécessaire d'imposer des contraintes sur certaines quantités du modèle, par exemple le fait que l'une des quantités reste bornée. Dans le cadre de la dynamique des gaz, la quantité représentant la densité (ou la fraction volumique) doit, dans certaines circonstances, être bornée. Pour la dynamique des gaz sans pression, des masses de Dirac peuvent apparaître même pour des données initiales régulières. Donc une contrainte de type $\alpha \leq 1$ devient impossible à prendre en compte telle quelle. Nous allons étudier, dans la quatrième partie de cette thèse, deux approches pour rendre compte des contraintes désirées et enlever ainsi cette impossibilité. Pour l'étude de ce problème dans le cadre incompressible et avec un terme de viscosité, on consultera [71]. Pour d'autres problèmes hyperboliques avec contraintes, on se reportera à [8], [50] et [68]. Pour une approche plus générale de ces problèmes, on consultera [41].

6.1 Modèle avec multiplicateur de type pression

On considère la dynamique des gaz sans pression, avec contrainte et avec un multiplicateur de type pression, à savoir :

$$\begin{cases} \partial_t \alpha + \partial_x(\alpha u) = 0, \\ \partial_t(\alpha u) + \partial_x(\alpha u^2 + \pi) = 0, \end{cases} \tag{6.1}$$

avec les contraintes

$$0 \leq \alpha \leq 1, \qquad \pi \geq 0. \tag{6.2}$$

On demande de plus que dans les régions où $\alpha < 1$, le terme π soit nul. Ainsi on prend en compte la relation d'exclusion de type multiplicateur de Lagrange

$$(1-\alpha)\pi = 0. \tag{6.3}$$

Le terme π peut être vu comme une pression qui oblige α à rester en dessous de 1. Ce modèle est, par exemple, le développement au premier ordre des équations de l'évolution du liquide dans un modèle diphasique constitué d'un liquide incompressible et d'un gaz.
Ce modèle fonctionne numériquement et on observe un principe du maximum. Pour approfondir ce modèle, on peut se référer à la quatrième partie de la thèse et à [23].

Un problème ouvert était alors l'étude théorique de ce modèle : existence de solutions globales faibles, principe du maximum, TVD, relation d'Oleinik et d'entropies. C'est à ce problème en particulier que va répondre notre travail.

Les équations (6.1)-(6.3) seront désignées comme constituant le modèle diphasique **I**. Pour ce modèle, il est possible de considérer des données $\alpha \in C_t([0,\infty[; L^\infty_{w*}(\mathbb{R}))$, mais des sauts en temps, même initialement, peuvent se présenter pour αu. Pour prendre en compte des données initiales, il est donc nécessaire de considérer l'équation d'évolution

$$\partial_t(\alpha u) + \partial_x(\alpha u^2 + \pi) = 0$$

sous la forme variationelle :

$$\int_{[0,+\infty[}\int_{\mathbb{R}}[\alpha u\partial_t\varphi + \alpha u^2\partial_x\varphi]\,dx\,dt + \int_{[0,+\infty[}\int_{\mathbb{R}}\partial_x\varphi\,\pi\; + \int_{\mathbb{R}}(\alpha^0u^0)(x)\varphi(0,x)\,dx = 0,$$

pour tout $\varphi \in \mathcal{D}([0,+\infty[\times\mathbb{R})$.

Approximation par bouchons collants

Notre étude repose tout d'abord sur l'existence de solutions particulières appelées bouchons collants de la forme

$$\alpha(t,x) = \sum_{i=1}^{n} \mathbb{1}_{a_i(t)<x<b_i(t)}, \quad \alpha(t,x)u(t,x) = \sum_{i=1}^{n} u_i(t)\mathbb{1}_{a_i(t)<x<b_i(t)}, \tag{6.4}$$

et qui sont une extension de la notion de particules collantes utilisées pour la dynamique des gaz sans pression. Il est à remarquer que les solutions bouchons collants qui ne prennent comme valeurs que 0 et 1 suffisent à obtenir toutes les données initiales dans L^1 telle que $0 \le \alpha \le 1$. On montre, en effet, en particulier, que toute fonction de L^1 à valeurs entre 0 et 1 p.p. est limite au sens des distributions de fonctions du type $\sum_{i=1}^{n} \mathbb{1}_{a_i(t)<x<b_i(t)}$.

Réécriture de la contrainte

On prouve également un résultat de stabilité sur les solutions faibles, ce qui permet de plus d'obtenir l'existence de solutions pour toute données initiales

$\alpha^0 \in L^1$ et $u \in L^\infty$. Ce résultat est très technique et utilise des espaces du type $L^\infty(\mathbb{R}_x, \mathcal{M}([0,T]))$ et $BV_t(\mathcal{D}'_x)$ afin d'obtenir l'existence du produit $\alpha\pi$ et sa stabilité faible. En effet, la contrainte $\alpha\pi = \pi$ soulève le problème du produit d'une fonction de $C_t([0,\infty[; L^\infty_{w*}(\mathbb{R}))$ par une mesure. On fournit pour cela une réécriture de $\alpha\pi$ qui est compatible avec la régularité des solutions. Cette réécriture est basée sur la décomposition de la compacité par compensation

$$\alpha\pi = (\alpha)(\alpha u^2 + \pi) - (\alpha u)(\alpha u).$$

Privilégiant la variable α, on définit M par $\partial_x M = \alpha$, $\partial_t M = -\alpha u$, et alors le produit s'écrit

$$\alpha\pi = \partial_t(\alpha u M) + \partial_x((\alpha u^2 + \pi)M).$$

Privilégiant la variable αu, on définit Q par $\partial_x Q = \alpha u$, $\partial_t Q = -(\alpha u^2 + \pi)$, et cette fois, on a

$$\alpha\pi = -\partial_t(\alpha Q) - \partial_x(\alpha u Q).$$

La contrainte peut alors se réécrire

$$\partial_t(\alpha u M) + \partial_x((\alpha u^2 + \pi)M) = -\partial_t(\alpha Q) - \partial_x(\alpha u Q) = \pi.$$

Ce calcul formel est vérifié en particulier pour les solutions régulières et pour les bouchons collants.
Nous proposons et étudions ensuite un autre modèle physique pour la dynamique des gaz avec contraintes.

6.2 Modèle avec multiplicateur de type perte de masse

Ce modèle est basé, lui, sur l'idée de retirer la masse qui dépasse du niveau un, tout comme l'eau déborde d'une rivière ou d'un récipient. Ce modèle est aussi basé sur l'idée d'un schéma numérique du type $\alpha' = \min(\alpha, 1)$, $u' = u$, $\alpha' - \alpha = Q \leq 0$ et $\alpha' u' - \alpha u = Qu$. Dès que α atteint 1, Q devient non nul et absorbe la masse en excès. Le modèle s'écrit

$$\begin{cases} \partial_t \alpha + \partial_x(\alpha u) = Q, \\ \partial_t(\alpha u) + \partial_x(\alpha u^2 + p(\alpha)) = Qu, \end{cases} \tag{6.5}$$

avec les contraintes

$$0 \leq \alpha \leq 1, \qquad Q \leq 0, \tag{6.6}$$

et la relation d'exclusion

$$(1 - \alpha)Q = 0. \tag{6.7}$$

On désigne par modèle diphasique $\mathbf{II_0}$ les équations (6.5)-(6.7) avec $p(\alpha) = 0$ (modèle sans pression) et par modèle diphasique $\mathbf{II}_\gamma$ les équations (6.5)-(6.7) avec $p(\alpha) = \kappa\alpha^\gamma$, $\gamma > 1$ (modèle avec pression isentropique).

Produit faible entropique

Pour ce modèle, deux produits posent cette fois problème : il s'agit des produits Qu et $Q\alpha$, car Q peut contenir des masses de Dirac et u et α peuvent être discontinues. On fournit donc une formulation faible, que nous appelons produit faible entropique. Elle consiste à introduire une inconnue supplémentaire $v \in L^\infty(Q)$ et à considérer le système suivant

$$\begin{cases} \partial_t \alpha + \partial_x(\alpha u) = Q, \\ \partial_t(\alpha u) + \partial_x(\alpha u^2 + p(\alpha)) = Qv, \end{cases} \tag{6.8}$$

avec

$$0 \leq \alpha \leq 1, \qquad Q \leq 0. \tag{6.9}$$

On prend $v \in L^\infty(Q)$, de façon à ce que le produit Qv soit bien défini. On doit donner un sens faible à $Qv = Qu$ et $Q\alpha = Q$. Pour cela, on demande que la famille d'inégalités suivantes soient satisfaites

$$\partial_t \, \eta_S(\alpha, u) + \partial_x \, G_S(\alpha, u) \leq Q \, \eta_S'(1, v) \cdot (1, v), \tag{6.10}$$

pour toute entropie convexe η_S d'une famille paramétrée par des fonctions convexes S, où G_S est le flux d'entropie associé, et la dérivation dans η_S' étant prise par rapport à $(\alpha, \alpha u)$. On vérifie que (6.8)-(6.10) est bien une formulation faible de (6.5)-(6.7). Pour le modèle $\mathbf{II}_\gamma$ avec $\gamma > 1$ et pour des solutions régulières, l'inégalité (6.10) avec $S(v) = v^2/2$ est équivalente à $Qu = Qv$ et $Q\alpha = Q$.

Méthode numérique BGK

Pour le modèle $\mathbf{II_0}$ et dans le cas sans pression, nous proposons une méthode numérique basée sur un splitting et un modèle BGK. Cette méthode numérique nous permet de trouver une dynamique pour les bouchons collants associée à ce problème. Ceci est à rattacher avec [22] qui montre les liens existant entre méthode numérique par splitting du premier ordre et modèles BGK cinétiques compatibles avec des familles d'entropie. Pour une approche numérique des modèles BGK, on consultera [58], [79], [3], [19] et [52].

Existence de solutions

Grâce à cette dynamique et à un résultat de stabilité faible des solutions, nous prouvons l'existence de solutions faibles pour toutes données initiales dans le cas sans pression (modèle $\mathbf{II_0}$). Pour le modèle $\mathbf{II}_\gamma$, nous utilisons des formulations cinétiques de type BGK, une pour $1 < \gamma < 3$ et une autre pour $\gamma = 3$. Grâce, en particulier, à la compacité par compensation (Théorème 2.1), nous prouvons la stabilité et l'existence de solutions faibles. Pour $\gamma = 3$, le système devient de rang un et il est possible d'utiliser un lemme de compacité en moyenne [46] plutôt que la compacité par compensation. On note que, pour les modèles sans pression $\mathbf{I}$ et $\mathbf{II_0}$, on obtient également un principe du maximum sur la vitesse

$$\operatorname{essinf} \, u^0(x) \leq u(t,x) \leq \operatorname{esssup} \, u^0(x), \qquad \forall t > 0, \tag{6.11}$$

la relation d'Oleinik

$$\partial_x u(t,x) \leq \frac{1}{t}, \qquad \forall t > 0, \tag{6.12}$$

et des inégalités d'entropie locales. La relation d'Oleinik est indispensable pour avoir la stabilité des solutions mais elle n'est pas suffisante ici pour assurer l'unicité. Par ailleurs, dans le cas avec pression isentropique, l'utilisation des domaines invariants permet d'avoir un principe du maximum et ainsi des bornes sur la vitesse.

7 Perspectives

Nous présentons dans cette section différentes directions de recherche supplémentaires qu'offre cette thèse.

7.1 Cinétique de la dynamique des gaz isentropiques

– Pour le modèle de relaxation vers la dynamique des gaz, le problème de passer aux dimensions supérieures en espace, typiquement $N = 3$, reste à ce jour encore ouvert. Pour cela, une approche consisterait dans un premier temps à prouver la relaxation en dimension un sans l'utilisation des domaines invariants. En effet, en dimension supérieure, les invariants de Riemann n'existent plus. Une approche de ce résultat est faite dans l'Annexe B où nous prouvons la convergence $f_\varepsilon - M[f_\varepsilon] \to 0$ dans L^1_{loc} lorsque $\varepsilon \to 0$ sans supposer que la donnée initiale est bornée et sans aucun recours aux domaines invariants. Il semble cependant, pour l'instant, bien délicat d'en déduire les inégalités (2.5) permettant d'appliquer la compacité par compensation selon [72]. On peut néanmoins consulter [91] pour une démarche dans une telle direction.

– Un autre problème ouvert, et non des moindres, est l'unicité des solutions entropiques faibles de la dynamique des gaz isentropiques unidimensionnels. Néanmoins, la formulation cinétique utilisée ici semble insuffisante pour l'obtenir, voir l'Annexe A.

7.2 Conditions de bord

– Un premier axe de recherche sur ce thème est l'existence d'une trace forte, problème relié à l'unicité précédente. Le cas d'une loi de conservation scalaire a été étudié par [101].

– Un second axe de recherche est l'approche par les problèmes de Riemann et les éventuelles correspondances entre les deux formulations faibles de la condition de bord pour notre système.

– Un troisième axe de recherche est de voir, dans le cas régulier, quelle information la formulation faible avec les inégalités d'entropie donne dans le cas où les invariants de Riemann sont de signes différents.

– L'aspect numérique du problème est également envisagé ainsi que la prise en compte de conditions de bord différentes de Dirichlet (par exemple condition de type Neumann, ...).

7.3 Modèles avec contraintes

– On peut envisager de s'intéresser, pour le modèle **I** au cas où une pression du type ρ^γ est prise en compte. Pour cela, la meilleure piste semble être de trouver un modèle BGK avec un terme de pression π_ξ pour ce problème.

– Une autre perspective ouverte est l'étude de l'unicité, très certainement à l'aide de la notion de dualité de [25], bien qu'ici le problème ne peut pas se ramener au cas scalaire, ce qui rend les choses plus délicates.

– Il est également envisageable d'étudier des méthodes numériques pour ces différents modèles (on en propose et on en utilise une ici, l'article [23] en propose également ; il est possible de les étudier en détail et de les améliorer si besoin).

– Finalement, on peut évoquer les extensions naturelles : modèles plus complexes, cas d'un problème en plusieurs dimensions.

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Deuxième partie

Relaxation d'un modèle cinétique vers la dynamique des gaz satisfaisant toutes les inégalités d'entropie

Chapitre 1

Solution d'énergie finie pour un système BGK relaxant vers la dynamique des gaz isentropiques

Solution with finite energy to a BGK system relaxing to isentropic gas dynamics

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Abstract

We consider a vector kinetic BGK equation leading to isentropic gas dynamics in the relaxation limit. We prove the existence of a weak solution satisfying a kinetic entropy inequality for any initial data with finite energy.

Key-words: isentropic gas dynamics – kinetic BGK model – kinetic entropy

Mathematics Subject Classification: 35L65, 76N15, 35B35, 82C40

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Contents

1 Introduction and main results

Relaxation models such as proposed in [6] have proved their efficiency in building numerical methods for conservation laws that are very easy to code and have nice properties. General strategies have been proposed in [9], [13], [1], [3] to build BGK approaches for systems of conservation laws. However, analysis of such models has mainly be achieved until now only when the relaxed equation is scalar (see [14]). Here we consider a BGK model for the one-dimensional system of isentropic gas dynamics

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \kappa\rho^\gamma) = 0, \end{cases} \tag{1.1}$$

with $\rho(t,x) \geq 0,\ u(t,x) \in \mathbf{R}$ and $\kappa > 0,\ 1 < \gamma < 3$. The kinetic model has been introduced in [3] and can be written

$$\partial_t f + \xi \partial_x f = \frac{M[f] - f}{\varepsilon} \quad \text{in }]0,\infty[\times \mathbf{R} \times \mathbf{R}, \tag{1.2}$$

where $f = f(t,x,\xi) \in \mathbf{R}^2$, $t \geq 0$, $x \in \mathbf{R}$, $\xi \in \mathbf{R}$,

$$f(t,x,\xi) \in D, \tag{1.3}$$

$$D = \{(f_0, f_1) \in \mathbf{R}^2,\ f_0 > 0 \text{ or } f_1 = f_0 = 0\}, \tag{1.4}$$

$$M[f](t,x,\xi) = M\left(\rho(t,x), u(t,x), \xi\right), \tag{1.5}$$

$$\rho(t,x) = \int_{\mathbf{R}} f_0(t,x,\xi)\, d\xi,\ \ \rho(t,x)u(t,x) = \int_{\mathbf{R}} f_1(t,x,\xi)\, d\xi, \tag{1.6}$$

and

$$M(\rho,u,\xi) = \Big(\chi(\rho, \xi - u), ((1-\theta)u + \theta\xi)\chi(\rho, \xi - u)\Big), \tag{1.7}$$

$$\chi(\rho,\xi) = c_{\gamma,\kappa} \left(\frac{4\gamma\kappa}{(\gamma-1)^2}\rho^{\gamma-1} - \xi^2 \right)_+^\lambda, \tag{1.8}$$

$$\theta = \frac{\gamma-1}{2},\ \lambda = \frac{1}{\gamma-1} - \frac{1}{2},\ c_{\gamma,\kappa} = \frac{1}{J_\lambda}\left(\frac{4\gamma\kappa}{(\gamma-1)^2}\right)^{-1/(\gamma-1)}, \tag{1.9}$$

$$J_\lambda = \int_{-1}^1 (1-z^2)^\lambda\, dz = \sqrt{\pi}\,\Gamma(\lambda+1)/\Gamma(\lambda+3/2). \tag{1.10}$$

The kinetic equilibrium χ in (1.8) has been introduced in [7] as a generating function for entropies, and has also been used in the stability analysis of [10]. This function is also involved in [12] for the so called kinetic formulation, which is different from the BGK equation considered here. Previous results for scalar equations can be found in [4], [11], [17] (see also [5]). The first existence result for a BGK equation was given in [15] (see also [16]).

The main result of this paper is the existence of a global solution to the system (1.2) with initial data

$$f(0,x,\xi) = f^0(x,\xi) \tag{1.11}$$

satisfying energy bounds. We recall that the energy for (1.1) is given by

$$\eta(\rho,u)=\rho u^2/2+\frac{\kappa}{\gamma-1}\rho^\gamma, \tag{1.12}$$

and is a mathematical entropy for (1.1). The corresponding kinetic entropy for (1.2) is given, for $f=(f_0,f_1)\in D$, by

$$\begin{aligned}H(f,\xi)&=\frac{\theta}{1-\theta}\frac{\xi^2}{2}f_0+\frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda}+\frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0}-\frac{\theta}{1-\theta}\xi f_1\\&=\frac{\theta}{1-\theta}\frac{1}{2}\left(\frac{f_1}{\sqrt{f_0}}-\xi\sqrt{f_0}\right)^2+\frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda}+\frac{1}{2}\frac{f_1^2}{f_0}\geq 0,\end{aligned} \tag{1.13}$$

$$H(0,\xi)=0. \tag{1.14}$$

We have the following theorems.

Theorem 1.1 *Assume that $f^0\in L^1(\mathbf{R}_x\times\mathbf{R}_\xi)$ satisfies*

$$f^0(x,\xi)\in D \text{ a.e. in } \mathbf{R}\times\mathbf{R}, \tag{1.15}$$

$$\iint\limits_{\mathbf{R}\times\mathbf{R}} H(f^0(x,\xi),\xi)\,dxd\xi=C_H^0<\infty, \tag{1.16}$$

and

$$\iint\limits_{\mathbf{R}\times\mathbf{R}} x^2(f^0)_0(x,\xi)\,dxd\xi=C_\Delta^0<\infty. \tag{1.17}$$

Then there exists a solution f to (1.2)-(1.11) satisfying

$$f\in C([0,\infty[,L^1(\mathbf{R}\times\mathbf{R})), \tag{1.18}$$

$$f\in L^\infty([0,\infty[,L^1(\mathbf{R}\times\mathbf{R})), \tag{1.19}$$

$$\forall t\geq 0,\ f(t,x,\xi)\in D \text{ a.e. in } \mathbf{R}\times\mathbf{R}, \tag{1.20}$$

$$\partial_t(\int_{\mathbf{R}} f\,d\xi)+\partial_x(\int_{\mathbf{R}}\xi f\,d\xi)=0, \tag{1.21}$$

$$\forall t\geq 0,\ \iint\limits_{\mathbf{R}\times\mathbf{R}} f(t,x,\xi)\,dxd\xi=\iint\limits_{\mathbf{R}\times\mathbf{R}} f^0(x,\xi)\,dxd\xi, \tag{1.22}$$

$$\forall t\geq 0,\ \iint\limits_{\mathbf{R}\times\mathbf{R}} H(f(t,x,\xi),\xi)\,dxd\xi\leq C_H^0, \tag{1.23}$$

$$\forall t\geq 0,\ \iint\limits_{\mathbf{R}\times\mathbf{R}} x^2 f_0(t,x,\xi)\,dxd\xi\leq \max\left(C_\Delta^0,\frac{4\varepsilon^2}{\theta}C_H^0\right)(1+t/\varepsilon)^2. \tag{1.24}$$

Theorem 1.2 *If (1.17) is not satisfied, we have the same result except (1.24).*

To obtain this result, we have to work with the Tychonoff-Schauder theorem instead of the usual Schauder theorem. The only significant change is for compactness and will be explain in the corresponding section.

Theorem 1.3 *The solution f obtained in Theorems 1.1 or 1.2 satisfies*

$$H(f,\xi) \in C([0,\infty[, L^1(\mathbf{R}\times\mathbf{R})), \tag{1.25}$$

$$\partial_t(H(f,\xi)) + \xi\partial_x(H(f,\xi)) = \frac{1}{\varepsilon}H'(f,\xi)\cdot(M[f]-f), \tag{1.26}$$

with

$$H'(f,\xi) = \left(\frac{\theta}{1-\theta}\frac{\xi^2}{2} + \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}f_0^{1/\lambda} - \frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0^2}, -\frac{\theta}{1-\theta}\xi + \frac{1}{1-\theta}\frac{f_1}{f_0}\right) \tag{1.27}$$

for $f \neq 0$, and $H'(0,\xi) = (\frac{\theta}{1-\theta}\frac{\xi^2}{2}, -\frac{\theta}{1-\theta}\xi)$ by convention. Moreover,

$$H'(f,\xi)\cdot(M[f]-f) \in L^1(]0,T[\times\mathbf{R}\times\mathbf{R}) \qquad \text{for any } T>0 \tag{1.28}$$

as well as each term from the decomposition (1.27). Besides, $\xi H(f(t,x,\xi),\xi) \in L^1(]0,T[\times B_R\times\mathbf{R})$ for any $T>0$, $R>0$,

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\mathbf{R}} H(f(t,x,\xi),\xi)\,d\xi + \frac{\partial}{\partial x}\int_{\mathbf{R}} \xi H(f(t,x,\xi),\xi)\,d\xi\\ &= \frac{1}{\varepsilon}\int_{\mathbf{R}}\left(H'(f,\xi) - (\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u)\right)\cdot(M[f]-f)\,d\xi \le 0,\end{aligned} \tag{1.29}$$

$$\begin{aligned}&\frac{d}{dt}\iint_{\mathbf{R}\times\mathbf{R}} H(f(t,x,\xi),\xi)\,dxd\xi\\ &= \frac{1}{\varepsilon}\iint_{\mathbf{R}\times\mathbf{R}}\left(H'(f,\xi) - (\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u)\right)\cdot(M[f]-f)\,dxd\xi \le 0,\end{aligned} \tag{1.30}$$

and the integrand in the right-hand sides of (1.29) and (1.30) is nonpositive.

Let us finally mention that we can study kinetic invariant domains and establish the relaxation limit to (1.1), this is done in [2].

2 Properties of the kinetic entropy

We recall the value of the moments of M,

$$\int_{\mathbf{R}} M(\rho,u,\xi)\,d\xi = (\rho,\rho u), \tag{2.1}$$

$$\int_{\mathbf{R}} \xi M(\rho,u,\xi)\,d\xi = (\rho u, \rho u^2 + \kappa\rho^\gamma), \tag{2.2}$$

and

$$\int_{\mathbf{R}} \frac{1}{2}\xi^2 M_0(\rho,u,\xi)\,d\xi = \frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma-1}\rho^\gamma = \eta(\rho,u) \tag{2.3}$$

for every $\rho \geq 0$ and $u \in \mathbf{R}$. We have the following identities which link the energy and the kinetic entropy. They can be obtained easily.

Proposition 2.1 *The energy η of (1.12) and the kinetic entropy H of (1.13) satisfy*

$$\int_{\mathbf{R}} H(M(\rho,u,\xi),\xi)\,d\xi = \eta(\rho,u) \quad \textit{for every } \rho \geq 0,\ u \in \mathbf{R}, \tag{2.4}$$

and

$$\begin{cases} \dfrac{\partial H}{\partial f_0}(M(\rho,u,\xi),\xi) = \dfrac{\partial \eta}{\partial \rho}(\rho,u) = \dfrac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - u^2/2, \\ \dfrac{\partial H}{\partial f_1}(M(\rho,u,\xi),\xi) = \dfrac{\partial \eta}{\partial(\rho u)}(\rho,u) = u, \end{cases} \tag{2.5}$$

for every $\rho > 0$, $u \in \mathbf{R}$ and $\xi \in \mathbf{R}$ such that $M_0(\rho,u,\xi) > 0$.

The function H has the following properties.

Proposition 2.2 *i) H is convex with respect to f in D,*
ii) H is continuous with respect to f in $\Gamma_A = \{(f_0,f_1) \in D\ ;\ |f_1| \leq Af_0\}$ for any $A > 0$.

Proof. *i*) In $]0,\infty[\times\mathbf{R}$, it is easy to compute the hessian and to check its nonnegativity. Then, on a straight line from the origin, f_1^2/f_0 is linear, thus H is convex in D.
ii) The only difficulty is at 0. Let $(f^n)_{n\geq 0}$ be a sequence in $\Gamma_A\backslash\{0\}$ which converges to 0. Then

$$\left|\frac{{(f^n)_1}^2}{(f^n)_0}\right| \leq A^2 (f^n)_0 \underset{n\to\infty}{\longrightarrow} 0,$$

thus H is continuous at 0 in Γ_A. $\square$

Lemma 2.3 (Coerciveness) *There exist $\varepsilon_0, \varepsilon_1 > 0$ such that for any $f \in D$, $\xi \in \mathbf{R}$, we have*

$$H(f,\xi) \geq \varepsilon_0 {f_0}^{p_0} + \varepsilon_1 |f_1|^{p_1}, \tag{2.6}$$

with

$$p_0 = 1 + 1/\lambda > 1, \qquad p_1 = 2(1+\lambda)/(1+2\lambda) > 1. \tag{2.7}$$

Proof. Let $f \in D$, $f \neq 0$. From identity (1.13), we deduce that

$$H(f,\xi) \geq \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}} \frac{f_0^{1+1/\lambda}}{1+1/\lambda} + \frac{1}{2}\frac{f_1^2}{f_0}.$$

We have $|f_1|^{p_1} = \left(\frac{|f_1|}{\sqrt{f_0}}\right)^{p_1} \left(\sqrt{f_0}\right)^{p_1}$, and using Young's inequality,

$$|f_1|^{p_1} \leq \frac{p_1}{2}\left(\frac{|f_1|}{\sqrt{f_0}}\right)^2 + \frac{p_1}{2+2/\lambda}{f_0}^{1+1/\lambda}.$$

Taking $0<\varepsilon<1$ such that $\varepsilon<\theta/c_{\gamma,\kappa}^{1/\lambda}$, and setting $\varepsilon_0=\left(\theta/c_{\gamma,\kappa}^{1/\lambda}-\varepsilon\right)/(2+2/\lambda)$ and $\varepsilon_1=\varepsilon/p_1$, we get

$$\begin{aligned} H(f,\xi) &\geq \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda}+\varepsilon\frac{1}{2}\frac{f_1^2}{f_0} \\ &\geq \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda}-\frac{\varepsilon}{2+2/\lambda}f_0^{1+1/\lambda}+\frac{\varepsilon}{p_1}|f_1|^{p_1} \\ &= \varepsilon_0 {f_0}^{p_0}+\varepsilon_1|f_1|^{p_1}. \ \square \end{aligned}$$

The key result of this section is the following.

Proposition 2.4 (Key convexity inequality) *For any $f\in D$, $\rho\geq 0$ and $u,\xi\in\mathbf{R}$, we have*

$$H(f,\xi)\geq H(M(\rho,u,\xi),\xi)+\left(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1}-\frac{u^2}{2},u\right)\cdot(f-M(\rho,u,\xi)). \quad (2.8)$$

Proof. Let us consider $\psi\ :\ D\to\mathbf{R}$ defined by

$$\psi(f)=H(f,\xi)-H(M(\rho,u,\xi),\xi)-\left(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1}-\frac{u^2}{2},u\right)\cdot(f-M(\rho,u,\xi)),$$

which is possible because $M(\rho,u,\xi)\in D$. We have to prove that $\psi\geq 0$.
– At infinity,
by Lemma 2.3, $\psi(f)\geq\varepsilon_0{f_0}^{p_0}+\varepsilon_1|f_1|^{p_1}+af_0+bf_1+e$ with $a,b,e\in\mathbf{R}$, and since $p_0,p_1>1$, $\psi(f)\to\infty$ when $f_0+|f_1|$ tends to infinity. Moreover, $\psi(f)\geq af_0+bf_1+e$, and we deduce that ψ is lower bounded in D.
– At the frontier $\{f_0=0,f_1\neq 0\}$,
$f_1^2/f_0\to\infty$ and $\psi(f)$ also.
– At 0,
If $M_0(\rho,u,\xi)>0$, then by Proposition 2.2(i) and (2.5), (2.8) holds true for any $f\in D$, thus $\underline{\lim}_{f\to 0}\psi(f)\geq 0$. On the contrary, if $M_0(\rho,u,\xi)=0$, then

$$\psi(f)=H(f,\xi)-\left(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1}-u^2/2,u\right)\cdot f.$$

Since $H(f,\xi)\geq 0$, this yields $\underline{\lim}_{f\to 0}\psi(f)\geq 0$, in any case.
– Inside D
It only remains to study possible minima of ψ in $int(D)$. But ψ is smooth there, and if $f_0>0$,

$$\frac{\partial\psi}{\partial f_0}(f,\xi)=\frac{\theta}{1-\theta}\frac{\xi^2}{2}+\frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}f_0^{1/\lambda}-\frac{1}{1-\theta}\frac{1}{2}\left(\frac{f_1}{f_0}\right)^2-\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1}+\frac{u^2}{2},$$

and

$$\frac{\partial\psi}{\partial f_1}(f,\xi)=\frac{1}{1-\theta}\frac{f_1}{f_0}-\frac{\theta}{1-\theta}\xi-u.$$

If there is a minimum of ψ in $int(D)$, at this point, its partial derivatives must vanish, thus

$$\frac{f_1}{f_0} = (1-\theta)u + \theta\xi,$$

and

$$\begin{aligned}\frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}} f_0^{1/\lambda} &= \frac{1}{1-\theta}\frac{1}{2}\left(\frac{f_1}{f_0}\right)^2 + \frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2} - \frac{\theta}{1-\theta}\frac{\xi^2}{2} \\ &= \frac{\theta}{2}\left(\frac{4\gamma\kappa}{(\gamma-1)^2}\rho^{\gamma-1} - (\xi-u)^2\right).\end{aligned}$$

We deduce that $M_0(\rho, u, \xi) > 0$ and $f = M(\rho, u, \xi)$. The value of ψ at this point is 0.
Putting all the steps together, we conclude that $\psi \geq 0$ in D. $\square$

An important consequence of this result is the entropy minimization principle.

Proposition 2.5 *Consider $f \in L^1(\mathbf{R}_\xi)$ such that $f \in D$ a.e. and*

$$\int_{\mathbf{R}} H(f(\xi), \xi)\, d\xi < \infty.$$

Then

$$\int_{\mathbf{R}} H(M[f](\xi), \xi)\, d\xi \leq \int_{\mathbf{R}} H(f(\xi), \xi)\, d\xi.$$

Proof. Take in (2.8)

$$\rho = \int_{\mathbf{R}} f_0(\xi)\, d\xi, \qquad u = \frac{\int_{\mathbf{R}} f_1(\xi)\, d\xi}{\int_{\mathbf{R}} f_0(\xi)\, d\xi}, \qquad \text{or } u = 0 \text{ if } \rho = 0.$$

Then $M[f](\xi) = M(\rho, u, \xi)$, the terms of the right-hand side are integrable with respect to ξ, and $\int_{\mathbf{R}} M[f](\xi)\, d\xi = \int_{\mathbf{R}} f(\xi)\, d\xi$. Thus the result is obtained by integration in ξ. $\square$

We can also deduce that the entropy dissipation in (1.29) has a sign.

Proposition 2.6 *For any $f \in D$, $\rho \geq 0$, $u, \xi \in \mathbf{R}$,*

$$\left(H'(f,\xi) - \left(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u\right)\right) \cdot (f - M(\rho, u, \xi)) \geq 0. \tag{2.9}$$

Proof. Since H is convex in D, continuous in the cones of Proposition 2.2 and smooth in $int(D)$, we have if $f \neq 0$

$$H(M(\rho, u, \xi), \xi) \geq H(f, \xi) + H'(f, \xi)(M(\rho, u, \xi) - f). \tag{2.10}$$

Letting $f \to 0$ on the axis $f_1 = 0$, we obtain using (1.27) that (2.10) is indeed true for any $f \in D$ with the convention

$$H'(0, \xi) = \left(\frac{\theta}{1-\theta}\frac{\xi^2}{2}, -\frac{\theta}{1-\theta}\xi\right). \tag{2.11}$$

The case $f=0$ can also be seen directly by the definition of H. Finally, by adding (2.10) to (2.8), we get (2.9). $\square$

Let us end this part by estimates deduced from the boundedness of the kinetic entropy.

Proposition 2.7 *Consider $f \in L^1(\mathbf{R}_x \times \mathbf{R}_\xi)$ such that $f \in D$ a.e., and*

$$\iint_{\mathbf{R}\times\mathbf{R}} H(f(x,\xi),\xi)\, dx d\xi \le C_H,$$

$$\iint_{\mathbf{R}\times\mathbf{R}} f_0(x,\xi)\, dx d\xi \le C_0.$$

Then

$$\iint_{\mathbf{R}\times\mathbf{R}} |f_1(x,\xi)|\, dx d\xi \le \sqrt{2C_0 C_H}, \tag{2.12}$$

$$\iint_{\mathbf{R}\times\mathbf{R}} \xi^2 f_0(x,\xi)\, dx d\xi \le \frac{4}{\theta} C_H, \tag{2.13}$$

$$\iint_{\mathbf{R}\times\mathbf{R}} |\xi||f_1(x,\xi)|\, dx d\xi \le \sqrt{\frac{8}{\theta}} C_H, \tag{2.14}$$

$$f_i \text{ is bounded in } L^{p_i}(\mathbf{R}\times\mathbf{R}) \text{ for } i=0,1. \tag{2.15}$$

Proof. We recall that p_0, p_1 are defined by (2.7). First, using (1.13), we have

$$\mathbb{1}_{f_0>0} \frac{f_1^2}{f_0} \le 2H(f,\xi). \tag{2.16}$$

Then, since f_1 vanishes where $f_0 = 0$,

$$|f_1| = |\mathbb{1}_{f_0>0} f_1| = |\mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} \times \sqrt{f_0}| \le \sqrt{2H(f,\xi) \times f_0}. \tag{2.17}$$

Thus, from the Cauchy-Schwarz inequality we obtain (2.12). Next, with (1.13) again,

$$\begin{aligned}
\xi^2 f_0 &= \left| \mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} - \xi\sqrt{f_0} - \mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} \right|^2 \\
&\le \left(\left| \mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} - \xi\sqrt{f_0} \right| + \left| \mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} \right| \right)^2 \\
&\le 2\left[\left(\mathbb{1}_{f_0>0} \frac{f_1}{\sqrt{f_0}} - \xi\sqrt{f_0} \right)^2 + \mathbb{1}_{f_0>0} \frac{f_1^2}{f_0} \right] \\
&\le 2\left[\frac{2(1-\theta)}{\theta} H(f,\xi) + 2H(f,\xi) \right] \\
&= \frac{4}{\theta} H(f,\xi),
\end{aligned}$$

and (2.13) follows. Then, we write

$$\left|\mathbb{1}_{f_0>0}\xi f_1\right| = \left|(\mathbb{1}_{f_0>0}f_1/\sqrt{f_0}) \times (\xi\sqrt{f_0})\right| \leq \sqrt{2H(f,\xi)} \times \sqrt{\frac{4}{\theta}H(f,\xi)},$$

and obtain (2.14). Finally, (2.15) follows obviously from Lemma 2.3. □

3 Stability

In this section, we extend the analysis of [15] and prove the stability of an approximate solution. We shall denote by $\varphi_n(x,\xi) \to \varphi(x,\xi)$ in $L^1(\mathbf{R}_{loc} \times \mathbf{R})$ the convergence in $L^1(\omega \times \mathbf{R})$ for any $\omega \subset\subset \mathbf{R}$, and we shall use several similar conventions.
Let us first recall the classical characteristics formula for (1.2).

Lemma 3.1 *Let $h \in L^1(]0,T[, L^1(\mathbf{R}\times\mathbf{R}))$ and $f^0 \in L^1(\mathbf{R}\times\mathbf{R})$. Then there exists a unique solution*

$$f \in C([0,T], L^1(\mathbf{R}\times\mathbf{R})) \tag{3.1}$$

to the problem

$$\begin{cases} \partial_t f + \xi\partial_x f = \dfrac{h-f}{\varepsilon}, \\ f(0,x,\xi) = f^0(x,\xi). \end{cases} \tag{3.2}$$

Furthermore,

$$f(t,x,\xi) = f^0(x-t\xi,\xi)e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon}h(t-s,x-s\xi,\xi)\,ds. \tag{3.3}$$

We now prove a stability result for this problem.

Proposition 3.2 *Consider $g, g_n \in L^\infty(]0,T[, L^1(\mathbf{R}\times\mathbf{R}))$ such that $g, g_n \in D$ a.e.,*

$$\iint_{\mathbf{R}\times\mathbf{R}} H(g(t,x,\xi),\xi)\,dxd\xi \leq C_H, \qquad \iint_{\mathbf{R}\times\mathbf{R}} g_0(t,x,\xi)\,dxd\xi \leq C_0,$$

$$\iint_{\mathbf{R}\times\mathbf{R}} H(g_n(t,x,\xi),\xi)\,dxd\xi \leq C_H, \qquad \iint_{\mathbf{R}\times\mathbf{R}} (g_n)_0(t,x,\xi)\,dxd\xi \leq C_0.$$

Set

$$(\rho(t,x), \rho u(t,x)) = \int_{\mathbf{R}} g(t,x,\xi)\,d\xi$$

and

$$(\rho_n(t,x), \rho_n u_n(t,x)) = \int_{\mathbf{R}} g_n(t,x,\xi)\,d\xi.$$

If $\rho_n \underset{n\to\infty}{\rightarrow} \rho$ *and* $\rho_n u_n \underset{n\to\infty}{\rightarrow} \rho u$ *in* $L^1(]0,T[\times \mathbf{R}_{loc})$*, then there exists a subsequence such that* $F(g_n) \underset{n\to\infty}{\rightarrow} F(g)$ *in* $C([0,T], L^1(\mathbf{R}_{loc}\times\mathbf{R}))$*, where* $F(g)$ *is the solution to (3.1)-(3.2) with* $h = M[g] = M(\rho,u,\xi)$ *and* $f^0 \in L^1(\mathbf{R}\times\mathbf{R})$ *is fixed. Moreover, if* $\rho_n \to \rho$ *in* $L^1(]0,T[\times\mathbf{R})$*, then* $F(g_n)\to F(g)$ *in* $C([0,T], L^1(\mathbf{R}\times\mathbf{R}))$*.*

Proof. At first, let us check that $F(g)$ is well defined. Using Proposition 2.5, we have

$$\iint_{\mathbf{R}\times\mathbf{R}} H(M[g](t,x,\xi),\xi)\,dxd\xi \le C_H.$$

Besides,

$$\iint_{\mathbf{R}\times\mathbf{R}} M[g]_0(t,x,\xi)\,dxd\xi = \iint_{\mathbf{R}\times\mathbf{R}} g_0(t,x,\xi)\,dxd\xi \le C_0.$$

Estimate (2.12) applied to $M[g]$ gives then that

$$\iint_{\mathbf{R}\times\mathbf{R}} |M[g]_1(t,x,\xi)|\,dxd\xi \le \sqrt{2C_0C_H}.$$

Therefore, we deduce that $M[g] \in L^\infty(]0,T[, L^1(\mathbf{R}\times\mathbf{R}))$, and we have the same result for g_n with uniform bounds.
We turn now to the stability. We can find a subsequence such that $\rho_n\to\rho$ and $\rho_n u_n \to \rho u$ a.e. t,x. Set $E = \{(t,x)\in[0,T]\times\mathbf{R};\ \rho(t,x)=0\}$. In E^c, $\rho_n\to\rho$ and $u_n\to u$ a.e. t,x, thus $M[g_n]\to M[g]$ a.e. $(t,x)\in E^c$, $\xi\in\mathbf{R}$. But since

$$\iint_{(t,x)\in E, x\in B_R} \int M[g_n]_0\,dtdxd\xi = \iint_{(t,x)\in E, x\in B_R} \rho_n\,dtdx \to 0,$$

we deduce with (2.17) and Cauchy-Schwarz inequality that $M[g_n]\to 0$ in $L^1((E\times\mathbf{R})\cap\{x\in B_R\})$. Therefore, after extraction of a subsequence, we finally obtain

$$M[g_n]\to M[g] \qquad \text{a.e. in }]0,T[\times\mathbf{R}\times\mathbf{R}. \tag{3.4}$$

Besides, the estimates of Proposition 2.7 applied to $M[g_n](t,.,.)$ give that $M[g_n]$ is weakly compact in $L^1(]0,T[\times\mathbf{R}_{loc}\times\mathbf{R})$. Therefore, we are able to apply the Vitali theorem, which gives that

$$M[g_n]\to M[g] \ \text{ in } L^1(]0,T[\times\mathbf{R}_{loc}\times\mathbf{R}).$$

Now

$$\begin{aligned}
&\iint_{|x|\le R, |\xi|\le S} |(F(g_n)-F(g))(t,x,\xi)|\,dxd\xi \\
\le\ & \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} \iint_{|x|\le R,|\xi|\le S} |(M[g_n]-M[g])(t-s,x-s\xi,\xi)|\,dxd\xi\,ds \\
\le\ & \frac{1}{\varepsilon}\|M[g_n]-M[g]\|_{L^1(]0,T[\times B_{R+TS}\times B_S)} \longrightarrow 0,
\end{aligned}$$

and

$$
\begin{aligned}
&\iint\limits_{|\xi|>S} |(F(g_n)-F(g))(t,x,\xi)|\,dxd\xi \\
\leq\; & \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} \iint\limits_{|\xi|>S} \frac{|\xi|}{S}|(M[g_n]-M[g])(t-s,x-s\xi,\xi)|\,dxd\xi\,ds \\
\leq\; & \frac{1}{\varepsilon S}\|\xi M[g_n]-\xi M[g]\|_{L^1(]0,T[\times\mathbf{R}\times\mathbf{R})}.
\end{aligned}
$$

Since the latter norm is bounded, we deduce easily that $F(g_n)\to F(g)$ in $C([0,T],L^1(\mathbf{R}_{loc}\times\mathbf{R}))$.
Finally, if we assume moreover that $\rho_n\to\rho$ in $L^1(]0,T[\times\mathbf{R})$, then

$$
\iiint\limits_{]0,T[\times\mathbf{R}\times\mathbf{R}} M[g_n]_0\,dtdxd\xi \longrightarrow \iiint\limits_{]0,T[\times\mathbf{R}\times\mathbf{R}} M[g]_0\,dtdxd\xi,
$$

and since $M[g_n]_0\geq 0$, we deduce that

$$
\iiint\limits_{|x|>R} M[g_n]_0\,dtdxd\xi \longrightarrow 0 \quad \text{as } R\to\infty,
$$

uniformly in n. With (2.17) and Cauchy-Schwarz inequality, we get also

$$
\iiint\limits_{|x|>R} |M[g_n]_1|\,dtdxd\xi \longrightarrow 0 \quad \text{as } R\to\infty,
$$

uniformly in n, and we conclude that $M[g_n]\to M[g]$ in $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$, and therefore $F(g_n)\to F(g)$ in $C([0,T],L^1(\mathbf{R}\times\mathbf{R}))$. $\square$

4 Existence of a solution

This section is devoted to the proof of Theorem 1.1. Let $f^0\in L^1(\mathbf{R}_x\times\mathbf{R}_\xi)$ satisfy the assumptions of Theorem 1.1, namely

$$
f^0(x,\xi)\in D \ a.e. \text{ in } \mathbf{R}\times\mathbf{R}, \tag{4.1}
$$

$$
\iint\limits_{\mathbf{R}\times\mathbf{R}} H(f^0(x,\xi),\xi)\,dxd\xi = C_H^0 < \infty, \tag{4.2}
$$

$$
\iint\limits_{\mathbf{R}\times\mathbf{R}} x^2(f^0)_0(x,\xi)\,dxd\xi = C_\Delta^0 < \infty, \tag{4.3}
$$

and set

$$
\iint\limits_{\mathbf{R}\times\mathbf{R}} (f^0)_0(x,\xi)\,dxd\xi = C_0. \tag{4.4}
$$

We take C_H and C_Δ as

$$C_H = C_H^0, \qquad C_\Delta = \max\left(C_\Delta^0; \frac{4\varepsilon^2}{\theta} C_H^0\right). \tag{4.5}$$

Define C to be the set of all $g \in L^\infty(]0,T[, L^1(\mathbf{R}\times\mathbf{R}))$ satisfying $(C1)-(C4)$ for a.e. $t \in [0,T]$, where

$$\begin{aligned}
&(C1) \quad g(t,x,\xi) \in D \quad \text{a.e. in } \mathbf{R}\times\mathbf{R},\\
&(C2) \quad \iint_{\mathbf{R}\times\mathbf{R}} H(g(t,x,\xi),\xi)\,dxd\xi \le C_H,\\
&(C3) \quad \iint_{\mathbf{R}\times\mathbf{R}} g_0(t,x,\xi)\,dxd\xi \le C_0,\\
&(C4) \quad \iint_{\mathbf{R}\times\mathbf{R}} x^2 g_0(t,x,\xi)\,dxd\xi \le C_\Delta\,(1+t/\varepsilon)^2.
\end{aligned} \tag{4.6}$$

Let us also introduce

$$\begin{aligned}
\tilde{C} = \Big\{ g \in C([0,T], L^1(\mathbf{R}\times\mathbf{R})) \text{ satisfying } (C5),\\
\text{and } (C1)-(C4) \text{ for all } t\in[0,T]\Big\},
\end{aligned} \tag{4.7}$$

with

$$(C5) \quad \partial_t g + \xi\partial_x g + \frac{g}{\varepsilon} \in \frac{C}{\varepsilon}\cdot$$

The initial data f^0 being fixed, for any $g \in C$, we denote by $F(g)$ the solution to (3.1)-(3.2) with $h = M[g]$.

Lemma 4.1 *If $g \in C$, then $M[g] \in C$ and $F(g) \in \tilde{C}$.*

Proof. Let $g \in C$. As in the proof of Proposition 3.2, $M[g] \in L^\infty(]0,T[, L^1(\mathbf{R}\times\mathbf{R}))$ and we get easily that $M[g] \in C$. Next, according to Lemma 3.1, $F(g) \in C([0,T], L^1(\mathbf{R}\times\mathbf{R}))$, and in order to prove that $F(g) \in \tilde{C}$, we need to prove $(C1)-(C5)$. Condition $(C5)$ is satisfied since $M[g] \in C$. Then, for $i = 0;1$, we have

$$F(g)_i(t,x,\xi) = (f^0)_i(x-t\xi,\xi)e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} M[g]_i(t-s,x-s\xi,\xi)\,ds. \tag{4.8}$$

Obviously, $F(g)_0 \ge 0$. Furthermore, if $F(g)_0(t,x,\xi) = 0$ then $(f^0)_0(x-t\xi,\xi) = 0$ and $\int_0^t e^{-s/\varepsilon} M[g]_0(t-s,x-s\xi,\xi)\,ds = 0$, thus $M[g]_0(t-s,x-s\xi,\xi) = 0$ a.e. $s \in]0,t[$. But $f^0 \in D$ and $M[g] \in D$, thus $(f^0)_1(x-t\xi,\xi) = 0$ and $M[g]_1(t-s,x-s\xi,\xi) = 0$ a.e. $s \in]0,t[$, and therefore $F(g)_1(t,x,\xi) = 0$. Hence $F[g] \in D$ $\forall t \in [0,T]$, a.e. x,ξ, and $(C1)$ is satisfied. Next, if $t > 0$, we can write

$$F(g)(t,x,\xi) = f^0(x-t\xi,\xi)e^{-t/\varepsilon} + (1-e^{-t/\varepsilon})\int_0^t M[g](t-s,x-s\xi,\xi)\frac{e^{-s/\varepsilon}\,ds}{\int_0^t e^{-\sigma/\varepsilon}\,d\sigma}, \tag{4.9}$$

and using Jensen's inequality with the convex function H, we get

$$\begin{aligned}
& H(F(g)(t,x,\xi),\xi) \\
\leq\ & H(f^0(x-t\xi,\xi),\xi)e^{-t/\varepsilon} \\
& +(1-e^{-t/\varepsilon})H\Big(\int_0^t M[g](t-s,x-s\xi,\xi)\frac{e^{-s/\varepsilon}\,ds}{\int_0^t e^{-\sigma/\varepsilon}\,d\sigma},\xi\Big) \\
\leq\ & H(f^0(x-t\xi,\xi),\xi)e^{-t/\varepsilon} \\
& +(1-e^{-t/\varepsilon})\int_0^t H(M[g](t-s,x-s\xi,\xi),\xi)\frac{e^{-s/\varepsilon}\,ds}{\int_0^t e^{-\sigma/\varepsilon}\,d\sigma} \\
=\ & H(f^0(x-t\xi,\xi),\xi)e^{-t/\varepsilon} \\
& +\frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon}H(M[g](t-s,x-s\xi,\xi),\xi)\,ds,
\end{aligned}$$

and thus

$$\iint_{\mathbf{R}\times\mathbf{R}} H(F(g)(t,x,\xi),\xi)\,dxd\xi \leq e^{-t/\varepsilon}C_H^0 + (1-e^{-t/\varepsilon})C_H \leq C_H,$$

and $(C2)$ is satisfied. Similarly, integrating (4.8) with $i=0$ with respect to (x,ξ) gives $(C3)$. It only remains now to prove $(C4)$. We have by (2.13)

$$\begin{aligned}
& \iint_{\mathbf{R}\times\mathbf{R}} x^2(f^0)_0(x-t\xi,\xi)\,dxd\xi \\
=\ & \iint (y+t\xi)^2(f^0)_0(y,\xi)\,dyd\xi \\
\leq\ & \left(\sqrt{\iint y^2(f^0)_0(y,\xi)\,dyd\xi} + t\sqrt{\iint \xi^2(f^0)_0(y,\xi)\,dyd\xi}\right)^2 \\
\leq\ & \left(\sqrt{C_\Delta^0} + t\sqrt{4C_H^0/\theta}\right)^2.
\end{aligned}$$

Then, since $M[g]\in C$, relation $(C4)$ gives

$$\iint_{\mathbf{R}\times\mathbf{R}} x^2 M[g]_0(s,x,\xi)\,dxd\xi \leq C_\Delta\,(1+s/\varepsilon)^2,$$

hence

$$\begin{aligned}
& \iint_{\mathbf{R}\times\mathbf{R}} x^2 M[g]_0(t-s,x-s\xi,\xi)\,dxd\xi \\
=\ & \iint (y+s\xi)^2 M[g]_0(t-s,y,\xi)\,dyd\xi \\
\leq\ & \left(\sqrt{\iint y^2 M[g]_0(t-s,y,\xi)\,dyd\xi} + s\sqrt{\iint \xi^2 M[g]_0(t-s,y,\xi)\,dyd\xi}\right)^2 \\
\leq\ & \left(\sqrt{C_\Delta\left(1+\frac{t-s}{\varepsilon}\right)^2} + s\sqrt{4C_H/\theta}\right)^2.
\end{aligned}$$

Therefore, with (4.5),

$$\begin{aligned}
&\iint\limits_{\mathbf{R}\times\mathbf{R}} x^2 F(g)_0(t,x,\xi)\,dxd\xi \\
\le\;& \left(\sqrt{C_\Delta}+t\sqrt{4C_H/\theta}\right)^2 e^{-t/\varepsilon} \\
&+\frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon}\left(\sqrt{C_\Delta}\left(1+\frac{t-s}{\varepsilon}\right)+s\sqrt{4C_H/\theta}\right)^2 ds \\
\le\;& C_\Delta\left(1+\frac{t}{\varepsilon}\right)^2 e^{-t/\varepsilon} \\
&+\frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} C_\Delta\left(1+\frac{t}{\varepsilon}\right)^2 ds \\
=\;& C_\Delta\left(1+\frac{t}{\varepsilon}\right)^2.
\end{aligned}$$

This ends the proof of the lemma. □

Since $\tilde{C}\subset C$, we are now able to define $F:\tilde{C}\to\tilde{C}$ and we shall use the Schauder theorem to prove Theorem 1.1.

Lemma 4.2 *The sets C and $\tilde{C}$ are convex and not empty, C is compact for the weak topology of $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$, and $\tilde{C}$ is closed in $C([0,T],L^1(\mathbf{R}\times\mathbf{R}))$.*

Proof. Since H is convex, it is obvious that C and $\tilde{C}$ are convex. Then, the constant f^0 belongs to C, and by Lemma 4.1, $F(f_0)\in\tilde{C}$. Thus C and $\tilde{C}$ are not empty. Next, from Proposition 2.7, $(C4)$, (2.17) and Dunford-Pettis' theorem, C is relatively weakly compact in $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$. Let us prove now that C is closed in weak $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$. Since C is convex, it is enough to prove that C is closed in strong $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$. Thus let $(g_n)_n$ be a sequence in C which converges to g in $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$. After extraction of a subsequence, $g_n(t,.)\to g(t,.)$ in $L^1(\mathbf{R}\times\mathbf{R})$ and a.e. for a.e. $t\in]0,T[$. First, $(g_n)_0\ge 0$ thus $g_0\ge 0$. Then, since by (2.17) $|(g_n)_1|\le\sqrt{2H(g_n,\xi)(g_n)_0}$, we get for a fixed time t and any Borel set $\mathcal{V}$ by Fatou's lemma and Cauchy-Schwarz inequality

$$\begin{aligned}
\iint\limits_{\mathcal{V}}|g_1(t,x,\xi)|\,dxd\xi \;&\le\; \underline{\lim}\iint\limits_{\mathcal{V}}\sqrt{2H(g_n,\xi)(g_n)_0}\,dxd\xi \\
&\le\; \underline{\lim}\sqrt{2C_H\iint\limits_{\mathcal{V}}(g_n)_0\,dxd\xi} \\
&=\; \sqrt{2C_H\iint\limits_{\mathcal{V}} g_0\,dxd\xi}.
\end{aligned}$$

Taking $\mathcal{V} = \{x, \xi\ ;\ g_0(t,x,\xi) = 0\}$, we obtain that $g_1(t,x,\xi)$ vanishes a.e. in $\mathcal{V}$, thus $g(t,x,\xi) \in D$ a.e. Then, we apply Fatou's lemma again, and get

$$\begin{aligned}\iint\limits_{\mathbf{R}\times\mathbf{R}} H(g(t,x,\xi),\xi)\,dxd\xi &= \iint\limits_{g_0>0} H(g(t,x,\xi),\xi)\,dxd\xi \\ &\leq \underline{\lim} \iint\limits_{g_0>0} H(g_n(t,x,\xi),\xi)\,dxd\xi \\ &\leq C_H.\end{aligned}$$

Similar applications of Fatou's lemma finally give that $g \in L^\infty(]0,T[, L^1(\mathbf{R}\times\mathbf{R}))$, and $g \in C$. The closedness of $\tilde{C}$ in $C([0,T], L^1(\mathbf{R}\times\mathbf{R}))$ is treated in a similar way, and $(C5)$ follows from the compactness of C. □

Lemma 4.3 *F is continuous in $\tilde{C}$ for the topology $C([0,T], L^1(\mathbf{R}\times\mathbf{R}))$.*

Proof. Let $g_n, g \in \tilde{C}$ with $g_n \to g$ in $C([0,T], L^1(\mathbf{R}\times\mathbf{R}))$. Then, with the notations of Proposition 3.2, $\rho_n \to \rho$ and $\rho_n u_n \to \rho u$ in $C([0,T], L^1(\mathbf{R}))$. Thus Proposition 3.2 gives the existence of a subsequence such that $F(g_n) \to F(g)$ in $C([0,T], L^1(\mathbf{R}\times\mathbf{R}))$, which is enough to prove the continuity of F. □

Lemma 4.4 *$F(\tilde{C})$ is relatively compact in $C([0,T], L^1(\mathbf{R}\times\mathbf{R}))$.*

Proof. Let $f_n = F(g_n)$, $g_n \in \tilde{C}$ be a sequence in $F(\tilde{C})$. Since $\tilde{C} \subset C$, by Lemma 4.2 there exist $g \in C$ and a subsequence such that $g_n \rightharpoonup g$ in weak $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$. Then with the notations of Proposition 3.2, $\rho_n \rightharpoonup \rho$, $\rho_n u_n \rightharpoonup \rho u$ in weak $L^1(]0,T[\times\mathbf{R})$. But by $(C5)$, $\partial_t g_n + \xi\partial_x g_n + g_n/\varepsilon = h_n/\varepsilon$, with $h_n \in C$. By the compactness averaging lemma of [8] and by (2.13)-(2.14), we deduce that $\int_{\mathbf{R}} g_n(t,x,\xi)\,d\xi$ is compact in $L^1_{loc}(]0,T[\times\mathbf{R})$. We conclude that $\rho_n \to \rho$ and $\rho_n u_n \to \rho u$ in $L^1(]0,T[\times\mathbf{R})$. Proposition 3.2 gives thus the existence of a subsequence such that $f_n = F(g_n) \to F(g)$ in $C([0,T], L^1(\mathbf{R}\times\mathbf{R}))$. □

Proof of Theorem 1.1. We apply Schauder's theorem in $C([0,T], L^1(\mathbf{R}\times\mathbf{R}))$ to the operator $F : \tilde{C} \to \tilde{C}$. Using all the results of this section, $\tilde{C}$ is convex, closed and non empty, F is continuous $\tilde{C} \to \tilde{C}$ and $F(\tilde{C})$ is relatively compact. Thus we conclude the existence of a fixed point $f \in \tilde{C}$ verifying $F(f) = f$. This gives a solution in $[0,T]$ for any $T > 0$, and by extraction of a diagonal subsequence, we obtain a solution in $[0,\infty[$. Relation (1.21) comes clearly from (1.2) because $\xi f(t,x,\xi) \in L^\infty(]0,\infty[, L^1(\mathbf{R}\times\mathbf{R}))$ by Proposition 2.7, and (1.22) is obtained by integration of (4.8). □

Proof of Theorem 1.2. If the assumption (1.17) is not satisfied, we have to use the Tychonoff-Schauder fixed point theorem in the locally convex topological vector space $C([0,T], L^1(\mathbf{R}_{loc,x}\times\mathbf{R}_\xi))$. Let us mention the changes that need to be performed to the results of Section 4. First, we have to remove condition $(C4)$ in the definitions of C and $\tilde{C}$. Then, in Lemma 4.2, C is compact for the weak topology of $L^1(]0,T[\times\mathbf{R}_{loc}\times\mathbf{R})$, and $\tilde{C}$ is closed in $C([0,T], L^1(\mathbf{R}_{loc}\times\mathbf{R})$. In Lemma 4.3, F is continuous in $\tilde{C}$ for the topology $C([0,T], L^1(\mathbf{R}_{loc}\times\mathbf{R}))$, and in Lemma 4.4, $F(\tilde{C})$ is relatively compact in $C([0,T], L^1(\mathbf{R}_{loc}\times\mathbf{R}))$. □

5 Entropy equation

In this section we prove that the entropy $H(f(t,x,\xi),\xi)$ satisfies a renormalized equation, as stated in Theorem 1.3. The difficulty is that H is not smooth at the origin, and we have to build a monotone approximation of it that is convex, smooth and that has slow growth at infinity.

5.1 Approximation of H

The most singular term in H in (1.13) is obviously

$$\Phi(f) = \mathbb{1}_{f_0>0} f_1^2/f_0, \qquad f \in D. \tag{5.1}$$

We begin by translating the singularity $f_0 = 0$ to $f_0 = -\delta$, by defining for $\delta > 0$

$$\Phi_\delta(f) = \frac{f_1^2}{f_0+\delta}, \qquad f \in D, \tag{5.2}$$

which is convex and smooth in D, hence

$$\Phi_\delta(f) = \sup_{g\in D} \left(\Phi_\delta(g) + \Phi_\delta'(g)\cdot(f-g)\right), \qquad f \in D. \tag{5.3}$$

In order to obtain a linear growth at infinity, we set

$$\begin{aligned} \varphi_\delta(f) &= \sup_{g\in D_\delta} \left(\Phi_\delta(g) + \Phi_\delta'(g)\cdot(f-g)\right), \qquad f \in D,\\ D_\delta &= \left\{ f \in D\ ;\ \Big|\frac{f_1}{f_0+\delta}\Big| \le c_\delta \right\}, \qquad c_\delta = 1/\delta^2. \end{aligned} \tag{5.4}$$

One can check that with this choice of c_δ,

$$\text{if} \quad 0 < \delta' \le \delta \quad \text{then} \quad D_\delta \subset D_{\delta'}. \tag{5.5}$$

We have obviously $\varphi_\delta(f) \le \Phi_\delta(f) \le \Phi(f)$.

Lemma 5.1 *i) The value of φ_δ is*

$$\varphi_\delta(f) = \begin{cases} \dfrac{f_1^2}{f_0+\delta} & \text{if } f \in D_\delta,\\ -c_\delta^2(f_0+\delta) + 2c_\delta|f_1| & \text{if } f \in D\setminus D_\delta, \end{cases} \tag{5.6}$$

ii) the function φ_δ is convex in D, nonnegative, $\varphi_\delta(0)=0$,
iii) we have $\varphi_\delta(f) \le 2c_\delta|f_1|$ for $f \in D$,
iv) if $0 < \delta' \le \delta$ then $\varphi_\delta(f) \le \varphi_{\delta'}(f)$ for $f \in D$, and $\varphi_\delta(f) \uparrow \Phi(f)$ as $\delta \to 0$,
v) the function φ_δ is C^1 in $\overline{D}$ and φ_δ' is bounded in $\overline{D}$.

Proof. *i)* If $f \in D_\delta$, obviously $\varphi_\delta(f) = \Phi_\delta(f)$. If $f \in D \setminus D_\delta$,

$$\Phi_\delta(g) + \Phi'_\delta(g)\cdot(f-g) = -\frac{g_1^2}{(g_0+\delta)^2}(f_0+\delta) + 2\frac{g_1}{g_0+\delta}f_1,$$

hence

$$\varphi_\delta(f) = \sup_{-c_\delta \le x \le c_\delta} (-x^2(f_0+\delta) + 2xf_1).$$

If $f_1 \geq 0$, the supremum is reached at $x = c_\delta$ because $c_\delta < f_1/(f_0+\delta)$, and if $f_1 \leq 0$, it is reached at $-c_\delta$, and this gives the result.
ii) The convexity is obvious from the definition, the nonnegativity also since $0 \in D_\delta$, and $\varphi_\delta(0)$ is computed by (5.6).
iii) If $f \in D_\delta$ then $\varphi_\delta(f) = f_1^2/(f_0+\delta) \leq c_\delta|f_1|$, and if $f \in D \setminus D_\delta$, then $\varphi_\delta(f) = -c_\delta^2(f_0+\delta) + 2c_\delta|f_1| \leq 2c_\delta|f_1|$.
iv) Let $0 < \delta' \leq \delta$. If $f \in D_\delta$ then $f \in D_{\delta'}$ by (5.5), thus $\varphi_\delta(f) = f_1^2/(f_0+\delta) \leq f_1^2/(f_0+\delta') = \varphi_{\delta'}(f)$. If $f \in D_{\delta'} \setminus D_\delta$, then $\varphi_\delta(f) \leq \Phi_\delta(f) \leq \Phi_{\delta'}(f) = \varphi_{\delta'}(f)$. If $f \in D \setminus D_{\delta'}$, i.e. $|f_1| > c_{\delta'}(f_0+\delta')$, then since $c_\delta \leq c_{\delta'}$,

$$\begin{aligned}
\varphi_\delta(f) - \varphi_{\delta'}(f) &= -c_\delta^2(f_0+\delta) + c_{\delta'}^2(f_0+\delta') + 2(c_\delta - c_{\delta'})|f_1| \\
&\leq -c_\delta^2(f_0+\delta') + c_{\delta'}^2(f_0+\delta') + 2(c_\delta - c_{\delta'})c_{\delta'}(f_0+\delta') \\
&= -(c_{\delta'} - c_\delta)^2(f_0+\delta') \\
&\leq 0.
\end{aligned}$$

Finally, if $f \in D$ is fixed, for small enough δ we have $f \in D_\delta$, $\varphi_\delta(f) = f_1^2/(f_0+\delta) \to \mathbb{1}_{f_0>0}f_1^2/f_0$ as $\delta \to 0$.
v) Formula (5.6) gives obviously two smooth functions. We have to check the continuity of the functions and their derivatives at the frontier where $|f_1| = c_\delta(f_0+\delta)$. The continuity of φ_δ is straightforward. For the derivatives, we have

$$\varphi'_\delta(f) = \begin{cases} \left(\dfrac{-f_1^2}{(f_0+\delta)^2}, \dfrac{2f_1}{f_0+\delta}\right), & f \in int(D_\delta), \\ (-c_\delta^2, 2c_\delta \operatorname{sgn} f_1), & f \in int(D \setminus D_\delta), \end{cases} \tag{5.7}$$

and we see that the two formula match when $|f_1| = c_\delta(f_0+\delta)$. We finally notice that $|\varphi'_\delta| \leq c_\delta^2 + 2c_\delta$. $\square$

5.2 Sketch of the proof

In order to prove the equation on $H(f(t,x,\xi),\xi)$, we decompose H as

$$H(f,\xi) = \frac{\theta}{1-\theta}\frac{\xi^2}{2}f_0 - \frac{\theta}{1-\theta}\xi f_1 + \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda} + \frac{1}{1-\theta}\frac{1}{2}\mathbb{1}_{f_0>0}\frac{f_1^2}{f_0}, \tag{5.8}$$

and we are going to consider each term successively. A crucial argument is the following lemma.

Lemma 5.2 *Assume that $g \in C([0,T], L^1_{loc}(\mathbf{R}_x \times \mathbf{R}_\xi))$ verifies*

$$g(0) \in L^1(\mathbf{R}\times\mathbf{R}), \qquad \text{and} \qquad \partial_t g + \xi\partial_x g = h \in L^1(]0,T[\times\mathbf{R}\times\mathbf{R}). \tag{5.9}$$

Then

$$g \in C([0,T], L^1(\mathbf{R} \times \mathbf{R})), \tag{5.10}$$

$$\xi g \in L^1(]0,T[\times B_R \times \mathbf{R}), \qquad R > 0, \tag{5.11}$$

$$g(t,x,\xi) = g(0, x - t\xi, \xi) + \int_0^t h(t-s, x - s\xi, \xi)\, ds, \tag{5.12}$$

$$\partial_t \int g\, d\xi + \partial_x \int \xi g\, d\xi = \int h\, d\xi \qquad in \quad]0,T[\times \mathbf{R}, \tag{5.13}$$

$$\frac{d}{dt} \iint\limits_{\mathbf{R}\times\mathbf{R}} g(t,x,\xi)\, dx d\xi = \iint\limits_{\mathbf{R}\times\mathbf{R}} h(t,x,\xi)\, dx d\xi \qquad in \quad]0,T[. \tag{5.14}$$

Proof. Define $\tilde{g}(t,x,\xi) = g(t, x + t\xi, \xi)$. Then $\tilde{g} \in C([0,T], L^1_{loc}(\mathbf{R} \times \mathbf{R}))$, and $\tilde{g}$ verifies $\tilde{g}(0) = g(0) \in L^1(\mathbf{R} \times \mathbf{R})$, $\partial_t \tilde{g} = h(t, x + t\xi, \xi) \in L^1(]0,T[\times \mathbf{R} \times \mathbf{R})$. Therefore,

$$\tilde{g}(t,x,\xi) = g(0,x,\xi) + \int_0^t h(s, x + s\xi, \xi)\, ds,$$

and we deduce (5.12) and (5.10). Then, (5.11) comes from the higher moment lemma of [15], and (5.13) follows by integration of (5.9). Finally, integrating (5.12) with respect to (x, ξ), we get for any $t \in [0,T]$

$$\iint g(t,x,\xi)\, dx d\xi = \iint g(0,x,\xi)\, dx d\xi + \int_0^t \iint\limits_{\mathbf{R}\times\mathbf{R}} h(s,x,\xi)\, dx d\xi ds, \tag{5.15}$$

which is the integral form of (5.14). $\square$

5.3 Linear terms

We are now able to treat the first two terms in (5.8). From now on, f denotes the solution obtained in Theorems 1.1 or 1.2. We recall that the estimates of Proposition 2.7 apply to $f(t,.)$ and to $M[f](t,.)$ for any $t \geq 0$. Thus we can apply Lemma 5.2 to $\xi^2 f_0(t,x,\xi)$ and $\xi f_1(t,x,\xi)$, and we conclude that

$$\xi^2 f_0(t,x,\xi), \quad \xi f_1(t,x,\xi) \quad \in C([0,\infty[, L^1(\mathbf{R} \times \mathbf{R})), \tag{5.16}$$

and all the conclusions of Lemma 5.2 are valid.

5.4 Nonlinear terms

Let us first consider the last term in (5.8), corresponding to $\Phi(f) = \mathbb{I}_{f_0>0} f_1^2/f_0$. We use the approximation $\varphi_\delta(f)$ defined in Section 5.1. Since φ_δ is C^1 and globally Lipschitz continuous, a vector adaptation of the classical renormalization property allows to multiply (1.2) by $\varphi'_\delta(f)$, which yields

$$\partial_t \varphi_\delta(f) + \xi \partial_x \varphi_\delta(f) = \varphi'_\delta(f) \frac{M[f] - f}{\varepsilon}. \tag{5.17}$$

Since $\varphi_\delta(f) \in C([0,T], L^1(\mathbf{R}\times\mathbf{R}))$, we can apply Lemma 5.2 to $\varphi_\delta(f)$. We wish now to let $\delta \to 0$ in (5.17). We notice that Φ in (5.1) is smooth outside the origin, and we define $\Phi'(0) = 0$ by convention. Thus

$$\Phi'(f) = \left(-\mathbb{1}_{f_0>0} \frac{f_1^2}{f_0^2}, \mathbb{1}_{f_0>0} \frac{2f_1}{f_0} \right), \qquad f \in D. \tag{5.18}$$

Lemma 5.3 *The terms $\partial_0\Phi(f)f_0$, $\partial_1\Phi(f)f_1$, $\partial_0\Phi(f)M_0[f]$, $\partial_1\Phi(f)M_1[f]$ lie in $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$. The right-hand side of (5.17) is bounded in $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$ uniformly in δ, and tends to $\Phi'(f)(M[f]-f)/\varepsilon$ in $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$ as $\delta \to 0$.*

Proof. Since $0 \le \Phi(f) \le 2H(f,\xi)$ for any ξ, we have

$$\Phi(f(t,x,\xi)), \quad \Phi(M[f](t,x,\xi)) \in L^\infty(]0,\infty[, L^1(\mathbf{R}\times\mathbf{R})). \tag{5.19}$$

Then, $\partial_0\Phi(f)f_0 = -\mathbb{1}_{f_0>0}f_1^2/f_0 = -\Phi(f) \in L^\infty(]0,\infty[, L^1(\mathbf{R}\times\mathbf{R}))$, $\partial_1\Phi(f)f_1 = \mathbb{1}_{f_0>0}2f_1^2/f_0 = 2\Phi(f) \in L^\infty(]0,\infty[, L^1(\mathbf{R}\times\mathbf{R}))$. Next, since $0 \le \varphi_\delta(f) \le \Phi(f)$, $\varphi_\delta(f)$ is bounded in $C([0,T], L^1(\mathbf{R}\times\mathbf{R}))$ independently of δ. Therefore, by (5.17) and by applying the identity (5.15) to $\varphi_\delta(f)$, we obtain that

$$\iiint_{]0,T[\times\mathbf{R}\times\mathbf{R}} \varphi'_\delta(f)\,(M[f]-f)\; dtdxd\xi \quad \text{is bounded independently of } T \text{ and } \delta. \tag{5.20}$$

Then, since φ_δ is convex in D,

$$\varphi_\delta(M[f]) - \varphi_\delta(f) - \varphi'_\delta(f)\,(M[f]-f) \ge 0. \tag{5.21}$$

According to (5.20), this function is bounded in $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$ independently of δ. Observing that with (5.7), $\varphi'_\delta(f) \to \Phi'(f)$ as $\delta \to 0$, by applying Fatou's lemma, we deduce that

$$\Phi(M[f]) - \Phi(f) - \Phi'(f)\,(M[f]-f) \quad \in L^1(]0,T[\times\mathbf{R}\times\mathbf{R}), \tag{5.22}$$

and therefore

$$\Phi'(f)\cdot M[f] = -\mathbb{1}_{f_0>0}\frac{f_1^2}{f_0^2}M_0[f] + \mathbb{1}_{f_0>0}\frac{2f_1}{f_0}M_1[f] \in L^1(]0,T[\times\mathbf{R}\times\mathbf{R}). \tag{5.23}$$

From the inequality

$$2\left|\frac{f_1}{f_0}M_1[f]\right| \le \frac{1}{4}\frac{f_1^2}{f_0^2}M_0[f] + 4\frac{M_1[f]^2}{M_0[f]}\mathbb{1}_{M_0[f]>0},$$

we deduce

$$\begin{aligned}\frac{f_1^2}{f_0^2}M_0[f] &= 2\frac{f_1}{f_0}M_1[f] - \left(-\frac{f_1^2}{f_0^2}M_0[f] + 2\frac{f_1}{f_0}M_1[f]\right)\\ &\le \frac{1}{4}\frac{f_1^2}{f_0^2}M_0[f] + 4\frac{M_1[f]^2}{M_0[f]}\mathbb{1}_{M_0[f]>0} - \left(-\frac{f_1^2}{f_0^2}M_0[f] + 2\frac{f_1}{f_0}M_1[f]\right),\end{aligned}$$

and thus

$$0 \le \mathbb{1}_{f_0>0}\frac{3}{4}\frac{f_1^2}{f_0^2}M_0[f] \le 4\Phi(M[f]) - \Phi'(f)\cdot M[f]. \tag{5.24}$$

We conclude that

$$\partial_0\Phi(f)M_0[f], \quad \partial_1\Phi(f)M_1[f] \quad \in L^1(]0,T[\times\mathbf{R}\times\mathbf{R}). \tag{5.25}$$

Finally, we observe with (5.7) that

$$|\partial_0\varphi_\delta(f)| \le |\partial_0\Phi(f)|, \qquad |\partial_1\varphi_\delta(f)| \le |\partial_1\Phi(f)|, \tag{5.26}$$

and we conclude with Lebesgue's theorem the convergence of each term of the right-hand side of (5.17) in $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$ as $\delta\to 0$. □

Now, we are able to conclude. For any $t\ge 0$, $\varphi_\delta(f(t,.))\to\Phi(f(t,.))$ in $L^1(\mathbf{R}\times\mathbf{R})$ as $\delta\to 0$ by Lebesgue's theorem. But we can write (5.12) for $\varphi_\delta(f)$,

$$\varphi_\delta(f(t,x,\xi)) = \varphi_\delta(f^0(x-t\xi,\xi)) + \int_0^t \varphi_\delta'(f)\frac{M[f]-f}{\varepsilon}(t-s,x-s\xi,\xi)\,ds.$$

By Lemma 5.3, we can pass to the limit in the right-hand side in $C([0,T],L^1(\mathbf{R}\times\mathbf{R}))$. Therefore,

$$\Phi(f(t,x,\xi)) = \Phi(f^0(x-t\xi,\xi)) + \int_0^t \Phi'(f)\frac{M[f]-f}{\varepsilon}(t-s,x-s\xi,\xi)\,ds, \tag{5.27}$$

$$\Phi(f)\in C([0,T],L^1(\mathbf{R}\times\mathbf{R})), \tag{5.28}$$

$\varphi_\delta(f)\to\Phi(f)$ in $C([0,T],L^1(\mathbf{R}\times\mathbf{R}))$, and

$$\partial_t\Phi(f)+\xi\partial_x\Phi(f) = \Phi'(f)\frac{M[f]-f}{\varepsilon}\cdot \tag{5.29}$$

For the last term $\psi(f_0) = f_0^{1+1/\lambda}/(1+1/\lambda)$ in (5.8), we can use the approximation

$$\psi_\delta(f_0) = \begin{cases} \psi(f_0) & \text{if } \ 0\le f_0\le 1/\delta, \\ \psi(\frac{1}{\delta}) + \psi'(\frac{1}{\delta})(f_0-\frac{1}{\delta}) & \text{if } \ f_0 \ge 1/\delta. \end{cases} \tag{5.30}$$

Then the argument is similar as above (even simpler, $\psi'(f_0)M_0[f]\in L^1$ is obtained directly since $f_0^{1/\lambda}\in L^{\lambda(1+1/\lambda)}$, $M_0[f]\in L^{1+1/\lambda}$), and we do not repeat the proof.

5.5 Proof of Theorem 1.3

According to the previous subsections, we can apply Lemma 5.2 to each term of the decomposition (5.8). Noticing that the convention on the value of $H'(0,\xi)$ corresponds to $\Phi'(0)=0$, we obtain (1.25)-(1.28). Moreover, $\xi H(f(t,x,\xi),\xi)\in L^1(]0,T[\times B_R\times\mathbf{R})$ and

$$\partial_t\int_{\mathbf{R}} H(f,\xi)\,d\xi + \partial_x\int_{\mathbf{R}}\xi H(f,\xi)\,d\xi = \frac{1}{\varepsilon}\int_{\mathbf{R}} H'(f,\xi)\cdot(M[f]-f)\,d\xi, \tag{5.31}$$

$$\frac{d}{dt}\iint\limits_{\mathbf{R}\times\mathbf{R}} H(f,\xi)\,dxd\xi = \frac{1}{\varepsilon}\iint\limits_{\mathbf{R}\times\mathbf{R}} H'(f,\xi)\cdot(M[f]-f)\,dxd\xi. \tag{5.32}$$

But by Proposition 2.6,

$$P(t,x,\xi) \equiv \left(H'(f,\xi) - \big(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u\big)\right)\cdot(f - M[f]) \geq 0 \quad \text{a.e.}, \tag{5.33}$$

and since $\int (f - M[f])\,d\xi = 0$,

$$\iiint\limits_{]0,T[\times\mathbf{R}\times\mathbf{R}} P(t,x,\xi)\,dtdxd\xi = \iiint\limits_{]0,T[\times\mathbf{R}\times\mathbf{R}} H'(f,\xi)\cdot(f-M[f])\ dtdxd\xi < \infty, \tag{5.34}$$

thus $P \in L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$. Finally, since

$$\int_{\mathbf{R}} P(t,x,\xi)\,d\xi = \int_{\mathbf{R}} H'(f,\xi)\cdot(f-M[f])\ d\xi \quad \text{a.e. } t,x, \tag{5.35}$$

from (5.31) and (5.32) we obtain (1.29) and (1.30). □

Remark 5.1 It is possible to prove directly that

$$\left(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u\right)\cdot(M[f]-f) \in L^\infty(]0,T[, L^1(\mathbf{R}\times\mathbf{R})). \tag{5.36}$$

Indeed by (2.4) and Proposition 2.5,

$$\frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma-1}\rho^\gamma = \eta(\rho,u) \leq \int H(f,\xi)\,d\xi, \tag{5.37}$$

thus

$$\int_{\mathbf{R}}\left(\frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma-1}\rho^\gamma\right)dx \leq C_H^0, \tag{5.38}$$

and ρu^2, $\rho^\gamma \in L^\infty(]0,\infty[, L^1(\mathbf{R}))$. Then, the inequality

$$\begin{aligned}&\left(\iint |u(t,x)f_1(t,x,\xi)|\ dxd\xi\right)^2\\ &\leq \left(\iint u(t,x)^2 f_0(t,x,\xi)\,dxd\xi\right)\times\left(\iint\limits_{f_0(t,x,\xi)>0} \frac{f_1(t,x,\xi)^2}{f_0(t,x,\xi)}\,dxd\xi\right)\end{aligned} \tag{5.39}$$

proves that $\|uf_1\|_{L^1(\mathbf{R}\times\mathbf{R})} \leq 2C_H^0$. A similar estimate holds for $uM_1[f]$, and the other terms in (5.36) are easy to estimate with (5.38).

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Chapitre 2

Domaines invariants cinétiques et relaxation d'un modèle BGK vers la dynamique des gaz isentropiques

Kinetic invariant domains and relaxation limit from a BGK model to isentropic gas dynamics

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Abstract

We obtain rigorously the correspondence between kinetic and macroscopic invariant domains for a vector kinetic BGK system considered in a previous paper of the authors. We establish the convergence, as the relaxation parameter tends to 0, to weak entropy solutions of isentropic gas dynamics equations.

Key-words: isentropic gas dynamics – kinetic BGK model – kinetic entropy – kinetic invariant domain – relaxation limit

Mathematics Subject Classification: 35L65, 76N15, 35B35, 82C40

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Contents

1 Introduction and main results

In this paper we consider the vector kinetic BGK model already studied in [3], which leads in the relaxation limit to the one-dimensional system of isentropic gas dynamics

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \kappa\rho^\gamma) = 0, \end{cases} \tag{1.1}$$

with $\rho(t,x) \geq 0,\ u(t,x) \in \mathbf{R}$ and $\kappa > 0,\ 1 < \gamma < 3$. The aim of this paper is to obtain rigorously the correspondence between kinetic invariant domains of the BGK model and the macroscopic ones of (1.1), to establish a maximum principle and to prove the convergence as the relaxation parameter tends to 0. The kinetic model has been introduced in [4], and can be written as

$$\partial_t f + \xi\partial_x f = \frac{M[f] - f}{\varepsilon} \quad \text{in }]0,\infty[\times\mathbf{R}\times\mathbf{R}, \tag{1.2}$$

where $f = f(t,x,\xi) \in \mathbf{R}^2$, $t > 0$, $x \in \mathbf{R}$, $\xi \in \mathbf{R}$,

$$f(t,x,\xi) \in D, \tag{1.3}$$

$$D = \{(f_0, f_1) \in \mathbf{R}^2,\ f_0 > 0 \text{ or } f_1 = f_0 = 0\}, \tag{1.4}$$

$$M[f](t,x,\xi) = M\left(\rho(t,x), u(t,x), \xi\right), \tag{1.5}$$

$$\rho(t,x) = \int_{\mathbf{R}} f_0(t,x,\xi)\,d\xi,\ \ \rho(t,x)u(t,x) = \int_{\mathbf{R}} f_1(t,x,\xi)\,d\xi, \tag{1.6}$$

and

$$M(\rho,u,\xi) = \Big(\chi(\rho,\xi-u), ((1-\theta)u + \theta\xi)\chi(\rho,\xi-u)\Big), \tag{1.7}$$

$$\chi(\rho,\xi) = c_{\gamma,\kappa}\left(\frac{4\gamma\kappa}{(\gamma-1)^2}\rho^{\gamma-1} - \xi^2\right)_+^\lambda, \tag{1.8}$$

$$\theta = \frac{\gamma-1}{2},\ \ \lambda = \frac{1}{\gamma-1} - \frac{1}{2},\ \ c_{\gamma,\kappa} = \frac{1}{J_\lambda}\left(\frac{4\gamma\kappa}{(\gamma-1)^2}\right)^{-1/(\gamma-1)}, \tag{1.9}$$

$$J_\lambda = \int_{-1}^{1}(1-z^2)^\lambda\,dz = \sqrt{\pi}\,\Gamma(\lambda+1)/\Gamma(\lambda+3/2). \tag{1.10}$$

The kinetic equilibrium χ in (1.8) has been introduced in [8] as a generating function for entropies, and has also been used in the stability analysis of [9]. This function is also involved in [10] for the so called kinetic formulation, which is slightly different from the BGK equation considered here. The aim of the present paper is to extend to the above case of isentropic gas dynamics the results obtained for scalar equations in [12]. We refer also to [6], [16], [17], [15], [13] for related works.

We consider the system (1.2) with initial data

$$f(0,x,\xi) = f^0(x,\xi) \tag{1.11}$$

satisfying energy bounds. We recall that the energy for (1.1) is given by

$$\eta(\rho,u)=\rho u^2/2+\frac{\kappa}{\gamma-1}\rho^\gamma, \tag{1.12}$$

and is a mathematical entropy for (1.1). The corresponding kinetic entropy for (1.2) is given, for $f=(f_0,f_1)\in D$, by

$$H(f,\xi)=\frac{\theta}{1-\theta}\frac{\xi^2}{2}f_0+\frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda}+\frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0}-\frac{\theta}{1-\theta}\xi f_1, \tag{1.13}$$

$$H(0,\xi)=0. \tag{1.14}$$

We recall that $H\geq 0$. In [3], we proved the existence of a weak solution satisfying a kinetic entropy inequality for any initial data with finite energy. Let us recall the main results of [3].

Theorem 1.1 *Assume that $f^0\in L^1(\mathbf{R}_x\times\mathbf{R}_\xi)$ satisfies*

$$f^0(x,\xi)\in D \text{ a.e. in } \mathbf{R}\times\mathbf{R} \tag{1.15}$$

and

$$\iint_{\mathbf{R}\times\mathbf{R}} H(f^0(x,\xi),\xi)\,dxd\xi=C_H^0<\infty. \tag{1.16}$$

Then there exists a solution f to (1.2)-(1.11) satisfying

$$f\in C([0,\infty[,L^1(\mathbf{R}\times\mathbf{R})), \tag{1.17}$$

$$f\in L^\infty([0,\infty[,L^1(\mathbf{R}\times\mathbf{R})), \tag{1.18}$$

$$\forall t\geq 0,\ f(t,x,\xi)\in D \text{ a.e. in } \mathbf{R}\times\mathbf{R}, \tag{1.19}$$

$$\partial_t\Big(\int_{\mathbf{R}} f\,d\xi\Big)+\partial_x\Big(\int_{\mathbf{R}}\xi f\,d\xi\Big)=0, \tag{1.20}$$

$$\forall t\geq 0,\ \iint_{\mathbf{R}\times\mathbf{R}} H(f(t,x,\xi),\xi)\,dxd\xi\leq C_H^0. \tag{1.21}$$

Theorem 1.2 *The solution f obtained in Theorem 1.1 satisfies*

$$\begin{aligned}&\left[\iint_{\mathbf{R}\times\mathbf{R}} H(f(t,x,\xi),\xi)\,dxd\xi\right]_0^T\\ &=\frac{1}{\varepsilon}\iiint_{]0,T[\times\mathbf{R}\times\mathbf{R}}\left(H'(f,\xi)-\Big(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1}-\frac{u^2}{2},u\Big)\right)\cdot(M[f]-f)\,dtdxd\xi,\end{aligned} \tag{1.22}$$

where the derivative $H'(f,\xi)$ of $H(f,\xi)$ with respect to f is given by

$$H'(f,\xi)=\left(\frac{\theta}{1-\theta}\frac{\xi^2}{2}+\frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}f_0^{1/\lambda}-\frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0^2},-\frac{\theta}{1-\theta}\xi+\frac{1}{1-\theta}\frac{f_1}{f_0}\right) \tag{1.23}$$

for $f \neq 0$*, and* $H'(0,\xi) = (\frac{\theta}{1-\theta}\frac{\xi^2}{2}, -\frac{\theta}{1-\theta}\xi)$ *by convention. Moreover,*

$$\left(H'(f,\xi) - (\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u)\right) \cdot (M[f] - f) \leq 0 \quad a.e. \tag{1.24}$$

Indeed, the definition of $H'(0,\xi)$ does not really matter in (1.22), since by applying the result of [5] to each component of f, we have

$$M[f] - f = 0 \quad \text{a.e. } t, x, \xi \text{ where } f = 0. \tag{1.25}$$

Following [14], let us now introduce the notion of kinetic invariant domains, which is the heart of this paper.

Definition 1.3 *A family* $\tilde{D}_\xi$ *of subsets of* $\mathbf{R}^2$ *is a family of kinetic invariant domains if, denoting by* f *the solution obtained in Theorem 1.1,*

$$f^0(x,\xi) \in \tilde{D}_\xi \text{ a.e. } x, \xi \Rightarrow \forall t \; f(t,x,\xi) \in \tilde{D}_\xi \text{ a.e. } x, \xi. \tag{1.26}$$

We have the following result.

Theorem 1.4 *For any* $\omega_{min} < \omega_{max}$*, the system (1.2) has the property that*

$$\tilde{D}_\xi = \{f \in D;\; f = 0 \text{ or } \omega_{min} \leq \omega_1(f,\xi) \leq \omega_2(f,\xi) \leq \omega_{max}\} \tag{1.27}$$

is a family of convex kinetic invariant domains, where

$$\omega_1(f,\xi) = u(f,\xi) - \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho(f,\xi)^{\frac{\gamma-1}{2}},\; \omega_2(f,\xi) = u(f,\xi) + \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho(f,\xi)^{\frac{\gamma-1}{2}}, \tag{1.28}$$

with

$$u(f,\xi) = \frac{f_1/f_0 - \theta\xi}{1-\theta}, \tag{1.29}$$

and

$$\rho(f,\xi) = \left(\frac{4\gamma\kappa}{(\gamma-1)^2}\right)^{-1/(\gamma-1)} \left(\left(\frac{f_1/f_0 - \xi}{1-\theta}\right)^2 + (f_0/c_{\gamma,\kappa})^{1/\lambda}\right)^{1/(\gamma-1)}. \tag{1.30}$$

Moreover, the set $\tilde{D}_\xi$ *is associated with the invariant domain of (1.1)*

$$\tilde{D} = \{(\rho,u) \in \mathbf{R}^+ \times \mathbf{R};\; \rho = 0 \text{ or } \omega_{min} \leq \omega_1 \leq \omega_2 \leq \omega_{max}\}, \tag{1.31}$$

where the Riemann invariants ω_1*,* ω_2 *are given by*

$$\omega_1 = u - \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}},\; \omega_2 = u + \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}, \tag{1.32}$$

in the sense that

$$\forall(\rho,u) \in \tilde{D},\; M(\rho,u,\xi) \in \tilde{D}_\xi \text{ a.e.} \xi, \tag{1.33}$$

and

$$\text{for any } f(\xi) \in L^1(\mathbf{R}_\xi) \text{ such that } f(\xi) \in \tilde{D}_\xi \text{ a.e. } \xi,$$
$$(\rho, u) \in \tilde{D} \text{ with } (\rho, \rho u) = \int_{\mathbf{R}} f(\xi)\, d\xi. \tag{1.34}$$

In particular, if the initial data of Theorem 1.1 satisfies $f^0(x,\xi) \in \tilde{D}_\xi$ *for a.e.* x, ξ, *then* (ρ, u) *defined by (1.6) verify* $\forall t \geq 0$, $(\rho(t,x), u(t,x)) \in \tilde{D}$ *for a.e.* x. *Besides* $f(t,x,\xi) \in \tilde{D}_\xi$ $\forall t \geq 0$ *and consequently* f *has compact support with respect to* ξ, $supp_\xi f \subset [\omega_{min}, \omega_{max}]$.

According to [9], we define entropy solutions to (1.1) as solutions with finite energy, i.e. $\sup_t \int \eta(\rho(t,x), u(t,x))\, dx < \infty$, which satisfy the entropy inequalities

$$\partial_t(\eta_S) + \partial_x(G_S) \leq 0 \tag{1.35}$$

where η_S and G_S are suitable entropy and entropy flux for (1.1) parametrized by a convex function S with subquadratic growth that will be defined precisely in the next section. We have the following convergence result.

Theorem 1.5 *Let us denote by* f_ε *the solution of Theorem 1.1 with the same initial data* $f^0(x,\xi) \in L^1(\mathbf{R} \times \mathbf{R})$ *that satisfies* $f^0(x,\xi) \in \tilde{D}_\xi$ *a.e. for some* $\omega_{min} < \omega_{max}$, *and the energy bound (1.16). Then* $(\rho_\varepsilon, u_\varepsilon)$ *defined by (1.6) are uniformly bounded in* L^∞, *and passing if necessary to subsequences,* $(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)$ *converge a.e. in* $]0,\infty[\times\mathbf{R}$ *when* $\varepsilon \to 0$ *to an entropy solution* $(\rho, \rho u)$ *to (1.1), with initial data* $(\rho^0, \rho^0 u^0) = \int f^0 d\xi$.

We have to notice that in particular, we can take as initial data

$$f^0(x,\xi) = M(\rho^0(x), u^0(x), \xi), \tag{1.36}$$

with

$$\begin{gathered} (\rho^0(x), u^0(x)) \in \tilde{D} \quad a.e., \\ \int_{\mathbf{R}} \rho^0(x)\, dx < \infty, \qquad \int_{\mathbf{R}} \eta(\rho^0(x), u^0(x))\, dx < \infty. \end{gathered} \tag{1.37}$$

The analysis relies on the stability results of [9] together with the use of the kinetic entropies $H_S(f,\xi)$ introduced in [4], that are associated to the entropies η_S via the properties

$$\int H_S(M(\rho,u,\xi),\xi)\, d\xi = \eta_S(\rho,u), \tag{1.38}$$

$$\eta_S(\rho,u) \leq \int H_S(f(\xi),\xi)\, d\xi, \qquad (\rho,\rho u) = \int f(\xi) d\xi. \tag{1.39}$$

They verify also the important relation

$$H_S'(M(\rho,u,\xi),\xi) = \eta_S'(\rho,u), \tag{1.40}$$

whenever $M(\rho,u,\xi) \neq 0$. In general, a weaker form holds with a subdifferential,

$$\partial H_S(M(\rho,u,\xi),\xi) \ni T_S(\rho,u), \tag{1.41}$$

where $T_S(\rho,u)$ is the Lagrange multiplier corresponding to the minimization problem associated to (1.39), and coincides with $\eta_S'(\rho,u)$ when $\rho>0$ (see Proposition 4.1).

Let us finally remark that the BGK system (1.2) leads formally, at first order in ε, to the viscous model

$$\begin{aligned} \partial_t\rho+\partial_x(\rho u)&=\varepsilon\mu\partial_x(\rho^{\gamma-1}\partial_x\rho),\\ \partial_t(\rho u)+\partial_x(\rho u^2+\kappa\rho^\gamma)&=\varepsilon\mu\partial_x(\rho^{\gamma-1}\partial_x(\rho u)), \end{aligned}\tag{1.42}$$

with $\mu=\kappa\gamma(3-\gamma)/(\gamma-1)$.

2 Notations

In order to define the kinetic entropies, we need to introduce some notations. We set

$$\Phi(\rho,u,\xi,v)=\frac{(1-\theta)^2}{\theta}\frac{c_{\gamma,\kappa}}{J_\lambda}\mathbb{I}_{\omega_1<\xi<\omega_2}\mathbb{I}_{\omega_1<v<\omega_2}|\xi-v|^{2\lambda-1}\Upsilon_{\lambda-1}(z),\tag{2.1}$$

where

$$z=\frac{(\xi+v)(\omega_1+\omega_2)-2(\omega_1\omega_2+\xi v)}{(\omega_2-\omega_1)|\xi-v|},\tag{2.2}$$

and

$$\Upsilon_{\lambda-1}(z)=\int_1^z(y^2-1)^{\lambda-1}\,dy,\qquad z\geq 1.\tag{2.3}$$

The z from (2.2) is indeed greater than 1, because

$$\begin{aligned} (\xi+v)(\omega_1+\omega_2)-2(\omega_1\omega_2+\xi v)-(\omega_2-\omega_1)(\xi-v)&=2(\omega_2-\xi)(v-\omega_1),\\ (\xi+v)(\omega_1+\omega_2)-2(\omega_1\omega_2+\xi v)+(\omega_2-\omega_1)(\xi-v)&=2(\xi-\omega_1)(\omega_2-v), \end{aligned}$$

which are both nonnegative in $\{\omega_1<\xi,v<\omega_2\}$. Also,

$$z^2-1=\frac{4(\omega_2-\xi)(\xi-\omega_1)(\omega_2-v)(v-\omega_1)}{(\omega_2-\omega_1)^2(\xi-v)^2}.\tag{2.4}$$

One can check that $\Phi\geq 0$,

$$\begin{aligned} \int\Phi(\rho,u,\xi,v)\,dv&=\chi(\rho,\xi-u),\\ \int v\Phi(\rho,u,\xi,v)\,dv&=[(1-\theta)u+\theta\xi)]\chi(\rho,\xi-u), \end{aligned}\tag{2.5}$$

and similar relations also hold for integration in ξ since Φ is symmetric in ξ,v. We now define for any convex function S the kinetic entropy

$$H_S(f,\xi)=\int_{\mathbf{R}}\Psi(f,\xi,v)S(v)\,dv,\ \text{for } f\neq 0,\qquad H_S(0,\xi)=0,\tag{2.6}$$

where

$$\Psi(f,\xi,v)=\Phi(\rho(f,\xi),u(f,\xi),\xi,v),\quad f\neq 0,\tag{2.7}$$

and its macroscopic version

$$G_S(\rho,u,\xi)=\int_{\mathbf{R}}\Phi(\rho,u,\xi,v)S(v)\,dv=H_S(M(\rho,u,\xi),\xi). \tag{2.8}$$

Then, the entropy and entropy flux for (1.1) are defined by

$$\eta_S(\rho,u)=\int_{\mathbf{R}}\chi(\rho,v-u)S(v)\,dv=\int G_S(\rho,u,\xi)\,d\xi, \tag{2.9}$$

$$G_S(\rho,u)=\int_{\mathbf{R}}[(1-\theta)u+\theta v)]\chi(\rho,v-u)S(v)\,dv=\int \xi\, G_S(\rho,u,\xi)\,d\xi. \tag{2.10}$$

Let us remark that for $S(v)=v^2/2$, we get $H_S=H$ and $\eta_S=\eta$. We shall prove the differentiability of H_S for $f\in D\setminus\{0\}$, and we use the convention $H_S'(0,\xi)=0$ for $S\in C^1(\mathbf{R},\mathbf{R})$ and not identically equal to $v\mapsto v^2/2$, while $H'(0,\xi)=(\frac{\theta}{1-\theta}\frac{\xi^2}{2},-\frac{\theta}{1-\theta}\xi)$ (but this is not really necessary, according to (1.25)). We finally define

$$A_\gamma=\frac{4\gamma\kappa}{(\gamma-1)^2}, \tag{2.11}$$

$$l(z)=u+\sqrt{A_\gamma}\rho^{(\gamma-1)/2}\,z, \tag{2.12}$$

$$m(z)=-u+\frac{\gamma-1}{2}\sqrt{A_\gamma}\rho^{(\gamma-1)/2}\,z, \tag{2.13}$$

and

$$T(\rho,u)=(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1}-u^2/2,u), \tag{2.14}$$

which coincides with $\eta'(\rho,u)$ whenever $\rho>0$. Here and all throughout the paper, the prime denotes differentiation with respect to conservative variables, $(\rho,q\equiv\rho u)$ here.

3 Preliminary Convexity results

In this section we prove that H_S is convex in D, and some technical regularity results. We notice that

$$M_0(\rho,u,\xi)=\chi(\rho,\xi-u)=c_{\gamma,\kappa}\left[(\omega_2-\xi)(\xi-\omega_1)\right]_+^\lambda, \tag{3.1}$$

with $\omega_1=u-\frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}$, $\omega_2=u+\frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}$.

Let us first invert the maxwellian function. Consider the relation $f=M(\rho,u,\xi)$ for some fixed $\xi\in\mathbf{R}$. Whenever $f_0>0$, this relation can be inverted in $u=u(f,\xi)$, $\rho=\rho(f,\xi)$ from (1.29), (1.30), and then $\omega=F(f)$ with

$$F(f)=\begin{pmatrix}\frac{f_1/f_0-\theta\xi}{1-\theta}-\sqrt{\left(\frac{f_1/f_0-\xi}{1-\theta}\right)^2+(f_0/c_{\gamma,\kappa})^{1/\lambda}}\\ \frac{f_1/f_0-\theta\xi}{1-\theta}+\sqrt{\left(\frac{f_1/f_0-\xi}{1-\theta}\right)^2+(f_0/c_{\gamma,\kappa})^{1/\lambda}}\end{pmatrix}. \tag{3.2}$$

The inverse relation is given by $f = R(\omega)$, with

$$R(\omega) = \begin{pmatrix} c_{\gamma,\kappa}\left(\left(\frac{\omega_2-\omega_1}{2}\right)^2 - \left(\xi - \frac{\omega_1+\omega_2}{2}\right)^2\right)_+^\lambda \\ \left((1-\theta)\frac{\omega_1+\omega_2}{2} + \theta\xi\right)c_{\gamma,\kappa}\left(\left(\frac{\omega_2-\omega_1}{2}\right)^2 - \left(\xi - \frac{\omega_1+\omega_2}{2}\right)^2\right)_+^\lambda \end{pmatrix}. \tag{3.3}$$

Lemma 3.1 *By these formulas, $\{f_0 > 0\}$ is in bijection with $\{\omega_1 < \xi < \omega_2\}$.*

Proof. Whenever $f_0 > 0$, we can check that $R(F(f)) = f$. Obviously, we have $|\frac{\omega_2-\omega_1}{2}| > |\xi - \frac{\omega_2+\omega_1}{2}| \Leftrightarrow \omega_1 < \xi < \omega_2$, thus $F(R(\omega)) = \omega$ in $\{\omega_1 < \xi < \omega_2\}$. Furthermore, $R(\omega)_0 > 0$ in $\{\omega_1 < \xi < \omega_2\}$ and

$$F(f)_1 - \xi = \frac{1}{1-\theta}(f_1/f_0 - \xi) - \sqrt{\left(\frac{f_1/f_0-\xi}{1-\theta}\right)^2 + (f_0/c_{\gamma,\kappa})^{1/\lambda}} < 0 \text{ if } f_0 > 0.$$

Finally we get a C^∞ change of variable. □

Lemma 3.2 *i) If $S : \mathbf{R} \to \mathbf{R}$ is of class C^k then the functions $(\rho, u) \mapsto \eta_S(\rho, u)$ and $(\rho, q) \mapsto \eta_S(\rho, u)$ with $q = \rho u$ are C^k in $\{\rho > 0\}$.*
ii) If $S : \mathbf{R} \to \mathbf{R}$ is bounded on compact sets, then the function $(\omega_1, \omega_2) \mapsto G_S(\rho, u, \xi)$ is continuously differentiable in $\{\omega_1 < \xi < \omega_2\}$, with

$$\frac{\partial G_S}{\partial \omega_i}(\rho, u, \xi) = \int_{\mathbf{R}} \frac{\partial \Phi}{\partial \omega_i}(\rho, u, \xi, v) S(v)\, dv \qquad \textit{for } i = 1, 2.$$

iii) If $S : \mathbf{R} \to \mathbf{R}$ is bounded on compact sets, then $H_S(., \xi)$ is continuous at 0 in $\{f \in D\ ;\ |f_1| \le Af_0\}$, for any $A > 0$.

Proof. *i)* By the change of variable $v - u = \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}} z$, we get

$$\eta_S(\rho, u) = \frac{\rho}{J_\lambda}\int_{\mathbf{R}} (1-z^2)_+^\lambda S\left(u + \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}} z\right) dz, \tag{3.4}$$

and we can obviously differentiate under the integral.
ii) Let us first show that G_S in (2.8) is well defined. Since we need $\omega_1 < v < \omega_2$ for Φ to be different from zero, we only have to check that $\int_{\omega_1}^{\omega_2} \Phi\, dv < \infty$. Obviously this integral vanishes if $\xi \notin]\omega_1, \omega_2[$, thus let us assume that $\omega_1 < \xi < \omega_2$. The only possible singularities of Φ in v are at the points ω_1, ω_2, and ξ. We have $\Upsilon_{\lambda-1}(z) \underset{z\to 1}{\sim} 2^{\lambda-1}(z-1)^\lambda/\lambda$, hence $\Upsilon_{\lambda-1}(z) \underset{v\to\omega_2}{\sim} L(\omega_2 - v)^\lambda$ with L a constant. Thus Φ is continuous at ω_2, and also at ω_1.
Next, if $\lambda > 1/2$, $\Upsilon_{\lambda-1}(z) \underset{z\to\infty}{\sim} z^{2\lambda-1}/(2\lambda-1)$, which yields $\Upsilon_{\lambda-1}(z) \underset{v\to\xi}{\sim} L'|\xi - v|^{-(2\lambda-1)}$ with L' a constant, and therefore $\lim_{v\to\xi} |\xi - v|^{2\lambda-1}\Upsilon_{\lambda-1}(z)$ is finite.
If $\lambda = 1/2$, we get $\Upsilon_{\lambda-1}(z) \underset{v\to\xi}{\sim} |\ln|\xi - v||$, which is integrable at ξ.
If $0 < \lambda < 1/2$, $\Upsilon_{\lambda-1}(z)$ has a finite limit when $v \to \xi$, and $|\xi - v|^{2\lambda-1}\Upsilon_{\lambda-1}(z)$

is integrable in v at ξ.
Finally, in any case, G_S is defined by a convergent integral.
More generally, we have that for any $A>0$,

$$v \mapsto |\xi - v|^{2\lambda-1}\Upsilon_{\lambda-1}\left(\frac{A}{|\xi-v|}\right) \text{ is locally integrable in } v. \tag{3.5}$$

In the set $\{\omega_1 < \xi < \omega_2\}$, we can locally upper estimate Φ by such a function, thus by Lebesgue's theorem, we obtain that G_S is continuous in $\{\omega_1 < \xi < \omega_2\}$.

Next, we have in distribution sense in $\{\omega_1 < \xi < \omega_2\}$, for any $v \neq \xi$,

$$\frac{\partial}{\partial \omega_1}(\mathbb{I}_{\omega_1<v<\omega_2}\Upsilon_{\lambda-1}(z)) = \mathbb{I}_{\omega_1<v<\omega_2}\frac{\partial}{\partial \omega_1}\Upsilon_{\lambda-1}(z),$$

because $\Upsilon_{\lambda-1}(z)$ vanishes for $\omega_1 = v$. Now

$$\frac{\partial}{\partial \omega_1}(\Upsilon_{\lambda-1}(z)) = (z^2-1)^{\lambda-1}\frac{\partial z}{\partial \omega_1},$$

with

$$\frac{\partial z}{\partial \omega_1} = -\frac{2(\omega_2-\xi)(\omega_2-v)}{|\xi-v|(\omega_2-\omega_1)^2},$$

thus

$$\begin{aligned} &\frac{\partial}{\partial \omega_1}(\mathbb{I}_{\omega_1<v<\omega_2}\Upsilon_{\lambda-1}(z)) \\ = \quad & -2\mathbb{I}_{\omega_1<v<\omega_2}\left[\frac{(\omega_2-\xi)(\omega_2-v)}{(\omega_2-\omega_1)^2}\right]^{\lambda}\frac{[4(\xi-\omega_1)(v-\omega_1)]^{\lambda-1}}{|\xi-v|^{2\lambda-1}}. \end{aligned}$$

Similarly, we have in distribution sense in $\{\omega_1 < \xi < \omega_2\}$, for any $v \neq \xi$,

$$\begin{aligned} &\frac{\partial}{\partial \omega_2}(\mathbb{I}_{\omega_1<v<\omega_2}\Upsilon_{\lambda-1}(z)) \\ = \quad & 2\mathbb{I}_{\omega_1<v<\omega_2}\left[\frac{(\xi-\omega_1)(v-\omega_1)}{(\omega_2-\omega_1)^2}\right]^{\lambda}\frac{[4(\omega_2-\xi)(\omega_2-v)]^{\lambda-1}}{|\xi-v|^{2\lambda-1}}. \end{aligned}$$

With this formula, we see that

$$\Lambda_i(\omega_1,\omega_2) \equiv \int_{\mathbf{R}}\frac{\partial \Phi}{\partial \omega_i}(\rho,u,\xi,v)S(v)\,dv$$

is well-defined for $\omega_1 < \xi < \omega_2$, and is locally bounded. Moreover, one can easily deduce that $\partial_{\omega_i}G_S = \Lambda_i$ in the sense of distributions in $\{\omega_1 < \xi < \omega_2\}$. Then, by the change of variable $\mu = v - \omega_1$, we see by Lebesgue's theorem that Λ_1 is continuous in $\{\omega_1 < \xi < \omega_2\}$. A similar result holds for Λ_2, thus we obtain the result.
iii) Let ξ be fixed. In $\{f \in D\ ;\ |f_1| \leq Af_0 \textit{ and } f_0 \leq B\}$, and if $f \neq 0$, setting

$(\omega_1, \omega_2) = F(f)$ from (3.2), we have $D \le \omega_1 < \xi < \omega_2 \le E$ with D, E some constants. Then, $S(v)$ is bounded in $[D, E]$, and thus

$$f_0 \inf S \le H_S(f, \xi) \le f_0 \sup S,$$

which yields that $H_S(f, \xi) \to 0$ as $f \to 0$ with $|f_1| \le A f_0$. □

Proposition 3.3 *If $S : \mathbf{R} \to \mathbf{R}$ is of class C^1 then we have $H'_S(M(\rho, u, \xi), \xi) = \eta'_S(\rho, u)$ whenever $(M(\rho, u, \xi))_0 > 0$.*

Proof. Since by Lemma 3.1 $H_S(f, \xi) = G_S(\rho(f, \xi), u(f, \xi), \xi)$ whenever $f_0 > 0$, we get from Lemma 3.2(ii) that H_S is C^1 in $\{f_0 > 0\}$. Furthermore, for $\omega_1 < \xi < \omega_2$,

$$\begin{aligned}\frac{\partial G_S}{\partial \omega_1}(\rho, u, \xi) = -2\frac{(1-\theta)^2}{\theta}\frac{c_{\gamma,\kappa}}{J_\lambda}4^{\lambda-1}\frac{(\omega_2-\xi)^\lambda(\xi-\omega_1)^{\lambda-1}}{(\omega_2-\omega_1)^{2\lambda}}\\ \times \int_{\mathbf{R}} \mathbb{I}_{\omega_1<v<\omega_2}(\omega_2-v)^\lambda(v-\omega_1)^{\lambda-1}S(v)\,dv,\end{aligned}$$

$$\begin{aligned}\frac{\partial G_S}{\partial \omega_2}(\rho, u, \xi) = 2\frac{(1-\theta)^2}{\theta}\frac{c_{\gamma,\kappa}}{J_\lambda}4^{\lambda-1}\frac{(\omega_2-\xi)^{\lambda-1}(\xi-\omega_1)^{\lambda}}{(\omega_2-\omega_1)^{2\lambda}}\\ \times \int_{\mathbf{R}} \mathbb{I}_{\omega_1<v<\omega_2}(\omega_2-v)^{\lambda-1}(v-\omega_1)^{\lambda}S(v)\,dv.\end{aligned}$$

Then, we have by Lemma 3.2(i) that for $\rho > 0$, η_S is C^1, and by differentiating directly (2.9), we get for $\rho > 0$

$$\begin{aligned}\frac{\partial \eta_S}{\partial \rho}(\rho, u) &= \frac{1}{\rho}\int c_{\gamma,\kappa}[(\omega_2-v)(v-\omega_1)]_+^{\lambda-1}\Big[(1-\theta)(\omega_2-\omega_1)^2/4\\ &\qquad -2\lambda u(v-u)\Big]S(v)dv,\end{aligned}$$

$$\frac{\partial \eta_S}{\partial q}(\rho, u) = \frac{1}{\rho}\int c_{\gamma,\kappa}[(\omega_2-v)(v-\omega_1)]_+^{\lambda-1}2\lambda(v-u)S(v)dv.$$

Next, we write

$$\begin{aligned}M_0(\rho, u, \xi) &= c_{\gamma,\kappa}\left[(\omega_2-\xi)(\xi-\omega_1)\right]_+^\lambda,\\ M_1(\rho, u, \xi) &= \left[(1-\theta)\frac{\omega_1+\omega_2}{2}+\theta\xi\right]M_0(\rho, u, \xi),\end{aligned}$$

and for $\omega_1 < \xi < \omega_2$, we get

$$\begin{aligned}\frac{\partial M_0}{\partial \omega_1} &= -\lambda c_{\gamma,\kappa}(\omega_2-\xi)^\lambda(\xi-\omega_1)^{\lambda-1},\\ \frac{\partial M_0}{\partial \omega_2} &= \lambda c_{\gamma,\kappa}(\omega_2-\xi)^{\lambda-1}(\xi-\omega_1)^{\lambda},\\ \frac{\partial M_1}{\partial \omega_1} &= -\lambda c_{\gamma,\kappa}(\omega_2-\xi)^\lambda(\xi-\omega_1)^{\lambda-1}\left[u-\theta(\omega_2-\omega_1)/2\right],\\ \frac{\partial M_1}{\partial \omega_2} &= \lambda c_{\gamma,\kappa}(\omega_2-\xi)^{\lambda-1}(\xi-\omega_1)^{\lambda}\left[u+\theta(\omega_2-\omega_1)/2\right].\end{aligned}$$

Finally, a computation gives that for $\omega_1 < \xi < \omega_2$,

$$\frac{\partial \eta_S}{\partial \rho}(\rho,u)\frac{\partial M_0}{\partial \omega_i}(\rho,u,\xi) + \frac{\partial \eta_S}{\partial q}(\rho,u)\frac{\partial M_1}{\partial \omega_i}(\rho,u,\xi) = \frac{\partial G_S}{\partial \omega_i}(\rho,u,\xi), \quad i=1,2.$$

This formula can be written equivalently

$$\eta_S'(\rho,u)M'(\rho,u,\xi) = G_S'(\rho,u,\xi), \quad \omega_1 < \xi < \omega_2.$$

Since for $\omega_1 < \xi < \omega_2$, $G_S(\rho,u,\xi) = H_S(M(\rho,u,\xi),\xi)$ and $M'(\rho,u,\xi)$ is invertible, this gives the result. □

Corollary 3.4 *If $S : \mathbf{R} \to \mathbf{R}$ is of class C^k then $H_S(.,\xi)$ is C^k in $\{f_0 > 0\}$.*

The main result of this section is the following.

Proposition 3.5
i) If $S : \mathbf{R} \to \mathbf{R}$ is convex, of class C^2, then η_S is convex in $\{\rho > 0\}$, and if $S'' > 0$ then $\eta_S'' > 0$.
ii) If $S : \mathbf{R} \to \mathbf{R}$ is convex and continuous, then $H_S(.,\xi)$ is convex in D.

Proof. The $i)$ was proved in [10]. Indeed, from (3.4) we get

$$\begin{aligned}&\eta_S''(\rho,u)\cdot X\cdot X\\ &= \frac{1}{J_\lambda}\int_{-1}^{1}(1-z^2)^\lambda\left\{\gamma\kappa\rho^{\gamma-2}(1-z^2)X_0^2 + \frac{(X_1+m(z)X_0)^2}{\rho}\right\}S''(l(z))\,dz,\end{aligned} \tag{3.6}$$

which yields the result.

We turn to the $ii)$. We can assume that S is C^2 and $S'' > 0$, since the general case of a continuous convex S can easily be obtained by approximating S by smooth strictly convex functions.

Let us first prove that H_S is convex in $\{f_0 > 0\}$. Since by Corollary 3.4, H_S is C^2 there, and according to Proposition 3.3, $H_S'(f,\xi) = \eta_S'(M^{-1}(f,\xi))$, we get by differentiation that $H_S''(f,\xi) = \eta_S''(M^{-1}(f,\xi))(M^{-1})'(f,\xi)$ for $f_0 > 0$. Then, we can apply Lemma 3.6 below with $B = (M^{-1})'$, $C = \eta_S''$, $D = H_S''$. We need only to prove that the eigenvalues of M' are positive, and this was done in [4].

Finally, we need to extend this convexity around 0. For $f, g \in D \setminus \{0\}$, $0 < \vartheta < 1$, we have the convexity inequality $H_S((1-\vartheta)f + \vartheta g,\xi) \le (1-\vartheta)H_S(f,\xi) + \vartheta H_S(g,\xi)$. Now letting f tend to 0 in the cones, $iii)$ of Lemma 3.2 gives $H_S(\vartheta g,\xi) \le \vartheta H_S(g,\xi)$, and it yields that H_S is convex in D. □

Lemma 3.6 *If $D = CB$ in $\mathcal{M}_n(\mathbf{R})$ with D symmetric, C symmetric positive definite, and $\sigma(B) \subset \mathbf{R}^+$, then D is positive semidefinite.*

Proof. C defines a scalar product in $\mathbf{R}^n$ by $(X|Y) = X^tCY$. Since $D = CB$ is symmetric, we have $(Y|BX) = Y^tDX = X^tDY = (X|BY)$, hence B is self-adjoint for $(.|.)$ and thus diagonalizable. Then, $(X|BX) = X^tDX$ gives that $\sigma(B) \subset \mathbf{R}^+$ if and only if D is positive semidefinite. □

4 The Key convexity inequality

We need now to generalize Proposition 2.4 of [3].

Proposition 4.1 *If $S : \mathbf{R} \to \mathbf{R}$ is convex, of class C^1, then for every $f \in D$, $\rho \geq 0$ and $u, \xi \in \mathbf{R}$, we have*

$$H_S(f,\xi) \geq H_S(M(\rho,u,\xi),\xi) + T_S(\rho,u)\cdot(f - M(\rho,u,\xi)), \tag{4.1}$$

with

$$T_S(\rho,u) = \frac{1}{J_\lambda}\int_{-1}^{1}(1-z^2)^\lambda \begin{pmatrix} S(l(z)) + m(z)S'(l(z)) \\ S'(l(z)) \end{pmatrix} dz, \tag{4.2}$$

which coincides with $\eta'_S(\rho,u)$ when $\rho > 0$.

We recall from (2.12) and (2.13) that $l(z) = u + \sqrt{A_\gamma}\rho^{(\gamma-1)/2}z$, and $m(z) = -u + \frac{\gamma-1}{2}\sqrt{A_\gamma}\rho^{(\gamma-1)/2}z$. We remark that $T_{v\mapsto v^2/2} = T$. Let us first state two lemmas that are left to the reader.

Lemma 4.2 $H_{v\mapsto 1}(f,\xi) = f_0$, $H_{Id}(f,\xi) = f_1$, $H_{v\mapsto v^2/2}(f,\xi) = H(f,\xi)$.

Lemma 4.3 *For $S : \mathbf{R} \to \mathbf{R}$ of class C^1, we have for any fixed $u \in \mathbf{R}$, $\eta'_S(\rho,u) = T_S(\rho,u) \underset{\rho\to 0+}{\to} T_S(0,u)$.*

Proof of Proposition 4.1. We only treat the case when S is of class C^2 such that $S'' \geq \mathcal{C} > 0$, since the general case can easily be deduced by approximation of S.

*If $M_0(\rho,u,\xi) \neq 0$.
Then $T_S(\rho,u) = \eta'_S(\rho,u)$. The convexity of H_S (Proposition 3.5) together with the identity of Proposition 3.3 give the result.

*If $M_0(\rho,u,\xi) = 0$ and $\rho \neq 0$.
Let us consider $\psi : D \mapsto \mathbf{R}$ defined by

$$\begin{aligned}\psi(f) &= H_S(f,\xi) - H_S(M(\rho,u,\xi),\xi) - T_S(\rho,u)\cdot(f - M(\rho,u,\xi)) \\ &= H_S(f,\xi) - T_S(\rho,u)\cdot f.\end{aligned}$$

We need to prove that ψ is nonnegative.
– At infinity,
We have $S(v) \geq a + bv + \mathcal{C}v^2/2$, thus from the definition (2.6) of H_S and since $\Phi \geq 0$, we obtain with Lemma 4.2

$$H_S(f,\xi) \geq af_0 + bf_1 + \mathcal{C}H(f,\xi). \tag{4.3}$$

Now $f \mapsto af_0 + bf_1 - T_S(\rho,u)\cdot f$ is linear and $\mathcal{C} > 0$ thus by the superlinearity of H (Lemma 2.3 of [3]),

$$H(f,\xi) \geq \varepsilon_0 f_0^{p_0} + \varepsilon_1|f_1|^{p_1},$$

with $p_0 = 1 + 1/\lambda$, $p_1 = 2(1+\lambda)/(1+2\lambda)$, we obtain that $\psi(f) \underset{f\to\infty}{\to} \infty$. Moreover, since $H \geq 0$, we deduce with (4.3) that ψ is lower bounded in D.

– At the frontier $\{f_0 = 0,\ f_1 \neq 0\}$,
The inequality (4.3) gives that $\psi(f) \to \infty$.
– At 0,
From (4.3) again and since $H \geq 0$, we obtain $\underline{\lim}_{f\to 0} \psi(f) \geq 0$.
– Inside D,
It only remains to study the possible minima of ψ in $int(D)$. By Proposition 3.3, ψ is C^1 there, and $\psi'(f) = \eta_S'(\rho(f,\xi), u(f,\xi)) - \eta_S'(\rho, u)$. From the identity

$$\Big(\eta_S'(Y) - \eta_S'(X)\Big)(Y - X) = \int_0^1 \eta_S''((1-\vartheta)X + \vartheta Y)\cdot(Y-X)\cdot(Y-X)\,d\vartheta,$$

and since η_S'' is positive definite, we conclude that if at some point f with $f_0 > 0$, $\psi'(f) = 0$, we must have $\rho(f,\xi) = \rho$, $u(f,\xi) = u$, and therefore $f = M(\rho, u, \xi)$. But since $M(\rho,u,\xi) = 0$, we get a contradiction. Therefore, there is no interior point that is a minimum, and we conclude that $\psi \geq 0$ in D.

*If $\rho = 0$.
We now know from the previous steps that (4.1) holds as soon as $\rho > 0$. Taking f, u and ξ fixed, we let $\rho \to 0$. We see obviously that $M(\rho,u,\xi) \to 0$, while remaining in a cone, and with Lemma 3.2 we get that $H_S(M(\rho,u,\xi),\xi) \to 0$. With Lemma 4.3, we can pass to the limit $\rho \to 0$ and get the result. □

We deduce obviously as in Proposition 2.5 of [3] an entropy minimization principle.

Corollary 4.4 *Assume that $S : \mathbf{R} \to \mathbf{R}$ is convex, of class C^1 and such that $|S(v)| \leq B(1+v^2)$ for some $B \geq 0$. Consider $f \in L^1(\mathbf{R}_\xi)$ such that $f \in D$ a.e. and*

$$\int_{\mathbf{R}} H(f(\xi),\xi)\,d\xi < \infty. \tag{4.4}$$

Then $H_S(f(\xi),\xi)$ and $H_S(M[f](\xi),\xi)$ lie in $L^1(\mathbf{R}_\xi)$ with

$$\int_{\mathbf{R}} H_S(M[f](\xi),\xi)\,d\xi \leq \int_{\mathbf{R}} H_S(f(\xi),\xi)\,d\xi. \tag{4.5}$$

Proof. Since $\Phi \geq 0$, we get from definition (2.6) that

$$|H_S(f,\xi)| \leq B(f_0 + 2H(f,\xi)), \tag{4.6}$$

thus $\int |H_S(f(\xi),\xi)|d\xi < \infty$, and $\int |H_S(M[f](\xi),\xi)|d\xi < \infty$. Then, integrating the inequality of Proposition 4.1 with respect to ξ, with $(\rho, \rho u) = \int f(\xi)d\xi$ gives the result. □

Another important consequence of Proposition 4.1 is

Proposition 4.5 *Assume that $S : \mathbf{R} \to \mathbf{R}$ is convex, of class C^1 and such that $|S(v)| \leq B(1+v^2)$ for some $B \geq 0$. Then, with f the solution of Theorem 1.1,*

$$\iint\limits_{\mathbf{R}\times\mathbf{R}} H_S(f(t,x,\xi),\xi)\,dxd\xi \leq \iint\limits_{\mathbf{R}\times\mathbf{R}} H_S(f^0(x,\xi),\xi)\,dxd\xi. \tag{4.7}$$

Proof. We apply Corollary 4.4 at fixed t and for a.e. x. We obtain

$$\int_{\mathbf{R}} H_S(M[f](t,x,\xi),\xi)\,d\xi \le \int_{\mathbf{R}} H_S(f(t,x,\xi),\xi)\,d\xi \qquad \forall t\ge 0,\ a.e.\ x\in\mathbf{R}.$$

Then, because of (4.6), we have obviously

$$\iint |H_S(f(t,x,\xi),\xi)|dxd\xi < \infty, \qquad \iint |H_S(M[f](t,x,\xi),\xi)|dxd\xi < \infty,$$

and therefore

$$\iint_{\mathbf{R}\times\mathbf{R}} H_S(M[f](t,x,\xi),\xi)\,dxd\xi \le \iint_{\mathbf{R}\times\mathbf{R}} H_S(f(t,x,\xi),\xi)\,dxd\xi, \qquad \forall t\ge 0. \quad (4.8)$$

But we have a representation of the solution f,

$$f(t,x,\xi) = f^0(x-t\xi,\xi)e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} M[f](t-s,x-s\xi,\xi)\,ds.$$

According to Jensen's inequality with the convex function H_S, we deduce as in [3] that

$$\begin{aligned} H_S(f(t,x,\xi),\xi) \le\ & H_S(f^0(x-t\xi,\xi),\xi)e^{-t/\varepsilon} \\ & + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} H_S(M[f](t-s,x-s\xi,\xi),\xi)\,ds. \end{aligned}$$

By integration in x and ξ, we obtain

$$\begin{aligned} & \iint_{\mathbf{R}\times\mathbf{R}} H_S(f(t,x,\xi),\xi)\,dxd\xi \\ \le & \iint_{\mathbf{R}\times\mathbf{R}} H_S(f^0(x-t\xi,\xi),\xi)\,dxd\xi\, e^{-t/\varepsilon} \\ & \quad + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} \iint_{\mathbf{R}\times\mathbf{R}} H_S(M[f](t-s,x-s\xi,\xi),\xi)\,dxd\xi\,ds \\ = & \iint_{\mathbf{R}\times\mathbf{R}} H_S(f^0(x,\xi),\xi)\,dxd\xi\, e^{-t/\varepsilon} \\ & \quad + \frac{1}{\varepsilon}\int_0^t e^{-(t-s)/\varepsilon} \iint_{\mathbf{R}\times\mathbf{R}} H_S(M[f](s,x,\xi),\xi)\,dxd\xi\,ds, \end{aligned}$$

and setting

$$F_S(t) = \iint_{\mathbf{R}\times\mathbf{R}} H_S(f(t,x,\xi),\xi)\,dxd\xi,$$

we get with (4.8)

$$F_S(t) \le F_S(0)e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t e^{-(t-s)/\varepsilon} F_S(s)\,ds.$$

Gronwall's lemma finally gives $F_S(t) \le F_S(0)$ and our result. □

5 Maximum principle

This short section is devoted to the proof of Theorem 1.4 giving the maximum principle on the Riemann invariants.

Proof of Theorem 1.4. We have first that $\tilde{D}_\xi$ is convex. Indeed, using $S_M(v) = (v - \omega_{max})_+^2$ and $S_m(v) = (\omega_{min} - v)_+^2$ which are convex and C^1, we notice that $H_{S_M} \geq 0$, $H_{S_m} \geq 0$, and we have for a fixed ξ

$$\begin{aligned} H_{S_M}(f,\xi) \leq 0 \;&\Leftrightarrow\; f = 0 \textit{ or } \Phi(\rho(f,\xi), u(f,\xi), \xi, v)(v - \omega_{max})_+^2 = 0 \textit{ a.e. } v \\ &\Leftrightarrow\; f = 0 \textit{ or } (v - \omega_{max})_+ = 0 \textit{ a.e. in }]\omega_1(f,\xi); \omega_2(f,\xi)[\\ &\Leftrightarrow\; f = 0 \textit{ or } \omega_2(f,\xi) \leq \omega_{max}. \end{aligned}$$

A similar result holds for S_m, thus finally

$$(H_{S_M}(f,\xi) \leq 0 \text{ and } H_{S_m}(f,\xi) \leq 0) \Leftrightarrow f \in \tilde{D}_\xi, \tag{5.1}$$

and we conclude with Proposition 3.5 that $\tilde{D}_\xi$ is convex.

Let us now denote by $f(t,x,\xi)$ the solution of Theorem 1.1, with $f^0(x,\xi) \in \tilde{D}_\xi$. Then by Proposition 4.5 $H_{S_M}(f(t,x,\xi),\xi) = H_{S_m}(f(t,x,\xi),\xi) = 0\ \forall t \geq 0$, *a.e.* x, ξ. This proves that $f \in \tilde{D}_\xi\ \forall t \geq 0$ and $\tilde{D}_\xi$ is kinetic invariant.

We turn now to properties (1.33) and (1.34) . Let $(\rho, u) \in \tilde{D}$. Then either $M(\rho,u,\xi) = 0 \in \tilde{D}_\xi$, or $M(\rho,u,\xi) \neq 0$, and in this case obviously $\omega_i(M(\rho,u,\xi),\xi) = \omega_i(\rho,u)$ for $i = 1; 2$. Therefore, $M(\rho,u,\xi) \in \tilde{D}_\xi$ also since $(\rho,u) \in \tilde{D}$ and $\rho > 0$. Thus (1.33) is proved.

Let now $f(\xi) \in L^1_\xi$ such that $f(\xi) \in \tilde{D}_\xi$ *a.e.* ξ. Setting $(\rho, \rho u) = \int_{\mathbf{R}} f(\xi)\, d\xi$, we have by integrating (4.1) for $S = S_M$ or S_m

$$0 \leq \int_{\mathbf{R}} H_S(M[f](\xi),\xi)\, d\xi \leq \int_{\mathbf{R}} H_S(f(\xi),\xi)\, d\xi = 0,$$

thus $H_S(M[f](\xi),\xi) = 0$ *a.e.* ξ. Therefore $M[f](\xi) = M(\rho,u,\xi) \in \tilde{D}_\xi$ for a.e. ξ. Then, either $\rho = 0$ and $(\rho,u) \in \tilde{D}$, or $\rho > 0$, and in this case, for $\omega_1(\rho,u) < \xi < \omega_2(\rho,u)$ we have $\omega_i(M[f](\xi),\xi) = \omega_i(\rho,u)$ for $i = 1; 2$, thus $\omega_{min} \leq \omega_1(\rho,u) \leq \omega_2(\rho,u) \leq \omega_{max}$, and $(\rho,u) \in \tilde{D}$, that is (1.34).

Finally, the compact support property comes directly from the fact that if $\xi \notin]\omega_{min}, \omega_{max}[$, we have $\tilde{D}_\xi = \{0\}$. Indeed, let us suppose that $\xi \notin]\omega_{min}, \omega_{max}[$, and let $f \in \tilde{D}_\xi$. If $f \neq 0$ then according to Lemma 3.1 $\omega_{min} \leq \omega_1(f,\xi) < \xi < \omega_2(f,\xi) \leq \omega_{max}$, which is impossible, thus $f = 0$. $\square$

6 Relaxation limit

Let f_ε be the solution of Theorem 1.1 with the same initial data $f^0(x,\xi)$ and $(\rho_\varepsilon, u_\varepsilon)$ be the approximate solutions to (1.1) defined by (1.6). In order to prove the convergence, as ε tends to 0, of ρ_ε and u_ε, we are going to use the stability result of [9]. Thus we need to prove that

$$\partial_t[\eta_S(\rho_\varepsilon, u_\varepsilon)] + \partial_x[G_S(\rho_\varepsilon, u_\varepsilon)] \text{ is compact in } H^{-1}_{loc}, \tag{6.1}$$

for any suitable entropy η_S and entropy flux G_S (see Proposition 6.5 for a more precise result). In order to obtain this, we shall need to prove first that $M[f_\varepsilon] - f_\varepsilon$ tends to 0 strongly. We refer to [1] and [2] and the references therein for other situations where a similar result can be obtained. Let us begin by an a priori bound.

Proposition 6.1 *If we suppose that $f^0(x,\xi) \in \tilde{D}_\xi$ a.e. x, ξ, then f_ε, $M[f_\varepsilon] \in \tilde{D}_\xi$ for any $t > 0$ and a.e. x, ξ, and ρ_ε, u_ε, f_ε, $M[f_\varepsilon]$ are uniformly bounded in L^∞.*

Proof. This is an immediate consequence of the maximum principle stated in Theorem 1.4 and of formula (3.3). □

In all this section, we shall suppose that

$$f^0(x,\xi) \in \tilde{D}_\xi \text{ for a.e. } x,\ \xi, \tag{6.2}$$

and we shall denote by F the a priori L^∞ bound. Thus we have

$$|f_\varepsilon| \le F, \qquad |M[f_\varepsilon]| \le F, \tag{6.3}$$

$$\mathrm{supp}_\xi f_\varepsilon \subset [\omega_{min}, \omega_{max}], \qquad \mathrm{supp}_\xi M[f_\varepsilon] \subset [\omega_{min}, \omega_{max}]. \tag{6.4}$$

From the definition of $\tilde{D}_\xi$, we see that we have also for some constant A the bounds

$$|(f_\varepsilon)_1| \le A (f_\varepsilon)_0, \qquad |(M[f_\varepsilon])_1| \le A (M[f_\varepsilon])_0. \tag{6.5}$$

Proposition 6.2 *After extraction of a subsequence, $f_\varepsilon - M[f_\varepsilon] \underset{\varepsilon \to 0}{\to} 0$ a.e. t, x, ξ.*

Proof. From the entropy dissipation identity

$$\iint_{\mathbf{R}\times\mathbf{R}} H(f_\varepsilon(T,x,\xi),\xi)\, dx d\xi + \iiint_{]0,T[\times\mathbf{R}\times\mathbf{R}} (H'(f_\varepsilon,\xi) - T(\rho_\varepsilon,u_\varepsilon)) \cdot \frac{f_\varepsilon - M[f_\varepsilon]}{\varepsilon}\, dt dx d\xi$$
$$= \iint_{\mathbf{R}\times\mathbf{R}} H(f^0(x,\xi),\xi)\, dx d\xi,$$

and since $H \ge 0$ and $(H'(f_\varepsilon,\xi) - T(\rho_\varepsilon,u_\varepsilon)) \cdot (f_\varepsilon - M[f_\varepsilon]) \ge 0$, we get

$$\iiint_{]0,T[\times\mathbf{R}\times\mathbf{R}} (H'(f_\varepsilon,\xi) - T(\rho_\varepsilon,u_\varepsilon)) \cdot (f_\varepsilon - M[f_\varepsilon])\, dt dx d\xi \le C_H^0 \varepsilon,$$

which yields that $(H'(f_\varepsilon,\xi) - T(\rho_\varepsilon,u_\varepsilon)) \cdot (f_\varepsilon - M[f_\varepsilon]) \to 0$ in $L^1(]0,T[\times\mathbf{R}\times\mathbf{R})$. Then, extracting a subsequence if necessary,

$$(H'(f_\varepsilon,\xi) - T(\rho_\varepsilon,u_\varepsilon)) \cdot (f_\varepsilon - M[f_\varepsilon]) \underset{\varepsilon \to 0}{\to} 0 \quad \text{a.e. } t, x, \xi. \tag{6.6}$$

Since by Proposition 3.3 $T(\rho_\varepsilon,u_\varepsilon) = H'(M[f_\varepsilon],\xi)$ whenever $M[f_\varepsilon] \ne 0$, in order to deduce the result, we need to get somehow a lower estimate on H''. We notice that for $X = (X_0, X_1) \in D \setminus \{0\}$, $Y = (Y_0, Y_1) \in \mathbf{R}^2$,

$$H''(X,\xi) \cdot Y \cdot Y = K_1 X_0^{1/\lambda - 1} Y_0^2 + K_2 (X_1 Y_0 - X_0 Y_1)^2 / X_0^3, \tag{6.7}$$

with $K_1 = \theta/2\lambda c_{\gamma,\kappa}^{1/\lambda}$ and $K_2 = 1/(1-\theta)$.

First step: let us prove that for $\lambda \geq 1$, $(f_\varepsilon)_0 - (M[f_\varepsilon])_0 \underset{\varepsilon\to 0}{\to} 0$ *a.e.* t, x, ξ. Consider a point (t,x,ξ) where $(M[f_\varepsilon])_0(t,x,\xi) \neq 0$ and $(f_\varepsilon)_0(t,x,\xi) \neq 0$. We set $X(\vartheta) = (1-\vartheta)f_\varepsilon + \vartheta M[f_\varepsilon]$. For every $\vartheta \in [0,1]$, $X(\vartheta)_0^{1/\lambda-1} \geq F^{1/\lambda-1}$ because $\lambda \geq 1$, thus with (6.7)

$$\begin{aligned} &(H'(M[f_\varepsilon],\xi) - H'(f_\varepsilon,\xi))\cdot(M[f_\varepsilon]-f_\varepsilon) \\ &= \int_0^1 H''(X(\vartheta),\xi)\cdot(M[f_\varepsilon]-f_\varepsilon)^2\,d\vartheta \\ &\geq K_1 F^{1/\lambda-1}((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2. \end{aligned} \tag{6.8}$$

Taking into account (1.25), we deduce that

$$\begin{aligned} &\mathbb{1}_{(M[f_\varepsilon])_0\neq 0}(H'(M[f_\varepsilon],\xi) - H'(f_\varepsilon,\xi))\cdot(M[f_\varepsilon]-f_\varepsilon) \\ \geq\ &\mathbb{1}_{(M[f_\varepsilon])_0\neq 0} K_1 F^{1/\lambda-1}((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2 \quad \text{a.e.}, \end{aligned}$$

and together with the entropy dissipation (6.6) we conclude that

$$\mathbb{1}_{(M[f_\varepsilon])_0\neq 0}((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2 \underset{\varepsilon\to 0}{\to} 0 \quad a.e.\ t, x, \xi. \tag{6.9}$$

But $\int_{\mathbf{R}} (f_\varepsilon)_0\,d\xi = \int_{\mathbf{R}} (M[f_\varepsilon])_0\,d\xi$ *a.e.* t, x, thus for any bounded domain B in (t,x),

$$\begin{aligned} &\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0\neq 0\}} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0)\,dtdxd\xi \\ = &\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0 = 0\}} ((f_\varepsilon)_0 - (M[f_\varepsilon])_0)\,dtdxd\xi \\ = &\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0 = 0\}} (f_\varepsilon)_0\,dtdxd\xi \\ = &\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0 = 0\}} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\,dtdxd\xi. \end{aligned}$$

But by dominated convergence and with (6.9),

$$\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0\neq 0\}} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0)\,dtdxd\xi \underset{\varepsilon\to 0}{\to} 0,$$

and we get

$$\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0 = 0\}} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\,dtdxd\xi \underset{\varepsilon\to 0}{\to} 0.$$

Extracting a subsequence if necessary, we conclude that

$$\mathbb{1}_{(M[f_\varepsilon])_0=0}((M[f_\varepsilon])_0 - (f_\varepsilon)_0) \underset{\varepsilon\to 0}{\to} 0 \quad a.e.\ t, x, \xi. \tag{6.10}$$

Finally, (6.9) and (6.10) give that

$$(f_\varepsilon)_0 - (M[f_\varepsilon])_0 \underset{\varepsilon\to 0}{\to} 0 \qquad a.e.\ t, x, \xi.$$

Second step: we prove that for $\lambda < 1$, $(f_\varepsilon)_0 - (M[f_\varepsilon])_0 \underset{\varepsilon\to 0}{\to} 0 \;\; a.e.\ t, x, \xi$.
Consider a point (t, x, ξ) where $(M[f_\varepsilon])_0(t, x, \xi) \neq 0$ and $(f_\varepsilon)_0(t, x, \xi) \neq 0$. We set $X(\vartheta) = (1-\vartheta)f_\varepsilon + \vartheta M[f_\varepsilon]$. For every $\vartheta \in [\frac{1}{2}, 1]$, $X(\vartheta)_0^{1/\lambda-1} \geq (1/2)^{1/\lambda-1}(M[f_\varepsilon])_0^{1/\lambda-1}$ because $\lambda < 1$. Thus

$$\begin{aligned}
&(H'(M[f_\varepsilon], \xi) - H'(f_\varepsilon, \xi)) \cdot (M[f_\varepsilon] - f_\varepsilon)\\
&\geq \int_{\frac{1}{2}}^1 K_1 X(\vartheta)_0^{1/\lambda-1}((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2\, d\vartheta\\
&\geq K_1(1/2)^{1/\lambda}(M[f_\varepsilon])_0^{1/\lambda-1}((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2,
\end{aligned}$$

and we deduce with (1.25) that for any $\alpha > 0$

$$\begin{aligned}
&\mathbb{1}_{(M[f_\varepsilon])_0>\alpha}(H'(M[f_\varepsilon], \xi) - H'(f_\varepsilon, \xi)) \cdot (M[f_\varepsilon] - f_\varepsilon)\\
&\geq K_1(1/2)^{1/\lambda}\alpha^{1/\lambda-1}((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2\mathbb{1}_{(M[f_\varepsilon])_0>\alpha} \quad \text{a.e.}
\end{aligned} \tag{6.11}$$

Together with (6.6), we deduce the convergence

$$\mathbb{1}_{(M[f_\varepsilon])_0>\alpha}((M[f_\varepsilon])_0 - (f_\varepsilon)_0) \underset{\varepsilon\to 0}{\to} 0 \quad a.e.\ t, x, \xi, \tag{6.12}$$

for every $\alpha > 0$. In order to treat the domain where $(M[f_\varepsilon])_0 \leq \alpha$, we use the conservation relation $\int_{\mathbf{R}}(f_\varepsilon)_0\, d\xi = \int_{\mathbf{R}}(M[f_\varepsilon])_0\, d\xi$ $a.e.\ t, x$, which gives

$$\begin{aligned}
&\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0\leq\alpha\}} (f_\varepsilon)_0\, dtdxd\xi\\
= &\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0>\alpha\}} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0)\, dtdxd\xi + \iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0\leq\alpha\}} (M[f_\varepsilon])_0\, dtdxd\xi.
\end{aligned}$$

Now, since $(M[f_\varepsilon])_0 \leq \alpha$ in the last integral, we can choose an α small enough so that this integral is less than an arbitrarily given $\mu > 0$. Then, from (6.12), the first integral can also be made less than μ if ε is small enough. Therefore, for such α and ε,

$$\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0\leq\alpha\}} (f_\varepsilon)_0\, dtdxd\xi \leq 2\mu,$$

$$\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0\leq\alpha\}} \Big((f_\varepsilon)_0 + (M[f_\varepsilon])_0\Big)\, dtdxd\xi \leq 3\mu.$$

Putting this together with (6.12), we conclude that for ε small enough,

$$\iiint_{(t,x)\in B} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\, dtdxd\xi \leq 4\mu.$$

This means that

$$\iiint_{(t,x)\in B} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\, dtdxd\xi \longrightarrow 0 \qquad \text{as } \varepsilon \to 0,$$

and extracting a subsequence if necessary, we get that

$$(f_\varepsilon)_0 - (M[f_\varepsilon])_0 \underset{\varepsilon\to 0}{\to} 0 \qquad a.e.\ t, x, \xi. \tag{6.13}$$

Third step: we prove that $(f_\varepsilon)_1 - (M[f_\varepsilon])_1 \underset{\varepsilon\to 0}{\to} 0$ $a.e.\ t, x, \xi$, for every $\lambda > 0$. Consider a point (t, x, ξ) where $(M[f_\varepsilon])_0(t, x, \xi) \neq 0$ and $(f_\varepsilon)_0(t, x, \xi) \neq 0$. By using the lower estimate from (6.7) $H''(X, \xi) \cdot Y \cdot Y \geq K_2(X_1Y_0 - X_0Y_1)^2/X_0^3$ with $X = X(\vartheta) = (1-\vartheta)f_\varepsilon + \vartheta M[f_\varepsilon]$ and $Y = M[f_\varepsilon] - f_\varepsilon$, we get

$$\begin{aligned} & (H'(M[f_\varepsilon], \xi) - H'(f_\varepsilon, \xi)) \cdot (M[f_\varepsilon] - f_\varepsilon) \\ \geq\ & K_2 \int_0^1 \frac{1}{X(\vartheta)_0^3} (X(\vartheta)_1 Y_0 - X(\vartheta)_0 Y_1)^2\, d\vartheta. \end{aligned}$$

But $X(\vartheta)_1 Y_0 - X(\vartheta)_0 Y_1 = (f_\varepsilon)_1 (M[f_\varepsilon])_0 - (f_\varepsilon)_0 (M[f_\varepsilon])_1$, and for $\vartheta \in [0, 1]$, we have the two upper estimates $X(\vartheta)_0 \leq (f_\varepsilon)_0 + (M[f_\varepsilon])_0$ and $X(\vartheta)_0 \leq F$, thus $X(\vartheta)_0^3 \leq F((f_\varepsilon)_0 + (M[f_\varepsilon])_0)^2$. Therefore we obtain

$$\begin{aligned} & \mathbb{1}_{(M[f_\varepsilon])_0\neq 0; (f_\varepsilon)_0 \neq 0} (H'(M[f_\varepsilon], \xi) - H'(f_\varepsilon, \xi)) \cdot (M[f_\varepsilon] - f_\varepsilon) \\ \geq\ & \frac{K_2}{F} \left(\frac{(f_\varepsilon)_1 (M[f_\varepsilon])_0 - (f_\varepsilon)_0 (M[f_\varepsilon])_1}{(f_\varepsilon)_0 + (M[f_\varepsilon])_0} \right)^2 \mathbb{1}_{(M[f_\varepsilon])_0\neq 0; (f_\varepsilon)_0 \neq 0}, \end{aligned}$$

and we conclude with (6.6) that

$$\mathbb{1}_{(M[f_\varepsilon])_0\neq 0} \frac{(f_\varepsilon)_1 (M[f_\varepsilon])_0 - (f_\varepsilon)_0 (M[f_\varepsilon])_1}{(f_\varepsilon)_0 + (M[f_\varepsilon])_0} \underset{\varepsilon\to 0}{\to} 0 \quad a.e.\ t, x, \xi. \tag{6.14}$$

Then, we remark that

$$\begin{aligned} & \mathbb{1}_{(M[f_\varepsilon])_0\neq 0} ((f_\varepsilon)_1 - (M[f_\varepsilon])_1) \\ = & \mathbb{1}_{(M[f_\varepsilon])_0\neq 0} \frac{(f_\varepsilon)_1 (M[f_\varepsilon])_0 - (f_\varepsilon)_0 (M[f_\varepsilon])_1}{(M[f_\varepsilon])_0} \\ & - \mathbb{1}_{(M[f_\varepsilon])_0\neq 0} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0) \frac{(M[f_\varepsilon])_1}{(M[f_\varepsilon])_0}, \end{aligned} \tag{6.15}$$

and that

$$\begin{aligned} & \mathbb{1}_{(f_\varepsilon)_0\neq 0} ((f_\varepsilon)_1 - (M[f_\varepsilon])_1) \\ = & \mathbb{1}_{(f_\varepsilon)_0\neq 0} \frac{(f_\varepsilon)_1 (M[f_\varepsilon])_0 - (f_\varepsilon)_0 (M[f_\varepsilon])_1}{(f_\varepsilon)_0} \\ & - \mathbb{1}_{(f_\varepsilon)_0\neq 0} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0) \frac{(f_\varepsilon)_1}{(f_\varepsilon)_0}. \end{aligned} \tag{6.16}$$

Multiplying (6.15) by $\mathbb{1}_{(f_\varepsilon)_0\neq 0}\mathbb{1}_{(f_\varepsilon)_0\leq (M[f_\varepsilon])_0}$, (6.16) by $\mathbb{1}_{(M[f_\varepsilon])_0\neq 0}\ \mathbb{1}_{(M[f_\varepsilon])_0<(f_\varepsilon)_0}$, and then adding the two, we get

$$\begin{aligned}
&\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;(f_\varepsilon)_0\neq 0}((f_\varepsilon)_1-(M[f_\varepsilon])_1)\\
&=\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;\ (f_\varepsilon)_0\neq 0;\ (f_\varepsilon)_0\leq (M[f_\varepsilon])_0}\frac{(f_\varepsilon)_1(M[f_\varepsilon])_0-(f_\varepsilon)_0(M[f_\varepsilon])_1}{(M[f_\varepsilon])_0}\\
&\quad+\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;\ (f_\varepsilon)_0\neq 0;\ (M[f_\varepsilon])_0<(f_\varepsilon)_0}\frac{(f_\varepsilon)_1(M[f_\varepsilon])_0-(f_\varepsilon)_0(M[f_\varepsilon])_1}{(f_\varepsilon)_0}\\
&\quad-\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;\ (f_\varepsilon)_0\neq 0;\ (f_\varepsilon)_0\leq (M[f_\varepsilon])_0}((M[f_\varepsilon])_0-(f_\varepsilon)_0)\frac{(M[f_\varepsilon])_1}{(M[f_\varepsilon])_0}\\
&\quad-\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;\ (f_\varepsilon)_0\neq 0;\ (M[f_\varepsilon])_0<(f_\varepsilon)_0}((M[f_\varepsilon])_0-(f_\varepsilon)_0)\frac{(f_\varepsilon)_1}{(f_\varepsilon)_0}.
\end{aligned}\tag{6.17}$$

From (6.5) and (6.13), the last two terms tend to 0 a.e. But according to (6.14), the two first terms also tend to 0 a.e., and we conclude that

$$\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;\ (f_\varepsilon)_0\neq 0}((f_\varepsilon)_1-(M[f_\varepsilon])_1)\underset{\varepsilon\to 0}{\to}0\quad a.e.\ t,x,\xi.$$

Then,

$$\begin{aligned}
&\mathbb{1}_{(M[f_\varepsilon])_0=0}|(f_\varepsilon)_1-(M[f_\varepsilon])_1|\\
&=\mathbb{1}_{(M[f_\varepsilon])_0=0}|(f_\varepsilon)_1|\\
&\leq A\mathbb{1}_{(M[f_\varepsilon])_0=0}(f_\varepsilon)_0\\
&=A\mathbb{1}_{(M[f_\varepsilon])_0=0}((f_\varepsilon)_0-(M[f_\varepsilon])_0)\underset{\varepsilon\to 0}{\to}0\quad a.e.
\end{aligned}\tag{6.18}$$

We perform a similar computation for $\mathbb{1}_{(f_\varepsilon)_0=0}|(f_\varepsilon)_1-(M[f_\varepsilon])_1|$, and we obtain finally

$$(f_\varepsilon)_1-(M[f_\varepsilon])_1\underset{\varepsilon\to 0}{\to}0\quad a.e.\ t,x,\xi,$$

which completes the proof of Proposition 6.2. □

Proposition 6.3 *For $S:\mathbf{R}\to\mathbf{R}$ convex, of class C^1, such that $|S(v)|\leq B(1+v^2)$ for some $B\geq 0$, we have*

$$\partial_t\int_{\mathbf{R}}H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi+\partial_x\int_{\mathbf{R}}\xi H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi\leq 0,\tag{6.19}$$

with equality if $S(v)=1$ or v.

Proof. Proposition 4.5 gives that $H_S(f_\varepsilon,\xi)\in L^\infty(]0,T[,L^1(\mathbf{R}\times\mathbf{R}))$. According to Proposition 3.3 and Proposition 6.1, $H'_S(f_\varepsilon,\xi)\in L^\infty$, since $H'_S(f_\varepsilon,\xi)=\eta'_S(\rho(f_\varepsilon,\xi),u(f_\varepsilon,\xi))$ whenever $(f_\varepsilon)_0>0$ (see (4.2)). Thus, according to [5],

$$\partial_t(H_S(f_\varepsilon,\xi))+\xi\partial_x(H_S(f_\varepsilon,\xi))=H'_S(f_\varepsilon,\xi)\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}.\tag{6.20}$$

From Proposition 4.1, we have

$$H_S(f_\varepsilon,\xi)\geq H_S(M[f_\varepsilon],\xi)+T_S(\rho_\varepsilon,u_\varepsilon)\cdot(f_\varepsilon-M[f_\varepsilon]).\tag{6.21}$$

Besides, since H_S is convex (Proposition 3.5),

$$H_S(M[f_\varepsilon],\xi) \geq H_S(f_\varepsilon,\xi) + H_S'(f_\varepsilon,\xi)\cdot(M[f_\varepsilon]-f_\varepsilon) \tag{6.22}$$

for $f_\varepsilon \neq 0$, but it is still valid where $f_\varepsilon = 0$ because there $M[f_\varepsilon]=0$ a.e., see (1.25). Thus by combining (6.21) and (6.22) we get

$$(H_S'(f_\varepsilon,\xi) - T_S(\rho_\varepsilon,u_\varepsilon))\cdot(M[f_\varepsilon]-f_\varepsilon) \leq 0. \tag{6.23}$$

Therefore, we obtain by integration of (6.20) with respect to ξ

$$\begin{aligned}
&\partial_t \int_{\mathbf{R}} H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi + \partial_x \int_{\mathbf{R}} \xi H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi \\
&= \int_{\mathbf{R}} H_S'(f_\varepsilon,\xi)\cdot \frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\,d\xi \\
&= \int_{\mathbf{R}} (H_S'(f_\varepsilon,\xi) - T_S(\rho_\varepsilon,u_\varepsilon))\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\,d\xi \;\leq\; 0.
\end{aligned} \tag{6.24}$$

If $S(v)=1$ or v, then $H_S(f,\xi)=f_0$ or f_1, thus $H_S'(f,\xi)=(1,0)$ or $(0,1)$, and (6.24) becomes an equality. □

Proposition 6.4 *For $S:\mathbf{R}\to\mathbf{R}$ convex, of class C^1, such that $|S(v)| \leq B(1+v^2)$ for some $B \geq 0$, we have*

$$\begin{aligned}
&\int_{\mathbf{R}} H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi - \int_{\mathbf{R}} H_S(M[f_\varepsilon](t,x,\xi),\xi)\,d\xi \underset{\varepsilon\to 0}{\to} 0,\\
&\int_{\mathbf{R}} \xi H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi - \int_{\mathbf{R}} \xi H_S(M[f_\varepsilon](t,x,\xi),\xi)\,d\xi \underset{\varepsilon\to 0}{\to} 0
\end{aligned} \tag{6.25}$$

in $L^1(]0,T[\times B_R)$ for any $R>0$.

Proof. Integrating (6.21) and (6.22), we get

$$0 \leq \int_{\mathbf{R}} H_S(f_\varepsilon,\xi)\,d\xi - \int_{\mathbf{R}} H_S(M[f_\varepsilon],\xi)\,d\xi \leq \int_{\mathbf{R}} H_S'(f_\varepsilon,\xi)\cdot(f_\varepsilon - M[f_\varepsilon])\,d\xi,$$

which is bounded by $C\varepsilon$ in $L^1(]0,\infty[\times\mathbf{R})$ according to (6.24), and we get the first convergence of (6.25). From (6.21) and (6.22), we also have

$$\begin{aligned}
&\left|\int_{\mathbf{R}} \xi H_S(f_\varepsilon,\xi)\,d\xi - \int_{\mathbf{R}} \xi H_S(M[f_\varepsilon],\xi)\,d\xi\right| \\
\leq\;& \int_{\mathbf{R}} \Big|\xi H_S(f_\varepsilon,\xi) - \xi H_S(M[f_\varepsilon],\xi)\Big|\,d\xi \\
\leq\;& \int_{\mathbf{R}} |\xi| \max\Big(|T_S(\rho_\varepsilon,u_\varepsilon)\cdot(f_\varepsilon - M[f_\varepsilon])|, |H_S'(f_\varepsilon,\xi)\cdot(f_\varepsilon - M[f_\varepsilon])|\Big)\,d\xi \\
\leq\;& \int_{\mathbf{R}} |\xi T_S(\rho_\varepsilon,u_\varepsilon)\cdot(f_\varepsilon - M[f_\varepsilon])|\,d\xi \\
&+ \int_{\mathbf{R}} |\xi|(H_S'(f_\varepsilon,\xi) - T_S(\rho_\varepsilon,u_\varepsilon))\cdot(f_\varepsilon - M[f_\varepsilon])\,d\xi.
\end{aligned}$$

Since ξ remains bounded, the second integral is of order ε in $L^1(]0,\infty[\times\mathbf{R})$ as before. For the first one, we conclude with Proposition 6.2 and by using that

$T_S(\rho_\varepsilon, u_\varepsilon)$ is bounded in $L^\infty(]0,T[\times\mathbf{R})$ according to Proposition 6.1. Therefore we get the second convergence of (6.25). $\square$

Proposition 6.5 *For $S:\mathbf{R}\to\mathbf{R}$ convex, of class C^1, such that $|S(v)|\le B(1+v^2)$ for some $B\ge 0$, we have that*

$$\partial_t[\eta_S(\rho_\varepsilon, u_\varepsilon)] + \partial_x[G_S(\rho_\varepsilon, u_\varepsilon)] \le R_{S,\varepsilon}(t,x) \tag{6.26}$$

with $R_{S,\varepsilon}\to 0$ in $W^{-1,p}_{loc}$ for any $1<p<\infty$ as $\varepsilon\to 0$. Moreover, (6.26) becomes an equality if $S(v)=1$ or v.

Proof. We have that

$$\int_{\mathbf{R}} H_S(M(\rho_\varepsilon, u_\varepsilon, \xi), \xi)\, d\xi = \eta_S(\rho_\varepsilon, u_\varepsilon),$$

$$\int_{\mathbf{R}} \xi H_S(M(\rho_\varepsilon, u_\varepsilon, \xi), \xi)\, d\xi = G_S(\rho_\varepsilon, u_\varepsilon).$$

Thus we can write

$$\begin{aligned} &\partial_t[\eta_S(\rho_\varepsilon, u_\varepsilon)] + \partial_x[G_S(\rho_\varepsilon, u_\varepsilon)] \\ = \;& \partial_t \int_{\mathbf{R}} H_S(M(\rho_\varepsilon, u_\varepsilon, \xi), \xi)\, d\xi + \partial_x \int_{\mathbf{R}} \xi H_S(M(\rho_\varepsilon, u_\varepsilon, \xi), \xi)\, d\xi \\ = \;& \partial_t W_\varepsilon + \partial_x Z_\varepsilon + \partial_t \int_{\mathbf{R}} H_S(f_\varepsilon, \xi)\, d\xi + \partial_x \int_{\mathbf{R}} \xi H_S(f_\varepsilon, \xi)\, d\xi, \end{aligned}$$

where

$$W_\varepsilon = \int_{\mathbf{R}} H_S(M[f_\varepsilon], \xi)\, d\xi - \int_{\mathbf{R}} H_S(f_\varepsilon, \xi)\, d\xi,$$

$$Z_\varepsilon = \int_{\mathbf{R}} \xi H_S(M[f_\varepsilon], \xi)\, d\xi - \int_{\mathbf{R}} \xi H_S(f_\varepsilon, \xi)\, d\xi$$

tend to 0 in L^p_{loc} according to Proposition 6.4. With Proposition 6.3, we obtain the result. $\square$

Finally, in order to obtain Theorem 1.5, we need only to apply the stability result of [9], which is valid for a sequence $\rho_\varepsilon, u_\varepsilon$ bounded in L^∞ satisfying the conclusion of Proposition 6.5.

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Troisième partie

Condition de bord

Conditions de bord entropiques faibles pour la dynamique des gaz isentropiques via la relaxation cinétique

Weak entropy boundary conditions for isentropic gas dynamics via kinetic relaxation

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Abstract

We consider a kinetic BGK model relaxing to isentropic gas dynamics previously introduced by the authors, but with Dirichlet boundary condition on the incoming velocities. We pass to the limit as the relaxation parameter tends to zero by compensated compactness inside the domain, and obtain that the limit satisfies entropy inequalities on the boundary involving weak traces of entropy fluxes. Our method is very general and could be applied to any entropy satisfying BGK model as soon as we have strong compactness of the macroscopic variables inside the domain.

Key-words: entropy boundary conditions – isentropic gas dynamics – kinetic BGK model – kinetic entropy – kinetic invariant domain – relaxation limit – weak trace

Mathematics Subject Classification: 35L50, 76N15, 35L65, 35B35, 82C40

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Contents

1 Introduction and main results

We consider the one-dimensional system of isentropic gas dynamics

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \kappa \rho^\gamma) = 0, \end{cases} \qquad t > 0, x > 0, \tag{1.1}$$

with $\rho(t,x) \geq 0,\ u(t,x) \in \mathbf{R}$ and $\kappa > 0,\ 1 < \gamma < 3$, with initial data

$$\rho(0,x) = \rho^0(x), \quad \rho(0,x)u(0,x) = \rho^0(x)u^0(x), \qquad x > 0, \tag{1.2}$$

and Dirichlet boundary conditions

$$\rho(t,0) = \rho^b(t), \quad \rho(t,0)u(t,0) = \rho^b(t)u^b(t), \qquad t > 0. \tag{1.3}$$

Entropy solutions are defined as solutions to (1.1) with finite energy, which satisfy the entropy inequalities

$$\partial_t(\eta_S) + \partial_x(G_S) \leq 0, \qquad \text{in }]0,\infty[_t \times]0,\infty[_x, \tag{1.4}$$

for any η_S and G_S suitable entropy and entropy flux for (1.1) parametrized by a convex function S with subquadratic growth that will be defined precisely in the next section.

It is well-known that (1.3) cannot be satisfied everywhere, therefore one has to introduce a weaker formulation. In [12], Dubois and LeFloch proposed two formulations for the boundary condition; one based on a Riemann problem in a half-space, and another based on entropy boundary inequalities derived from a viscosity approximation. Here we consider boundary conditions of the second kind, which are defined as

$$\overline{G_S(\rho,u)} - G_S(\rho^b,u^b) - \eta_S'(\rho^b,u^b) \cdot (\overline{F(\rho,u)} - F(\rho^b,u^b)) \leq 0, \quad \text{in }]0,\infty[_t \times \{0\}_x, \tag{1.5}$$

where $F(\rho,u) = (\rho u, \rho u^2 + \kappa\rho^\gamma)$ is the flux of the system, overbars denote weak traces (and will be defined later on), and prime denotes differentiation with respect to conservative variables $(\rho, q \equiv \rho u)$. The weak formulation of (1.4)-(1.5) is

$$\begin{aligned} &- \iint\limits_{]0,\infty[\times]0,\infty[} \eta_S(\rho,u)\partial_t\varphi \, dtdx - \iint\limits_{]0,\infty[\times]0,\infty[} G_S(\rho,u)\partial_x\varphi \, dtdx \\ &- \int_{]0,\infty[} [G_S(\rho^b,u^b) + \eta_S'(\rho^b,u^b) \cdot (\overline{F(\rho,u)} - F(\rho^b,u^b))]\varphi(t,0) \, dt \leq 0 \end{aligned} \tag{1.6}$$

for any $\varphi \in \mathcal{D}(]0,\infty[_t \times [0,\infty[_x)$, $\varphi \geq 0$. We notice that according to [1], (1.1) allows to define the weak trace of $\overline{F(\rho,u)}$, and the entropy inequalities (1.4) allow to define the weak traces of $G_S(\rho,u)$ for any S (see Section 6). Since we consider data that can contain vacuum $\rho^b = 0$, and since η_S is not differentiable at 0, this formula has to be generalized, see (1.22)-(1.23). We recall that in the scalar case, the weak entropy boundary conditions (1.5) give a unique solution, as proved by Otto [26].

The aim of this paper is to obtain the existence of a solution to the initial-boundary value problem, for arbitrary L^∞ initial-boundary value data. Indeed we are able to obtain it from the relaxation of an entropy compatible BGK model with Dirichlet condition on the incoming velocities, by a very general method that uses mainly the subdifferential inequality that characterizes the entropy compatibility of the kinetic model, as proved in [7]. We need only to know some a priori bound on the macroscopic variables, and their strong compactness. Here it is obtained by the existence of kinetic invariant domains and by compensated compactness. Our method could be used, for example, in the context of [25] to prove that the relaxation limit gives the right boundary conditions.

In the BV context, the gas dynamics in lagrangian coordinates was considered in [20]. The existence of a strong trace for BV functions is used for boundary conditions since [2], in which they got the first existence and uniqueness result in the scalar case. Other hyperbolic systems with boundary conditions in the BV context can be found in [14] for small data and together with a relaxation approximation in [32] for a uniqueness result, and in [23] for regular solutions. BGK approaches for scalar conservation laws with boundary conditions can be found in [21] and [25]. In the less restrictive context of L^1 or L^∞ functions, the existence of a strong trace is a difficult problem that has only be solved in the scalar case in [31], and in general weak trace formulations have to be taken into account, as introduced in [26].

Relations between the two sets of expected boundary values of [12] is an interesting problem. The fact that these sets are not equal was established in [4], in which they give a condition for the coincidence for 2×2 systems. General systems of conservations laws are considered in [16], where they give existence from Godunov schemes and viscous approximation and also prove in this context that the boundary values derived from the viscous approximation contains the one derived in terms of the boundary Riemann problems, and the falsity of the converse in general. The p-system in the interval $[0,1]$ is studied in [24] via a solution in $\mathbf{R}^+ \times [0,1]$ and in [3] as the limit of an associated parabolic equation. A problem arising in chemical engineering using compensated compactness methods is studied in [15]. These problems use the entropy formulation. In [13], the formulation with Godunov schemes is used for hyperbolic systems in one space variable.

The kinetic model we use here is the one of [5], [6] with Dirichlet conditions on incoming velocities and can be written as

$$\partial_t f + \xi \partial_x f = \frac{M[f] - f}{\varepsilon} \quad \text{in }]0,\infty[_t \times]0,\infty[_x \times \mathbf{R}_\xi, \tag{1.7}$$

where $f = f(t,x,\xi) \in \mathbf{R}^2$, $t > 0$, $x > 0$, $\xi \in \mathbf{R}$, $M[f]$ is defined by (2.1)-(2.6),

$$f(t,x,\xi) \in D = \{(f_0, f_1) \in \mathbf{R}^2,\ f_0 > 0 \text{ or } f_1 = f_0 = 0\}, \tag{1.8}$$

with the initial data

$$f(0,x,\xi) = f^0(x,\xi), \qquad x > 0,\ \xi \in \mathbf{R}, \tag{1.9}$$

and the boundary condition on incoming velocities

$$f(t,0,\xi) = f^b(t,\xi), \qquad t > 0,\ \xi > 0. \tag{1.10}$$

It is well-known that the natural space associated to the traces of the kinetic equation (1.7) is the space L^1_μ with

$$d\mu = |\xi| d\xi dt. \tag{1.11}$$

We have the following existence theorem for the BGK model.

Theorem 1.1 *Assume that* $f^0 \in L^1(]0,\infty[_x \times \mathbf{R}_\xi)$, $f^b \in L^1_\mu(]0,T[_t \times]0,\infty[_\xi)$ *satisfy*

$$f^0(x,\xi) \in D, \; f^b(t,\xi) \in D \qquad a.e., \tag{1.12}$$

and with H *the kinetic entropy (2.14) associated with the physical energy,*

$$\iint_{]0,\infty[\times\mathbf{R}} H(f^0(x,\xi),\xi)\, dxd\xi < \infty, \tag{1.13}$$

$$\iint_{]0,T[\times]0,\infty[} H(f^b(t,\xi),\xi)\, d\mu(t,\xi) < \infty. \tag{1.14}$$

Then there exists a solution f *to (1.7)-(1.10) satisfying*

$$f \in C_t([0,T], L^1(]0,\infty[_x \times \mathbf{R}_\xi)) \cap C_x([0,\infty[, L^1_\mu(]0,T[_t \times \mathbf{R}_\xi)), \tag{1.15}$$

$$\forall t \geq 0, \; f(t,x,\xi) \in D \; a.e. \; in \;]0,\infty[_x \times \mathbf{R}_\xi, \tag{1.16}$$

$$H(f(t,x,\xi),\xi) \in L^\infty_t(]0,T[, L^1(]0,\infty[_x \times \mathbf{R}_\xi)). \tag{1.17}$$

Furthermore, f *satisfies for any* $t \geq 0$ *the estimates*

$$\iint_{]0,\infty[\times\mathbf{R}} f_0(t,x,\xi)\, dxd\xi \leq \iint_{]0,\infty[\times\mathbf{R}} f_0^0(x,\xi)\, dxd\xi + \iint_{]0,t[\times]0,\infty[} f_0^b(s,\xi)\, d\mu(s,\xi), \tag{1.18}$$

$$\begin{aligned}\iint_{]0,\infty[\times\mathbf{R}} H(f(t,x,\xi),\xi)\, dxd\xi \leq & \iint_{]0,\infty[\times\mathbf{R}} H(f^0(x,\xi),\xi)\, dxd\xi \\ & + \iint_{]0,t[\times]0,\infty[} H(f^b(s,\xi),\xi)\, d\mu(s,\xi).\end{aligned} \tag{1.19}$$

We obtain for these solutions a maximum principle and a compact support property. In order to pass to the limit, we impose the boundary condition to be maxwellian. We also assume that initial data and boundary conditions lie in an invariant domain $\tilde{D}$ of the system,

$$\tilde{D} = \{(\rho,u) \in \mathbf{R}^+ \times \mathbf{R};\; \rho = 0 \; or \; \omega_{min} \leq \omega_1 \leq \omega_2 \leq \omega_{max}\}, \tag{1.20}$$

where ω_1 and ω_2 are the Riemann invariants of the system, defined by

$$\omega_1 = u - \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}, \qquad \omega_2 = u + \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}. \tag{1.21}$$

We refer to the next section for more details and for the expression of $\tilde{D}_\xi$ the kinetic invariant domains associated to $\tilde{D}$. We have the following convergence result.

Theorem 1.2 *Let ρ^b, $u^b \in L^\infty(]0,\infty[_t)$, $\rho^b \geq 0$, and let us denote by f_ε the solution of Theorem 1.1 with the same initial data $f^0(x,\xi) \in L^1(]0,\infty[\times\mathbf{R})$ and the same boundary condition $f^b(t,\xi) = M(\rho^b(t), u^b(t), \xi)$ that satisfy $f^0(x,\xi) \in \tilde{D}_\xi$ a.e. x, ξ, $\rho^b, u^b \in \tilde{D}$ a.e. t for some $\omega_{min} < \omega_{max}$, and the energy bound (1.13). Then $(\rho_\varepsilon, u_\varepsilon)$ defined by (2.2) lie in $\tilde{D}$ and thus are uniformly bounded in L^∞, and passing if necessary to subsequences, $(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)$ converge a.e. in $]0,\infty[\times]0,\infty[$ when $\varepsilon \to 0$ to an entropy solution $(\rho, \rho u)$ to (1.1), (1.4) remaining in $\tilde{D}$ with initial data $(\rho^0, \rho^0 u^0) = \int f^0 d\xi$ and the boundary condition*

$$\overline{G_S(\rho,u)} - G_S(\rho^b,u^b) - T_S(\rho^b,u^b)\cdot(\overline{F(\rho,u)} - F(\rho^b,u^b)) \leq 0, \quad \text{in }]0,\infty[_t\times\{0\}_x, \tag{1.22}$$

where

$$T_S(\rho^b,u^b) \text{ is defined by (2.17) and coincides with } \eta_S'(\rho^b,u^b) \text{ when } \rho^b > 0. \tag{1.23}$$

In particular, we can take for initial data in Theorem 1.2 any Maxwellian $f^0(x,\xi) = M(\rho^0(x), u^0(x), \xi)$ with $(\rho^0, u^0) \in \tilde{D}$ a.e. such that $\int_0^\infty \rho^0 dx < \infty$.

In Section 2, we define precisely the kinetic model. We prove in Section 3 the existence of a BGK solution with boundary condition. In Section 4, we give uniform bounds for the obtained sequence of solutions in order to pass to the limit in Section 5 and to get the boundary condition (1.22). Finally, in Section 6, we give basic features on weak traces that are used in this paper.

2 Kinetic model

In this section, we present in more detail the kinetic model we consider. This BGK model has been introduced in [7] and studied in [5], [6]. The Maxwellian is defined by

$$M[f](t,x,\xi) = M\left(\rho(t,x), u(t,x), \xi\right), \tag{2.1}$$

with

$$\rho(t,x) = \int_{\mathbf{R}} f_0(t,x,\xi)\, d\xi, \qquad \rho(t,x)u(t,x) = \int_{\mathbf{R}} f_1(t,x,\xi)\, d\xi, \tag{2.2}$$

and

$$M(\rho,u,\xi) = \Big(\chi(\rho,\xi-u), ((1-\theta)u + \theta\xi)\chi(\rho,\xi-u)\Big), \tag{2.3}$$

$$\chi(\rho,\xi) = c_{\gamma,\kappa}\left(a_\gamma^2\rho^{\gamma-1} - \xi^2\right)_+^\lambda, \tag{2.4}$$

$$\theta = \frac{\gamma-1}{2}, \quad \lambda = \frac{1}{\gamma-1} - \frac{1}{2}, \quad c_{\gamma,\kappa} = a_\gamma^{-2/(\gamma-1)}/J_\lambda, \tag{2.5}$$

$$J_\lambda = \int_{-1}^1 (1-z^2)^\lambda\, dz = \sqrt{\pi}\Gamma(\lambda+1)/\Gamma(\lambda+3/2), \quad a_\gamma = \frac{2\sqrt{\gamma\kappa}}{\gamma-1}. \tag{2.6}$$

The kinetic equilibrium χ has been introduced in [11] as a generating function for entropies and has been used in the stability analysis of [17]. This function is also involved in [19] for the so-called kinetic formulation. The Maxwellian has the moment properties

$$\int_{\mathbf{R}} M(\rho,u,\xi)\,d\xi=(\rho,\rho u),\qquad \int_{\mathbf{R}} \xi M(\rho,u,\xi)\,d\xi=(\rho u,\rho u^2+\kappa\rho^\gamma)=F(\rho,u), \tag{2.7}$$

for every $\rho\geq 0$ and $u\in\mathbf{R}$. Kinetic entropies are parametrized by convex functions S and are defined from a kernel by

$$H_S(f,\xi)=\int_{\mathbf{R}}\Phi(\rho(f,\xi),u(f,\xi),\xi,v)S(v)\,dv,\ \text{for } f\neq 0,\quad H_S(0,\xi)=0, \tag{2.8}$$

where

$$u(f,\xi)=\frac{f_1/f_0-\theta\xi}{1-\theta},\quad \rho(f,\xi)=a_\gamma^{-\frac{2}{\gamma-1}}\left(\left(\frac{f_1/f_0-\xi}{1-\theta}\right)^2+(f_0/c_{\gamma,\kappa})^{1/\lambda}\right)^{\frac{1}{\gamma-1}} \tag{2.9}$$

is the inverse relation for $f=M(\rho,u,\xi)$. The kernel Φ is symmetric in ξ,v and satisfies in particular $\Phi\geq 0$ and $\int_{\mathbf{R}}(1,v)\cdot\Phi(\rho,u,\xi,v)\,dv=M(\rho,u,\xi)$. The entropy and entropy flux for (1.1) are defined by

$$\eta_S(\rho,u)=\int_{\mathbf{R}}\chi(\rho,v-u)S(v)\,dv=\int_{\mathbf{R}}H_S(M(\rho,u,\xi),\xi)\,d\xi, \tag{2.10}$$

$$G_S(\rho,u)=\int_{\mathbf{R}}[(1-\theta)u+\theta v)]\chi(\rho,v-u)S(v)\,dv=\int_{\mathbf{R}}\xi\,H_S(M(\rho,u,\xi),\xi)\,d\xi. \tag{2.11}$$

The kinetic Riemann invariants are defined for $f\neq 0$ by

$$\omega_1(f,\xi)=u(f,\xi)-a_\gamma\rho(f,\xi)^{\frac{\gamma-1}{2}},\qquad \omega_2(f,\xi)=u(f,\xi)+a_\gamma\rho(f,\xi)^{\frac{\gamma-1}{2}}, \tag{2.12}$$

and the kinetic invariant domains by

$$\tilde{D}_\xi=\{f\in D;\ f=0\ \mathit{or}\ \omega_{min}\leq\omega_1(f,\xi)\leq\omega_2(f,\xi)\leq\omega_{max}\}. \tag{2.13}$$

We notice that for $S(v)=v^2/2$, we get $H_S=H$ and $\eta_S=\eta$ with

$$H(f,\xi)=\frac{\theta}{1-\theta}\frac{\xi^2}{2}f_0+\frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda}+\frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0}-\frac{\theta}{1-\theta}\xi f_1, \tag{2.14}$$

and

$$\eta(\rho,u)=\rho u^2/2+\frac{\kappa}{\gamma-1}\rho^\gamma, \tag{2.15}$$

the physical energy for (1.1). Let us recall the following properties for this kinetic model, that are proved in [6].

Proposition 2.1 *If $S:\mathbf{R}\to\mathbf{R}$ is of class C^1 then we have $H_S'(M(\rho,u,\xi),\xi)=\eta_S'(\rho,u)$ whenever $(M(\rho,u,\xi))_0>0$.*

Proposition 2.2 *i) If* $S : \mathbf{R} \to \mathbf{R}$ *is convex and continuous, then* $H_S(.,\xi)$ *is convex in* D.
ii) If $S : \mathbf{R} \to \mathbf{R}$ *is bounded on compact sets, then* $H_S(.,\xi)$ *is continuous at 0 in* $\{f \in D \; ; \; |f_1| \leq Af_0\}$*, for any* $A > 0$.

Proposition 2.3 (Subdifferential inequality) *If* $S : \mathbf{R} \to \mathbf{R}$ *is convex, of class* C^1*, then for every* $f \in D$, $\rho \geq 0$ *and* $u, \xi \in \mathbf{R}$*, we have*

$$H_S(f,\xi) \geq H_S(M(\rho,u,\xi),\xi) + T_S(\rho,u) \cdot (f - M(\rho,u,\xi)), \tag{2.16}$$

with

$$T_S(\rho,u) = \frac{1}{J_\lambda} \int_{-1}^{1} (1-z^2)^\lambda \begin{pmatrix} S(u + a_\gamma \rho^\theta z) + (\theta a_\gamma \rho^\theta z - u) S'(u + a_\gamma \rho^\theta z)) \\ S'(u + a_\gamma \rho^\theta z)) \end{pmatrix} dz, \tag{2.17}$$

which coincides with $\eta_S'(\rho,u)$ *when* $\rho > 0$. *We also have if* $f \neq 0$,

$$(H_S'(f,\xi) - T_S(\rho,u)) \cdot (M(\rho,u,\xi) - f) \leq 0. \tag{2.18}$$

Corollary 2.4 (Entropy minimization principle) *Assume that* $S : \mathbf{R} \to \mathbf{R}$ *is convex, of class* C^1 *and such that* $|S(v)| \leq B(1 + v^2)$ *for some* $B \geq 0$. *Consider* $f \in L^1(\mathbf{R}_\xi)$ *such that* $f \in D$ *a.e. and* $\int_{\mathbf{R}} H(f(\xi),\xi)\, d\xi < \infty$. *Then* $H_S(f(\xi),\xi)$ *and* $H_S(M[f](\xi),\xi)$ *lie in* $L^1(\mathbf{R}_\xi)$ *with*

$$\int_{\mathbf{R}} H_S(M[f](\xi),\xi)\, d\xi \leq \int_{\mathbf{R}} H_S(f(\xi),\xi)\, d\xi. \tag{2.19}$$

Following [29] and [6], we consider the notion of kinetic invariant domain.

Proposition 2.5 1) *The convex sets* $\tilde{D}_\xi$ *are associated to the domains* $\tilde{D}$ *in the sense that*
i) For any $(\rho,u) \in \tilde{D}$, $M(\rho,u,\xi) \in \tilde{D}_\xi$ *a.e.* ξ,
ii) For any $f \in L^1(\mathbf{R}_\xi)$ *such that* $f(\xi) \in \tilde{D}_\xi$ *a.e.* ξ*, the averages* $(\rho, \rho u) = \int_{\mathbf{R}} f(\xi)\, d\xi$ *verify* $(\rho,u) \in \tilde{D}$.
2) *If* $\xi \notin [\omega_{min}, \omega_{max}]$ *then* $\tilde{D}_\xi = \{0\}$.

For more details about this model, we refer to [6]. For kinetic relaxation models and kinetic formulations, we refer in particular to [7], [29], [22], [27], [30], [18], [9], [19] and [28].

3 Solution to the BGK model

This section is devoted to the existence result for the BGK model with initial-boundary condition. We notice first the following characteristics formula for (1.7).

Lemma 3.1 *Let* $h \in L^1(]0,T[, L^1(]0,\infty[\times\mathbf{R}))$, $f^0 \in L^1(]0,\infty[\times\mathbf{R})$ *and* $f^b \in L^1_\mu(]0,T[\times]0,\infty[)$. *Then there exists a unique solution*

$$f \in C_t([0,T], L^1(]0,\infty[_x\times\mathbf{R}_\xi)) \cap C_x([0,\infty[, L^1_\mu(]0,T[_t\times\mathbf{R}_\xi)) \tag{3.1}$$

to the problem

$$\begin{cases} \partial_t f + \xi \partial_x f = \dfrac{h-f}{\varepsilon}, & x > 0,\ t > 0,\ \xi \in \mathbf{R}, \\ f(0,x,\xi) = f^0(x,\xi), & x > 0,\ \xi \in \mathbf{R}, \\ f(t,0,\xi) = f^b(t,\xi), & t > 0,\ \xi > 0. \end{cases} \tag{3.2}$$

Furthermore, for any $t \geq 0$*, a.e.* $x > 0$*, a.e.* $\xi \in \mathbf{R}$*,*

$$\begin{aligned} f(t,x,\xi) = &\left[f^0(x-t\xi,\xi) e^{-t/\varepsilon} + \frac{1}{\varepsilon} \int_0^t e^{-s/\varepsilon} h(t-s, x-s\xi, \xi)\, ds \right] \mathbb{I}_{x>t\xi} \\ &+ \left[f^b(t-x/\xi,\xi) e^{-x/(\varepsilon\xi)} + \frac{1}{\varepsilon} \int_0^{x/\xi} e^{-s/\varepsilon} h(t-s, x-s\xi, \xi)\, ds \right] \mathbb{I}_{\xi>0, x<t\xi}, \end{aligned} \tag{3.3}$$

and

$$\|f\|_{C_x([0,\infty[, L^1_\mu(]0,T[_t \times \mathbf{R}_\xi))} \leq \|f^0\|_{L^1} + \|f^b\|_{L^1_\mu} + \frac{1}{\varepsilon}\|h\|_{L^1}. \tag{3.4}$$

Now, we explain briefly how we obtain Theorem 1.1. Fixing $T > 0$, $f^0 \in L^1(]0,\infty[_x \times \mathbf{R}_\xi)$, $f^b \in L^1_\mu(]0,T[_t \times]0,\infty[_\xi)$ such that (1.12) and (1.13), (1.14) are satisfied, we set

$$C_H(t) = \iint_{]0,\infty[\times\mathbf{R}} H(f^0(x,\xi),\xi)\, dx d\xi + \iint_{]0,t[\times]0,\infty[} H(f^b(s,\xi),\xi)\, d\mu(s,\xi), \tag{3.5}$$

$$C_0(t) = \iint_{]0,\infty[\times\mathbf{R}} f^0_0(x,\xi)\, dx d\xi + \iint_{]0,t[\times]0,\infty[} f^b_0(s,\xi)\, d\mu(s,\xi). \tag{3.6}$$

Proceeding as in [5], we use the Tychonoff-Schauder fixed point theorem in the locally convex topological vector space $C([0,T], L^1(]0,\infty[_{loc,x} \times \mathbf{R}_\xi))$ and with the following convex $\tilde{C}$ and the following function F. Define C to be the set of all $g \in L^\infty(]0,T[, L^1(]0,\infty[\times\mathbf{R}))$ satisfying $(C1) - (C2)$ for a.e. $t \in [0,T]$, where

$(C1)$ $\quad g(t,x,\xi) \in D \quad$ *a.e.* in $]0,\infty[\times\mathbf{R}$,

$(C2)$ $\quad \displaystyle\iint_{]0,\infty[\times\mathbf{R}} H(g(t,x,\xi),\xi)\, dx d\xi \leq C_H(t), \quad \iint_{]0,\infty[\times\mathbf{R}} g_0(t,x,\xi)\, dx d\xi \leq C_0(t).$

Let us also introduce

$$\begin{aligned} \tilde{C} = \Big\{ g \in C([0,T], L^1(]0,\infty[\times\mathbf{R})) \text{ satisfying } \partial_t g + \xi \partial_x g + \frac{g}{\varepsilon} \in \frac{C}{\varepsilon}, \\ \text{and } (C1) - (C2) \text{ for all } t \in [0,T] \Big\}. \end{aligned} \tag{3.7}$$

For any $g \in C$, we denote by $F(g)$ the solution to (3.1)-(3.2) with $h = M[g]$. Using Jensen's inequality with the convex function H, and with changes of variables, we get

$$\begin{aligned}
&\iint_{]0,\infty[\times\mathbf{R}} H(F(g)(t,x,\xi),\xi)\,dxd\xi \\
\leq\ &\iint_{\substack{x>0\\ x>t\xi}} H(f^0(x-t\xi,\xi),\xi)e^{-t/\varepsilon}\,dxd\xi + \iint_{\substack{\xi>0\\ 0<x<t\xi}} H(f^b(t-\frac{x}{\xi},\xi),\xi)e^{-\frac{x}{\varepsilon\xi}}\,dxd\xi \\
&+\frac{1}{\varepsilon}\iint_{]0,\infty[\times\mathbf{R}}\int_0^{\min(t,x/\xi_+)} e^{-s/\varepsilon}H(M[g](t-s,x-s\xi,\xi),\xi)\,dsdxd\xi \\
\leq\ &\left(\iint_{]0,\infty[\times\mathbf{R}} H(f^0(x,\xi),\xi)\,dxd\xi + \iint_{]0,t[\times]0,\infty[} H(f^b(s,\xi),\xi)e^{s/\varepsilon}\,d\mu(s,\xi)\right)e^{-t/\varepsilon} \\
&+\frac{1}{\varepsilon}\int_0^t\iint_{]0,\infty[\times\mathbf{R}} e^{-s/\varepsilon}H(M[g](t-s,x,\xi),\xi)\,dxd\xi ds.
\end{aligned}$$

With the entropy minimization principle, in the last integral we can replace $M[g]$ by g, and using $(C2)$ we finally get that $F(g)$ also satisfies $(C2)$. The estimation of the mass by $C_0(t)$ works the same. It gives the stability of the set $\tilde{C}$ by F. We have a similar compactness result than in [5], furthermore the definition of $\tilde{C}$ allows to apply compactness averaging lemma and to finally get the existence of a solution to (1.7)-(1.9) in $C([0,T],L^1(]0,\infty[_{loc}\times\mathbf{R}))$. Since $F(f)=f$, the function $F(f)$ is a solution with the desired regularity and we get Theorem 1.1.

4 Maximum principle

In this section, we obtain the uniform bounds for the solution of the BGK model. Let us first introduce the notion of kinetic invariant domains for an initial-boundary problem.

Definition 4.1 *A family $\tilde{D}_\xi$ of subsets of $\mathbf{R}^2$ is a family of kinetic invariant domains for an initial-boundary problem if, denoting by f the solution obtained in Theorem 1.1,*

$$f^0(x,\xi)\in\tilde{D}_\xi \text{ a.e. } x,\xi,\ f^b(t,\xi)\in\tilde{D}_\xi \text{ a.e. } t,\xi,\Rightarrow \forall t\ f(t,x,\xi)\in\tilde{D}_\xi \text{ a.e. } x,\xi.$$

Proposition 4.2 *For any $\omega_{min}<\omega_{max}$, define $\tilde{D}_\xi$ by (2.13). Assume that the initial-boundary condition of Theorem 1.1 satisfies $f^0(x,\xi)\in\tilde{D}_\xi$, $f^b(t,\xi)\in\tilde{D}_\xi$ a.e. Then the system (1.7) has the property that $\tilde{D}_\xi$ is a family of convex kinetic invariant domains for the initial-boundary problem. Furthermore (ρ,u) defined by (2.2) verify $(\rho(t,x),u(t,x))\in\tilde{D}$ $\forall t\geq 0$ and $M[f](t,x,\xi)\in\tilde{D}_\xi$ $\forall t\geq 0$. Consequently f has compact support with respect to ξ, $supp_\xi f\subset[\omega_{min},\omega_{max}]$.*

Proof. We consider functions $S_M(v) = (v - \omega_{max})_+^2$ and $S_m(v) = (\omega_{min} - v)_+^2$ which are convex and C^1. These functions allow to give a correspondence between kinetic entropies and kinetic invariant domains, namely, for a fixed ξ,

$$f \in \tilde{D}_\xi \Leftrightarrow (H_{S_M}(f,\xi) \leq 0 \text{ and } H_{S_m}(f,\xi) \leq 0), \tag{4.1}$$

see [6] for this result. Then similarly as we did for H, we can obtain the inequality (1.19) with H_S instead of H, with $S = S_M$ or $S = S_m$. Now, supposing that f^0 and f^b are in $\tilde{D}_\xi$ a.e., it gives $H_S(f^0(x,\xi),\xi) = H_S(f^b(t,\xi),\xi) = 0$ a.e., thus $H_S(f(t,x,\xi),\xi) = 0$ a.e., therefore $f \in \tilde{D}_\xi$ $\forall t \geq 0$ and $\tilde{D}_\xi$ is kinetic invariant. Finally, Proposition 2.5 allows to conclude. □

It gives the following bounds with the formula (2.9).

Proposition 4.3 *If we suppose that $f^0(x,\xi) \in \tilde{D}_\xi$ a.e. x,ξ and $f^b(t,\xi) \in \tilde{D}_\xi$ a.e. t,ξ, then ρ_ε, u_ε, f_ε, $M[f_\varepsilon]$ are uniformly bounded in L^∞. Furthermore we have $supp_\xi f_\varepsilon \subset [\omega_{min}, \omega_{max}]$, $supp_\xi M[f_\varepsilon] \subset [\omega_{min}, \omega_{max}]$ and $|(f_\varepsilon)_1| \leq A(f_\varepsilon)_0$ for some A.*

Corollary 4.4 *i) The sequence of functions $(t,x,\xi) \mapsto H(f_\varepsilon(t,x,\xi),\xi)$ are bounded in $L_t^\infty(]0,T[, L^1(]0,\infty[_x \times \mathbf{R}_\xi))$.*
ii) Functions $H_S(f_\varepsilon(t,x,\xi),\xi)$ are bounded in $C_t([0,T], L^1(]0,\infty[_x \times \mathbf{R}_\xi))$ for any $S : \mathbf{R} \to \mathbf{R}$ convex, of class C^1 and such that $|S(v)| \leq B(1+v^2)$ for some $B \geq 0$.

Proof. The $i)$ comes from the definition of H and the previous proposition. Now, the estimate $|H_S(f,\xi)| \leq B(f_0 + 2H(f,\xi))$ and the bound on $(f_\varepsilon)_0$ give that $H_S(f_\varepsilon(t,x,\xi),\xi)$ is bounded in $L_t^\infty(]0,T[, L^1(]0,\infty[_x \times \mathbf{R}_\xi))$. Using Lebesgue's theorem and the continuity of H_S in D_A of Proposition 2.2, we conclude the $ii)$. □

5 Relaxation to the boundary condition

Let us now consider the limit $\varepsilon \to 0$. At first, we have that as in [6], $H'_S(f_\varepsilon,\xi) \in L^\infty(]0,T[\times]0,\infty[\times\mathbf{R})$, thus according to [8] and Corollary 4.4, $M[f_\varepsilon] - f_\varepsilon = 0$ a.e. where $f_\varepsilon = 0$, and

$$\partial_t(H_S(f_\varepsilon,\xi)) + \xi\partial_x(H_S(f_\varepsilon,\xi)) = H'_S(f_\varepsilon,\xi) \cdot \frac{M[f_\varepsilon] - f_\varepsilon}{\varepsilon}. \tag{5.1}$$

Let $\varphi \in \mathcal{D}([0,\infty[_t \times [0,\infty[_x)$. Using the continuity of $t \mapsto H_S(f_\varepsilon(t,x,\xi),\xi)$ stated in Corollary 4.4 and the similar fact that $t \mapsto H_S(f_\varepsilon(t,x,\xi),\xi) \in$

$C_x([0,\infty[, L^1_\mu(]0,T[_t\times\mathbf{R}_\xi))$, we get

$$\begin{aligned}
&-\iiint\limits_{]0,\infty[^2\times\mathbf{R}} H_S(f_\varepsilon,\xi)\partial_t\varphi\,dtdxd\xi - \iint\limits_{]0,\infty[\times\mathbf{R}} H_S(f_\varepsilon,\xi)(t=0)\varphi(0,x)\,dxd\xi\\
&-\iiint\limits_{]0,\infty[^2\times\mathbf{R}} \xi H_S(f_\varepsilon,\xi)\partial_x\varphi\,dtdxd\xi - \iint\limits_{]0,\infty[\times\mathbf{R}} \xi H_S(f_\varepsilon,\xi)(x=0)\varphi(t,0)\,dtd\xi\\
=&\iiint\limits_{]0,\infty[^2\times\mathbf{R}} H'_S(f_\varepsilon,\xi)\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\varphi\,dtdxd\xi\\
=&\iiint\limits_{]0,\infty[^2\times\mathbf{R}} (H'_S(f_\varepsilon,\xi)-T_S(\rho_\varepsilon,u_\varepsilon))\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\varphi\,dtdxd\xi.
\end{aligned}\tag{5.2}$$

Using the sign of the entropy dissipation for $S(v)=v^2/2$ (Proposition 2.3), we obtain

$$\begin{aligned}
&-\iiint\limits_{]0,\infty[^2\times\mathbf{R}} H_S(f_\varepsilon,\xi)\partial_t\varphi\,dtdxd\xi - \iiint\limits_{]0,\infty[^2\times\mathbf{R}} \xi H_S(f_\varepsilon,\xi)\partial_x\varphi\,dtdxd\xi\\
&\qquad-\iint\limits_{]0,\infty[\times\mathbf{R}} \xi H_S(f_\varepsilon,\xi)(x=0)\varphi(t,0)\,dtd\xi\le 0
\end{aligned}\tag{5.3}$$

for $\varphi\in\mathcal{D}(]0,\infty[_t\times[0,\infty[_x)$, $\varphi\ge 0$.

We notice that $H_S(f_\varepsilon,\xi)(x=0)=H_S(f_\varepsilon(x=0),\xi)$. We use now the hypothesis on the boundary condition to be maxwellian, this means that

$$f_\varepsilon(t,0,\xi)=f^b(t,\xi)=M^b(t,\xi),\qquad t>0,\ \xi>0,\tag{5.4}$$

where

$$M^b(t,\xi)=M(\rho^b(t),u^b(t),\xi),\qquad t>0,\ \xi\in\mathbf{R}.\tag{5.5}$$

Using the subdifferential inequality (Proposition 2.3), we have for a.e. $t>0$, $\xi\in\mathbf{R}$,

$$H_S(f_\varepsilon(x=0),\xi)\ge H_S(M^b,\xi)+T_S(\rho^b,u^b)\cdot(f_\varepsilon(x=0)-M^b),\tag{5.6}$$

with equality if $\xi>0$, thus

$$\int_{\mathbf{R}}\xi H_S(f_\varepsilon(x=0),\xi)d\xi\le\int_{\mathbf{R}}\xi H_S(M^b,\xi)d\xi+T_S(\rho^b,u^b)\cdot\int_{\mathbf{R}}\xi(f_\varepsilon(x=0)-M^b)d\xi,$$

and with (5.3) it yields

$$\begin{aligned}
&-\iiint\limits_{]0,\infty[^2\times\mathbf{R}} H_S(f_\varepsilon,\xi)\partial_t\varphi\,dtdxd\xi - \iiint\limits_{]0,\infty[^2\times\mathbf{R}} \xi H_S(f_\varepsilon,\xi)\partial_x\varphi\,dtdxd\xi\\
&-\iint\limits_{]0,\infty[\times\mathbf{R}} \xi H_S(M^b,\xi)\varphi(t,0)\,dtd\xi\\
&-\int_{]0,\infty[} T_S(\rho^b,u^b)\cdot\int_{\mathbf{R}}\xi(f_\varepsilon(x=0)-M^b)\,d\xi\varphi(t,0)\,dt\le 0,
\end{aligned}\tag{5.7}$$

that is to say by comparing f_ε to $M[f_\varepsilon]$

$$\begin{aligned}
&-\iint\limits_{]0,\infty[^2} \eta_S(\rho_\varepsilon,u_\varepsilon)\partial_t\varphi\,dtdx-\iint\limits_{]0,\infty[^2} G_S(\rho_\varepsilon,u_\varepsilon)\partial_x\varphi\,dtdx\\
&-<R_{S,\varepsilon},\varphi>-\int_0^\infty G_S(\rho^b,u^b)\varphi(t,0)\,dt\\
&-\int_{]0,\infty[}T_S(\rho^b,u^b)\cdot\Big(\int_{\mathbf{R}}\xi f_\varepsilon(x=0)\,d\xi-F(\rho^b,u^b)\Big)\varphi(t,0)\,dt\le 0,
\end{aligned}\tag{5.8}$$

with

$$\begin{aligned}
<R_{S,\varepsilon},\varphi>=&\iiint\limits_{]0,\infty[^2\times\mathbf{R}}(H_S(f_\varepsilon,\xi)-H_S(M[f_\varepsilon],\xi))\partial_t\varphi\,dtdxd\xi\\
&+\iiint\limits_{]0,\infty[^2\times\mathbf{R}}\xi(H_S(f_\varepsilon,\xi)-H_S(M[f_\varepsilon],\xi))\partial_x\varphi\,dtdxd\xi.
\end{aligned}\tag{5.9}$$

But (5.2) gives

$$\begin{aligned}
&\iint\limits_{]0,\infty[\times\mathbf{R}} H_S(f_\varepsilon,\xi)(t=T)\,dxd\xi-\iint\limits_{]0,\infty[\times\mathbf{R}} H_S(f_\varepsilon,\xi)(t=0)\,dxd\xi\\
&\quad-\iint\limits_{]0,T[\times\mathbf{R}}\xi H_S(f_\varepsilon,\xi)(x=0)\,dtd\xi\\
&=\iiint\limits_{]0,T[\times]0,\infty[\times\mathbf{R}}(H_S'(f_\varepsilon,\xi)-T_S(\rho_\varepsilon,u_\varepsilon))\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\,dtdxd\xi,
\end{aligned}\tag{5.10}$$

therefore

$$\iiint\limits_{]0,T[\times]0,\infty[\times\mathbf{R}}(H'(f_\varepsilon,\xi)-T_{\frac{v^2}{2}}(\rho_\varepsilon,u_\varepsilon))\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\,dtdxd\xi\quad\text{is bounded},\tag{5.11}$$

uniformly in ε. This result, together with the fact that f_ε, $M[f_\varepsilon]$ are bounded in $L^\infty_{t,x,\xi}$ and the property of uniform compact support allow to apply the dissipation result of [6] and to get that $f_\varepsilon-M[f_\varepsilon]\to 0$ a.e. t,x,ξ. Thus $<R_{S,\varepsilon},\varphi>\to 0$. Then, (5.8) gives in particular, for $\varphi\in\mathcal{D}(]0,\infty[^2_{t,x})$,

$$\partial_t\eta_S(\rho_\varepsilon,u_\varepsilon)+\partial_x G_S(\rho_\varepsilon,u_\varepsilon)\le R_{S,\varepsilon}\to 0 \text{ in } W^{-1,p}_{loc}\tag{5.12}$$

for any $1<p<\infty$, as $\varepsilon\to 0$, for $S:\mathbf{R}\to\mathbf{R}$ convex, of class C^1. Moreover, (5.12) becomes an equality if $S(v)=1$ or v. Since ρ_ε and u_ε are bounded in L^∞, we can then apply the stability result of [17] which gives the first part of Theorem 1.2: up to a subsequence, $(\rho_\varepsilon,\rho_\varepsilon u_\varepsilon)$ converge a.e. in $]0,\infty[\times\mathbf{R}$ when $\varepsilon\to 0$ to an entropy solution $(\rho,\rho u)$ to (1.1), (1.4) remaining in $\tilde{D}$, with initial data $(\rho^0,\rho^0u^0)=\int f^0 d\xi$.
It only remains to pass to the limit in the boundary condition (5.8). The only nontrivial term is the one involving

$$\psi_\varepsilon(t)=\int_{\mathbf{R}}\xi f_\varepsilon(x=0)\,d\xi.\tag{5.13}$$

The sequence $(\psi_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(]0,\infty[)$, thus there exists $\psi \in L^\infty$ such that for a subsequence

$$\psi_\varepsilon \rightharpoonup \psi \text{ in } L^\infty_{w*}(]0,\infty[), \qquad \text{as } \varepsilon \to 0. \tag{5.14}$$

We have from (1.7)

$$-\iiint_{]0,\infty[^2\times\mathbf{R}} (f_\varepsilon \partial_t \varphi + \xi f_\varepsilon \partial_x \varphi)\, dt dx d\xi - \iint_{]0,\infty[\times\mathbf{R}} \xi f_\varepsilon(x=0)\varphi(t,0)\, dt d\xi = 0 \tag{5.15}$$

for any $\varphi \in \mathcal{D}(]0,\infty[_t \times [0,\infty[_x)$, that is to say

$$\begin{aligned} -\iint_{]0,\infty[^2} (\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)\partial_t\varphi\, dtdx - \iint_{]0,\infty[^2} F(\rho_\varepsilon, u_\varepsilon)\partial_x\varphi\, dtdx - \int_0^\infty \psi_\varepsilon(t)\varphi(t,0)\, dt \\ + \iiint_{]0,\infty[^2\times\mathbf{R}} \xi(M[f_\varepsilon] - f_\varepsilon)\partial_x\varphi\, dtdxd\xi = 0. \end{aligned} \tag{5.16}$$

Passing to the limit $\varepsilon \to 0$, we obtain

$$-\iint_{]0,\infty[^2} (\rho, \rho u)\partial_t\varphi\, dtdx - \iint_{]0,\infty[^2} F(\rho,u)\partial_x\varphi\, dtdx - \int_0^\infty \psi\varphi(t,0)\, dt = 0. \tag{5.17}$$

Now, according to Section 6, the relation $\partial_t(\rho,\rho u) + \partial_x F(\rho,u) = 0$ allows to consider the weak trace $\overline{F(\rho,u)}$, such that

$$-\iint_{]0,\infty[^2} (\rho, \rho u)\partial_t\varphi\, dtdx - \iint_{]0,\infty[^2} F(\rho,u)\partial_x\varphi\, dtdx - \int_{]0,\infty[} \overline{F(\rho,u)}\varphi(t,0)\, dt = 0. \tag{5.18}$$

By uniqueness of the weak trace, we obtain $\psi = \overline{F(\rho,u)}$, which yields

$$\psi_\varepsilon \rightharpoonup \overline{F(\rho,u)} \text{ in } L^\infty_{w*}(]0,\infty[), \qquad \text{as } \varepsilon \to 0. \tag{5.19}$$

Passing to the limit in (5.8), we finally obtain

$$\begin{aligned} -\iint_{]0,\infty[\times]0,\infty[} \eta_S(\rho,u)\partial_t\varphi\, dtdx - \iint_{]0,\infty[\times]0,\infty[} G_S(\rho,u)\partial_x\varphi\, dtdx \\ -\int_{]0,\infty[} [G_S(\rho^b,u^b) + T_S(\rho^b,u^b)\cdot(\overline{F(\rho,u)} - F(\rho^b,u^b))]\varphi(t,0)\, dt \le 0 \end{aligned} \tag{5.20}$$

for any $\varphi \in \mathcal{D}(]0,\infty[_t \times [0,\infty[_x)$, $\varphi \ge 0$.

In order to complete the proof of Theorem 1.2, it only remains to prove the equivalence of the weak formulation (5.20) with (1.4), (1.22), which is done in Proposition 6.2.

6 Weak trace

Let us recall here, with an elementary proof, the existence result for the weak trace of [1] and [10] in the case of a quarter space in $\mathbf{R}^2$.

Theorem 6.1 *Let $V = (V_0, V_1) \in L^\infty(]0,\infty[^2)$ be a vector field such that $\operatorname{div}_{t,x} V \in \mathcal{M}(]t_1,t_2[\times]0,R[)$ for any $0 < t_1 < t_2 < \infty$ and $R > 0$. Then there exists a unique solution $\overline{V_1} \in L^\infty_t(]0,\infty[)$ to*

$$-\iint_{]0,\infty[^2} \varphi \operatorname{div} V - \iint_{]0,\infty[^2} V_0 \partial_t \varphi \, dtdx - \iint_{]0,\infty[^2} V_1 \partial_x \varphi \, dtdx - \int_{]0,\infty[} \overline{V_1} \varphi(t,0)\, dt = 0 \tag{6.1}$$

for any $\varphi \in C^\infty_c(]0,\infty[_t \times [0,\infty[_x)$. In fact $\overline{V_1}$ depends only of V_1 (see (6.2) below) and satisfies the bound $\|\overline{V_1}\|_\infty \leq \|V_1\|_\infty$.

Proof. We take a smooth function φ_δ nonincreasing on $[0,\infty[$ and such that $\varphi_\delta(x) = 1$ for $x \leq \delta/2$, $\varphi_\delta(x) = 0$ for $x \geq \delta$ and $|\varphi'_\delta(x)| \leq C/\delta$. We have the decomposition

$$\begin{aligned}
& - \iint_{]0,\infty[^2} \varphi \operatorname{div} V - \iint_{]0,\infty[^2} V_0 \partial_t \varphi - \iint_{]0,\infty[^2} V_1 \partial_x \varphi \\
= & - \iint_{]0,\infty[^2} \varphi\varphi_\delta \operatorname{div} V - \iint_{]0,\infty[^2} V_0 (\partial_t \varphi)\varphi_\delta - \iint_{]0,\infty[^2} V_1 (\partial_x \varphi)\varphi_\delta - \iint_{]0,\infty[^2} V_1 \varphi (\partial_x \varphi_\delta) \\
& - \iint_{]0,\infty[^2} \varphi(1-\varphi_\delta) \operatorname{div} V - \iint_{]0,\infty[^2} V_0 \partial_t (\varphi(1-\varphi_\delta)) - \iint_{]0,\infty[^2} V_1 \partial_x(\varphi(1-\varphi_\delta)),
\end{aligned}$$

and the sum of the last three terms is 0 because $\varphi(1-\varphi_\delta) \in \mathcal{D}(]0,\infty[^2)$. Now

$$\iint_{]0,\infty[^2} V_0 (\partial_t\varphi)\varphi_\delta \, dtdx \to 0, \qquad \iint_{]0,\infty[^2} V_1 (\partial_x\varphi)\varphi_\delta \, dtdx \to 0,$$

as $\delta \to 0$, and using the dominated convergence theorem with the measure $\operatorname{div} V$, we have also

$$\iint_{]0,\infty[^2} \varphi\varphi_\delta \operatorname{div} V \to 0, \qquad \text{as } \delta \to 0.$$

The sequence $(\int_0^\infty V_1(t,x)\varphi'_\delta(x)\,dx)_{\delta>0}$ is bounded in $L^\infty(]0,\infty[)$ by $\|V_1\|_{L^\infty}$, thus there exists $\overline{V_1} \in L^\infty_t(]0,\infty[)$ such that, up to a subsequence,

$$\int_0^\infty V_1(t,x)\varphi'_\delta(x)\,dx \rightharpoonup \overline{V_1}(t) \text{ in } L^\infty_{w*}(]0,\infty[). \tag{6.2}$$

Now

$$\begin{aligned}
\iint_{]0,\infty[^2} V_1(t,x)\varphi(t,x)\varphi'_\delta(x)\,dtdx = & \iint_{]0,\infty[^2} V_1(t,x)\varphi'_\delta(x)(\varphi(t,x)-\varphi(t,0))\,dtdx \\
& + \int_{]0,\infty[} (\int_0^\infty V_1(t,x)\varphi'_\delta(x)\,dx)\varphi(t,0)\,dt,
\end{aligned} \tag{6.3}$$

which yields

$$\iint\limits_{]0,\infty[^2} V_1(t,x)\varphi(t,x)\varphi'_\delta(x)\,dtdx \to \int_{]0,\infty[} \overline{V_1}(t)\varphi(t,0)\,dt, \qquad \text{as } \delta\to 0, \tag{6.4}$$

and this ends the proof of existence. Uniqueness is obvious, and the bound follows from (6.2). □

Remark 6.1 In particular, it allows to define the weak trace of $G_S(\rho,u)$ for every S with subquadratic growth, by applying our result with $V=(\eta_S,G_S)$ thanks to (1.4).

Proposition 6.2 *The formulation (1.4), (1.22) is equivalent to (5.20).*

Proof. It is clear by (6.1) that (1.4), (1.22) implies (5.20). Let us assume that (5.20) holds. Then (1.4) comes from the use of a test function with support in $]0,\infty[^2$. Now, using the function φ_δ of the previous theorem, the weak formulation (5.20) with the test function $\varphi(t,x)=\psi(t)\varphi_\delta(x)$ gives for any $\psi\in C_c^\infty(]0,\infty[)$, $\psi\geq 0$,

$$\begin{aligned} &-\iint\limits_{]0,\infty[\times]0,\infty[} \eta_S(\rho,u)\psi'\varphi_\delta\,dtdx - \iint\limits_{]0,\infty[\times]0,\infty[} G_S(\rho,u)\psi\varphi'_\delta\,dtdx \\ &-\int_{]0,\infty[} [G_S(\rho^b,u^b)+T_S(\rho^b,u^b)\cdot(\overline{F(\rho,u)}-F(\rho^b,u^b))]\psi\,dt\leq 0. \end{aligned} \tag{6.5}$$

By using (6.1), it gives

$$\begin{aligned} &\iint\limits_{]0,\infty[\times]0,\infty[} \psi\varphi_\delta\,\mathrm{div}(\eta_S(\rho,u),G_S(\rho,u)) \\ &+\int\limits_{]0,\infty[} [\overline{G_S(\rho,u)}-G_S(\rho^b,u^b)-T_S(\rho^b,u^b)\cdot(\overline{F(\rho,u)}-F(\rho^b,u^b))]\psi\,dt\leq 0. \end{aligned} \tag{6.6}$$

We finally get the boundary condition (1.22) as $\delta\to 0$ by using the dominated convergence theorem with the measure $\mathrm{div}(\eta_S(\rho,u),G_S(\rho,u))$. □

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Quatrième partie

Modèles avec contraintes unilatérales

Chapitre 1

Existence et stabilité faible pour un modèle sans pression avec contrainte unilatérale

Existence and weak stability for a pressureless model with unilateral constraint

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Abstract

We consider a two-phase model described by a pressureless gas system with unilateral constraint. We prove weak stability and existence of weak solutions by passing to the limit in the sticky-blocks dynamics. We obtain the maximum principle on the velocity, the Oleinik entropy condition and local entropy inequalities. Initial data are taken in a very weak sense since the solution can jump initially in time.

Key-words: two-phase flow – conservation laws with constraint – pressureless gas – sticky blocks – jump in time

Mathematics Subject Classification: 35L85, 76T10, 35L65, 76N15, 35A35

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1 Introduction and main results

1.1 The model

Let us consider a two-phase flow of gas and liquid. We denote by $\alpha_l = \alpha$ the volume fraction of liquid, $u_l = u$ its velocity, ρ_l its density and by $\alpha_g = 1-\alpha$ the volume fraction of gas, u_g its velocity and ρ_g its density. Following [11], we can write two mass conservation equations and two momentum conservation equations,

$$\begin{cases} \partial_t(\alpha_i\rho_i) + \partial_x(\alpha_i\rho_i u_i) = 0, \\ \partial_t(\alpha_i\rho_i u_i) + \partial_x(\alpha_i\rho_i u_i^2) + \alpha_i\partial_x p + \tau_i = M_i^D, \end{cases}$$

for $i = l, g$. The right-hand sides M_l^D and M_g^D are source terms reflecting inter-phase drag and τ_l and τ_g are phase pressure fluctuations around p the common pressure. As was explain in [3], a simplified model can be derived for an incompressible liquid, and an infinitely light gas. The procedure is as follows. We choose the standard simplified closure laws

$$M_l^D = -M_g^D = \mu\alpha(1-\alpha)\rho_l(u_g - u), \tag{1.1}$$

$$\tau_l = C_p\rho_l(u_g-u)^2\partial_x\alpha, \qquad \tau_g = 0, \tag{1.2}$$

and we assume the liquid to be incompressible, $\rho_l = \textit{constant}$, and the pressure to be governed by the law $p = p(\rho_g) = \kappa\rho_g^\gamma$. We then introduce a scaling of the gas density $\tilde{\rho}_g = \rho_g/\rho_0$, where ρ_0 is the average of ρ_g. We also set $\tilde{p} = p/\rho_0^\gamma$. In the new variable $\tilde{\rho_g}$ (we shall drop the tilde for convenience), we perform the Chapman-Enskog expansion in $\varepsilon = \rho_0/\rho_l << 1$, supposing that $u_g - u = O(\varepsilon)$, and we get the model

$$\partial_t \begin{pmatrix} (1-\alpha)\rho_g \\ \alpha \\ \alpha u \end{pmatrix} + \partial_x \begin{pmatrix} (1-\alpha)\rho_g u \\ \alpha u \\ \alpha u^2 + \varepsilon\rho_0^{\gamma-1}p(\rho_g) \end{pmatrix} = \varepsilon \begin{pmatrix} D(\alpha,u,\rho_g) \\ 0 \\ 0 \end{pmatrix},$$

where $D(\alpha, u, \rho_g)$ is a diffusive second-order term.
If we let $\varepsilon \to 0$ formally, we get the system of pressureless gases

$$\begin{cases} \partial_t\alpha + \partial_x(\alpha u) = 0, \\ \partial_t(\alpha u) + \partial_x(\alpha u^2) = 0. \end{cases}$$

It is known ([2], [7], [4]) that this system gives Dirac distributions on α in finite time, even for smooth initial data. It is clearly incompatible with the constraint $\alpha \le 1$. Thus, we cannot neglect the pressure term and we make the more realistic assumption $\varepsilon\rho_0^{\gamma-1}p(\rho_g) \rightharpoonup \pi$ as $\varepsilon \to 0$, supposing that this term appears only when $\alpha = 1$.
We finally get the model

$$\begin{cases} \partial_t\alpha + \partial_x(\alpha u) = 0, \\ \partial_t(\alpha u) + \partial_x(\alpha u^2 + \pi) = 0, \end{cases} \tag{1.3}$$

with the constraints

$$0 \leq \alpha \leq 1, \tag{1.4}$$

$$\pi \geq 0, \tag{1.5}$$

and the exclusion relation

$$(1-\alpha)\pi = 0. \tag{1.6}$$

The pressure-type term π is a Lagrange multiplier, it represents the residual pressure of the gas which has infinite density and volume fraction $1-\alpha = 0$. For more detail, we refer to [3].
The aim of this paper is to analyse the model (1.3)- (1.6), but before coming to the point, let us also propose another interpretation of this system.
The Saint-Venant equations describe the water flow in a channel. Denoting by α the water height and by u its velocity, these equations write in one dimension with flat bottom

$$\begin{cases} \partial_t \alpha + \partial_x(\alpha u) = 0, \\ \partial_t(\alpha u) + \partial_x(\alpha u^2 + p(\alpha)) = 0, \end{cases}$$

with $p(\alpha) = \kappa \alpha^2$, the gravity term. In order to describe the flow in a pipe of height equal to 1, we need to replace the second the equation by

$$\partial_t(\alpha u) + \partial_x(\alpha u^2 + p(\alpha) + \pi) = 0,$$

with the constraint $\alpha \leq 1$, the Lagrange multiplier $\pi \geq 0$ standing for the roof pressure, and the exclusion condition $(1-\alpha)\pi = 0$. The simplified model $p(\alpha) = 0$, obtained by neglecting the gravity, gives the previous model (1.3)-(1.6).
The system (1.3)-(1.6) is a system of pressureless gas with constraint. In the classical pressureless model, α contains Dirac distributions, and the difficulty is to define the products αu and αu^2. On the contrary, in the present case with constraint $\alpha \leq 1$, this is not a problem since still $u \in L^\infty$. However, the new term π is a measure, and thus the important difficulty is to express the exclusion relation $\alpha\pi = \pi$ which roughly speaking sets π working when $\alpha = 1$, and makes the liquid fraction stay between 0 and 1. Unfortunately, the product of α and π is not well-defined with π a measure and α in L^∞. Besides, if we could apply the classical compensated-compactness theory, we would get with standard notations $\overline{\alpha_n(\alpha_n u_n^2 + \pi_n) - (\alpha_n u_n)^2} = \overline{\alpha_n} \cdot \overline{\alpha_n u_n^2 + \pi_n} - \left(\overline{\alpha_n u_n}\right)^2$, which gives $\overline{\alpha_n \pi_n} = \overline{\alpha_n} \cdot \overline{\pi_n}$ if u_n is known to be compact. The fact that π is only a measure is an obstacle to this computation and to overcome this problem, we decompose $\alpha(\alpha u^2 + \pi) - (\alpha u)^2$ in two different ways whether we privilege α or αu. Thus we consider the constraint

$$R_1 = R_2 = \pi, \tag{1.7}$$

where R_1 and R_2 are defined by

$$R_1 = -\partial_t(\alpha Q) - \partial_x(\alpha u Q), \tag{1.8}$$

$$\partial_x Q = \alpha u, \ \partial_t Q = -(\alpha u^2 + \pi), \tag{1.9}$$

and

$$R_2 = \partial_t(\alpha u M) + \partial_x((\alpha u^2 + \pi)M), \tag{1.10}$$

$$\partial_x M = \alpha,\ \partial_t M = -\alpha u. \tag{1.11}$$

We have to notice that in the case of smooth solutions to (1.3), we have $R_1 = R_2 = \alpha\pi$. We shall come back to this point in Section 6. The reference [13] considered the same problem but with viscosity. We refer to [11] for more details about modelling and two phase-flows. Other hyperbolic problems with constraints have been studied in [1], [9] and [12]. Existence and properties of the system of pressureless gas have been studied in [10], [8], [7], [4] and [6].
We give initial conditions for our problem,

$$\alpha(0,x) = \alpha^0(x),\ \alpha(0,x)u(0,x) = \alpha^0(x)u^0(x). \tag{1.12}$$

Let us define precisely the weak solutions we shall consider. We look for solutions satisfying

$$\alpha \in L^\infty_t(0,\infty; L^\infty_x(\mathbf{R}) \cap L^1_x(\mathbf{R})) \cap C_t([0,\infty[; L^\infty_{w*}(\mathbf{R})), \tag{1.13}$$

$$u \in L^\infty_t(0,\infty; L^\infty_x(\mathbf{R})), \tag{1.14}$$

$$\pi \in \mathcal{M}_{loc}([0,\infty[\times\mathbf{R}) \cap L^\infty(\mathbf{R}_x, \mathcal{M}([0,T])). \tag{1.15}$$

Hence, (1.3), (1.12) must be satisfied in the sense of distributions:
for all $\varphi \in \mathcal{D}([0,+\infty[\times\mathbf{R})$,

$$\int_{[0,+\infty[}\int_{\mathbf{R}} [\alpha\partial_t\varphi + \alpha u\partial_x\varphi]\,dx\,dt + \int_{\mathbf{R}} \alpha^0(x)\varphi(0,x)\,dx = 0, \tag{1.16}$$

$$\int_{[0,+\infty[}\int_{\mathbf{R}} [\alpha u\partial_t\varphi + \alpha u^2\partial_x\varphi]\,dx\,dt + \int_{[0,+\infty[}\int_{\mathbf{R}} \partial_x\varphi\,\pi + \int_{\mathbf{R}} (\alpha^0 u^0)(x)\varphi(0,x)\,dx = 0. \tag{1.17}$$

The fact that α lies in $C_t([0,\infty[; L^\infty_{w*}(\mathbf{R}))$ implies with (1.16) that $\alpha(0,x) = \alpha^0(x)$. It is not the case for u and in general αu can have jumps in time, even initially, so that $\alpha(0,x)u(0,x) \neq \alpha^0(x)u^0(x)$. Indeed, $\alpha(t,x) = \mathbb{1}_{[-1,1]}(x)$, $u(t,x) = 0$, $\pi(t,x) = (1-|x|)\mathbb{1}_{-1<x<1}\delta(t)$ is a solution for initial conditions $\alpha^0 = \mathbb{1}_{[-1,1]}$, $\alpha^0 u^0 = \mathbb{1}_{[-1,0]} - \mathbb{1}_{[0,1]}$. This represents a collision between two sticky blocks at time 0 and it gives a Dirac distribution at this time. As a result, (1.12) will be considered only in this weak sense (1.17). Similarly, the constraint (1.7) has to be understood as

$$\int_{[0,+\infty[}\int_{\mathbf{R}} [\alpha Q\partial_t\varphi + \alpha uQ\partial_x\varphi]\,dx\,dt + \int_{\mathbf{R}} (\alpha^0 Q^0)(x)\varphi(0,x)\,dx = \int_{[0,+\infty[}\int_{\mathbf{R}} \varphi\,\pi \tag{1.18}$$

and

$$\begin{aligned} -\int_{[0,+\infty[}\int_{\mathbf{R}} [\alpha uM\partial_t\varphi + \alpha u^2 M\partial_x\varphi]\,dx\,dt - \int_{[0,+\infty[}\int_{\mathbf{R}} M\partial_x\varphi\,\pi \\ - \int_{\mathbf{R}} (\alpha^0 u^0 M^0)(x)\varphi(0,x)\,dx = \int_{[0,+\infty[}\int_{\mathbf{R}} \varphi\,\pi \end{aligned} \tag{1.19}$$

for all $\varphi \in \mathcal{D}([0,+\infty[\times\mathbf{R})$.

Remark 1.1 Distributions Q and M are defined up to an additive constant. In (1.8) and (1.10), it is clearly not a trouble thanks to (1.3), and the same remark holds for the weak formulations (1.18) and (1.19) thanks to (1.16) and (1.17).

We also notice that the relation (1.11) with α, αu in L^∞ gives that M is Lipschitz continuous, so that $M^0 = M(0,.)$ is well-defined and the above writing makes sense. However, the meaning of Q^0 will be precised in Proposition 2.5.

Remark 1.2 There is no uniqueness without further conditions. Indeed, if we consider the initial datum $\alpha^0 = \mathbb{1}_{[-1,1]}$, $\alpha^0 u^0 = 0$, we can define

$$\begin{aligned} \alpha(t,x) &= \mathbb{1}_{-1-t<x<-t} + \mathbb{1}_{t<x<1+t}, \\ \alpha u(t,x) &= -\mathbb{1}_{-1-t<x<-t} + \mathbb{1}_{t<x<1+t} \end{aligned}$$

and $\pi = \delta(t)(1-|x|)\mathbb{1}_{-1<x<1}$. Then (α, u, π) solves (1.16)-(1.19). This represents a block which is divided in two at time 0. *We also have the trivial solution $\alpha = \mathbb{1}_{[-1,1]}$, $\alpha u = 0$.*

1.2 Main results

The main results we prove in this paper are the two following theorems.

Theorem 1.1 *If $\alpha^0 \in L^1(\mathbf{R})$ and $u^0 \in L^\infty(\mathbf{R})$, $0 \le \alpha^0 \le 1$, then there exists (α, u, π) with regularities (1.13) -(1.15), satisfying (1.16)-(1.17) with the constraints (1.4),(1.5),(1.18), (1.19).*

Theorem 1.2 *In addition, the solution (α, u, π) of Theorem 1.1 satisfies*

$$\partial_x u(t,x) \le \tfrac{1}{t}, \tag{1.20}$$

$$\textit{essinf}\; u^0(x) \le u(t,x) \le \textit{esssup}\; u^0(x), \tag{1.21}$$

$$TV_{[a,b]}(u(t,.)) \le \tfrac{2(b-a)}{t} + 2\|u^0\|_{L^\infty} \quad \forall a<b, \tag{1.22}$$

$$\partial_t(\alpha S(u)) + \partial_x(\alpha u S(u) + \pi_S) \le 0 \;\; \textit{in}\;]0,\infty[\times\mathbf{R} \tag{1.23}$$

for every $S \in C^1(\mathbf{R})$ convex and Lipschitz continuous, where π_S lies in $L^\infty(\mathbf{R}_x, \mathcal{M}([0,T]))$ and satisfies

$$|\pi_S| \le Lip(S)\pi. \tag{1.24}$$

The inequality (1.22) is classically a consequence of (1.20) and (1.21). The last inequality (1.23) is a rigorous analogue of the formal computation

$$\partial_t(\alpha S(u)) + \partial_x(\alpha u S(u) + S'(u)\pi) = S''(u)\pi\partial_x u \le 0.$$

In Section 2, we establish the compactness (or stability) of sequences of solutions. In Section 3, we study a class of special solutions, called the sticky blocks. Using the convergence of those functions, we prove in Section 4 our existence theorem. Finally in Section 5, we prove the entropy inequalities of Theorem 1.2 and discuss the meaning of the relation $\alpha\pi = \pi$ in Section 6.

2 Compactness

The result we prove in this section has two specific purposes. First, it will prove the existence of solutions by passage to the limit in some particular solutions which initial data can approximate every initial data in $L^1(\mathbf{R})$. Secondly, it will prove stability of the obtained solutions.

Theorem 2.1 *Let us consider a sequence of solutions (α_n, u_n, π_n) with regularities (1.13) -(1.15), satisfying (1.16)-(1.17) with the constraints (1.4), (1.5), (1.18), (1.19). Initial data α_n^0, u_n^0 are supposed to satisfy*

$$0 \leq \alpha_n^0 \leq 1, \qquad (\alpha_n^0)_{n\geq 0} \text{ is bounded in } L^1(\mathbf{R}), \tag{2.1}$$

$$(u_n^0)_{n\geq 0} \text{ is bounded in } L^\infty(\mathbf{R}). \tag{2.2}$$

We also suppose that (α_n, u_n, π_n) satisfies (1.20), (1.21), and that

$$(\pi_n)_{n\geq 0} \text{ is bounded in } \mathcal{M}_{loc}([0,\infty[\times\mathbf{R}) \cap L^\infty(\mathbf{R}, \mathcal{M}([0,T])). \tag{2.3}$$

Then, extracting a subsequence if necessary, as $n \to \infty$, $(\alpha_n, u_n, \pi_n) \rightharpoonup (\alpha, u, \pi)$ in the sense of (2.4)-(2.5) and Proposition 2.4, where (α, u, π) has regularities (1.13) -(1.15) and satisfies (1.16)-(1.17) with the constraints (1.4),(1.5),(1.18), (1.19), and where α^0 and u^0 are defined by (2.6).

What follows in the section is the proof of this result. To achieve it, we suppose that the theorem's hypothesis are satisfied. Therefore, Properties (2.2) and (1.21) imply that $(u_n)_{n\geq 0}$ is bounded in $L^\infty_{t,x}(]0,\infty[\times\mathbf{R})$. Then, since the first equation in (1.3) gives time compactness, extracting a subsequence if necessary, we can assume that as $n \to \infty$,

$$\alpha_n \to \alpha \text{ in } C_t([0,T]; L^\infty_{w*}(\mathbf{R})), \qquad u_n \rightharpoonup u \text{ in } L^\infty_{w*}(]0,\infty[\times\mathbf{R}), \tag{2.4}$$

$$\pi_n \rightharpoonup \pi \text{ in } \mathcal{M}_{loc}([0,\infty[\times\mathbf{R})w*, \tag{2.5}$$

$$\alpha_n^0 \rightharpoonup \alpha^0 \text{ in } L^\infty_{w*}(\mathbf{R}), \quad \text{and} \quad \alpha_n^0 u_n^0 \rightharpoonup m^0 \equiv \alpha^0 u^0 \text{ in } L^\infty_{w*}(\mathbf{R}). \tag{2.6}$$

Passing to the limit in $\mathcal{D}'$, the limit u also satisfies the Oleinik entropy condition

$$\partial_x u(t,x) \leq 1/t.$$

The only non-trivial facts that need to be proved are (1.16), (1.17), (1.18) and (1.19). Let ζ_ε be a sequence of mollifiers in $\mathbf{R}$. We shall use in all this section the notation $z^\varepsilon = z \underset{x}{*} \zeta_\varepsilon$. We have the following technical result that allows to pass to the limit in most of the products we are going to deal with.

Lemma 2.2 *Let us suppose that $(\gamma_n)_{n\in\mathbf{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbf{R})$ that tends to γ in L^∞_{w*}, and satisfies for any $\Gamma \in \mathcal{D}(\mathbf{R})$*

$$\int_{\mathbf{R}} (\gamma_n - \gamma)(t,x)\Gamma(x)\,dx \to 0, \quad n \to \infty, \tag{2.7}$$

either i) a.e. $t \in]0,T[$ or ii) in $L^1_t(]0,T[)$. Let us also suppose that $(\omega_n)_{n\in\mathbf{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbf{R})$ that tends to ω in L^∞_{w*} and such that for all compact interval $K = [a,b]$, there exists $C > 0$ such that for $\beta = \omega$ or $\beta = \omega_n$,

$$TV_K(\beta(t,.)) \leq f(t) \quad a.e.\ t, \tag{2.8}$$

where $f(t) = C(\frac{1}{t} + 1)$. Then $\gamma_n\omega_n \rightharpoonup \gamma\omega$ in L^∞_{w*}, as $n \to \infty$.

We first prove a partial result.

Lemma 2.3 *Let $\varepsilon > 0$. Let us suppose that $(\gamma_n)_{n\in\mathbf{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbf{R})$ that tends to γ in L^∞_{w*} and satisfies (2.7), either i) a.e. $t\in]0,T[$ or ii) in $L^1_t(]0,T[)$. Let us also suppose that $(\omega_n)_{n\in\mathbf{N}}$ is a bounded family in $L^\infty(]0,T[\times\mathbf{R})$. Then we have $(\gamma_n-\gamma)\omega_n^\varepsilon \rightharpoonup 0$ in L^∞_{w*}, as $n\to\infty$.*

Proof of Lemma 2.3. We treat the case $i)$, the proof is very similar in the other case. We only need to consider a test function $\varphi \in C_c^\infty(]0,T[\times\mathbf{R})$, $\varphi(t,x) = \varphi_t(t)\varphi_x(x)$, $\varphi_t \in C_c^\infty(]0,T[)$, $\varphi_x \in C_c^\infty(\mathbf{R})$. In order to prove that

$$\int_0^T \int_{\mathbf{R}} (\gamma_n-\gamma)(t,x)\omega_n^\varepsilon(t,x)\varphi(t,x)\,dx\,dt \to 0, \quad n\to\infty,$$

we write this integral as $\int_{[0,T]}\int_{\mathbf{R}} I_n(t,y)\,dy\,dt$ where

$$I_n(t,y) = \omega_n(t,y)\left(\int_{\mathbf{R}} (\gamma_n-\gamma)(t,x)\zeta_\varepsilon(x-y)\varphi(t,x)\,dx\right).$$

We are going to prove the convergence of this integral using Lebesgue's theorem. Since, at y beeing fixed, $x\mapsto \zeta_\varepsilon(x-y)\varphi_x(x) \in C_c^\infty(\mathbf{R})$ and together with the fact that $(\omega_n)_{n\geq 0}$ is bounded in $L^\infty(]0,T[\times\mathbf{R})$, we deduce from the property $i)$ of γ, that for *a.e.* t,y, $I_n(t,y)\to 0$ as $n\to\infty$. We also have the following estimate,

$$|I_n(t,y)| \leq \sup_n \|\omega_n\|_{L^\infty_{t,x}}(\sup_n \|\gamma_n\|_{L^\infty_{t,x}} + \|\gamma\|_{L^\infty_{t,x}}) \int_{\mathbf{R}} \zeta_\varepsilon(x-y)|\varphi(t,x)|\,dx,$$

where $(t,y)\mapsto \int_{\mathbf{R}}\zeta_\varepsilon(x-y)|\varphi(t,x)|\,dx \in L^1(]0,T[\times\mathbf{R})$. Therefore, by dominated convergence, we have that $I_n(t,y)\to 0$ in $L^1_{t,y}(]0,T[\times\mathbf{R})$, which gives the desired convergence. $\square$

Proof of Lemma 2.2. Let us write the decomposition

$$\gamma_n\omega_n - \gamma\omega = \gamma_n(\omega_n-\omega_n^\varepsilon) + (\gamma_n-\gamma)\omega_n^\varepsilon + \gamma(\omega_n^\varepsilon-\omega^\varepsilon) + \gamma(\omega^\varepsilon-\omega). \tag{2.9}$$

We are first going to control the first and fourth terms of this decomposition for ε small enough and uniformly in n. Then, fixing ε, we shall pass to the limit, when n tends towards infinity, in the second and third terms. Let $\varphi \in C_c^\infty(]0,T[\times\mathbf{R})$.

– Let $\eta > 0$. The term $\int_0^T\int_{\mathbf{R}}\gamma(\omega^\varepsilon-\omega)\varphi\,dx\,dt$ is controlled in the following way. Since $\omega\in BV_x$, we have

$$\int_{[a,b]} |\omega(t,x-y)-\omega(t,x)|\,dx \leq |y| TV_{[a-2|y|,b+2|y|]}(\omega(t,.)),$$

hence for $\varepsilon < 1$,

$$\begin{aligned} \int_a^b \left|\int_{\mathbf{R}} (\omega(t,x-y)-\omega(t,x))\zeta_\varepsilon(y)\,dy\right|dx &\leq \int_{-\varepsilon}^{\varepsilon} |y|C(\frac{1}{t}+1)\zeta_\varepsilon(y)\,dy \\ &\leq \varepsilon C(1/t+1), \end{aligned}$$

where C is a constant independent of ε and t, obtained from the hypothesis with the compact $[a-2, b+2]$. Then, $K = [a,b]$ beeing a fixed compact containing the support in x of φ, we get

$$\|\omega^\varepsilon - \omega\|_{L^1_x(K)} \leq C(1/t+1)\varepsilon, \quad \text{a.e. } 0 < t \leq T.$$

There exists $\delta > 0$ such that $supp_t \varphi \subset [\delta, T]$, thus

$$\begin{aligned}
|\int_0^T \int_{\mathbf{R}} \gamma(\omega^\varepsilon - \omega)\varphi \, dx \, dt| &\leq \int_\delta^T \int_{\mathbf{R}} |\gamma(\omega^\varepsilon - \omega)\varphi| \, dx \, dt \\
&\leq \int_\delta^T C\varepsilon(1/t+1) \, dt \|\varphi\|_\infty \|\gamma\|_\infty \\
&\leq \varepsilon C(1/\delta^2 + T)\|\varphi\|_\infty \|\gamma\|_\infty.
\end{aligned}$$

This is less than η if ε is small enough. We have the same bound uniformly in n for $(\gamma_n)_{n\geq 0}$ and $(\omega_n)_{n\geq 0}$, thus for such ε,

$$|\int_0^T \int_{\mathbf{R}} \gamma(\omega^\varepsilon - \omega)\varphi \, dx \, dt| \leq \eta \;\text{ and }\; |\int_0^T \int_{\mathbf{R}} \gamma_n(\omega_n^\varepsilon - \omega_n)\varphi \, dx \, dt| \leq \eta, \quad \forall n \in \mathbf{N}.$$

– Let now ε be fixed as above. For the third term of the decomposition (2.9), obviously $\omega_n^\varepsilon - \omega^\varepsilon \rightharpoonup 0$ in $L^\infty_{w*}(]0,T[\times\mathbf{R})$, thus $\gamma(\omega_n^\varepsilon - \omega^\varepsilon) \rightharpoonup 0$. Lemma 2.3 establishes the convergence $(\gamma_n - \gamma)\omega_n^\varepsilon \rightharpoonup 0$ in $L^\infty_{w*}(]0,T[\times\mathbf{R})$, thus we have proved that $\gamma_n\omega_n - \gamma\omega \rightharpoonup 0$ in $L^\infty_{w*}(]0,T[\times\mathbf{R})$. □

We prove now that the limit (α, u, π) is a solution to (1.16)-(1.17).

Proposition 2.4 *We have the following weak convergences:*

$$\alpha_n u_n \rightharpoonup \alpha u \quad \textit{and} \quad \alpha_n u_n^2 \rightharpoonup \alpha u^2 \quad \textit{in } L^\infty_{w*}(]0,T[\times\mathbf{R}), \quad n \to \infty.$$

Proof. From the convergence $\alpha_n \to \alpha$ in $C_t([0,T]; L^\infty_{w*}(\mathbf{R}))$, we have

$$\sup_{t\in[0,T]} \left| \int_{\mathbf{R}} (\alpha_n - \alpha)(t,x)\Gamma(x) \, dx \right| \to 0, \quad \forall \Gamma \in L^1(\mathbf{R}).$$

From Oleinik's inequality on u, we get

$$\int_{[a,b]} |\partial_x u| \leq 2\frac{b-a}{t} + 2\|u_0\|_\infty.$$

A similar result holds for u_n, namely $\int_{[a,b]} |\partial_x u_n| \leq 2\frac{b-a}{t} + 2\sup_n \|u_n^0\|_\infty$.
We apply Proposition 2.2 with $\gamma_n = \alpha_n$ and $\omega_n = u_n$ and we get that $\alpha_n u_n - \alpha u \rightharpoonup 0$ in $L^\infty_{w*}(]0,T[\times\mathbf{R})$.
We turn now to the proof of the convergence $\alpha_n u_n^2 - \alpha u^2 \rightharpoonup 0$.
Let $\varphi \in \mathcal{D}(\mathbf{R})$. We have

$$\frac{d}{dt}\int_{\mathbf{R}} (\alpha_n u_n)(t,x)\varphi(x) \, dx - \int_{\mathbf{R}} \alpha_n u_n^2 \varphi'(x) \, dx - \int_{\mathbf{R}} \varphi'(x)\pi_n = 0,$$

thus the sequence $(\int_{\mathbf{R}}(\alpha_n u_n)(t,x)\varphi(x)\,dx)_{n\in\mathbf{N}}$ is bounded in BV_t. Therefore we can extract a subsequence such that

$$\int_{\mathbf{R}}(\alpha_n u_n)(t,x)\varphi(x)\,dx \to \int_{\mathbf{R}}(\alpha u)(t,x)\varphi(x)\,dx \text{ in } L^1(]0,T[),\ \forall\varphi\in\mathcal{D}(\mathbf{R}).$$

We can now apply Proposition 2.2 in the case $ii)$ with $\gamma_n = \alpha_n u_n$, $\omega_n = u_n$, and get that for a subsequence $(\alpha_n u_n)u_n \rightharpoonup (\alpha u)u$ in $L^\infty_{w*}(]0,T[\times\mathbf{R})$. $\square$
Combining the previous Proposition and convergences (2.4)-(2.6), we get that (α, u, π) is a solution to (1.16)-(1.17). It remains now to prove the weak form of (1.18)-(1.19).

Proposition 2.5 *There exists $Q_n \in \mathcal{D}'_{t,x}$ such that $\alpha_n u_n = \partial_x Q_n$, $\partial_t Q_n = -\alpha_n u_n^2 - \pi_n$ and such that $(Q_n)_{n\in\mathbf{N}}$ is bounded in $L^\infty(]0,T[\times\mathbf{R})\cap BV_{loc}([0,\infty[\times\mathbf{R})$.*

Proof. We take $Q_n \in \mathcal{D}'_{t,x}$ such that $\alpha_n u_n = \partial_x Q_n$ and $\alpha_n u_n^2 + \pi_n = -\partial_t Q_n$. The distribution Q_n is defined up to an additive constant that will be fixed later on. We have $\partial_x Q_n = \alpha_n u_n$ and $(\alpha_n u_n)_{n\in\mathbf{N}}$ is bounded in $L^\infty_{t,x}$. Then $\partial_t Q_n = -\alpha_n u_n^2 - \pi_n$ gives that $(Q_n)_{n\in\mathbf{N}}$ is bounded in BV_{loc}. We set

$$\widetilde{Q_n} = Q_n - \int_0^x \alpha_n u_n\,dy.$$

Then $\partial_x\widetilde{Q_n} = 0$, thus $\widetilde{Q_n}$ is a function of t only, and

$$\partial_t\widetilde{Q_n} = \partial_t Q_n - \partial_t\int_0^x \alpha_n u_n\,dy = -\alpha_n u_n^2 - \pi_n - \partial_t\int_0^x \alpha_n u_n\,dy.$$

Let $\varphi\in\mathcal{D}(\mathbf{R})$ such that $\int_{\mathbf{R}}\varphi(x)\,dx = 1$. Then $\partial_t\widetilde{Q_n} = \int_{\mathbf{R}}\partial_t\widetilde{Q_n}\varphi(x)\,dx$, and

$$\partial_t\widetilde{Q_n} = -\int_{\mathbf{R}}\alpha_n u_n^2\varphi(x)\,dx - \int_{\mathbf{R}}\varphi(x)\pi_n - \partial_t(\int_{\mathbf{R}}\int_0^x \alpha_n u_n\,dy\varphi(x)\,dx).$$

We obtain, fixing now the unknow constant, that

$$\widetilde{Q_n} = -\int_0^t\int_{\mathbf{R}}\alpha_n u_n^2\varphi\,dx\,ds - \int_0^t\int_{\mathbf{R}}\varphi\pi_n - \int_{\mathbf{R}}(\int_0^x \alpha_n u_n\,dy)\varphi\,dx.$$

We have

$$|\int_0^t\int_{\mathbf{R}}\alpha_n u_n^2\varphi(x)\,dx\,ds| \le T\ meas(\text{supp}\varphi)\ (\sup_n\|u_n^0\|_\infty)^2\|\varphi\|_\infty < +\infty,$$

$$|\int_0^t\int_{\mathbf{R}}\varphi(x)\pi_n| \le \|\varphi\|_\infty \sup_n\int_0^T\int_{\text{supp}\varphi}|\pi_n| < +\infty,$$

hence we get that $(\widetilde{Q_n})_{n\in\mathbf{N}}$ is bounded in $L^\infty_t(]0,T[)$. Now, according to the conservation relation $\int_{\mathbf{R}}\alpha_n(t,y)\,dy = \int_{\mathbf{R}}\alpha_n^0(y)\,dy$, we get

$$|Q_n(t,x)| \le \sup_n\|\widetilde{Q_n}\|_\infty + \sup_n\|u_n^0\|_\infty\int_{\mathbf{R}}\alpha_n^0(y)\,dy,$$

and therefore $(Q_n)_{n\in\mathbf{N}}$ is bounded in $L^\infty(]0,T[\times\mathbf{R})$ because of (2.1) and (2.2).
□
We can now precise the meaning of Q_n^0, which is defined by the relation

$$\int_{[0,+\infty[}\int_{\mathbf{R}} Q_n\partial_t\varphi\,dx\,dt+\int_{\mathbf{R}} Q_n^0(x)\varphi(0,x)\,dx=\int_{[0,+\infty[}\int_{\mathbf{R}}\varphi\,(\alpha_n u_n^2+\pi_n) \quad (2.10)$$

for any $\varphi\in\mathcal{D}([0,\infty[\times\mathbf{R})$. Since π_n is bounded in $L^\infty(\mathbf{R},\mathcal{M}([0,T]))$, we get that Q_n^0 is bounded in $L^\infty(\mathbf{R})$.
As in Proposition 2.5 and with the fact that $\partial_x M_n=\alpha_n$ and $\partial_t M_n=-\alpha_n u_n$ are bounded uniformly in n, we have the following result.

Proposition 2.6 *There exists $M_n\in\mathcal{D}'_{t,x}$ such that $\partial_x M_n=\alpha_n$, $\partial_t M_n=-\alpha_n u_n$ and such that $(M_n)_{n\in\mathbf{N}}$ is bounded in $L^\infty(]0,T[\times\mathbf{R})\cap Lip(]0,\infty[\times\mathbf{R})$.*

Remark 2.1 From the relations $\partial_x Q_n=\alpha_n u_n$ and $\partial_x M_n=\alpha_n$, we get that

$$\int_{[a,b]}|\partial_x Q_n|\le\sup_n\|u_n^0\|_\infty(b-a)\quad\text{and}\quad\int_{[a,b]}|\partial_x M_n|\le(b-a).$$

This gives property (2.8) with $f(t)=C$.

The following result is devoted to the proof of some results that deals with initial regularities.

Proposition 2.7 *i) Let us consider a sequence of solutions (α_n,u_n,π_n) with regularities (1.13)-(1.15) satisfying hypothesis of Theorem 2.1, in particular $(\alpha_n)_{n\in\mathbf{N}}$, $(u_n)_{n\in\mathbf{N}}$ are bounded sequences in L^∞ and $(\pi_n)_{n\in\mathbf{N}}$ is a bounded sequence in $L^\infty(\mathbf{R}_x,\mathcal{M}([0,T]))$. We consider the distribution Q_n obtained in Proposition 2.5 and Q_n^0 defined by relation (2.10). Then $(Q_n^0)_{n\in\mathbf{N}}$ is bounded in $Lip(\mathbf{R})$.*
ii) Let us assume we have some functions $\alpha^0\in L^1(\mathbf{R})$, $u^0\in L^\infty(\mathbf{R})$, $0\le\alpha^0\le 1$ and (α,u,π) with regularities (1.13) -(1.15), satisfying (1.16)-(1.17) with the constraints (1.4),(1.5),(1.18), (1.19). Then we have the relation $\alpha^0(x)\pi(\{0\},x)=\pi(\{0\},x)$ a.e. x and the equality $\frac{dQ^0}{dx}=\alpha^0u^0$ holds in $\mathcal{D}'$.

Proof. 0) We suppose that (α,u,π) is a solution to (1.17) with regularities (1.13)-(1.15). Let us consider a test function $\varphi(t,x)=\chi(t)\psi(x)$, $\chi\in C_c^\infty([0,\infty[)$, $\psi\in C_c^\infty(\mathbf{R})$. The relation (1.17) gives

$$\int_{\mathbf{R}}\left(\int_0^\infty\alpha u\chi'(t)\,dt+\alpha^0u^0\chi(0)\right)\psi(x)\,dx$$
$$+\int_{\mathbf{R}}\left(\int_0^\infty\alpha u^2\chi(t)\,dt+\int_{t\in[0,\infty[}\chi(t)\pi(\,dt,x)\right)\psi'(x)\,dx=0$$

for any ψ, therefore in $\mathcal{D}'(\mathbf{R})$ we have

$$\frac{d}{dx}f_\chi(x)=\int_0^\infty\alpha u\chi'(t)\,dt+\alpha^0(x)u^0(x)\chi(0), \quad (2.11)$$

where

$$f_\chi(x) = \int_0^\infty \alpha u^2 \chi(t)\, dt + \int_{t\in[0,\infty[} \chi(t)\pi(\,dt, x).$$

We notice that $f_\chi \in Lip(\mathbf{R})$ with $Lip(f_\chi) \le C(|\chi(0)| + \int_0^\infty |\chi'(t)|\, dt)$.
$i)$ The relation (2.10) means

$$Q_n^0(x) = Q_n(0^+, x) + \pi_n(\{0\}, x). \tag{2.12}$$

We have the regularity $\pi_n \in L^\infty(\mathbf{R}_x, \mathcal{M}([0,T]))$, thus for a.e. x, $\pi_n(.,x)$ is a measure and $\pi_n(\{0\},x) = \int_{t\in\{0\}} \pi_n(\,dt, x) \in L^\infty_x$.
First, according to $\partial_x Q_n(0^+,x) = (\alpha_n u_n)(0^+,x)$, we have $(Q_n(0^+,x))_{n\in\mathbf{N}}$ bounded in $Lip(\mathbf{R})$. Thus we need to control $\pi_n(\{0\},x)$ in $Lip(\mathbf{R})$.
We use 0) with χ_ε nonincreasing on $[\varepsilon/2, \varepsilon]$ and such that $\chi_\varepsilon(t) = 1$ on $[0,\varepsilon/2]$, $\chi_\varepsilon(t) = 0$ for $t \ge \varepsilon$. We have $\int_0^\infty |\chi_\varepsilon'(t)|\, dt = 1$, thus $(f_{\chi_\varepsilon})_{\varepsilon>0}$ is bounded in $Lip(\mathbf{R}) \cap L^\infty(\mathbf{R})$. Now, $f_{\chi_\varepsilon} \to \int_{t\in\{0\}} \pi(\,dt, x)$ a.e. as $\varepsilon \to 0$, therefore $\int_{t\in\{0\}} \pi(\,dt,x) \in Lip(\mathbf{R})$ with $Lip(\int_{t\in\{0\}} \pi(\,dt,x)) \le 2C$. Using (2.12), we get that $(Q_n^0)_{n\in\mathbf{N}}$ is bounded in $Lip(\mathbf{R})$.
$ii)$ The relation (1.18) yields

$$\frac{d}{dx}\left(\int_0^\infty \alpha u Q \chi_\varepsilon(t)\, dt\right) = \alpha^0(x)Q^0(x) + \int_0^\infty \alpha Q \chi_\varepsilon'(t)\, dt - \int_{[0,\infty[} \chi_\varepsilon(t)\pi(\,dt, x),$$

thus, when $\varepsilon \to 0$, we get

$$\alpha^0(x)Q^0(x) - \alpha^0(x)Q(0^+,x) = \int_{t\in\{0\}} \pi(\,dt, x),$$

using (2.12), it means

$$\alpha^0(x)\pi(\{0\},x) = \pi(\{0\},x).$$

The relation (2.11) gives

$$\frac{df_{\chi_\varepsilon}}{dx} \to -(\alpha u)(0^+,x) + \alpha^0(x)u^0(x), \qquad \text{as } \varepsilon \to 0,$$

thus

$$\frac{d}{dx}\int_{t\in\{0\}} \pi(\,dt,x) = \alpha^0(x)u^0(x) - (\alpha u)(0^+,x),$$

and

$$\frac{dQ^0}{dx} = \alpha^0 u^0. \ \square$$

Using these various regularities, we now pass to the limit in the constraints (1.18)-(1.19). Since $(Q_n)_{n\in\mathbf{N}}$ is bounded in $BV_{loc} \cap L^\infty_{t,x}$ and $(\alpha_n^0 Q_n^0)_{n\in\mathbf{N}}$ is bounded in $L^\infty(\mathbf{R})$, there exists $Q \in BV_{loc} \cap L^\infty_{t,x}$, $P^0 \in L^\infty(\mathbf{R})$ and a subsequence still noted $(Q_n)_{n\in\mathbf{N}}$ such that, as $n \to \infty$,

$$Q_n \to Q \text{ in } L^1_{loc},\ Q_n \rightharpoonup Q \text{ in } L^\infty_{w*} \text{ and } \alpha_n^0 Q_n^0 \rightharpoonup P^0 \text{ in } L^\infty_{w*}.$$

We notice that $\partial_x Q = \alpha u$ and $\partial_t Q = -\alpha u^2 - \pi$. We can also pass to the limit in (2.10) and get $Q_n^0 \rightharpoonup Q^0$ in L^∞_{w*}. Then since $Q_n \to Q$ strongly, we can pass to the limit in (1.18), but we need to prove that $P^0 = \alpha^0 Q^0$. This last point comes from the fact that $(Q_n^0)_{n\in\mathbf{N}}$ is bounded in $Lip(\mathbf{R})$ which is proved in Proposition 2.7. Now, concerning R_2, the proof is straightforward because $M_n \to M$ locally uniformly. This ends the proof of Theorem 2.1.

3 Sticky blocks

In this section, we are going to provide explicit solutions for some particular initial data. Nevertheless, we shall have enough solutions in the weak sense to pass to the limit by using the stability result of the previous section. With these solutions, we shall be able to prove the existence theorem for arbitrary initial data. The class of initial data which gives simple solutions are the sticky blocks. These sticky blocks generalize the sticky particles involved in the pressureless gas system of [7] and were introduced in [3].
Sticky particles are formal limits of sticky blocks when the height of the blocks goes to infinity. In other words, taking a constraint $\alpha \leq \alpha_{max}$ for the blocks and letting $\alpha_{max} \to \infty$, we find sticky particles.
Let us consider a volume fraction $\alpha(t,x)$ and a momentum density $\alpha(t,x)u(t,x)$ given by

$$\alpha(t,x) = \sum_{i=1}^{n} \mathbb{1}_{a_i(t)<x<b_i(t)}, \quad \alpha(t,x)u(t,x) = \sum_{i=1}^{n} u_i(t)\mathbb{1}_{a_i(t)<x<b_i(t)}, \tag{3.1}$$

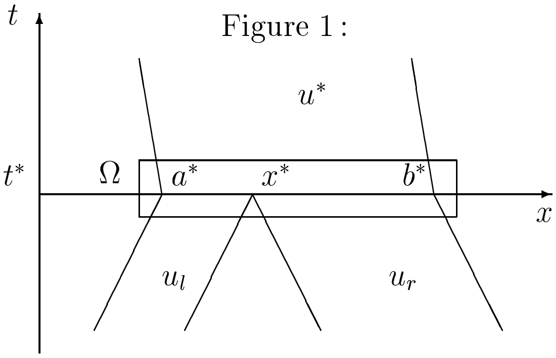
with $a_1(t) < b_1(t) < a_2(t) < b_2(t) < \cdots < b_n(t)$. The time evolution is defined as follows. The number of blocks n indeed depends on t, but is piecewise constant. As long as the blocks do not meet, they move at constant velocity $u_i(t)$. When two or more blocks collide, they get stuck, building a new block, with volume the sum of the volumes, and with momentum the sum of momenta. Obviously at most $n^0 - 1$ collisions can occur, where n^0 is the initial number of blocks. We have to underline that we only consider in this section initial data such that u^0 is constant in each block, so that π contains no Dirac at $t = 0$. Generalized sticky blocks with non constant u^0 in a block, such as in the example of Section 1.1, can nevertheless appear in the limit of Theorem 2.1. In [3], the following Theorem was obtained.

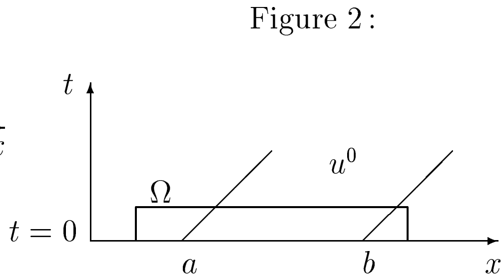
Theorem 3.1 *There exists a nonnegative measure $\pi(t,x)$ such that with $\alpha(t,x)$ and $u(t,x)$ defined by (3.1), and with the above defined dynamics, we get a solution to (1.3)-(1.5).*

The expression of π on a shock at time t^* is given by the formula

$$\pi(t,x) = \begin{cases} \delta(t-t^*)(u_l - u^*)(x - a^*) & if\ a^* \leq x \leq x^*, \\ \delta(t-t^*)(u^* - u_r)(b^* - x) & if\ x^* \leq x \leq b^*, \\ 0 & otherwise, \end{cases} \tag{3.2}$$

with the notations of Figure 1.





We also set $a(t) = a^* + u^*(t - t^*)$, $b(t) = b^* + u^*(t - t^*)$, $a_l(t) = a^* + u_l(t - t^*)$, $b_l(t) = x^* + u_l(t - t^*)$, $a_r(t) = x^* + u_r(t - t^*)$ and $b_r(t) = b^* + u_r(t - t^*)$.
We define the value of $u(t, x)$ for all x by asking that u is Lipschitz continuous, takes the value $u_i(t)$ in each block i and is extended linearly between each block, and by a constant at infinity. As was proved in [3], the sticky blocks dynamics satisfies Oleinik's condition.

Proposition 3.2 *For any $t > 0$, we have $\partial_x u(t, x) \leq 1/t$.*

Notice that we also have the properties (1.4), (1.5), (1.21), (1.13), (1.14), (1.15). We prove now that the sticky blocks satisfy the weak formulation (1.16)-(1.17) and the constraint $R_1 = R_2 = \pi$ in the sense of (1.18)-(1.19).

Proposition 3.3 *The solution (α, u, π) satisfies (1.16)-(1.19).*

Proof. First, according to Theorem 3.1, we only have to prove the result (1.16)-(1.17) for $\varphi \in \mathcal{D}(\Omega)$ where $\Omega = [0; \varepsilon[\times]a - \eta; b + \eta[$ where $]a; b[$ is a block at $t = 0$ and ε such that there is no collisions between $t = 0$ and $t = \varepsilon$. The situation is displayed on Figure 2. We have

$$\begin{aligned}
&\int_{[0,+\infty[}\int_{\mathbf{R}}[-\alpha u \partial_t \varphi - (\alpha u^2 + \pi)\partial_x \varphi]\, dx\, dt \\
= &\; -\int_0^{\varepsilon}\int_{a(t)}^{b(t)}(u^0 \partial_t \varphi + (u^0)^2 \partial_x \varphi)\, dx\, dt \\
= &\; -\int_0^{\varepsilon}\frac{d}{dt}(u^0 \int_{a(t)}^{b(t)} \varphi\, dx)\, dt + \int_0^{\varepsilon}(\varphi(t, b(t)) - \varphi(t, a(t)))(u^0)^2\, dt \\
&\; -\int_0^{\varepsilon}(\varphi(t, b(t)) - \varphi(t, a(t)))(u^0)^2\, dt \\
= &\; u^0 \int_{a(0)}^{b(0)} \varphi(0, x)\, dx - u^0 \int_{a(\varepsilon)}^{b(\varepsilon)} \varphi(\varepsilon, x)\, dx \\
= &\; \int_a^b u^0 \varphi(0, x)\, dx = \int_{\mathbf{R}}(\alpha^0 u^0)(x)\varphi(0, x)\, dx,
\end{aligned}$$

that is (1.17). We have similarly (1.16). Concerning (1.18) and (1.19), we have to work with Figure 1 and Figure 2. We shall detail the case R_1 with Figure 2, the case R_2 with Figure 1 and the two similar cases are left to the reader.
– About R_1. A useful expression of $Q(t, x)$ is

$$Q(t, x) = \sum_{i=1}^{n} u_i(t) T_{a_i(t), b_i(t)}(x), \quad \text{where} \quad T_{a,b}(x) = \begin{cases} 0 & if\ x \leq a \\ x - a & if\ a \leq x \leq b \\ b - a & if\ x \geq b. \end{cases}$$

In the neighbourhood of the block, up to a constant, we can take $Q(t, x) = u^0 T_{a(t), b(t)}(x)$. We compute R_1,

$$\begin{aligned}
< R_1, \varphi > \;=\; & \int_0^{\varepsilon}\int_{\mathbf{R}} \mathbb{I}_{a(t)<x<b(t)} Q(t, x)\partial_t \varphi\, dx\, dt + \\
& \int_0^{\varepsilon}\int_{\mathbf{R}} \mathbb{I}_{a(t)<x<b(t)} u(t, x) Q(t, x)\partial_x \varphi\, dx\, dt + \int_{\mathbf{R}} (\alpha Q)^0 \varphi(0, x)\, dx.
\end{aligned}$$

Then the expression of R_1 is $< R_1, \varphi >= A + A^0$, where

$$A = \int_0^\varepsilon \int_{a(t)}^{b(t)} u^0(x - a(t))\partial_t\varphi \, dx \, dt + \int_0^\varepsilon \int_{a(t)}^{b(t)} (u^0)^2(x - a(t))\partial_x\varphi \, dx \, dt,$$
$$A^0 = \int_a^b u_i^0(x - a)\varphi^0 \, dx.$$

We have

$$\begin{aligned} A &= \int_0^\varepsilon u^0 \Bigg[\frac{d}{dt}\int_{a(t)}^{b(t)} x\varphi(t,x)\,dx - [b(t)\varphi(t,b(t)) - a(t)\varphi(t,a(t))]u^0 \\ &\quad -a(t)\frac{d}{dt}\int_{a(t)}^{b(t)} \varphi(t,x)\,dx + [b(t)\varphi(t,b(t)) - a(t)\varphi(t,a(t))]u^0\Bigg]\,dt \\ &\quad + \int_0^\varepsilon (u^0)^2[b(t)\varphi(t,b(t)) - a(t)\varphi(t,a(t)) - \int_{a(t)}^{b(t)} \varphi(t,x)\,dx \\ &\quad -a(t)[\varphi(t,b(t)) - \varphi(t,a(t))]]\,dt \\ &= -u^0\int_{a(0)}^{b(0)} x\varphi(0,x)\,dx - (u^0)^2\int_0^\varepsilon (b(t) - a(t))\varphi(t,b(t))\,dt \\ &\quad -u^0\int_0^\varepsilon (a^0 + tu^0)\left(\frac{d}{dt}\int_{a(t)}^{b(t)} \varphi(t,x)\,dx\right)\,dt \\ &\quad +(u^0)^2\int_0^\varepsilon (b(t) - a(t))\varphi(t,b(t))\,dt - (u_0)^2\int_0^\varepsilon\int_{a(t)}^{b(t)} \varphi(t,x)\,dx\,dt, \end{aligned}$$

now using

$$\begin{aligned} &\int_0^\varepsilon (a^0 + tu^0)\left(\frac{d}{dt}\int_{a(t)}^{b(t)} \varphi(t,x)\,dx\right)\,dt \\ &= -a^0\int_{a(0)}^{b(0)} \varphi(0,x)\,dx + u^0\left[t\int_{a(t)}^{b(t)} \varphi(t,x)\,dx\right]_0^\varepsilon - u^0\int_0^\varepsilon\int_{a(t)}^{b(t)} \varphi(t,x)\,dx\,dt \\ &= -a^0\int_a^b \varphi(0,x)\,dx - u^0\int_0^\varepsilon\int_{a(t)}^{b(t)} \varphi(t,x)\,dx\,dt, \end{aligned}$$

the expression A finally simplifies in $\int_a^b u^0(a - x)\varphi(0,x)\,dx$. Thus we have $< R_1, \varphi >= 0 =< \pi, \varphi > \forall\varphi \in \mathcal{D}(\Omega)$ with Ω as in Figure 2.
– About R_2. We use now $M(t,x) = \sum_{i=1}^n T_{a_i(t),b_i(t)}(x)$. Let $\varphi \in \mathcal{D}(\Omega)$ with Ω as in Figure 1. In this domain, we can use

$$M(t,x) = \begin{cases} T_{a_l(t),b_l(t)}(x) + T_{a_r(t),b_r(t)}(x) & if\ t < t^*, \\ T_{a(t),b(t)}(x) & if\ t > t^*. \end{cases}$$

With similar technics, we obtain successively that

$$\iint_\Omega \alpha u^2 M\partial_x\varphi\,dx\,dt + \iint_\Omega \alpha u M\partial_t\varphi\,dx\,dt$$

$$\begin{aligned}
= \;& \int^{t^*} \left(\int_{a_l(t)}^{b_l(t)} (x - a_l(t)) \left[u_l^2 \partial_x \varphi + u_l \partial_t \varphi \right] \, dx \right) \, dt \\
& + \int^{t^*} \left(\int_{a_r(t)}^{b_r(t)} (b_l(t) - a_l(t) + x - a_r(t)) \left[u_r^2 \partial_x \varphi + u_r \partial_t \varphi \right] \, dx \right) \, dt \\
& + \int_{t^*} \left(\int_{a(t)}^{b(t)} (x - a(t)) \left[(u^*)^2 \partial_x \varphi + u^* \partial_t \varphi \right] \, dx \right) \, dt \\
= \;& -u_l^2 \int^{t^*} \int_{a_l(t)}^{b_l(t)} \varphi(t,x) \, dx \, dt - \int^{t^*} a_l(t) u_l \left[\frac{d}{dt} \int_{a_l(t)}^{b_l(t)} \varphi(t,x) \, dx \right] \, dt \\
& + \int^{t^*} u_l \left[\frac{d}{dt} \int_{a_l(t)}^{b_l(t)} x\varphi(t,x) \, dx \right] \, dt - u_r^2 \int^{t^*} \int_{a_r(t)}^{b_r(t)} \varphi(t,x) \, dx \, dt \\
& + \int^{t^*} (b_l(t) - a_l(t) - a_r(t)) u_r \left[\frac{d}{dt} \int_{a_r(t)}^{b_r(t)} \varphi(t,x) \, dx \right] \, dt \\
& + \int^{t^*} u_r \left[\frac{d}{dt} \int_{a_r(t)}^{b_r(t)} x\varphi(t,x) \, dx \right] \, dt - (u^*)^2 \int_{t^*} \int_{a(t)}^{b(t)} \varphi(t,x) \, dx \, dt \\
& - \int_{t^*} a(t) u^* \left[\frac{d}{dt} \int_{a(t)}^{b(t)} \varphi(t,x) \, dx \right] \, dt + \int_{t^*} u^* \left[\frac{d}{dt} \int_{a(t)}^{b(t)} x\varphi(t,x) \, dx \right] \, dt \\
= \;& -a^* \left[u_l \int_{a^*}^{x^*} \varphi(t^*,x) \, dx + u_r \int_{x^*}^{b^*} \varphi(t^*,x) \, dx - u^* \int_{a^*}^{b^*} \varphi(t^*,x) \, dx \right] \\
& + u_l \int_{a^*}^{x^*} x\varphi(t^*,x) \, dx + u_r \int_{x^*}^{b^*} x\varphi(t^*,x) \, dx - u^* \int_{a^*}^{b^*} x\varphi(t^*,x) \, dx.
\end{aligned}$$

On an other hand, a computation gives that the expression of $\int_\Omega M\pi\partial_x\varphi$ is

$$- \int_{\mathbf{R}} [2(x - a^*)(u_l - u^*) \mathbb{1}_{[a^*,x^*]}(x) + (u^* - u_r)(b^* + a^* - 2x) \mathbb{1}_{[x^*,b^*]}(x)] \varphi(t^*,x) \, dx,$$

and finally after simplifications we get

$$\iint_\Omega M\alpha u^2 \partial_x \varphi \, dx \, dt + \iint_\Omega \alpha u M \partial_t \varphi \, dx \, dt + \iint_\Omega M\pi \partial_x \varphi = - \iint_\Omega \varphi\pi. \; \square$$

4 Existence of a solution

In this section, we prove the existence of a solution. To reach this aim, we prove that every initial data can be approched by blocks in $\mathcal{D}'(\mathbf{R})$ and then we pass to the limit in the sticky blocks solution of Section 3 using the compactness result of Section 2. We first prove a lemma about weak limit of sticky blocks.

Lemma 4.1 *Let $\alpha^0 \in L^1(\mathbf{R})$ such that $0 \leq \alpha^0 \leq 1$ and $u^0 \in L^\infty(\mathbf{R})$, then there exist block initial data $(\alpha_n^0)_{n\geq 0}$ and $(\alpha_n^0 u_n^0)_{n\geq 0}$ such that $\int_{\mathbf{R}} \alpha_n^0(x)\, dx \leq \int_{\mathbf{R}} \alpha^0(x)\, dx$ and essinf $u^0 \leq u_n^0 \leq$ esssup u^0 for which the convergences $\alpha_n^0 \rightharpoonup \alpha^0$ and $\alpha_n^0 u_n^0 \rightharpoonup \alpha^0 u^0$ hold in $\mathcal{D}'_x$.*

Proof. We set

$$\alpha_n^0(x) = \sum_{\substack{i=-n^2 \\ m_{in}>0}}^{n^2} \mathbb{I}_{]\frac{i}{n},\frac{i}{n}+m_{in}[} \qquad \text{and} \qquad \alpha_n^0 u_n^0(x) = \sum_{\substack{i=-n^2 \\ m_{in}>0}}^{n^2} u_{in}\mathbb{I}_{]\frac{i}{n},\frac{i}{n}+m_{in}[},$$

where

$$m_{in} = \int_{i/n}^{(i+1)/n-1/n^2} \alpha^0(x)\,dx \qquad \text{and} \qquad u_{in} = \frac{1}{m_{in}}\int_{i/n}^{(i+1)/n-1/n^2} \alpha^0(x)u^0(x)\,dx.$$

We notice that $0 \leq m_{in} \leq 1/n - 1/n^2 < 1/n$ and that u_{in} is defined when $m_{in} > 0$. Let $\varphi \in \mathcal{D}(\mathbf{R})$, let n_0 be an integer such that $supp\varphi \subset [-n_0, n_0]$, we have

$$\begin{aligned}\int_{\mathbf{R}} \alpha_n^0(x)\varphi(x)\,dx &= \sum_{i=-n^2}^{n^2} \int_{i/n}^{i/n+m_{in}} \varphi(x)\,dx = \sum_{i=-n^2}^{n^2} \Big[\varphi(i/n)m_{in} + \varphi'(x_n^i)m_{in}^2/2\Big] \\ &= \sum_{i=-n^2}^{n^2} \int_{i/n}^{(i+1)/n-1/n^2} \alpha^0(x)\varphi(i/n)\,dx + \sum_{i=-n^2}^{n^2} \varphi'(x_n^i)m_{in}^2/2,\end{aligned}$$

where $x_n^i \in [i/n; i/n + m_{in}]$. Thus we get for $n > n_0$,

$$\begin{aligned}\left|\int_{\mathbf{R}} \alpha_n^0(x)\varphi(x)\,dx - \int_{\mathbf{R}} \alpha^0(x)\varphi(x)\,dx\right| \;\leq\; & \sum_{i=-nn_0}^{nn_0-1} \int_{i/n}^{(i+1)/n-1/n^2} |\varphi(i/n) - \varphi(x)|\,dx \\ &+ \sum_{i=-nn_0}^{nn_0-1} \int_{(i+1)/n-1/n^2}^{(i+1)/n} |\varphi(x)|\,dx \\ &+ \sum_{i=-nn_0}^{nn_0-1} \frac{1}{2}|\varphi'(x_n^i)|m_{in}^2,\end{aligned}$$

according to $[i/n, (i+1)/n] \subset [-n_0, n_0] \Leftrightarrow -nn_0 \leq i \leq nn_0 - 1$. We finally get that, as $n \to \infty$,

$$< \alpha_n^0, \varphi > \to < \alpha^0, \varphi > \qquad \text{and} \qquad < \alpha_n^0 u_n^0, \varphi > \to < \alpha^0 u^0, \varphi > . \; \square$$

Proof of Theorem 1.1. Let $\alpha^0 \in L^1(\mathbf{R})$, $0 \leq \alpha^0 \leq 1$, $u^0 \in L^\infty(\mathbf{R})$. Let α_n^0 and $\alpha_n^0 u_n^0$ be the block initial data of Lemma 4.1. Then using Section 3, we get (α_n, u_n, π_n) with regularity (1.13)-(1.15) solution to (1.16)-(1.19) such that these solutions satisfy (1.20), (1.21), (1.4), (1.5). From the inequality $essinf\ u^0 \leq u_n^0 \leq esssup\ u^0$, we get that $essinf\ u^0 \leq u_n(t,x) \leq esssup\ u^0$. Thus, taking the limit (2.4), we obtain the maximum principle (1.21). We have (2.1), (2.2) and (1.21), we get that the nonnegative measures $(\pi_n)_{n\geq 0}$ are bounded in $\mathcal{D}'_{t,x}$ thus bounded in $\mathcal{M}_{loc}([0,\infty[\times\mathbf{R})$. Using now (1.17), we get (2.3). Thus we can apply Theorem 2.1. $\square$

5 Entropy inequalities

We first prove results on sticky blocks, then we shall pass to the limit.

5.1 The case of sticky blocks

Proposition 5.1 *The sticky blocks solution of Section 3 satisfies*

$$TV(u(t,.)) \leq TV(u^0) \quad \forall t \geq 0, \tag{5.1}$$

$$\partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)) + \partial_x(\pi_S) \leq 0 \tag{5.2}$$

for every $S \in C^1(\mathbf{R})$ *convex Lipschitz,*
where $\pi_S(t,x) = \begin{cases} S'(u^*)\pi(t,x) & \text{on a shock (see (3.2))}, \\ 0 & \text{otherwise}, \end{cases}$ *and* $|\pi_S| \leq Lip(S)\pi$.

Proof. *i*) We have $TV(u^0) = \sum_{i=2}^{n^0} |u_i^0 - u_{i-1}^0|$ initially. When a collision happens, for exemple at time t^* between blocks k and $k+1$ to simplify the presentation, we have after the collision $u^* = u_k^0 \Delta_a/\Delta + u_{k+1}^0 \Delta_b/\Delta$, where $\Delta_a = x^* - a^*$, $\Delta_b = b^* - x^*$ and $\Delta = b^* - a^* = \Delta_a + \Delta_b$ using the notation of Figure 1. As

$$TV(u(t^*)) = \sum_{i=2}^{k-1} |u_i^0 - u_{i-1}^0| + |u^* - u_{k-1}^0| + |u_{k+2}^0 - u^*| + \sum_{i=k+3}^{n^0} |u_i^0 - u_{i-1}^0|,$$

in order to obtain $TV(u(t^*)) \leq TV(u^0)$, we have to prove that

$$|u^* - u_{k-1}^0| + |u_{k+2}^0 - u^*| \leq |u_k^0 - u_{k-1}^0| + |u_{k+1}^0 - u_k^0| + |u_{k+2}^0 - u_{k+1}^0|.$$

First, we have

$$\begin{aligned} |u_{k+2}^0 - u^*| &\leq |u_{k+2}^0 - u_{k+1}^0| + |u_{k+1}^0 - u^*| \\ &\leq |u_{k+2}^0 - u_{k+1}^0| + |u_{k+1}^0 \frac{\Delta_a}{\Delta} + u_{k+1}^0 \frac{\Delta_b}{\Delta} - u_k^0 \frac{\Delta_a}{\Delta} - u_{k+1}^0 \frac{\Delta_b}{\Delta}| \\ &\leq |u_{k+2}^0 - u_{k+1}^0| + \frac{\Delta_a}{\Delta} |u_{k+1}^0 - u_k^0|. \end{aligned}$$

Similarly, we get $|u^* - u_{k-1}^0| \leq |u_k^0 - u_{k-1}^0| + \frac{\Delta_b}{\Delta} |u_{k+1}^0 - u_k^0|$, and by adding these two inequalities, we get the desired inequality. Finally, collision after collision, the total variation is nonincreasing.
ii) We have to prove the entropy inequality on a shock. We use again the notation of Figure 1. Let $\varphi(t,x)$ be a smooth function with support in Ω. We compute as in [3] that

$$\begin{aligned} &< \partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)), \varphi > \\ = & - \int_{a^*}^{x^*} S(u_l)\varphi(t^*,x)\,dx - \int_{x^*}^{b^*} S(u_r)\varphi(t^*,x)\,dx + \int_{a^*}^{b^*} S(u^*)\varphi(t^*,x)\,dx. \end{aligned}$$

Besides we also have

$$< \partial_x \pi_S, \varphi >= S'(u^*) \left[u_l \int_{a^*}^{x^*} \varphi(t^*,x)\,dx + u_r \int_{x^*}^{b^*} \varphi(t^*,x)\,dx - u^* \int_{a^*}^{b^*} \varphi(t^*,x)\,dx \right]$$

thanks to (3.2). Thus we get

$$\begin{aligned}
&< \partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)) + \partial_x \pi_S, \varphi > \\
= &\int_{a^*}^{x^*} [S(u^*) - S(u_l) - S'(u^*)\cdot(u^* - u_l)]\,\varphi(t^*, x)\,dx \\
&+ \int_{x^*}^{b^*} [S(u^*) - S(u_r) - S'(u^*)\cdot(u^* - u_r)]\,\varphi(t^*, x)\,dx.
\end{aligned}$$

Now by convexity of S, for every $\varphi \geq 0$, we get

$$< \partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)) + \partial_x \pi_S, \varphi > \leq 0,$$

that is $\partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)) + \partial_x \pi_S \leq 0$ in $\mathcal{D}'$. □

5.2 Entropy inequalities for a general solution

We want to pass to the limit in (5.2). Let (α, u, π) be the solution of Theorem 1.1 and (α_n, u_n, π_n) the sticky blocks that tends to (α, u, π) as in Section 4.

Lemma 5.2 *For any $S : \mathbf{R} \to \mathbf{R}$ continuous, $\alpha_n S(u_n) \rightharpoonup \alpha S(u)$.*

Proof. Obviously, it is enough to get the result for $S_k(u) = u^k$, $k \in \mathbf{N}$. We have, for $S \in C^1(\mathbf{R})$ convex,

$$\partial_t(\alpha_n S(u_n)) + \partial_x(\alpha_n u_n S(u_n)) + \partial_x \pi_S^n \leq 0$$

and $(\pi_S^n)_{n\in\mathbf{N}}$ is bounded in $\mathcal{M}_{t,x}$, thus we get that $(\alpha_n S(u_n))_{n\in\mathbf{N}}$ is bounded in $BV_t(\mathcal{D}'_x)$.
For $S \in C^1(\mathbf{R})$ concave, we have $\partial_t(\alpha_n S(u_n)) + \partial_x(\alpha_n u_n S(u_n)) + \partial_x \pi_S^n \geq 0$. Hence, similarly, $(\alpha_n S(u_n))_{n\in\mathbf{N}}$ is bounded in $BV_t(\mathcal{D}'_x)$.
Now, $x \mapsto x^{2k}$ is convex, of class C^1 on $\mathbf{R}$ for every $k \geq 0$, therefore

$$(\alpha_n u_n^{2k})_{n\in\mathbf{N}} \text{ is bounded in } BV_t(\mathcal{D}'_x).$$

For $k \geq 0$, $x^{2k+1} = S_1(x) + S_2(x)$ where $S_1(x) = x^{2k+1}\mathbb{1}_{x\geq 0}$ and $S_2(x) = x^{2k+1}\mathbb{1}_{x<0}$. The function S_1 is convex of class C^1 on $\mathbf{R}$ and S_2 is concave of class C^1 on $\mathbf{R}$, hence $(\alpha_n S_1(u_n))_{n\in\mathbf{N}}$ is bounded in $BV_t(\mathcal{D}'_x)$ and $(\alpha_n S_2(u_n))_{n\in\mathbf{N}}$ is bounded in $BV_t(\mathcal{D}'_x)$, thus

$$(\alpha_n u_n^{2k+1})_{n\in\mathbf{N}} \text{ is bounded in } BV_t(\mathcal{D}'_x).$$

Finally,

$$(\alpha_n u_n^{k})_{n\in\mathbf{N}} \text{ is bounded in } BV_t(\mathcal{D}'_x), \quad \forall k \geq 0.$$

We may now proceed by induction on k and apply Lemma 2.2. Thus we obtain

$$\alpha_n u_n^k \rightharpoonup \alpha u^k, \text{ as } n \to \infty, \text{ for every } k \geq 0. \ \square$$

Proof of Theorem 1.2. It only remains to prove the entropy inequality (1.23). For $S \in C^1(\mathbf{R})$ convex, we can write the inequality

$$\partial_t(\alpha_n S(u_n)) + \partial_x(\alpha_n u_n S(u_n)) + \partial_x \pi_S^n \leq 0.$$

By Lemma 5.2, $\alpha_n S(u_n) \rightharpoonup \alpha S(u)$ and $(\alpha_n S(u_n))u_n \rightharpoonup (\alpha S(u))u$. Now $(\pi_S^n)_{n\in\mathbf{N}}$ is bounded in $\mathcal{M}_{t,x}$ hence for a subsequence, $\pi_S^n \rightharpoonup \pi_S$ and we get, passing to the limit, the relation

$$\partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)) + \partial_x \pi_S \leq 0.$$

Finally, from the estimate $|\pi_S^n| \leq C\pi^n$ that is to say $-<C\pi^n,\varphi> \leq <\pi_S^n,\varphi> \leq <C\pi^n,\varphi>$ for every $\varphi \in C_c^\infty$, $\varphi \geq 0$, by passing to the limit, we get the localisation $|\pi_S| \leq C\pi$. $\square$

6 The exclusion relation $\alpha\pi = \pi$

The aim of this section is to give a more classical sense than that of (1.18)-(1.19) to the relation $\alpha\pi = \pi$ in a special class of solutions.

6.1 Definition of $\alpha\pi$ for π in the class $\mathcal{ML}$

We say that π is in $\mathcal{ML}$ if π lies in $\mathcal{M}_{loc}([0,\infty[\times\mathbf{R})$ and if there exists C such that

$$\left|\int_{[0,\infty[}\int_{\mathbf{R}} \phi(t,x)\pi\,dt\,dx\right| \leq C\|\phi\|_{L_t^\infty(]0,\infty[,L_x^1(\mathbf{R}))}, \quad \forall\phi \in C_c([0,\infty[\times\mathbf{R}). \qquad (6.1)$$

If π is in $\mathcal{ML}$, then thanks to the density of $C_c([0,\infty[\times\mathbf{R})$ in $C_c([0,\infty[,L^1(\mathbf{R}))$ for $\|.\|_{L_t^\infty(]0,\infty[,L_x^1(\mathbf{R}))}$, we can define $<\pi,\phi>$ for $\phi \in C_c([0,\infty[,L^1(\mathbf{R}))$ by classical argument. We obtain besides that

$$|<\pi,\phi>| \leq C\|\phi\|_{L_t^\infty(]0,\infty[,L_x^1(\mathbf{R}))}, \quad \forall\phi \in C_c([0,\infty[,L^1(\mathbf{R})).$$

Definition 6.1 *Let $\alpha \in C_t([0,\infty[,L^1_{loc}(\mathbf{R}))$ and $\pi \in \mathcal{ML}$. Then the product $\alpha\pi$ is defined as a measure by $<\alpha\pi,\phi> = <\pi,\alpha\phi>$ for $\phi \in C_c([0,\infty[\times\mathbf{R})$.*

We notice that if $\pi \in \mathcal{M}_{loc}([0,\infty[\times\mathbf{R})$ satisfies

$$\int_{[0,\infty[}\int_{\mathbf{R}} |\phi(t,x)\pi|\,dt\,dx \leq C\|\phi\|_{L_t^\infty(]0,\infty[,L_x^1(\mathbf{R}))}, \quad \forall\phi \in C_c([0,\infty[,L^1(\mathbf{R})), \quad (6.2)$$

then $\pi \in \mathcal{ML}$ and

$$<\pi,\phi> = \int_{[0,\infty[}\int_{\mathbf{R}} \phi(t,x)\pi\,dt\,dx, \quad \forall\phi \in C_c([0,\infty[,L^1(\mathbf{R})).$$

We also remark that $\mathcal{ML}$ looks like $\mathcal{M}_t(L_x^\infty)$ but we shall not discuss here the relations furthermore. The two following Propositions prove that the sticky blocks of Section 3 have this regularity.

Proposition 6.2 *For the sticky blocks of Section 3, the pressure π satisfies (6.2).*

Proof. This is obvious with (3.2) and we obtain (6.2) with a constant C which depends on the number of blocks and on the size of the support of the solution. □

Proposition 6.3 *For the sticky blocks, we have $\alpha \in C([0,\infty[, L^1(\mathbf{R}))$ and $\alpha\pi = \pi$ in $\mathcal{M}_{loc}([0,\infty[\times\mathbf{R})$.*

Proof. With the notations of Figure 1, for any $\varphi \in C_c(\Omega)$, we have

$$\begin{aligned} <\alpha\pi, \varphi> &= <\pi, \alpha\varphi> = \iint_\Omega \pi\alpha\varphi = \int_{a^*}^{b^*} \alpha(t^*, x)\varphi(t^*, x)\pi(t^*, x)\, dx \\ &= \int_{a^*}^{b^*} \varphi(t^*, x)\pi(t^*, x)\, dx = <\pi, \varphi>, \end{aligned}$$

and it is enough to get our result. □

6.2 Approximation in $\mathcal{ML}$

We prove here different technical lemma about approximation of measures with the following sequence of mollifiers ζ_ε : $\zeta_\varepsilon(t,x) = \zeta_\varepsilon^1(t)\zeta_\varepsilon^2(x)$ where $supp(\zeta_\varepsilon^1) \subset]0, \varepsilon[$ and $supp(\zeta_\varepsilon^2) \subset]-\varepsilon, \varepsilon[$. We shall use this sequence in all Section 6.

Lemma 6.4 *Let $\mu \in \mathcal{ML}$ and set $\mu_\varepsilon(t,x) = \int_{[0,\infty[}\int_{\mathbf{R}} \zeta_\varepsilon(t-s, x-y)\, d\mu(s,y)$. Then $\mu_\varepsilon \in \mathcal{ML}$ with a constant independant of ε.*

Proof. Let $\phi \in C_c([0,\infty[\times\mathbf{R})$. We set $\phi_\varepsilon(s,y) = \int_0^\infty \int_{\mathbf{R}} \zeta_\varepsilon(t-s, x-y)\phi(t,x)\, dt\, dx$. We notice that

$$\begin{aligned} \int_{\mathbf{R}} |\phi_\varepsilon(s,y)|\, dy &= \int_{\mathbf{R}} |\int_0^\infty \int_{\mathbf{R}} \zeta_\varepsilon(t-s, x-y)\phi(t,x)\, dt\, dx|\, dy \\ &\le \int_0^\infty \int_{\mathbf{R}} |\phi(t,x)|\zeta_\varepsilon^1(t-s)\, dx\, dt. \end{aligned}$$

Now from the estimate $\int_0^\infty F(t)\zeta_\varepsilon^1(t-s)\, dt \le \sup\limits_{t\ge 0} F(t)$ with $F(t) = \int_{\mathbf{R}} |\phi(t,x)|\, dx$, and taking the supremum over $s \ge 0$, we obtain that

$$\sup_{s\ge 0} \int_{\mathbf{R}} |\phi_\varepsilon(s,y)|\, dy \le \sup_{t\ge 0} \int_{\mathbf{R}} |\phi(t,x)|\, dx,$$

that is to say $\|\phi_\varepsilon\|_{L_t^\infty(L_x^1)} \le \|\phi\|_{L_t^\infty(L_x^1)}$. Since $\phi_\varepsilon \in C_c([0,\infty[\times\mathbf{R})$, we have

$$\begin{aligned} |\int_{[0,\infty[}\int_{\mathbf{R}} \phi(t,x)\mu_\varepsilon(t,x)\, dt\, dx| &= |\int_{[0,\infty[}\int_{\mathbf{R}} \phi_\varepsilon(s,y)\, d\mu(s,y)| \\ &\le C\|\phi_\varepsilon\|_{L_t^\infty(L_x^1)} \le C\|\phi\|_{L_t^\infty(L_x^1)}, \end{aligned}$$

thus $\mu_\varepsilon \in \mathcal{ML}$. □

Lemma 6.5 *Let $\mu \in \mathcal{ML}$ and set $\mu_\varepsilon(t,x) = \int_{[0,\infty[}\int_{\mathbf{R}} \zeta_\varepsilon(t-s, x-y)\, d\mu(s,y)$. Then $<\mu_\varepsilon, \varphi> \to <\mu, \varphi>$ as $\varepsilon \to 0$, for any $\varphi \in C_c([0,\infty[, L^1(\mathbf{R}))$.*

Proof. Let $\varphi \in C_c([0,\infty[, L^1_x(\mathbf{R}))$. Thanks to the density of $C_c([0,\infty[\times\mathbf{R})$ in $C_c([0,\infty[, L^1(\mathbf{R}))$ for the norm $\|.\|_{L^\infty_t(L^1_x)}$, there exists $\phi(t,x)$ in $C_c([0,\infty[\times\mathbf{R})$ such that $\|\varphi-\phi\|_{L^\infty_t(L^1_x)} \le \eta$. We write

$$< \mu_\varepsilon, \phi >= \int_{[0,\infty[} \int_{\mathbf{R}} \phi_\varepsilon(s,y)\, d\mu(s,y),$$

where $\phi_\varepsilon(s,y) = \int_0^\infty \int_{\mathbf{R}} \zeta_\varepsilon(t-s,x-y)\phi(t,x)\, dt\, dx$. For $s \ge 0$, $y \in \mathbf{R}$, we have

$$\phi_\varepsilon(s,y) - \phi(s,y) = \int_0^\varepsilon \int_{-\varepsilon}^\varepsilon (\phi(u+s,y+z) - \phi(s,y))\zeta_\varepsilon(u,z))\, du\, dz. \tag{6.3}$$

Thus $\phi_\varepsilon(s,y) \to \phi(s,y)$, as $\varepsilon \to 0$, uniformly for $(s,y) \in [0,\infty[\times\mathbf{R}$, and we get

$$\int_{[0,\infty[} \int_{\mathbf{R}} \phi_\varepsilon(s,y)\, d\mu(s,y) \to \int_{[0,\infty[} \int_{\mathbf{R}} \phi(s,y)\, d\mu(s,y), \quad \varepsilon \to 0.$$

We use now the decomposition

$$< \mu_\varepsilon, \varphi > - < \mu, \varphi >=< \mu_\varepsilon, \varphi-\phi > + < \mu_\varepsilon, \phi > - < \mu, \phi > + < \mu, \phi-\varphi > .$$

We have just proved that $< \mu_\varepsilon, \phi > - < \mu, \phi > \underset{\varepsilon\to 0}{\to} 0$. Now $\phi - \varphi \in C_c([0,\infty[, L^1(\mathbf{R}))$, therefore

$$| < \mu, \phi-\varphi > | \le C\|\phi-\varphi\|_{L^\infty_t(L^1_x)} \le C\eta.$$

Using Lemma 6.4, we have similarly $| < \mu_\varepsilon, \varphi-\phi > | \le C\eta$.
Finally $| < \mu_\varepsilon, \varphi > - < \mu, \varphi > | \le 3C\eta$, for ε small enough, that is our result. □

6.3 Relation $\alpha\pi = \pi$ for solutions with π in the class $\mathcal{ML}$

Proposition 6.6 *If (α, u, π) is a solution to (1.3) -(1.6), (1.12) in the sense of Theorem 1.1 and if we suppose furthermore that $\alpha \in C_t([0,\infty[, L^1_{loc}(\mathbf{R}))$ and $\pi \in \mathcal{ML}$, then the exclusion relation $\alpha\pi = \pi$ holds in the sense of definition 6.1.*

Proof. We set $\alpha_\varepsilon = \alpha * \zeta_\varepsilon$, $(\alpha u)_\varepsilon = (\alpha u) * \zeta_\varepsilon$, $(\alpha u^2)_\varepsilon = (\alpha u^2) * \zeta_\varepsilon$ where $g * \zeta_\varepsilon(t,x) = \int_0^\infty \int_{\mathbf{R}} \zeta_\varepsilon(t-s,x-y)g(s,y)\, ds\, dy$, and $\pi_\varepsilon(t,x) = \int_{[0,\infty[} \int_{\mathbf{R}} \zeta_\varepsilon(t-s,x-y)\, d\pi(s,y)$. We have from (1.16)-(1.17)

$$\left\{ \begin{array}{l} \partial_t \alpha_\varepsilon + \partial_x(\alpha u)_\varepsilon = (\delta(t)\alpha^0(x)) * \zeta_\varepsilon, \\ \partial_t(\alpha u)_\varepsilon + \partial_x((\alpha u^2)_\varepsilon + \pi_\varepsilon) = (\delta(t)\alpha^0(x)u^0(x)) * \zeta_\varepsilon. \end{array} \right. \tag{6.4}$$

We define Q_ε in $]\varepsilon,\infty[\times\mathbf{R}$ by $\partial_x Q_\varepsilon = (\alpha u)_\varepsilon$, $\partial_t Q_\varepsilon = -(\alpha u^2)_\varepsilon - \pi_\varepsilon$. Since all functions are smooth, we have

$$\begin{aligned} \partial_t(\alpha_\varepsilon Q_\varepsilon) + \partial_x((\alpha u)_\varepsilon Q_\varepsilon) &= \alpha_\varepsilon \partial_t Q_\varepsilon + Q_\varepsilon \partial_t \alpha_\varepsilon + (\partial_x(\alpha u)_\varepsilon)Q_\varepsilon + (\partial_x Q_\varepsilon)(\alpha u)_\varepsilon \\ &= -\alpha_\varepsilon(\alpha u^2)_\varepsilon - \alpha_\varepsilon \pi_\varepsilon + (\alpha u)^2_\varepsilon + Q_\varepsilon(\partial_t \alpha_\varepsilon + \partial_x(\alpha u)_\varepsilon), \end{aligned}$$

and thanks to (6.4), we obtain

$$\partial_t(\alpha_\varepsilon Q_\varepsilon) + \partial_x((\alpha u)_\varepsilon Q_\varepsilon) = -\alpha_\varepsilon(\alpha u^2)_\varepsilon - \alpha_\varepsilon\pi_\varepsilon + (\alpha u)^2_\varepsilon. \tag{6.5}$$

From the definition of Q_ε, we have that $(Q_\varepsilon)_{\varepsilon>0}$ is bounded in $BV_{t,x}$. Therefore, passing if necessary to a subsequence and adding eventually a constant, Q_ε converges in L^1_{loc} to some Q satisfying $\partial_x Q = \alpha u$, $\partial_t Q = -\alpha u^2 - \pi$. Then, in (6.5) all the terms pass to the limit trivially except $\alpha_\varepsilon\pi_\varepsilon$. For $\varphi \in \mathcal{D}(]0, +\infty[\times\mathbf{R})$, we get, passing to the limit in (1.18) and using (6.5), that

$$\int_{[0,\infty[}\int_{\mathbf{R}} \varphi\pi = \lim_{\varepsilon\to 0}\int_{[0,\infty[}\int_{\mathbf{R}} \alpha_\varepsilon\pi_\varepsilon\varphi = \lim_{\varepsilon\to 0} < \pi_\varepsilon, \alpha_\varepsilon\varphi > .$$

We can write, since $\pi \in \mathcal{ML}$ and $\alpha \in C([0,\infty[, L^1_{loc}(\mathbf{R}))$,

$$< \pi_\varepsilon, \alpha_\varepsilon\varphi > - < \pi, \alpha\varphi > = < \pi_\varepsilon, (\alpha_\varepsilon - \alpha)\varphi > + < \pi_\varepsilon - \pi, \alpha\varphi > .$$

On one hand, $\alpha\varphi \in C_c([0,\infty[, L^1_x(\mathbf{R}))$ hence, by Lemma 6.5, $< \pi_\varepsilon - \pi, \alpha\varphi > \underset{\varepsilon\to 0}{\to} 0$. On the other hand,

$$\begin{aligned} | < \pi_\varepsilon, (\alpha_\varepsilon - \alpha)\varphi > | &\leq C \sup_t \int_{\mathbf{R}} |(\alpha_\varepsilon - \alpha)\varphi|\, dx \\ &\leq C\|\varphi\|_{L^\infty_{t,x}}\|\alpha_\varepsilon - \alpha\|_{L^\infty_t(L^1_{loc,x})} \to 0 \end{aligned}$$

because $\alpha \in C_t([0,\infty[, L^1_x(\mathbf{R}))$. Therefore we get $\lim_{\varepsilon\to 0} < \pi_\varepsilon, \alpha_\varepsilon\varphi > = < \alpha\pi, \varphi >$. Finally, we have proved that $\int_{[0,\infty[}\int_{\mathbf{R}} \varphi\pi = < \alpha\pi, \varphi >$, $\forall\varphi \in \mathcal{D}(]0, +\infty[\times\mathbf{R})$ and thus $\pi = \alpha\pi$ in $]0, +\infty[\times\mathbf{R}$. We can extend the result to 0 because for $t = 0$, the property is always satisfied according to Proposition 2.7. □

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Chapitre 2

Solutions faibles pour un système hyperbolique avec contrainte unilatérale et perte de masse

Weak solutions for a hyperbolic system with unilateral constraint and mass loss

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Abstract

We consider isentropic gas dynamics equations with unilateral constraint on the density and mass loss. The γ and pressureless pressure laws are considered. We propose an entropy weak formulation of the system that incorporates the constraint and Lagrange multiplier, for which we prove weak stability and existence of solutions. The nonzero pressure model is approximated by a kinetic BGK relaxation model, while the pressureless model is approximated by a sticky-blocks dynamics with mass loss. It is proved that the exclusion relation is lost in weak limits for the pressureless model.

Key-words: conservation laws with constraint – mass loss – entropy weak produc – pressureless gas – sticky blocks

Mathematics Subject Classification: 76T10, 35L65, 35L85, 76N15, 35A35

Contents

1 Introduction and models

1.1 The models

The aim of this paper is to introduce a weak formulation and establish weak stability and existence for solutions to some one-dimensional systems of conservations laws with unilateral constraint. Such system arises for example in the modeling of two-phase flows, see [9], as

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho) + \pi) = 0, \end{cases} \tag{1.1}$$

with constraint and pressure Lagrange multiplier

$$0 \leq \rho \leq 1, \qquad \pi \geq 0, \tag{1.2}$$

and exclusion relation

$$(1-\rho)\pi = 0. \tag{1.3}$$

This system was studied in [21] with viscosity. Existence and weak stability of suitable weak solutions is obtained in [2] in the pressureless case $p(\rho) = 0$, but however, the general nonzero pressure case remains open.

Here we are going to consider a slightly different model with mass loss, that can be written as

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = Q, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = Qu, \end{cases} \tag{1.4}$$

with constraint and mass loss rate Lagrange multiplier

$$0 \leq \rho \leq 1, \qquad Q \leq 0, \tag{1.5}$$

and exclusion relation

$$(1-\rho)Q = 0. \tag{1.6}$$

This model is based on the physical idea to remove from the density ρ what overflows with 1 as rain could make overflow a reservoir or a river. The term Qu on the right-hand side of the momentum equation expresses that the overflowing matter travels at velocity u. This interpretation is especially relevant for the Saint-Venant equations when $p(\rho) = \kappa\rho^2$. We are going to consider here pressure laws of the form

$$p(\rho) = \kappa\rho^\gamma, \tag{1.7}$$

with $1 < \gamma \leq 3$, and $\kappa \geq 0$ ($\kappa = 0$ in the pressureless case).

The main difficulty in this model is to give a suitable sense to the products Qu in (1.4) and ρQ in (1.6), because Q is only a measure, and ρ, u can be discontinuous. Indeed, since Q can be nonzero only where $\rho = 1$, we have formal formulas obtained by frozing $\rho = 1$ in (1.4),

$$Q = \mathbb{1}_{\rho=1}\partial_x u, \qquad Qu = \mathbb{1}_{\rho=1}(\partial_t u + \partial_x u^2), \tag{1.8}$$

but of course this is again meaningless. Therefore, we provide the following entropy weak formulation of the problem, that involves what we call entropy weak products. The idea is to introduce a different velocity $v(t,x)$ for the lost matter, and write the system

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = Q, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = Qv, \end{cases} \tag{1.9}$$

with as before

$$0 \le \rho \le 1, \qquad Q \le 0. \tag{1.10}$$

We take $v \in L^\infty(Q)$, so that the product Qv is well-defined as a measure. We need then to formulate in a weak sense that $Qv = Qu$, and that $Q\rho = Q$. In order to do so, we require the family of entropy inequalities

$$\partial_t\, \eta_S(\rho,u) + \partial_x\, G_S(\rho,u) \le Q\, \eta_S'(1,v)\cdot(1,v), \tag{1.11}$$

for any convex entropy η_S in a suitable family parametrized by a convex function S, where G_S is its entropy flux, and η_S' is its derivative with respect to $(\rho,\rho u)$. Since v, by definition, is defined Q a.e., the term on the right-hand side of (1.11) is well-defined. In order to see that (1.9)-(1.11) is a weak formulation of (1.4)-(1.6), we observe first that any suitable solution to (1.4)-(1.6) also solves (1.9)-(1.11) with $v = u$. Conversely, if we have a sufficiently smooth solution to (1.9)-(1.11), then multiplying (1.9) by $\eta_S'(\rho,u)$ and comparing with (1.11), we get

$$Q\, \eta_S'(\rho,u)\cdot(1,v) \le Q\, \eta_S'(1,v)\cdot(1,v). \tag{1.12}$$

If we take for the entropy the physical energy

$$\eta(\rho,u) = \rho u^2/2 + \frac{\kappa}{\gamma-1}\rho^\gamma, \tag{1.13}$$

we have $\eta'(\rho,u) = (\gamma\kappa\rho^{\gamma-1}/(\gamma-1) - u^2/2, u)$, and (1.12) gives

$$Q\left[\frac{\gamma\kappa}{\gamma-1}(\rho^{\gamma-1}-1) - (v-u)^2/2\right] \le 0. \tag{1.14}$$

Together with the constraints (1.10), we deduce that $Qv = Qu$ and $Q\rho = Q$, except in the pressureless case $\kappa = 0$, in which we can only conclude $Qv = Qu$.

Our main result is that the entropy formulation of the system (1.9)-(1.11) is weakly stable. We are able to prove a priori estimates and compactness of suitable approximations, that lead to existence for the Cauchy problem. For the nonzero pressure model, the approximate solutions are obtained by a kinetic BGK equation with additional projection to enforce the constraint $\rho \le 1$. For the pressureless model, the approximation is based on the notion of sticky blocks that has been introduced in [9] and used in [2], but here with a different dynamics based on mass loss.
We look for solutions with regularities

$$\rho \in L_t^\infty(0,\infty; L_x^\infty(\mathbf{R}) \cap L_x^1(\mathbf{R})), \tag{1.15}$$

$$u \in L_t^\infty(0,T; L_x^\infty(\mathbf{R})), \tag{1.16}$$

$$Q \in \mathcal{M}_{loc}([0,\infty[\times\mathbf{R}), \qquad Qv \in \mathcal{M}_{loc}([0,\infty[\times\mathbf{R}). \tag{1.17}$$

Thus, we consider weak solutions: for all $\varphi \in \mathcal{D}([0,+\infty[\times\mathbf{R})$,

$$\int_{[0,+\infty[}\int_{\mathbf{R}}[\rho\partial_t\varphi+\rho u\partial_x\varphi]\,dx\,dt+\int_{\mathbf{R}}\rho^0(x)\varphi(0,x)\,dx=-\int_{[0,+\infty[}\int_{\mathbf{R}}\varphi Q\ , \quad (1.18)$$

$$\int_{[0,+\infty[}\int_{\mathbf{R}}[\rho u\partial_t\varphi+(\rho u^2+p(\rho))\partial_x\varphi]\,dx\,dt+\int_{\mathbf{R}}(\rho^0u^0)(x)\varphi(0,x)\,dx=-\int_{[0,+\infty[}\int_{\mathbf{R}}\varphi Qu\ . \quad (1.19)$$

It allows us to consider initial data

$$\rho(0,x)=\rho^0(x),\ \ \rho(0,x)u(0,x)=\rho^0(x)u^0(x). \quad (1.20)$$

We notice that a solution satisfying (1.11) is an entropy solution thanks to the Lemma:

Lemma 1.1 *The function $v\mapsto\eta'_S(1,v)\cdot(1,v)$ is convex for $S:\mathbf{R}\to\mathbf{R}$ convex. Furthermore, it is a nonnegative function as soon as $S\geq 0$.*

1.2 The results

We formule our main results. The first one is a compactness result and is valid for the two models.

Theorem 1.2 *Let us consider a sequence of solutions (ρ_n,u_n,Q_n,v_n) with regularities (1.15)-(1.17) satisfying (1.18)-(1.19) with the constraints (1.10)-(1.11). Initial data ρ_n^0, u_n^0 are supposed to satisfy*

$$0\leq\rho_n^0\leq 1,\qquad(\rho_n^0)_{n\geq0}\ \textit{is bounded in}\ L^1(\mathbf{R}), \quad (1.21)$$

$$(u_n^0)_{n\geq0}\ \textit{is bounded in}\ L^\infty(\mathbf{R}). \quad (1.22)$$

We also suppose that (ρ_n,u_n,Q_n,v_n) satisfies

$$(Q_n)_{n\geq0}\ \textit{is bounded in}\ \mathcal{M}_{loc}([0,\infty[\times\mathbf{R}),\ \textit{and}\ (\|v_n\|_{L^\infty(Q_n)})_{n\geq0}\ \textit{is bounded}\ , \quad (1.23)$$

and, in the pressureless case, that the Oleinik relation (1.27) holds. Then, up to a subsequence, as $n\to\infty$, $(\rho_n,u_n,Q_n,Q_nv_n)\rightharpoonup(\rho,u,Q,Qv)$ in the following sense

$$\rho_n\rightharpoonup\rho,\qquad u_n\rightharpoonup u\qquad \textit{in}\ L^\infty_{w*}(]0,\infty[\times\mathbf{R}), \quad (1.24)$$

$$Q_n\rightharpoonup Q,\qquad Q_nv_n\rightharpoonup Qv\qquad \textit{in}\ \mathcal{M}_{loc}([0,\infty[\times\mathbf{R})w*, \quad (1.25)$$

where (ρ,u,Q,v) is a solution of (1.18)-(1.19) with the constraints (1.10)-(1.11) with initial data ρ^0, u^0 defined by

$$\rho_n^0\rightharpoonup\rho^0\ \textit{in}\ L^\infty_{w*}(\mathbf{R}),\quad \textit{and}\quad \rho_n^0u_n^0\rightharpoonup m^0\equiv\rho^0u^0\ \textit{in}\ L^\infty_{w*}(\mathbf{R}). \quad (1.26)$$

In the pressureless case, we also get (1.27). Furthermore, in this case, if we suppose that the sequence of solutions satisfies the maximum principle (1.28), then the limit also satisfies it.

We turn now to existence results. For the isentropic model with nonzero pressure, we have:

Theorem 1.3 *Let $\rho^0 \in L^1(\mathbf{R})$ such that $0 \leq \rho^0 \leq 1$ and $u^0 \in L^\infty(\mathbf{R})$, then there exists (ρ, u, Q, v) with regularities (2.1)-(2.3) satisfying (1.18)-(1.19) with the constraints (1.10)-(1.11).*

In the pressureless context, we have the following results:

Theorem 1.4 *Let $\rho^0 \in L^1(\mathbf{R})$ such that $0 \leq \rho^0 \leq 1$ and $u^0 \in L^\infty(\mathbf{R})$, then there exists (ρ, u, Q, v) with regularities (1.15) -(1.17) satisfying (1.18)-(1.19) with the constraints (1.10)-(1.11).*

Theorem 1.5 *In addition, the solution (ρ, u, Q, v) of Theorem 1.4 satisfies*

$$\partial_x u(t,x) \leq \tfrac{1}{t}, \tag{1.27}$$

$$\operatorname{essinf} u^0(x) \leq u(t,x) \leq \operatorname{esssup} u^0(x), \tag{1.28}$$

$$TV_{[a,b]}(u(t,.)) \leq \tfrac{2(b-a)}{t} + 2\|u^0\|_{L^\infty} \quad \forall a < b, \tag{1.29}$$

$$\partial_t(\rho S(u)) + \partial_x(\rho u S(u)) = Q_S \ \textit{in}\ ]0,\infty[\times\mathbf{R} \tag{1.30}$$

for every $S \in C^0(\mathbf{R})$, where $Q_S \in \mathcal{M}_{loc}([0,\infty[\times\mathbf{R})$ satisfies

$$|Q_S| \leq \|S\|_{L^\infty}|Q|. \tag{1.31}$$

In Section 2, we prove the existence of solutions for a BGK model which relaxation leads to the nonzero pressure case. We also prove entropy inequalities and the existence of kinetic invariant domains. We prove the convergence, when $\varepsilon \to 0$, towards the nonzero pressure gas dynamics model with loss of mass. Then we prove the weak stability of the solutions using compensated compactness. Finally, we study the relaxation in the particular case $\gamma = 3$ using averaging lemma. In Section 3, we explain the sticky blocks dynamics which fits in with the pressureless model. We also give inequalities which generalizes the entropy ones that hold for pressureless gases. Finally, we prove that the (strong) exclusion relation is lost in weak limits for the pressureless model. In Section 4, we propose a numerical simulation in the pressureless context.

For kinetic invariant domains, we follow [24]. In [4], kinetic invariant domains for $1 < \gamma < 3$ was studied for the first time. We refer to [1], [20], [17], [21] and [2] for other hyperbolic problems with constraints. For some formulations for such problems, we refer to [14]. For averaging lemma with measure and kinetic equations, we refer to [15], [12], [23], [25] and [26]. Existence and properties of the system of pressureless gas have been studied in [19], [10], [13], [16] and [11]. For more details about BGK models, we refer to [6] and [7]. Finally, for numerical schemes, we refer to [18] and [7].

2 Isentropic model with nonzero pressure

In this section, after the presentation of the BGK models we use, we study in detail the results we get at this level, then at the limit of the relaxation parameter $\varepsilon \to 0$. First at all, existence of solutions for the BGK model (2.4) is proved, then thanks to kinetic invariant domains, the relaxation of the model towards an entropy solution of (1.9) is obtained. The compactness of ρ_ε and $\rho_\varepsilon u_\varepsilon$ is obtained via compensated compactness for $1 < \gamma \leq 3$. For the special case $\gamma = 3$, we can rather use averaging lemma.

2.1 The BGK isentropic model with nonzero pressure

We consider here a pressure law of the form $p(\rho) = \kappa\rho^\gamma$ with $\kappa > 0$, $1 < \gamma \leq 3$. We look for solutions with the regularities

$$\rho \in (C \cup L^\infty)_t(0, \infty; L^\infty_x(\mathbf{R}) \cap L^1_x(\mathbf{R})), \tag{2.1}$$

$$u \in L^\infty_t(0, T; L^\infty_x(\mathbf{R})), \qquad \rho u \in C_t(0, \infty; L^1_x(\mathbf{R})), \tag{2.2}$$

$$Q \in \mathcal{M}_{loc}([0, \infty[\times\mathbf{R}), \qquad Qv \in \mathcal{M}_{loc}([0, \infty[\times\mathbf{R}). \tag{2.3}$$

Nevertheless, for the stability result of solutions, we will not take into account the continuity in time which can be lost for ρu.
The model is studied as a relaxation of the following kinetic one which accounts for the projection done at the numerical level (see Section 4),

$$\partial_t f + \xi \partial_x f = \frac{M^*[f] - f}{\varepsilon} \quad \text{in }]0, \infty[\times\mathbf{R} \times \mathbf{R}, \tag{2.4}$$

where $f = f(t, x, \xi) \in \mathbf{R}^2$, $t > 0$, $x \in \mathbf{R}$, $\xi \in \mathbf{R}$,

$$f(t, x, \xi) \in D_\xi, \tag{2.5}$$

$$M^*[f](t, x, \xi) = M^* \left(\rho(t, x), u(t, x), \xi\right), \tag{2.6}$$

$$\rho(t, x) = \int_{\mathbf{R}} f_0(t, x, \xi)\, d\xi, \;\; \rho(t, x)u(t, x) = \int_{\mathbf{R}} f_1(t, x, \xi)\, d\xi, \tag{2.7}$$

$$M^*(\rho, u, \xi) = M(\min(1, \rho), u, \xi), \tag{2.8}$$

where M is the Maxwellian.
We have existence of a global solution to this system with initial data

$$f(0, x, \xi) = f^0(x, \xi), \tag{2.9}$$

satisfying energy bounds.
Given some kinetic entropies $H_S(f, \xi)$ for the kinetic system, the entropy and entropy flux are defined by

$$\eta_S(\rho, u) = \int_{\mathbf{R}} H_S(M(\rho, u, \xi), \xi)\, d\xi, \qquad G_S(\rho, u) = \int_{\mathbf{R}} \xi\, H_S(M(\rho, u, \xi), \xi)\, d\xi. \tag{2.10}$$

Following [22], we define entropy solutions to (1.9) as solutions with finite energy, i.e. $\sup_t \int \eta(\rho(t, x), u(t, x))\, dx < \infty$, which satisfy the entropy inequalities

$$\partial_t(\eta_S) + \partial_x(G_S) \leq 0, \tag{2.11}$$

parametrized by a convex function S with subquadratic growth.
We have to notice that [6] suggest the natural model $\partial_t f + \xi\partial_x f = \frac{M[f]-f}{\varepsilon}$, here we replace this model by (2.4) in order to force $\rho \leq 1$ at the relaxation limit.
For $1 < \gamma < 3$, we take

$$D_\xi = D = \{(f_0, f_1) \in \mathbf{R}^2, \;\; f_0 > 0 \text{ or } f_1 = f_0 = 0\}, \tag{2.12}$$

$$M(\rho,u,\xi) = \Big(\chi(\rho,\xi-u), ((1-\theta)u+\theta\xi)\chi(\rho,\xi-u)\Big), \tag{2.13}$$

$$\chi(\rho,\xi) = c_{\gamma,\kappa}\left(a_\gamma^2\rho^{\gamma-1}-\xi^2\right)_+^\lambda, \tag{2.14}$$

$$\theta = \frac{\gamma-1}{2}, \quad \lambda = \frac{1}{\gamma-1}-\frac{1}{2}, \quad c_{\gamma,\kappa} = a_\gamma^{-2/(\gamma-1)}/J_\lambda, \tag{2.15}$$

$$J_\lambda = \int_{-1}^{1}(1-z^2)^\lambda\,dz = \sqrt{\pi}\Gamma(\lambda+1)/\Gamma(\lambda+3/2), \quad a_\gamma = \frac{2\sqrt{\gamma\kappa}}{\gamma-1}. \tag{2.16}$$

The kinetic entropies are parametrized by convex functions S and are defined from a kernel by

$$H_S(f,\xi) = \int_{\mathbf{R}}\Phi(\rho(f,\xi),u(f,\xi),\xi,v)S(v)\,dv, \text{ for } f\neq 0, \quad H_S(0,\xi)=0, \tag{2.17}$$

where

$$u(f,\xi) = \frac{f_1/f_0-\theta\xi}{1-\theta}, \quad \rho(f,\xi) = a_\gamma^{-\frac{2}{\gamma-1}}\left(\left(\frac{f_1/f_0-\xi}{1-\theta}\right)^2 + (f_0/c_{\gamma,\kappa})^{1/\lambda}\right)^{\frac{1}{\gamma-1}} \tag{2.18}$$

is the inverse relation for $f = M(\rho,u,\xi)$. The kernel Φ is symmetric in ξ, v and satisfies in particular $\Phi \geq 0$ and $\int_{\mathbf{R}}(1,v)\cdot\Phi(\rho,u,\xi,v)\,dv = M(\rho,u,\xi)$. For more details about this model, we refer to [4].
For $\gamma=3$, we take

$$D_\xi = \{f=(f_0,\xi f_0)\in\mathbf{R}^2;\ 0\leq f_0(\xi)\leq \frac{1}{2\sqrt{3\kappa}}\}, \tag{2.19}$$

the Maxwellian is

$$M(\rho,u,\xi) = \frac{1}{2\sqrt{3\kappa}}\mathbb{I}_{(\xi-u)^2<3\kappa\rho^2}K(\xi), \qquad K(\xi)=(1,\xi), \tag{2.20}$$

and the kinetic entropy, for S convex, is given for $f=(f_0,\xi f_0)$, by

$$H_S(f,\xi) = S(\xi)f_0. \tag{2.21}$$

We will prove that the following sets are kinetic invariant domains for (2.4) and invariant domains for (1.9) : in the case $1<\gamma<3$,

$$\tilde{D}_\xi = \{f\in D;\ f=0 \text{ } or \text{ } \omega_{min}\leq\omega_1(f,\xi)\leq\omega_2(f,\xi)\leq\omega_{max}\}, \tag{2.22}$$

where

$$\omega_1(f,\xi) = u(f,\xi)-a_\gamma\rho(f,\xi)^{\frac{\gamma-1}{2}},\ \omega_2(f,\xi) = u(f,\xi)+a_\gamma\rho(f,\xi)^{\frac{\gamma-1}{2}}, \tag{2.23}$$

and

$$\tilde{D} = \{(\rho,u)\in\mathbf{R}^+\times\mathbf{R};\ \rho=0 \text{ } or \text{ } \omega_{min}\leq\omega_1\leq\omega_2\leq\omega_{max}\}, \tag{2.24}$$

where the Riemann invariants ω_1, ω_2 are given by

$$\omega_1 = u - a_\gamma \rho^{\frac{\gamma-1}{2}},\ \omega_2 = u + a_\gamma \rho^{\frac{\gamma-1}{2}}, \tag{2.25}$$

and in the case $\gamma = 3$,

$$\tilde{D}_\xi = \{f \in D_\xi;\ 0 \le f_0(\xi) \le \frac{1}{2\sqrt{3\kappa}} \mathbb{1}_{\omega_{min} < \xi < \omega_{max}}\}, \tag{2.26}$$

and

$$\tilde{D} = \{(\rho, u) \in \mathbf{R}^+ \times \mathbf{R};\ \rho = 0 \text{ or } \omega_{min} \le u - \sqrt{3\kappa}\rho \le u + \sqrt{3\kappa}\rho \le \omega_{max}\}. \tag{2.27}$$

2.2 Properties of the kinetic entropy

We first notice that the Lemma 1.1 becomes clear when we express the desired quantity, that is for $1 < \gamma < 3$,

$$\eta'_S(1, v) \cdot (1, v) = a_\gamma^2 (1 - \theta) c_{\gamma,\kappa} \int_{\mathbf{R}} (a_\gamma^2 - (v - \sigma)^2)_+^{\lambda - 1} S(\sigma)\, d\sigma,$$

and for $\gamma = 3$,

$$\eta'_S(1, v) \cdot (1, v) = (S(v + \sqrt{3\kappa}) + S(v - \sqrt{3\kappa}))/2.$$

The following properties are valid for the two cases: $1 < \gamma < 3$ and $\gamma = 3$. In sections 2.2-2.4, we give the proof only for the second case; in the first one, proofs can be found in [3] and [4]. We recall the value of the moments of M,

$$\int_{\mathbf{R}} M(\rho, u, \xi)\, d\xi = (\rho, \rho u), \qquad \int_{\mathbf{R}} \xi M(\rho, u, \xi)\, d\xi = (\rho u, \rho u^2 + \kappa \rho^\gamma),$$

and

$$\int_{\mathbf{R}} \frac{1}{2} \xi^2 M_0(\rho, u, \xi)\, d\xi = \frac{1}{2} \rho u^2 + \frac{\kappa}{\gamma - 1} \rho^\gamma = \eta(\rho, u),$$

for every $\rho \ge 0$ and $u \in \mathbf{R}$. We get easily the following majoration.

Lemma 2.1 *For every $\rho \ge 0$ and $u, \xi \in \mathbf{R}$, we have $0 \le M_0^*(\rho, u, \xi) \le M_0(\rho, u, \xi)$.*

The kinetic entropy associated to the physical energy is noted by $H = H_{v^2/2}$. As in [3], we need a convexity inequality in order to prove stability and existence, namely

Proposition 2.2 (Subdifferential inequality) *For any $f \in D_\xi$, $\rho \ge 0$ and $u, \xi \in \mathbf{R}$, we have*

$$H(f, \xi) \ge H(M(\rho, u, \xi), \xi) + \left(\frac{\gamma\kappa}{\gamma - 1} \rho^{\gamma - 1} - \frac{u^2}{2}, u \right) \cdot (f - M(\rho, u, \xi)), \tag{2.28}$$

and

$$(H'(f,\xi) - (\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - \frac{u^2}{2}, u)) \cdot (f - M(\rho,u,\xi)) \geq 0,$$

which is, for $\gamma = 3$,

$$[(u-\xi)^2 - 3\kappa\rho^2][f_0 - M_0(\rho,u,\xi)] \geq 0.$$

Proof. We compute that

$$\begin{aligned} A &= H(f,\xi) - H(M(\rho,u,\xi),\xi) - \left(\frac{3\kappa}{2}\rho^2 - \frac{u^2}{2}, u\right) \cdot (f - M(\rho,u,\xi)) \\ &= \frac{1}{2}[(u-\xi)^2 - 3\kappa\rho^2][f_0 - M_0(\rho,u,\xi)]. \end{aligned}$$

If $(u-\xi)^2 \geq 3\kappa\rho^2$, then $M_0(\rho,u,\xi) = 0$, thus $A \geq 0$. If $(u-\xi)^2 < 3\kappa\rho^2$, then $M_0(\rho,u,\xi) = \frac{1}{2\sqrt{3\kappa}} \geq f_0$, thus $A \geq 0$ also. □
Using the two previous results, we get the decrease of the kinetic entropy.

Proposition 2.3 *Consider $f \in L^1(\mathbf{R}_\xi)$ such that $f \in D_\xi$ a.e. and*

$$\int_{\mathbf{R}} H(f(\xi),\xi)\, d\xi < \infty.$$

Then, setting $(\rho, \rho u) = \int_{\mathbf{R}} f\, d\xi$, we have

$$\int_{\mathbf{R}} H(M^*[f](\xi),\xi)\, d\xi + (\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} + u^2/2)(\rho-1)_+ \leq \int_{\mathbf{R}} H(f(\xi),\xi)\, d\xi.$$

2.3 Stability and existence for the BGK model

In the previous subsection, we got sufficient bounds so as to proceed as in [3] in order to apply Schauder's theorem and to get the existence of global solutions. We notice that proofs are simplified for $\gamma = 3$ because $H(.) = \xi^2/2\ (.)$ is linear and because the kinetic system becomes in fact a rank-one model, and as a consequence we mainly only study the convergence of the first component which is positive. We thus get the following result.

Theorem 2.4 *Assume that $f^0 \in L^1(\mathbf{R}_x \times \mathbf{R}_\xi)$ satisfies $f^0(x,\xi) \in D_\xi$ a.e. in $\mathbf{R} \times \mathbf{R}$ and*

$$\iint_{\mathbf{R}\times\mathbf{R}} H(f^0(x,\xi),\xi)\, dx d\xi = C_H^0 < \infty. \tag{2.29}$$

Then there exists a solution f to (2.4)-(2.9) satisfying

$$f \in (C \cap L^\infty)_t([0,\infty[, L^1(\mathbf{R}_x \times \mathbf{R}_\xi)), \tag{2.30}$$

$$\forall t \geq 0,\ f(t,x,\xi) \in D_\xi \text{ a.e. in } \mathbf{R}\times\mathbf{R}, \tag{2.31}$$

$$\forall t \geq 0,\ \iint_{\mathbf{R}\times\mathbf{R}} f(t,x,\xi)\, dx d\xi \leq \iint_{\mathbf{R}\times\mathbf{R}} f^0(x,\xi)\, dx d\xi, \tag{2.32}$$

$$\partial_t(\int_{\mathbf{R}} f_0\, d\xi) + \partial_x(\int_{\mathbf{R}} \xi f_0\, d\xi) = \frac{1}{\varepsilon}\int_{\mathbf{R}}(M^*[f]_0 - f_0)\, d\xi \equiv Q \le 0, \tag{2.33}$$

$$\begin{aligned} & \frac{d}{dt}\iint\limits_{\mathbf{R}\times\mathbf{R}} H(f(t,x,\xi),\xi)\, dx d\xi \\ = \quad & -\frac{1}{\varepsilon}\iint\limits_{\mathbf{R}\times\mathbf{R}} (H'(f,\xi) - (\frac{\gamma\kappa}{\gamma-1}(\rho^*)^{\gamma-1} - \frac{u^2}{2}, u))\cdot(f - M^*[f])\, dx d\xi \\ & -\frac{1}{\varepsilon}(\frac{\gamma\kappa}{\gamma-1}(\rho^*)^{\gamma-1} + \frac{u^2}{2})(\rho-1)_+ \\ \le \quad & 0, \end{aligned} \tag{2.34}$$

where $(\rho, \rho u) = \int_{\mathbf{R}} f\, d\xi$ *and* $\rho^* = \min(1, \rho)$.

We notice that we have a representation of the solution f,

$$f(t,x,\xi) = f^0(x - t\xi, \xi)e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} M^*[f](t-s, x-s\xi, \xi)\, ds. \tag{2.35}$$

2.4 Kinetic invariant domains

It results from the definition of a convex function that

Lemma 2.5 *For every* $S : \mathbf{R} \to \mathbf{R}$ *convex, we have*

$$(\omega_2-\omega_1)S(\xi)+(\omega_1-\xi)S(\omega_2)+(\xi-\omega_2)S(\omega_1) \begin{cases} \ge 0 \text{ if } \omega_2 \ge \omega_1 \ge \xi \text{ or } \xi \ge \omega_2 \ge \omega_1, \\ \le 0 \text{ if } \omega_2 > \xi > \omega_1. \end{cases}$$

We generalize now the subdifferential inequality.

Proposition 2.6 *If* $S : \mathbf{R} \to \mathbf{R}$ *is convex, then for every* $f \in D_\xi$, $\rho \ge 0$ *and* $u, \xi \in \mathbf{R}$, *we have*

$$H_S(f,\xi) \ge H_S(M(\rho,u,\xi),\xi) + T_S(\rho,u)\cdot(f - M(\rho,u,\xi)), \tag{2.36}$$

with $T_S(0,u) = 0$ *and for* $\rho \ne 0$,

$$T_S(\rho,u) = \frac{1}{2\sqrt{3\kappa}\rho}\begin{pmatrix} (\sqrt{3\kappa}\rho - u)S(u+\sqrt{3\kappa}\rho) + (\sqrt{3\kappa}\rho + u)S(u - \sqrt{3\kappa}\rho) \\ S(u+\sqrt{3\kappa}\rho) - S(u-\sqrt{3\kappa}\rho) \end{pmatrix}, \tag{2.37}$$

which coincides with $\eta_S'(\rho,u)$ *when* $\rho > 0$, *the prime denotes differentiation with respect to the conservative variables* $(\rho, q \equiv \rho u)$.

Proof. We set $\omega_1 = u - \sqrt{3\kappa}\rho$, $\omega_2 = u + \sqrt{3\kappa}\rho$, we have $\omega_2 - \omega_1 = 2\sqrt{3\kappa}\rho \ge 0$. We compute that

$$\begin{aligned} A &= H_S(f,\xi) - H_S(M(\rho,u,\xi),\xi) - T_S(\rho,u)\cdot(f - M(\rho,u,\xi)) \\ &= [f_0 - M_0(\rho,u,\xi)][(\omega_2-\omega_1)S(\xi) + (\omega_1-\xi)S(\omega_2) + (\xi-\omega_2)S(\omega_1)]. \end{aligned}$$

Proceeding as in Proposition 2.2 and thanks to Lemma 2.5, we get the result. □
For S of class C^1, a more precise relation holds at 0, namely we can take

$$T_S(0,u) = \begin{pmatrix} S(u) - uS'(u) \\ S'(u) \end{pmatrix},$$

because here

$$H_S(f,\xi) - T_S(0,u)\cdot f = (S(\xi) - S(u) - (\xi-u)S'(u))f_0 \geq 0.$$

Two important consequences of Proposition 2.6 are the following results.

Corollary 2.7 *Assume that $S : \mathbf{R} \to \mathbf{R}$ is convex and such that $0 \leq S(v) \leq B(1+v^2)$ for some $B \geq 0$. Consider $f \in L^1(\mathbf{R}_\xi)$ such that $f \in D_\xi$ a.e. and*

$$\int_{\mathbf{R}} H(f(\xi),\xi)\, d\xi < \infty. \tag{2.38}$$

Then $H_S(f(\xi),\xi)$ and $H_S(M^*[f](\xi),\xi)$ lie in $L^1(\mathbf{R}_\xi)$, and setting $(\rho,\rho u) = \int_{\mathbf{R}} f\, d\xi$ we have

$$\int_{\mathbf{R}} H_S(M^*[f](\xi),\xi)\, d\xi + T_S(\min(1,\rho),u)\cdot(1,u)(\rho-1)_+ \leq \int_{\mathbf{R}} H_S(f(\xi),\xi)\, d\xi. \tag{2.39}$$

Corollary 2.8 *Assume that $S : \mathbf{R} \to \mathbf{R}$ is convex and such that $0 \leq S(v) \leq B(1+v^2)$ for some $B \geq 0$. Then, with f the solution of Theorem 2.4,*

$$\iint_{\mathbf{R}\times\mathbf{R}} H_S(f(t,x,\xi),\xi)\, dx d\xi \leq \iint_{\mathbf{R}\times\mathbf{R}} H_S(f^0(x,\xi),\xi)\, dx d\xi. \tag{2.40}$$

These results allow to obtain invariant domains and bounds for ρ_ε, u_ε, f_ε and $M[f_\varepsilon]$.

Theorem 2.9 *For any $\omega_{min} < \omega_{max}$, the system (2.4) has the property that $\tilde{D}_\xi$ is a family of convex kinetic invariant domains. Moreover, the set $\tilde{D}_\xi$ is associated with the invariant domain $\tilde{D}$ in the sense that*

$$\forall (\rho,u) \in \tilde{D},\ \ M(\rho,u,\xi) \in \tilde{D}_\xi \ \text{a.e.}\xi, \tag{2.41}$$

and

$$\text{for any } f(\xi) \in L^1(\mathbf{R}_\xi) \text{ such that } f(\xi) \in \tilde{D}_\xi \text{ a.e. } \xi,$$
$$(\rho,u) \in \tilde{D} \text{ with } (\rho,\rho u) = \int_{\mathbf{R}} f(\xi)\, d\xi. \tag{2.42}$$

In particular, if the initial data of Theorem 2.4 satisfies $f^0(x,\xi) \in \tilde{D}_\xi$ for a.e. x, ξ, then, denoting by f the solution obtained in Theorem 2.4, (ρ,u) defined by (2.7) verify $\forall t \geq 0$, $(\rho(t,x),u(t,x)) \in \tilde{D}$ for a.e. x.
Besides $f(t,x,\xi) \in \tilde{D}_\xi$ $\forall t \geq 0$ and consequently f has compact support with respect to ξ, $supp_\xi f \subset [\omega_{min},\omega_{max}]$.
Furthermore $\rho_\varepsilon, u_\varepsilon, f_\varepsilon, M[f_\varepsilon]$ and $M^*[f_\varepsilon]$ are uniformy bounded in L^∞.

Proof. Using the functions $S(v) = (v-\omega_{max})_+^2$, $S(v) = (\omega_{min} - v)_+^2$ and the fact that $\mathbb{1}_{(\xi-u)^2<3\kappa\rho^2} \leq \mathbb{1}_{\omega_{min}<\xi<\omega_{max}} \Leftrightarrow]u-\sqrt{3\kappa}\rho, u+\sqrt{3\kappa}\rho[\subset]\omega_{min},\omega_{max}[$, an adaptation of the proof of a similar result in [4] gives the result. Uniform bound for $f_1 = \xi f_0$ comes from the compact support with respect to ξ. □

2.5 Stability via compensated compactness

In this Section, we prove the stability's Theorem 1.2 in the case of nonzero pressure and the existence's Theorem 1.4. We first prove the result about convergence of the entropy weak products.

Lemma 2.10 *Let Q_n be nonpositive measures and $v_n \in L^\infty(Q_n)$. If $(Q_n)_{n\geq 0}$ is a sequence bounded in $\mathcal{M}_{loc}([0,\infty[\times\mathbf{R})$ and $(\|v_n\|_{L^\infty(Q_n)})_{n\geq 0}$ is bounded, then there exists a measure Q and a function $v \in L^\infty(Q)$ such that $Q_n \rightharpoonup Q$, $Q_n v_n \rightharpoonup Qv$ and $Q_n\varphi(v_n) \rightharpoonup Q^\varphi \leq Q\varphi(v)$.*

Proof. Measures Q_n are bounded in $\mathcal{M}_{loc}$, thus for a subsequence $Q_n \rightharpoonup Q$ in $\mathcal{M}_{loc}w*$. By diagonal extraction, there exists a subsequence such that

$$Q_n\varphi(v_n) \rightharpoonup Q^\varphi$$

for every φ continuous. We have $|Q_n v_n| \leq C|Q_n|$, thus at the limit $|Q^{Id}| \leq -CQ$ and therefore there exists $v \in L^\infty(Q)$ such that $Q^{Id} = Qv$.
We compare now Q^φ and $Q\varphi(v)$ for φ convex. A convex function φ can be written as

$$\varphi(v) = \sup\{av + b\ ;\ a, b \text{ such that } \varphi \geq aId + b\}.$$

Let φ be a convex function and let $a, b \in \mathbf{R}$ such that $\varphi \geq aId + b$. The measure Q_n is nonpositive thus $Q_n\varphi(v_n) \leq Q_n(av_n + b)$, which gives at the limit $Q^\varphi \leq Q(av + b)$. Now, for a fixed value of v, there exists a, b such that the value of $\varphi(v)$ is reach by $av+b$ with $\varphi \geq aId+b$ and it gives $Q^\varphi \leq Q\varphi(v)$. □

Proof of Theorem 1.2 for nonzero presssure.
Let (ρ_n, u_n, Q_n, v_n) satisfy the assumption of Theorem 1.2. The relation

$$\partial_t\eta_S(\rho_n, u_n) + \partial_x G_S(\rho_n, u_n) \leq Q\eta_S'(1, v_n)\cdot(1, v_n), \tag{2.43}$$

with Lemma 1.1, gives

$$\frac{d}{dt}\int_{\mathbf{R}} \eta_S(\rho_n, u_n)\,dx \leq 0. \tag{2.44}$$

The sequences $(\rho_n^0)_{n\geq 0}$ and $(u_n^0)_{n\geq 0}$ are bounded in L^∞, thus there exists ω_{min} and ω_{max} such that $(\rho_n^0, u_n^0) \in \tilde{D}$ for all n. The decreasing (2.43) with $S(v) = (v - \omega_{max})_+^2$, $S(v) = (\omega_{min} - v)_+^2$ yields the maximum principle

$$\omega_{min} \leq u_n - a_\gamma\rho_n^\theta \leq u_n + a_\gamma\rho_n^\theta \leq \omega_{max}, \qquad \forall n,$$

therefore u_n is bounded in L^∞. Furthermore the last term of (2.43) is bounded in $W_{loc}^{-1,\infty}$ and in $\mathcal{M}_{loc}$, thus we can apply the compensated compactness result of [22] and it gives that, up to a subsequence, $(\rho_n, \rho_n u_n)$ converge a.e. in $]0,\infty[\times\mathbf{R}$ when $n \to \infty$ to some functions $(\rho, \rho u)$. Using the previous Lemma, we conclude the proof of the stability result. □

We turn now to the existence result (Theorem 1.3) and prove first the relaxation of (2.4) to (1.9)-(1.11).

Theorem 2.11 *Let us denote by f_ε the solution of Theorem 2.4 with the same initial data $f^0(x,\xi) \in L^1(\mathbf{R}\times\mathbf{R})$ that satisfies $f^0(x,\xi) \in \tilde{D}_\xi$ a.e. for some $\omega_{min} < \omega_{max}$, and the energy bound (2.29). Then $(\rho_\varepsilon, u_\varepsilon)$ defined by (2.7) are uniformly bounded in L^∞, and passing if necessary to subsequences, $(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)$ converge a.e. in $]0,\infty[\times\mathbf{R}$ when $\varepsilon \to 0$ to an entropy solution $(\rho, \rho u)$ to (1.9) -(1.11), with initial data $(\rho^0, \rho^0 u^0) = \int f^0 d\xi$.*

Proof. Using the bounds of Theorem 2.9 and the renormalisation result for a transport equation of [8], we have

$$\partial_t H_S(f_\varepsilon,\xi) + \xi\partial_x H_S(f_\varepsilon,\xi) = H_S'(f_\varepsilon,\xi)\cdot(M^*[f_\varepsilon]-f_\varepsilon).$$

By integration, it yields to

$$\begin{gathered}\partial_t \textstyle\int_{\mathbf{R}} H_S(f_\varepsilon,\xi)\,d\xi + \partial_x \int_{\mathbf{R}} \xi H_S(f_\varepsilon,\xi)\,d\xi = \\ \tfrac{1}{\varepsilon}\textstyle\int_{\mathbf{R}}(H_S'(f_\varepsilon,\xi) - T_S(\rho_\varepsilon^*, u_\varepsilon))\cdot(M^*[f_\varepsilon]-f_\varepsilon)\,d\xi + T_S(\rho_\varepsilon^*,u_\varepsilon)\cdot\int_{\mathbf{R}}\frac{M^*[f_\varepsilon]-f_\varepsilon}{\varepsilon}\,d\xi,\end{gathered}$$

with $\rho^* = \min(1,\rho)$ and $Q_\varepsilon = -\frac{(\rho_\varepsilon-1)_+}{\varepsilon} = \int_{\mathbf{R}} \frac{(M^*[f_\varepsilon])_0-(f_\varepsilon)_0}{\varepsilon}\,d\xi$, thus

$$\begin{aligned}&\partial_t\eta_S(\rho_\varepsilon^*,u_\varepsilon) + \partial_x G_S(\rho_\varepsilon^*,u_\varepsilon)\\ \le\ &\partial_t\int_{\mathbf{R}}(H_S(f_\varepsilon,\xi) - H_S(M^*[f_\varepsilon],\xi))\,d\xi + \partial_x\int_{\mathbf{R}}\xi(H_S(f_\varepsilon,\xi)-H_S(M^*[f_\varepsilon],\xi))\,d\xi\\ &+Q_\varepsilon\eta_S'(1,u_\varepsilon)\cdot(1,u_\varepsilon),\end{aligned} \tag{2.45}$$

since

$$Q_\varepsilon T_S(\rho_\varepsilon^*,u_\varepsilon) = Q_\varepsilon\eta_S'(1,u_\varepsilon). \tag{2.46}$$

From (2.34), we have

$$\iiint\limits_{]0,T[\times]0,\infty[\times\mathbf{R}} (H'(f_\varepsilon,\xi) - T_{\frac{v^2}{2}}(\rho_\varepsilon^*,u_\varepsilon))\cdot\frac{M^*[f_\varepsilon]-f_\varepsilon}{\varepsilon}\,dtdxd\xi \tag{2.47}$$

bounded uniformly in ε. Furthermore, f_ε, $M^*[f_\varepsilon]$ are bounded in $L^\infty_{t,x,\xi}$ and we have uniform compact support property. Thus, we can adapt the dissipation result of [4] and get that $f_\varepsilon - M^*[f_\varepsilon] \to 0$ a.e. t,x,ξ, replacing the use of $\int_{\mathbf{R}}(f_\varepsilon)_0\,d\xi = \int_{\mathbf{R}}(M^*[f_\varepsilon])_0\,d\xi$ by

$$0 \le \iiint\limits_{(t,x)\in B}((f_\varepsilon)_0 - (M^*[f_\varepsilon])_0)\,dtdxd\xi \le K\varepsilon, \qquad \forall B.$$

The last relation comes from

$$\iint -Q_\varepsilon dxdt \le \int_{\mathbf{R}}\alpha(0,x)\,dx, \tag{2.48}$$

which is a consequence of (2.33). The study of [4] is for $1<\gamma<3$; for $\gamma=3$, the study is a little different and we refer to the next section for such a dissipative

result.
Thus $\int_{\mathbf{R}}(H_S(f_\varepsilon,\xi)-H_S(M[f_\varepsilon],\xi))\,d\xi$ and $\int_{\mathbf{R}}\xi(H_S(f_\varepsilon,\xi)-H_S(M[f_\varepsilon],\xi))\,d\xi$ go to 0 as $\varepsilon\to 0$. The last term of (2.45) is negative (Lemma 1.1) and we obtain

$$\partial_t\eta_S(\rho_\varepsilon,u_\varepsilon)+\partial_x G_S(\rho_\varepsilon,u_\varepsilon)\le R_{S,\varepsilon}\to 0 \text{ in } W^{-1,p}_{loc},\ \text{ as } \varepsilon\to 0, \tag{2.49}$$

for any $1<p<\infty$, for $S:\mathbf{R}\to\mathbf{R}$ convex, of class C^1. Besides, for $S(v)=1,v$, we have

$$\partial_t\eta_1(\rho_\varepsilon,u_\varepsilon)+\partial_x G_1(\rho_\varepsilon,u_\varepsilon)=R_{1,\varepsilon}+Q_\varepsilon,$$
$$\partial_t\eta_{Id}(\rho_\varepsilon,u_\varepsilon)+\partial_x G_{Id}(\rho_\varepsilon,u_\varepsilon)=R_{Id,\varepsilon}+Q_\varepsilon u_\varepsilon,$$

where Q_ε and $Q_\varepsilon u_\varepsilon$ are bounded in $W^{-1,\infty}_{loc}$ and in $\mathcal{M}_{loc}$, thanks to (2.48) and u_ε bounded in L^∞.
Since furthermore ρ^*_ε and u_ε are bounded in L^∞, we can then apply the stability result of [22] which gives that up to a subsequence, $(\rho^*_\varepsilon,\rho^*_\varepsilon u_\varepsilon)$ converge a.e. in $]0,\infty[\times\mathbf{R}$ when $\varepsilon\to 0$ to some $(\rho,\rho u)$. The majoration (2.48) gives $\rho^*_\varepsilon-\rho_\varepsilon\to 0$, thus $\rho_\varepsilon\to\rho$ with $\rho\le 1$. The measures Q_ε are bounded and we can apply the Lemma 2.10 to get the entropy weak product by passing to the limit in (2.45). □

Proof of Theorem 1.4. Let ρ^0, u^0 satisfy the assumptions of the Theorem. There exists ω_{min}, ω_{max} such that $(\rho^0,u^0)\in\tilde D$. We take $f^0(x,\xi)=M(\rho^0(x),u^0(x),\xi)\in\tilde D_\xi$. Since

$$\iint\limits_{\mathbf{R}\times\mathbf{R}} H(M(\rho^0,u^0,\xi),\xi)dxd\xi=\int_{\mathbf{R}}\eta(\rho^0,u^0)dx\le(\frac{1}{2}\|u^0\|^2_\infty+\frac{\kappa}{\gamma-1})\|\rho^0\|_1<+\infty,$$

we can apply the Theorem 2.11 and we get the result. □

2.6 Relaxation limit for $\gamma=3$ via averaging lemma

In this section, we give a more elementary proof of the relaxation limit for $\gamma=3$ using a compactness averaging lemma instead of compensated compactness. Furthermore, we study the entropy dissipation term in this case. Let f_ε be the solution of Theorem 2.4 with the same initial data $f^0(x,\xi)$ and $(\rho_\varepsilon,u_\varepsilon)$ be the approximate solutions to (1.9) defined by (2.7). In order to prove the convergence, as ε tends to 0, of ρ_ε and u_ε, we are going to use the compactness averaging lemma of [15] which is:

Proposition 2.12 *Let $g_\varepsilon\in L^\infty(]0,\infty[\times\mathbf{R}\times\mathbf{R})$ satisfying equation*

$$\partial_t g_\varepsilon+\xi\partial_x g_\varepsilon=-\lambda_\varepsilon-\partial_{\xi\xi}\mu_\varepsilon, \tag{2.50}$$

for some nonnegative measures λ_ε, μ_ε bounded uniformly in ε. If g_ε is bounded in L^∞ uniformly in ε, then $\int_{\mathbf{R}} g_\varepsilon(t,x,\xi)\psi(\xi)\,d\xi$ belongs to a compact set of $L^p_{loc}(]0,\infty[\times\mathbf{R})$, $1<p<\infty$, for any $\psi\in L^{p'}_c(\mathbf{R})$.

In order to use this result, we need to prove that:

Proposition 2.13 *The solution f_ε of Theorem 2.4 satisfies*

$$\partial_t(f_\varepsilon)_0 + \xi\partial_x(f_\varepsilon)_0 = -\lambda_\varepsilon - \partial_{\xi\xi}\mu_\varepsilon, \tag{2.51}$$

where $(\lambda_\varepsilon)_{\varepsilon>0}$ and $(\mu_\varepsilon)_{\varepsilon>0}$ are nonnegative measures bounded uniformly in ε.

Proof. We set $\lambda_\varepsilon = (M_0[f_\varepsilon] - M_0^*[f_\varepsilon])/\varepsilon$ and $h_\varepsilon = (M_0[f_\varepsilon] - (f_\varepsilon)_0)/\varepsilon$. We have $\lambda_\varepsilon \geq 0$ thanks to Lemma 2.1. Since $\int_{\mathbf{R}} h_\varepsilon \, d\xi = 0$ and $\int_{\mathbf{R}} \xi h_\varepsilon \, d\xi = 0$, there exists a distribution μ_ε with compact support such that $h_\varepsilon = -\partial_{\xi\xi}\mu_\varepsilon$. Thus we have (2.51). We integrate this equality and get

$$\iiint\limits_{]0,T[\times\mathbf{R}\times\mathbf{R}} \lambda_\varepsilon \;\leq \iint\limits_{\mathbf{R}\times\mathbf{R}} f_0^0(x,\xi)\, dx d\xi.$$

For $\varphi(t,x,\xi) = \varphi_1(t,x)\varphi_2(\xi)$ with φ_1, φ_2 nonnegative and of class C_c^∞, and taking $\phi \in C^\infty$ such that $\varphi_2 = \partial_{\xi\xi}\phi$, we have

$$< \mu_\varepsilon, \varphi > = < \partial_{\xi\xi}\mu_\varepsilon, \varphi_1\phi > = - \iiint\limits_{]0,T[\times\mathbf{R}\times\mathbf{R}} \varphi_1\phi h_\varepsilon \;\geq 0,$$

thus $\mu_\varepsilon \geq 0$. We integrate now (2.51) after having multiplicate it by ξ^2, and we get for any compact set K of $\mathbf{R}$,

$$2\int_{]0,T[}\int_{\mathbf{R}}\int_{\xi\in K} \mu_\varepsilon \;\leq \iint\limits_{\mathbf{R}\times\mathbf{R}} \xi^2 f_0^0(x,\xi)\, dx d\xi. \;\square$$

Furthermore we have f_ε bounded in L^∞, then for a subsequence, we get the weak convergences

$$f_\varepsilon \rightharpoonup f \text{ in } L^\infty_{w*}, \qquad \lambda_\varepsilon \rightharpoonup \lambda \text{ measure}, \qquad \mu_\varepsilon \rightharpoonup \mu \text{ measure}.$$

If we suppose that the initial data f^0 satisfies $f^0(x,\xi) \in \tilde{D}_\xi$ a.e. for some $\omega_{min} < \omega_{max}$, then $\mathrm{supp}_\xi f_\varepsilon \subset [\omega_{min}, \omega_{max}]$ and we can apply Proposition 2.12 with $\psi(v) = 1$ and $\psi(v) = v$ and we obtain, up to an extraction, that

$$\rho_\varepsilon = \textstyle\int_{\mathbf{R}} (f_\varepsilon)_0 \, d\xi, \;\; \rho_\varepsilon u_\varepsilon = \int_{\mathbf{R}} \xi (f_\varepsilon)_0 \, d\xi$$
$$\text{converge in } L^p_{loc} \text{ to } \rho = \textstyle\int_{\mathbf{R}} f_0 \, d\xi, \;\; \int_{\mathbf{R}} \xi f_0 \, d\xi \equiv \rho u.$$

For a subsequence, we have the convergences

$$\rho_\varepsilon \to \rho, \qquad \rho_\varepsilon u_\varepsilon \to \rho u \qquad \text{a.e. } t,x, \tag{2.52}$$

and thus $M_0(\rho_\varepsilon, u_\varepsilon, \xi) \to M_0(\rho, u, \xi)$ and $M_0^*(\rho_\varepsilon, u_\varepsilon, \xi) \to M_0^*(\rho, u, \xi)$ a.e. t, x, ξ and in L^1_{loc} by Lebesgue's theorem. We integrate (2.34) and we get that

$$\varepsilon \iint\limits_{\mathbf{R}\times\mathbf{R}} H(f^0(x,\xi),\xi)\, dx d\xi \geq \iiint\limits_{]0,T[\times\mathbf{R}\times\mathbf{R}} \frac{\xi^2}{2}((f_\varepsilon)_0(t,x,\xi) - M_0^*[f_\varepsilon](t,x,\xi))\, dt dx d\xi.$$

Since $\int_{\mathbf{R}} f_\varepsilon \, d\xi = \int_{\mathbf{R}} M[f_\varepsilon] \, d\xi$, we have $\int_{\mathbf{R}} (f_\varepsilon)_0 \, d\xi = \int_{\mathbf{R}} M_0[f_\varepsilon] \, d\xi$, $\int_{\mathbf{R}} \xi (f_\varepsilon)_0 \, d\xi = \int_{\mathbf{R}} \xi M_0[f_\varepsilon] \, d\xi$, and it yields $\int_{\mathbf{R}} (\frac{1}{2}(u_\varepsilon)^2 - \frac{3\kappa}{2}(\rho_\varepsilon)^2 - \xi u_\varepsilon)((f_\varepsilon)_0 - M_0^*[f_\varepsilon]) \, d\xi = 0$, from which we deduce

$$\begin{aligned} &\varepsilon \iint\limits_{\mathbf{R}\times\mathbf{R}} H(f^0(x,\xi),\xi) \, dx d\xi \\ \geq \; & \iiint\limits_{]0,T[\times\mathbf{R}\times\mathbf{R}} \xi^2/2(M_0[f_\varepsilon] - M_0^*[f_\varepsilon]) \, dt dx d\xi \\ &+ \frac{1}{2} \iiint\limits_{]0,T[\times\mathbf{R}\times\mathbf{R}} [(u_\varepsilon - \xi)^2 - 3\kappa(\rho_\varepsilon)^2][(f_\varepsilon)_0 - M_0(\rho_\varepsilon, u_\varepsilon, \xi)] \, dt dx d\xi. \end{aligned}$$

Nonnegativity of the last two terms under the integral implies that, for a subsequence,

$$\xi^2(M_0[f_\varepsilon] - M_0^*[f_\varepsilon]) \to 0 \qquad \text{a.e. } t, x, \xi, \text{ as } \varepsilon \to 0,$$

and

$$[(u_\varepsilon - \xi)^2 - 3\kappa(\rho_\varepsilon)^2][(f_\varepsilon)_0 - M_0(\rho_\varepsilon, u_\varepsilon, \xi)] \to 0 \qquad \text{a.e. } t, x, \xi, \text{ as } \varepsilon \to 0.$$

With the previous convergences, it gives $M_0^*(\rho, u, \xi) = M_0(\rho, u, \xi)$ a.e. and

$$[(u - \xi)^2 - 3\kappa\rho^2][(f_\varepsilon)_0 - M_0(\rho, u, \xi)] \to 0 \qquad \text{a.e. } t, x, \xi, \text{ as } \varepsilon \to 0.$$

By integration of $M_0^*(\rho, u, \xi) = M_0(\rho, u, \xi)$ with respect to ξ, we get that $\min(1, \rho) = \rho$, that is to say $\rho \leq 1$.

We set $E = \{(t,x) \in [0,T] \times \mathbf{R};\ \rho(t,x) \neq 0\}$, and, at (t,x) beeing fixed, $F(t,x) = \{\xi \in \mathbf{R};\ (u(t,x) - \xi)^2 \neq 3\kappa\rho^2(t,x)\}$. On E, we have $u_\varepsilon \to u$ a.e. t, x. If $(t,x) \in E$, then $(f_\varepsilon)_0 \to M_0(\rho, u, \xi)$ a.e. $\xi \in F(t,x)$. By dominated convergence, we have for any bounded domain B in (t,x),

$$\iint\limits_{(t,x)\in B\cap E} \int_{\xi\in F(t,x)} ((f_\varepsilon)_0 - M_0[f_\varepsilon])(t,x,\xi) \, dt dx d\xi \to 0, \qquad \text{as } \varepsilon \to 0.$$

Passing to the limit as $\varepsilon \to 0$ in

$$\begin{aligned} &\iint\limits_{B\cap E} \int_{\xi\in F(t,x)} (f_\varepsilon)_0 \, dt dx d\xi + \iint\limits_{B\cap E} \int_{\xi\notin F(t,x)} (f_\varepsilon)_0 \, dt dx d\xi \\ = \; & \iint\limits_{B\cap E} \int_{\xi\in F(t,x)} M_0[f_\varepsilon] \, dt dx d\xi + \iint\limits_{B\cap E} \int_{\xi\notin F(t,x)} M_0[f_\varepsilon] \, dt dx d\xi, \end{aligned}$$

we get that

$$\iint\limits_{B\cap E} \int_{\xi\notin F(t,x)} (f_\varepsilon)_0(t,x,\xi) \, dt dx d\xi \to \iint\limits_{B\cap E} \int_{\xi\notin F(t,x)} M_0(\rho, u, \xi) \, dt dx d\xi = 0.$$

Finally we have, up to an extraction,

$$\mathbb{1}_E(f_\varepsilon)_0 = \mathbb{1}_{\{\rho\neq 0\}}(f_\varepsilon)_0 \to M_0(\rho,u,\xi) \qquad \text{a.e. } t,x,\xi, \text{ as } \varepsilon\to 0. \tag{2.53}$$

Now, dominated convergence and convergence (2.53) gives that

$$\iint_{(t,x)\in B\cap E} \int_{\xi\in\mathbf{R}} ((f_\varepsilon)_0 - M_0[f_\varepsilon])(t,x,\xi)\,dtdxd\xi \to 0, \qquad \text{as } \varepsilon\to 0.$$

Then, passing to the limit as $\varepsilon\to 0$ in

$$\begin{aligned}&\iint_{(t,x)\in B\cap E} \int_{\xi\in\mathbf{R}} (f_\varepsilon)_0\,dtdxd\xi + \iint_{\substack{(t,x)\in B\\(t,x)\notin E}} \int_{\xi\in\mathbf{R}} (f_\varepsilon)_0\,dtdxd\xi\\ =&\iint_{(t,x)\in B\cap E} \int_{\xi\in\mathbf{R}} M_0[f_\varepsilon]\,dtdxd\xi + \iint_{\substack{(t,x)\in B\\(t,x)\notin E}} \int_{\xi\in\mathbf{R}} M_0[f_\varepsilon]\,dtdxd\xi,\end{aligned}$$

we get that

$$\iint_{\substack{(t,x)\in B\\(t,x)\notin E}} \int_{\xi\in\mathbf{R}} (f_\varepsilon)_0(t,x,\xi)\,dtdxd\xi \to \iint_{\substack{(t,x)\in B\\(t,x)\notin E}} \int_{\xi\in\mathbf{R}} M_0(\rho,u,\xi)\,dtdxd\xi = 0.$$

Finally we get that, up to an extraction,

$$(f_\varepsilon)_0 \to M_0(\rho,u,\xi) \qquad \text{a.e. } t,x,\xi, \text{ as } \varepsilon\to 0. \tag{2.54}$$

Since $(f_\varepsilon)_1 = \xi(f_\varepsilon)_0$, we also have $(f_\varepsilon)_1 \to \xi M_0(\rho,u,\xi) = M_1(\rho,u,\xi)$ a.e. t,x,ξ. We integrate (2.4) in ξ and we obtain

$$\left\{\begin{array}{l}\partial_t\rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = Q_\varepsilon\\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\int_{\mathbf{R}} \xi^2 (f_\varepsilon)_0\,d\xi) = Q_\varepsilon u_\varepsilon,\end{array}\right. \tag{2.55}$$

where

$$Q_\varepsilon = \int_{\mathbf{R}} (M_0^*[f_\varepsilon] - M_0[f_\varepsilon])/\varepsilon\,d\xi = (\min(1,\rho_\varepsilon)-\rho_\varepsilon)/\varepsilon,$$

and

$$\int_{\mathbf{R}} \xi(M_0^*[f_\varepsilon] - M_0[f_\varepsilon])/\varepsilon\,d\xi = u_\varepsilon(\min(1,\rho_\varepsilon)-\rho_\varepsilon)/\varepsilon = Q_\varepsilon u_\varepsilon.$$

Measures Q_ε are nonnegatives and bounded in $\mathcal{D}'_{t,x}$, thus they are bounded in $\mathcal{M}_{loc}$. Since $|Q_\varepsilon u_\varepsilon| \le \|u_\varepsilon\|_\infty |Q_\varepsilon|$, the measures $Q_\varepsilon u_\varepsilon$ are also bounded in $\mathcal{M}_{loc}$. Consequently, there exists Q in $\mathcal{M}_{loc}$ and $v\in L^\infty(Q)$ such that for a subsequence

$$Q_\varepsilon \rightharpoonup Q, \qquad Q_\varepsilon u_\varepsilon \rightharpoonup Qv \qquad \text{in } \mathcal{M}_{loc}([0,\infty[\times\mathbf{R})w*.$$

Furthermore, by dominated convergence $\int_{\mathbf{R}} \xi^2 (f_\varepsilon)_0 \, d\xi \to \int_{\mathbf{R}} \xi^2 M_0(\rho, u, \xi) \, d\xi$ in L^1_{loc} and finally we obtain that (ρ, u, Q, v) is a solution to (1.9) with the constraints (1.10).

Now we prove that the solution satisfies the entropy weak products. We multiply (2.4) by $S(\xi)$, for $S : \mathbf{R} \to \mathbf{R}$ convex and such that $0 \leq S(v) \leq B(1+v^2)$ for some $B \geq 0$, then we integrate it with respect to ξ, and we get that

$$\begin{aligned} \partial_t \int_{\mathbf{R}} S(\xi)(f_\varepsilon)_0 \, d\xi + \partial_x \int_{\mathbf{R}} \xi S(\xi)(f_\varepsilon)_0 \, d\xi &= \int_{\mathbf{R}} S(\xi)(M_0^*[f_\varepsilon] - (f_\varepsilon)_0)/\varepsilon \, d\xi \\ &\leq Q_\varepsilon \eta_S'(1, u_\varepsilon) \cdot (1, u_\varepsilon), \end{aligned}$$

thanks to Corollary 2.7. According to (2.52), we have by dominated convergence $\eta_S(\rho_\varepsilon, u_\varepsilon) \to \eta_S(\rho, u)$ and $G_S(\rho_\varepsilon, u_\varepsilon) \to G_S(\rho, u)$ in L^p_{loc} as $\varepsilon \to 0$, thus we have

$$\begin{aligned} R_{S,\varepsilon} &= \partial_t[\eta_S(\rho, u)] + \partial_x[G_S(\rho, u)] - (\partial_t[\eta_S(\rho_\varepsilon, u_\varepsilon)] + \partial_x[G_S(\rho_\varepsilon, u_\varepsilon)]) \\ &\underset{\varepsilon \to 0}{\to} 0 \text{ in } W^{-1,p}_{loc} \text{ for any } 1 < p < \infty. \end{aligned}$$

Therefore

$$\partial_t[\eta_S(\rho, u)] + \partial_x[G_S(\rho, u)] \leq R_{S,\varepsilon} + Q_\varepsilon \eta_S'(1, u_\varepsilon) \cdot (1, u_\varepsilon),$$

and we obtain the result thanks to Lemma 2.10.

3 Pressureless model

In this section, we give a sticky blocks dynamics that solve the system (1.9) with $p = 0$ under all the needed constraints. Our choice is motivated by the numerical simulations of Section 4.

3.1 Some remarks on the pressureless model

The analysis in this case is similar to that of [2], and differs from the one for the system of pressureless gases without constraint, that gives Dirac distributions on ρ in finite time (see [5], [10], [13]).
We look for solutions ρ, u to (1.9)-(1.11) with $p \equiv 0$, satisfying (1.15)-(1.17) in the sense of (1.18)-(1.19). The regularity (1.15) comes from the a priori bound $0 \leq \rho \leq 1$ and the following decrease $\int_{\mathbf{R}} \rho(t,x) \, dx \leq \int_{\mathbf{R}} \rho^0(x) \, dx$. The volume fraction ρ and the momentum density ρu are not continuous with respect to time, because Q can contain Dirac functions.
We notice that for smooth solutions, we have entropy inequalities

$$\partial_t(\eta_S(\rho, u)) + \partial_x(G_S(\rho, u)) = QS(u),$$

where $\eta_S(\rho, u) = \rho S(u)$ and $G_S(\rho, u) = \rho u S(u)$. Furthermore, the entropies satisfy

$$\eta_S'(1, v) \cdot (1, v) = S(v),$$

thus the Lemma 1.1 is valid in the pressureless case.

3.2 Sticky blocks dynamics

Let us consider a volume fraction $\rho(t,x)$ and a momentum density $\rho(t,x)u(t,x)$ given by

$$\rho(t,x) = \sum_{i=1}^{n} \mathbb{1}_{a_i(t)<x<b_i(t)}, \quad \rho(t,x)u(t,x) = \sum_{i=1}^{n} u_i(t)\mathbb{1}_{a_i(t)<x<b_i(t)}, \tag{3.1}$$

with $a_1(t) < b_1(t) < a_2(t) < b_2(t) < \cdots < b_n(t)$. The time evolution is defined as follows. The number of blocks n indeed depends on t, but is piecewise constant. As long as the blocks do not meet, they move at constant velocity $u_i(t)$. When two blocks collide at a time t^*, the volume fraction ρ is given locally by

$$\rho(t,x) = \begin{cases} \mathbb{1}_{a_l(t)<x<b_l(t)} + \mathbb{1}_{a_r(t)<x<b_r(t)} & \text{if } t < t^*, \\ \mathbb{1}_{a_l(t)<x<b_r(t)} & \text{if } t^* \le t \le t_f, \\ \mathbb{1}_{a(t)<x<b(t)} & \text{if } t > t_f. \end{cases} \tag{3.2}$$

and the momentum density ρu by

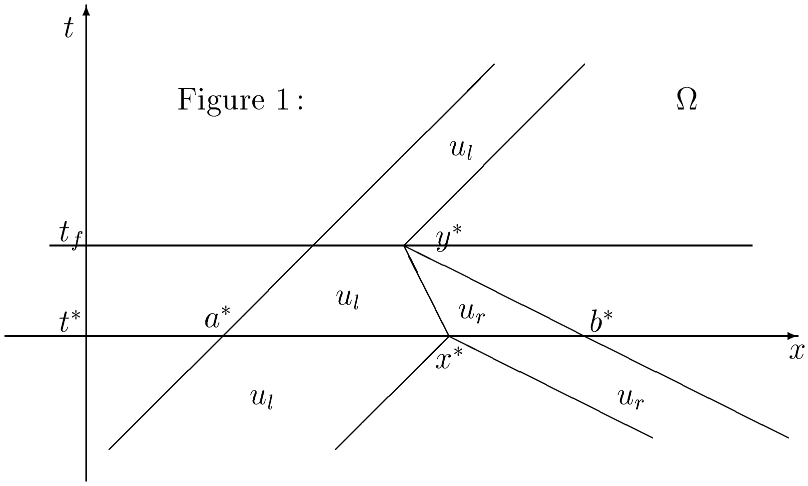
$$\rho u(t,x) = \begin{cases} u_l\mathbb{1}_{a_l(t)<x<b_l(t)} + u_r\mathbb{1}_{a_r(t)<x<b_r(t)} & \text{if } t < t^*, \\ u_l\mathbb{1}_{a_l(t)<x<c(t)} + u_r\mathbb{1}_{c(t)<x<b_l(t)} & \text{if } t^* \le t \le t_f, \\ u^*\mathbb{1}_{a(t)<x<b(t)} & \text{if } t > t_f. \end{cases} \tag{3.3}$$

We have $a_l(t) = a^* + u_l(t-t^*)$, $b_l(t) = x^* + u_l(t-t^*)$, $a_r(t) = x^* + u_r(t-t^*)$, $b_r(t) = b^* + u_r(t-t^*)$, $c(t) = x^* + u_{loss}(t-t^*)$, $a(t) = a_l(t_f) + u^*(t-t_f)$ and $b(t) = b_r(t_f) + u^*(t-t_f)$, with

$$u_{loss} = \frac{u_l + u_r}{2}, \tag{3.4}$$

$$(u^*, t_f) = \begin{cases} (u_l, t^* + 2(b^* - x^*)/(u_l - u_r)) & \text{if } x^* - a^* > b^* - x^*, \\ (u_r, t^* + 2(x^* - a^*)/(u_l - u_r)) & \text{if } x^* - a^* < b^* - x^*, \end{cases} \tag{3.5}$$

The dynamics is exhibited in the following Figure.



When $x^* - a^* > b^* - x^*$ (resp. $x^* - a^* < b^* - x^*$) the right (resp. the left) sticky block disappears. In other words, the sticky block of smallest volume vanishes. In the case where the two blocks have the same volume, that is to say $x^* - a^* = b^* - x^*$, the two sticky blocks annihilate at time $t_f = t^* + 2(b^* - x^*)/(u_l - u_r) = t^* + 2(x^* - a^*)/(u_l - u_r)$. When three blocks collide at a time t^*, we treat independently the dynamics of blocks 1 and 2 and blocks 2 and 3, where the block 2 is the intermediary one. At the time when one of the three blocks disappears, we are back to the case of two blocks.
If a block collides at a time t_1 with two blocks that are colliding since a time t^* with $t^* < t_1 < t_f$, at this time t_1 the situation is this described before. We extend easily to the case of any number of collisions.

3.3 Properties of sticky blocks

We have the following existence result.

Theorem 3.1 *There exists a negative measure $Q(t,x)$ such that with $\rho(t,x)$ and $u(t,x)$ defined by (3.1), and with the above defined dynamics, we get a solution to the weak formulation (1.18)-(1.19) with constraints (1.10)-(1.11).*

Proof. As long as there is no collision, each block moves at the constant velocity u_i, and (ρ, u) solves the pressureless Euler system. In a neighbourhood of initial data, the proof is given in [2]. We look at the case of a collision of two blocks at a time t^*. Let $\varphi(t,x)$ be a smooth function with support in Ω of Figure 1 such that there is no other contact in Ω. We have

$$\begin{aligned}
& \int_{t^*}^{t_f} \int_{a_l(t)}^{c(t)} [S(u_l)\partial_t\varphi(t,x) + S(u_l)u_l\partial_x\varphi(t,x)]\; dx\, dt \\
= \;& S(u_l)\left\{ \int_{t^*}^{t_f} \left[\frac{d}{dt}\int_{a_l(t)}^{c(t)} \varphi(t,x)\, dx - u_{loss}\varphi(t,c(t)) + u_l\varphi(t,a_l(t)) \right] dt \right. \\
& \left. + \int_{t_f}^{t^*} u_l(\varphi(t,c(t)) - \varphi(t,a_l(t)))\, dt \right\} \\
= \;& S(u_l)\left\{ \int_{a_l(t_f)}^{b_r(t_f)} \varphi(t_f,x)\, dx - \int_{a^*}^{x^*} \varphi(t^*,x)\, dx + \int_{t^*}^{t_f} \frac{u_l - u_r}{2}\varphi(t,c(t))\, dt \right\}.
\end{aligned}$$

Now

$$\begin{aligned}
& < \partial_t(\rho S(u)) + \partial_x(\rho u S(u)), \varphi > \\
= \;& - < \rho S(u), \partial_t\varphi > - < \rho u S(u), \partial_x\varphi > \\
= \;& - \int^{t^*} \left(\int_{a_l(t)}^{b_l(t)} S(u_l)(\partial_t\varphi + u_l\partial_x\varphi)\, dx + \int_{a_r(t)}^{b_r(t)} S(u_r)(\partial_t\varphi + u_r\partial_x\varphi)\, dx \right) dt \\
& - \int_{t^*}^{t_f} \left(\int_{a_l(t)}^{c(t)} S(u_l)(\partial_t\varphi + u_l\partial_x\varphi)\, dx + \int_{c(t)}^{b_r(t)} S(u_r)(\partial_t\varphi + u_r\partial_x\varphi)\, dx \right) dt \\
& - \int_{t_f} \int_{a(t)}^{b(t)} S(u^*)(\partial_t\varphi + u^*\partial_x\varphi)\, dx dt.
\end{aligned}$$

Every term beeing computed as before, we get

$$< \partial_t(\rho S(u)) + \partial_x(\rho u S(u)), \varphi >= -\int_{t^*}^{t_f} \frac{S(u_r)+S(u_l)}{2}(u_l-u_r)\varphi(t,c(t))\,dt.$$

Thus we have (1.30) with

$$Q_S = -\left[\frac{S(u_r)+S(u_l)}{2}\right](u_l-u_r)\mathbb{1}_{[t^*,t_f]}(t)\delta(x-c(t)).$$

For $S(u)=1$, we get

$$Q(t,x) = -(u_l-u_r)\mathbb{1}_{[t^*,t_f]}(t)\delta(x-c(t)), \tag{3.6}$$

and for $S(u)=u$, we get

$$(Qu)(t,x) = -(u_l-u_r)u_{loss}\mathbb{1}_{[t^*,t_f]}(t)\delta(x-c(t)). \tag{3.7}$$

Since $u_l > u_r$ is necessary to have the collision, we have $Q \leq 0$. We prove now the entropy weak products: since $\eta_S'(1,u)\cdot(1,u) = S(u)$ and

$$< QS(u), \varphi >= \int_{t^*}^{t_f} -(u_l-u_r)S(u_{loss})\varphi(t,c(t))\,dt,$$

we obtain, for S convex, that

$$\begin{aligned} &< \partial_t(\rho S(u)) + \partial_x(\rho u S(u)) - Q\eta_S'(1,u)\cdot(1,u), \varphi > \\ = \;& -\int_{t^*}^{t_f}\left[\frac{S(u_r)+S(u_l)}{2} - S\left(\frac{u_r+u_l}{2}\right)\right](u_l-u_r)\varphi(t,c(t))\,dt \leq 0. \;\Box \end{aligned}$$

Similarly to the model with the term of pression (see [9]), we have

$$\frac{u_i(t)-u_{i-1}(t)}{a_i(t)-b_i(t)} \leq \frac{1}{t} \text{ for } 2 \leq i \leq n.$$

Defining the value of $u(t,x)$ for all x by asking that u is Lipschitz continuous, takes the value $u_i(t)$ in each block i and is extended linearly between each block, and by a constant at infinity, we get that the sticky blocks dynamics satisfies Oleinik's condition (1.27). Notice that we also have the maximum principle (1.28) and the entropy equality (1.30) with

$$Q_S(t,x) = \frac{S(u_l)+S(u_r)}{2}Q(t,x).$$

Thus, we get (1.31) for every S continuous.
We notice that if we take a dynamics with $u_{loss} = u_l + u_r$, then we get a dynamics with loss of mass and with conservation of the momentum density ($Qu = 0$). But in return, we do not have maximum principle.

3.4 Existence of a solution

We first prove the stability result of Theorem 1.2 in the pressureless case.
As for the nonzero pressure case, we have (2.44). We use this decreasing with $S(v) = (v - \sup_m \operatorname{esssup}_x u_m^0)_+^2$ and $S(v) = (\inf_m \operatorname{essinf}_x u_m^0 - v)_+^2$, and we get

$$\inf_m \operatorname{essinf}_x u_m^0 \leq u_n(t,x) \leq \sup_m \operatorname{esssup}_x u_m^0,$$

therefore $(u_n)_{n\geq 0}$ is bounded in L^∞. After that, the proof is close as for the similar result of [2], the signifiant change here is the loss of $(\rho_n)_{n\geq 0}$ bounded in $C_t([0,T]; L^\infty_{w*}(\mathbf{R}))$. Instead of, we use the entropy inequality

$$\frac{d}{dt}\int_{\mathbf{R}} \rho_n(t,x)\varphi(x)\,dx - \int_{\mathbf{R}} \rho_n u_n \varphi'(x)\,dx = \int_{\mathbf{R}} \varphi(x) Q_n,$$

which yields that $(\int_{\mathbf{R}} \rho_n(t,x)\varphi(x)\,dx)_{n\in\mathbf{N}}$ is bounded in BV_t for any $\varphi \in \mathcal{D}(\mathbf{R})$ and the result is kept. Furthermore Lemma 2.10 gives entropy weak products.
We turn now to the proof of Theorem 1.4.
As in [2], we use sticky blocks to get a solution to (1.18)-(1.19) with constraints (1.10)-(1.11) with $p = 0$. Let $\rho^0 \in L^1(\mathbf{R})$, $0 \leq \rho^0 \leq 1$, $u^0 \in L^\infty(\mathbf{R})$. We have the following lemma of sticky blocks approximation.

Lemma 3.2 *Let $\rho^0 \in L^1(\mathbf{R})$ such that $0 \leq \rho^0 \leq 1$ and $u^0 \in L^\infty(\mathbf{R})$, then there exist block initial data $(\rho_n^0)_{n\geq 0}$ and $(\rho_n^0 u_n^0)_{n\geq 0}$ such that $\int_{\mathbf{R}} \rho_n^0(x)\,dx \leq \int_{\mathbf{R}} \rho^0(x)\,dx$ and $\operatorname{essinf} u^0 \leq u_n^0 \leq \operatorname{esssup} u^0$ for which the convergences $\rho_n^0 \rightharpoonup \rho^0$ and $\rho_n^0 u_n^0 \rightharpoonup \rho^0 u^0$ hold in $\mathcal{D}'_x$.*

We have to notice that this result, which is proved in [2], is independent of the choosen dynamics. Then, we use the stability result to get the solution. The other results (Theorem 1.5) can be obtain from [2].

Remark 3.1 In Theorem 1.2, we also have the existence of $Q_S \in \mathcal{M}_{loc}([0,\infty[\times\mathbf{R})$ such that (1.30) and (1.31) are satisfied.

3.5 The exclusion relation is lost in weak limit for the pressureless model

We consider two blocks with a height of 1/2 :

$$\rho^0(x) = \frac{1}{2}(\mathbb{1}_{0<x<1} + \mathbb{1}_{2<x<3}), \qquad \rho^0 u^0(x) = \frac{1}{2}(\mathbb{1}_{0<x<1} - \mathbb{1}_{2<x<3}).$$

We use the following approximation of this initial data

$$\rho_n^0(x) = \sum_{k=1}^n \mathbb{1}_{\frac{2k-1}{2n}<x<\frac{2k}{2n}} + \sum_{k=1}^n \mathbb{1}_{2+\frac{2(k-1)}{2n}<x<2+\frac{2k-1}{2n}},$$

$$\rho_n^0 u_n^0(x) = \sum_{k=1}^n \mathbb{1}_{\frac{2k-1}{2n}<x<\frac{2k}{2n}} - \sum_{k=1}^n \mathbb{1}_{2+\frac{2(k-1)}{2n}<x<2+\frac{2k-1}{2n}}.$$

Using the dynamics of Section 3.2, we get a solution (ρ_n, u_n, Q_n, v_n) to (1.18)-(1.19) with the constraints (1.10)-(1.11) and with regularities (1.15)-(1.17), which is given by

$$\rho_n(t,x) = \sum_{k=1}^{n} \mathbb{I}_{t+\frac{2k-1}{2n}<x<t+\frac{2k}{2n}} \mathbb{I}_{x<3/2} + \sum_{k=1}^{n} \mathbb{I}_{2-t+\frac{2(k-1)}{2n}<x<2-t+\frac{2k-1}{2n}} \mathbb{I}_{x>3/2},$$

$$\rho_n u_n(t,x) = \sum_{k=1}^{n} \mathbb{I}_{t+\frac{2k-1}{2n}<x<t+\frac{2k}{2n}} \mathbb{I}_{x<3/2} - \sum_{k=1}^{n} \mathbb{I}_{2-t+\frac{2(k-1)}{2n}<x<2-t+\frac{2k-1}{2n}} \mathbb{I}_{x>3/2},$$

$$Q_n(t,x) = -2\sum_{k=1}^{n} \mathbb{I}_{[\frac{3}{2}+\frac{2(k-1)}{2n},\frac{3}{2}+\frac{2(k-1)}{2n}+\frac{1}{n}]}(t)\delta(x-\frac{3}{2}), \qquad v_n = 0.$$

We study now the convergence of ρ_n and Q_n when $n \to \infty$.
We have, for a test function φ,

$$\begin{aligned} < Q_n, \varphi > &= -2\sum_{k=1}^{n} \int_{\frac{3}{2}+\frac{2(k-1)}{2n}}^{\frac{3}{2}+\frac{2(k-1)}{2n}+\frac{1}{n}} \varphi(t,\frac{3}{2})\,dt \\ &= -2\sum_{k=1}^{n} \varphi(\frac{3}{2}+\frac{k-1}{n},\frac{3}{2})\frac{1}{n} - 2\sum_{k=1}^{n} \varphi'(c_k,\frac{3}{2})\frac{1}{2n^2}, \end{aligned}$$

where $c_k \in [\frac{3}{2}+\frac{2(k-1)}{2n}, \frac{3}{2}+\frac{2(k-1)}{2n}+\frac{1}{n}]$. Now

$$\sum_{k=1}^{n} \varphi(\frac{3}{2}+\frac{k-1}{n},\frac{3}{2})\frac{1}{n} \to \int_{\frac{3}{2}}^{\frac{5}{2}} \varphi(t,\frac{3}{2})\,dt, \text{ as } n\to\infty,$$

therefore

$$< Q_n, \varphi > \to -2\int_{\frac{3}{2}}^{\frac{5}{2}} \varphi(t,\frac{3}{2})\,dt, \text{ as } n\to\infty,$$

that is to say

$$Q_n \rightharpoonup -2\mathbb{I}_{[\frac{3}{2},\frac{5}{2}]}(t)\delta(x=\frac{3}{2}), \text{ as } n\to\infty.$$

Setting

$$r_n(t,x) = \sum_{k=1}^{n} \mathbb{I}_{t+\frac{2k-1}{2n}<x<t+\frac{2k}{2n}} \mathbb{I}_{x<3/2},$$

we have, for a test function φ,

$$< r_n, \varphi > = \sum_{k=1}^{n} \int_{\frac{2k-1}{2n}}^{\frac{2k}{2n}} \left(\int_{y}^{\frac{3}{2}} \varphi(x-y,x)\,dx \right) dy,$$

by the change $x-y=t$. Now,

$$\begin{aligned} < r_n, \varphi > &= \sum_{k=1}^{n} \left(\int_{\frac{2k-1}{2n}}^{\frac{3}{2}} \varphi(x-\frac{2k-1}{2n},x)\,dx \frac{1}{2n} + \underset{n\to\infty}{o}(\frac{1}{n}) \right) \\ &= \frac{1}{2}\sum_{k=1}^{n} \left(\int_{\frac{2k-1}{2n}}^{\frac{3}{2}} \varphi(x-\frac{2k-1}{2n},x)\,dx \right) \frac{1}{n} + \underset{n\to\infty}{o}(1) \\ &\underset{n\to\infty}{\to} \frac{1}{2}\int_0^1 \left(\int_y^{\frac{3}{2}} \varphi(x-y,x)\,dx \right) dy = \iint_A \varphi(t,x)\,dtdx, \end{aligned}$$

where

$$A = \{(t,x) \in \mathbf{R}^+ \times \mathbf{R}\ ;\ t < x < t+1,\ x < \frac{3}{2}\}.$$

Thus, we obtain

$$\rho_n \rightharpoonup \rho = \mathbb{1}_{t<x<t+1}\mathbb{1}_{x<\frac{3}{2}} + \mathbb{1}_{2-t<x<3-t}\mathbb{1}_{x>\frac{3}{2}},\ \text{as } n \to \infty,$$

and

$$\rho_n u_n \rightharpoonup \rho u = \mathbb{1}_{t<x<t+1}\mathbb{1}_{x<\frac{3}{2}} - \mathbb{1}_{2-t<x<3-t}\mathbb{1}_{x>\frac{3}{2}},\ \text{as } n \to \infty.$$

We use the stability result and we get that $(\rho, u, Q, 0)$ is a solution to (1.18)-(1.19) with the constraints (1.10)-(1.11) and with regularities (1.15)-(1.17) for the initial data ρ^0 and $\rho^0 u^0$. Evering happens as with sticky blocks of height 1, but 1/2 is playing the role of 1. The two blocks with a height of 1/2 lose their mass when they collide, all the mass disappears though ρ is staying less than 1/2. The exclusion relation $(1-\rho)Q = 0$ is lost because we get here $(\frac{1}{2} - \rho)Q = 0$.

4 Numerical Scheme

In this section, we expose the numerical method we use to get the dynamics of section 3.

4.1 The scheme

The scheme we propose here is based on a splitting method. We first solve the pressureless system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0, \end{cases} \tag{4.1}$$

and then we perform the constraint problem

$$\begin{cases} \partial_t \rho = Q, \\ \partial_t(\rho u) = Qu, \end{cases} \qquad 1 \le \rho \le 1, \quad Q \le 0, \quad (1-\rho)Q = 0. \tag{4.2}$$

For the first step, we use a flux vector splitting scheme, and for the second step we just replace ρ by $\min(\rho, 1)$, while u is unchanged. Let us detail more exactly the process.

First step: We use the notation $U = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}$. The flux of the system is $F(U) = \begin{pmatrix} \rho u \\ \rho u^2 \end{pmatrix}$. We consider the flux vector splitting scheme proposed in [5], which has the property to be entropy compatible. We refer to [7] for a general analysis of discrete entropy inequalities for flux vector splittings.

$$\begin{pmatrix} \rho \\ \rho u \end{pmatrix}_i^{n+1} - \begin{pmatrix} \rho \\ \rho u \end{pmatrix}_i^{n} + \frac{\Delta t}{\Delta x}(F^n_{i+1/2} - F^n_{i-1/2}) = 0, \tag{4.3}$$

with the flux

$$F^n_{i+1/2} = F^+(U^n_i) + F^-(U^n_{i+1}), \tag{4.4}$$

where

$$F^+(U) = F(U)\mathbb{1}_{u>0}, \qquad F^-(U) = F(U)\mathbb{1}_{u<0}. \tag{4.5}$$

We notice that $F(U) = F^+(U) + F^-(U)$.
Second step: We take

$$\begin{cases} \rho_i^{n+1} = \min(\rho_i^n, 1), \\ u_i^{n+1} = u_i^n. \end{cases} \tag{4.6}$$

It is consistant with the system (4.2) because

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = Q_i^n \leq 0,$$

and

$$\frac{\rho_i^{n+1} u_i^{n+1} - \rho_i^n u_i^n}{\Delta t} = Q_i^n u_i^n.$$

Process: To pass from the step n to the step $n+1$, we compute (4.3), then we process (4.6).

4.2 Numerical results

In order to establish the sticky blocks dynamics of Section 3, we propose some results which allow to define a suitable one.

- First, to illustrate the case where the two blocks have different volumes and opposite velocity, we take Riemann initial data, with

$$(\rho)_{t=0} = \mathbb{1}_{[0.5,1]} + \mathbb{1}_{[1.25,1.5]}, \qquad (\rho u)_{t=0} = \mathbb{1}_{[0.5,1]} - \mathbb{1}_{[1.25,1.5]}.$$

  The collision is located at $t^* = 0.125$ and the end of the shock is at $t_f = 0.375$. The computational domain is $[0,2]$ and the results are represented at $t = 0.25$ (Figure 2) and at t_f (Figure 3). The exact solution from the dynamics of Section 2 is given by

$$(\rho)_{t=0.25} = \mathbb{1}_{[0.75,1.25]}, \qquad (\rho u)_{t=0.25} = \mathbb{1}_{[0.75,1.125]} - \mathbb{1}_{[1.125,1.25]},$$

$$(\rho)_{t=0.375} = (\rho u)_{t=0.375} = \mathbb{1}_{[0.875,1.125]}.$$

- To illustrate the case where two blocks have the same volume and different velocity's modules, we take

$$(\rho)_{t=0} = \mathbb{1}_{[0.25,0.5]} + \mathbb{1}_{[1.25,1.5]}, \qquad (\rho u)_{t=0} = \mathbb{1}_{[0.25,0.5]} - \mathbb{1}_{[1.25,1.5]}/2.$$

  The collision starts at $t^* = 0.5$, the results are represented at $t = 0.75$ (Figure 4) and at $t = t_f = 0.875$ (Figure 5). The solution is

$$(\rho)_{t=0.75} = \mathbb{1}_{[1,1.125]}, \quad (\rho u)_{t=0.75} = \mathbb{1}_{[1,1.0625]} - \mathbb{1}_{[1.0625,1.125]}/2,$$

$$(\rho)_{t=0.875} = \mathbb{1}_{0.625}.$$

– Finally, to situate the velocity's effect with regard to the first example, we consider the case where

$$(\rho)_{t=0} = \mathbb{1}_{[0.25,0.75]} + \mathbb{1}_{[1.5,1.75]}, \qquad (\rho u)_{t=0} = \mathbb{1}_{[0.25,0.75]}/2 - \mathbb{1}_{[1.5,1.75]},$$

$t^* = 0.5$ and the results are represented at $t = 0.75$ (Figure 6) and at $t = t_f = 5/6$ (Figure 7). The solution is

$$(\rho)_{t=0.75} = \mathbb{1}_{[0.625,1]}, \qquad (\rho u)_{t=0.75} = \mathbb{1}_{[0.625,0.9375]}/2 - \mathbb{1}_{[0.9375,1]},$$

$$(\rho)_{t=5/6} = \mathbb{1}_{[2/3,11/12]}, \qquad (\rho u)_{t=5/6} = \mathbb{1}_{[2/3,11/12]}/2.$$

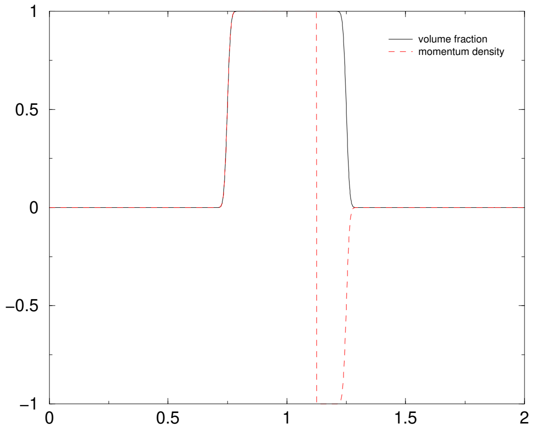


Figure 2

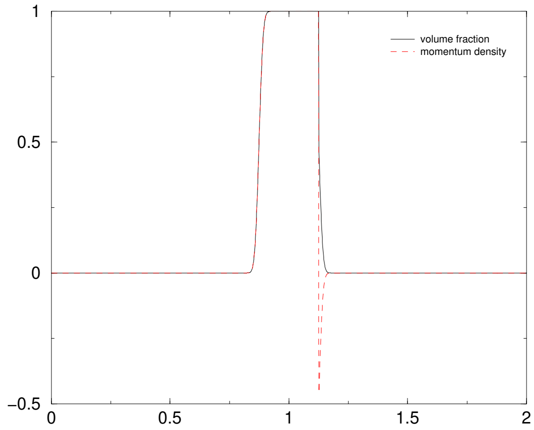


Figure 3

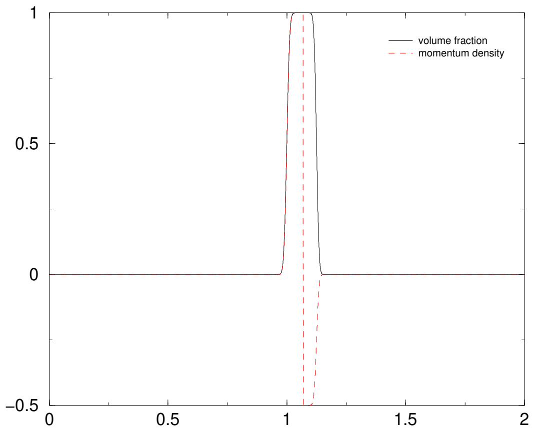


Figure 4

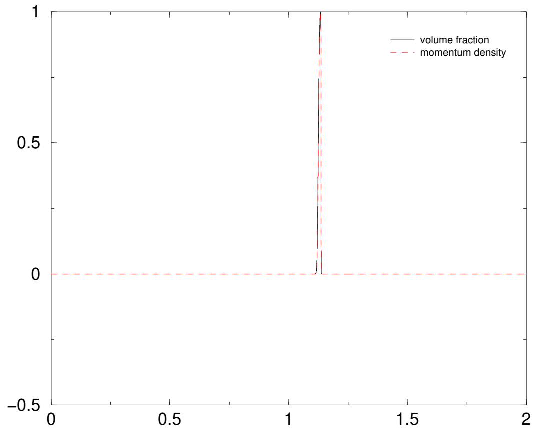


Figure 5

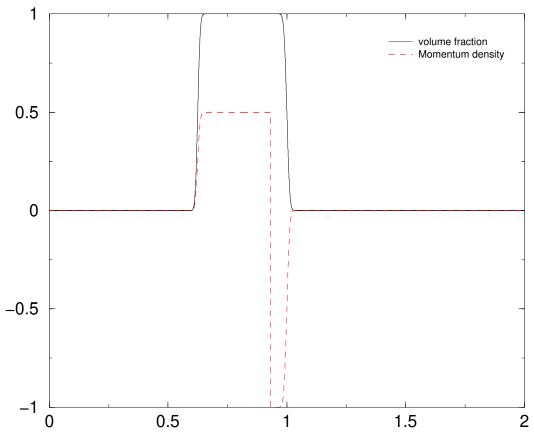


Figure 6

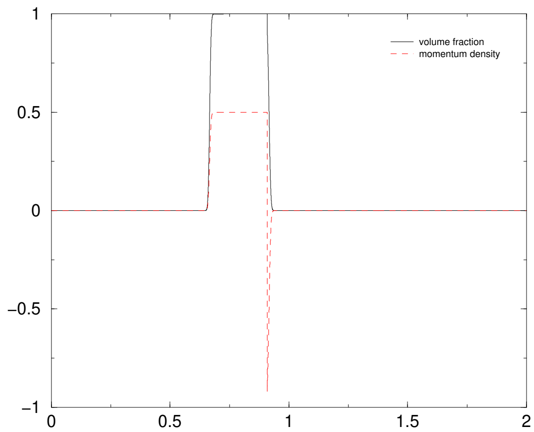


Figure 7

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Cinquième partie

Annexes

Annexe A

Correspondances cinétiques

On parle de correspondance cinétique pour désigner une correspondance entre les solutions d'un système de lois de conservation et une équation de type cinétique. Une correspondance cinétique pour les lois de conservation scalaire multidimensionnelles a été étudié dans [2] et une correspondance dans le cas des gaz isentropiques et pour le p-système se trouve dans [3]. Cette dernière correspondance n'est pas purement cinétique dans le sens où l'équation de transport décrivant les quantités microscopiques a une vitesse faisant intervenir des variables macroscopiques. Dans cette Annexe, nous allons obtenir une correspondance cinétique pour la dynamique des gaz qui est purement cinétique, la vitesse de l'équation de transport décrivant les quantités microscopiques étant ξ.
On établit ainsi une correspondance entre les solutions du système de la dynamique des gaz unidimensionnelle

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \kappa\rho^\gamma) = 0, \end{cases} \tag{0.1}$$

d'énergie finie $\sup\limits_t \int_{\mathbf{R}} \eta(\rho, u)\, dx < +\infty$ et satisfaisant les conditions d'entropie

$$\partial_t \eta_S(\rho, u) + \partial_x G_S(\rho, u) \leq 0 \text{ pour toute } S \text{ convexe sous-quadratique}, \tag{0.2}$$

et les solutions de l'équation cinétique

$$\partial_t f + \xi \partial_x f = -\partial_{vv} m, \tag{0.3}$$

avec

$$f(t, x, \xi, v) = \Phi(\rho(t, x), u(t, x), \xi, v) \text{ pour des fonctions } \rho \text{ et } u, \tag{0.4}$$

et

$$\begin{gathered} < m, 1 >_\xi \text{ une mesure en } (t, x, v), \\ \text{positive, bornée sur } \mathbf{R}^+ \times \mathbf{R}^2 \text{ et à support compact en } v. \end{gathered} \tag{0.5}$$

On rappelle pour cela que les invariants de Riemann sont définis par

$$\omega_1 = u - \frac{2\sqrt{\gamma\kappa}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}, \ \omega_2 = u + \frac{2\sqrt{\gamma\kappa}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}},$$

et la distribution cinétique de notre modèle BGK par

$$\Phi(\rho,u,\xi,v)=\frac{(1-\theta)^2}{\theta}\frac{c_{\gamma,\kappa}}{J_\lambda}\mathbb{I}_{\omega_1<\xi<\omega_2}\mathbb{I}_{\omega_1<v<\omega_2}|\xi-v|^{2\lambda-1}\Upsilon_{\lambda-1}(z), \qquad (0.6)$$

où

$$z=\frac{(\xi+v)(\omega_1+\omega_2)-2(\omega_1\omega_2+\xi v)}{(\omega_2-\omega_1)|\xi-v|}, \qquad (0.7)$$

et

$$\Upsilon_{\lambda-1}(z)=\int_1^z (y^2-1)^{\lambda-1}\,dy, \qquad z\geq 1.$$

Les entropies, flux d'entropie sont définis par

$$\eta_S(\rho,u)=\iint \Phi(\rho,u,\xi,v)S(v)\,d\xi\,dv, \qquad G_S(\rho,u)=\iint \xi\Phi(\rho,u,\xi,v)S(v)\,d\xi\,dv,$$

et l'énergie physique s'écrit simplement $\eta=\eta_{v^2/2}$, $G=G_{v^2/2}$.
Le noyau Φ est continue par rapport à ω_1, ω_2. On va tout d'abord améliorer l'étude faite dans [1] afin d'obtenir le lemme suivant, utile pour la continuité en temps du noyau.

Lemme 1 *La fonction $(\xi,v)\mapsto|\xi-v|^{2\lambda-1}\Upsilon_{\lambda-1}(\frac{A}{|\xi-v|})$ est localement intégrable.*

Preuve. Pour $\lambda>1/2$, $\Upsilon_{\lambda-1}(z)\underset{z\to\infty}{\sim}z^{2\lambda-1}/(2\lambda-1)$, d'où $\Upsilon_{\lambda-1}(\frac{A}{|\xi-v|})\underset{|\xi-v|\to 0}{\sim}L|\xi-v|^{-(2\lambda-1)}$ où L est une constante indépendante de ξ et v, ainsi $\lim\limits_{|\xi-v|\to 0}|\xi-v|^{2\lambda-1}\Upsilon_{\lambda-1}(z)$ est finie.
Pour $\lambda=1/2$, $\Upsilon_{\lambda-1}(\frac{A}{|\xi-v|})\underset{|\xi-v|\to 0}{\sim}|\ln|\xi-v||$ et $(x,y)\mapsto\ln|x-y|$ est intégrable.
Pour $0<\lambda<1/2$, $\Upsilon_{\lambda-1}(z)$ a une limite finie lorsque $|\xi-v|\to 0$ et $|\xi-v|^{2\lambda-1}\Upsilon_{\lambda-1}(\frac{A}{|\xi-v|})$ est intégrable car $(x,y)\mapsto 1/|y-x|^l$ l'est pour $l\in]0,1[$. $\square$

On a le théorème de correspondance suivant :

Théorème 0.1
1) Soient $\rho,u\in L^\infty(]0,\infty[\times\mathbf{R})$ telles que $\rho\geq 0$, $\rho,\rho u\in C_t([0,\infty[,L^1_{loc\ x}(\mathbf{R}))$ solution de (0.1)-(0.2). Alors notant $f(t,x,\xi,v)=\Phi(\rho(t,x),u(t,x),\xi,v)$, $(f,vf)\in C_t([0,\infty[,L^1_{(loc\ x),\xi,v}(\mathbf{R}^3))\cap L^\infty_{t,x}(]0,\infty[\times\mathbf{R},L^1_{\xi,v}(\mathbf{R}^2))$ satisfait (0.3) pour une certaine distribution m satisfaisant (0.5).
2) Soit la fonction $f=\Phi(\rho,u,\xi,v)$ telle que $(f,vf)\in C_t([0,\infty[,L^1_{(loc\ x),\xi,v}(\mathbf{R}^3))$ $\cap\ L^\infty_{t,x}(]0,\infty[\times\mathbf{R},L^1_{\xi,v}(\mathbf{R}^2))$ satisfaisant (0.3) pour une certaine distribution (0.5). Alors $\rho,\rho u\in L^\infty_{t,x}(]0,\infty[\times\mathbf{R})\cap C_t([0,\infty[,L^1_{loc}(\mathbf{R}))$ satisfont (0.1)-(0.2).

Preuve du 1). Montrons tout d'abord que $f\in C_t([0,\infty[,L^1_{(loc\ x),\xi,v}(\mathbf{R}^3))$. Cela revient à montrer que si $\rho_n\to\rho$, $\rho_n u_n\to\rho u$ dans $L^1_{loc}(\mathbf{R})$, alors

$$\Phi(\rho_n,u_n,\xi,v)\to\Phi(\rho,u,\xi,v) \text{ dans } L^1_{(loc\ x),\xi,v}(\mathbf{R}^3). \qquad (0.8)$$

Supposons donc que $\rho_n \to \rho$, $\rho_n u_n \to \rho u$ dans $L^1_{loc}(\mathbf{R})$. Alors, pour une sous-suite, on a $\rho_n \to \rho$ et $\rho_n u_n \to \rho u$ p.p. Notons $E = \{x\ ;\ \rho(x) = 0\}$. Sur E^C, $\Phi(\rho_n(x), u_n(x), \xi, v) \to \Phi(\rho(x), u(x), \xi, v)$ p.p. car Φ est continue par rapport à ω_1, ω_2 et ω_1, ω_2 sont continues par rapport à ρ et u. Sur E, $\rho = 0$ c'est-à-dire $\omega_1 = \omega_2$, d'où $\Phi(\rho, u, \xi, v) = 0$ p.p. Puisque

$$\int_E \int_{\mathbf{R}} \int_{\mathbf{R}} \Phi(\rho_n, u_n, \xi, v)\, d\xi\, dv\, dx \quad = \quad \int_E \rho_n\, dx \to 0, \text{ lorsque } n \to +\infty,$$

on obtient

$$\int_E \int_{\mathbf{R}} \int_{\mathbf{R}} |\Phi(\rho_n, u_n, \xi, v) - \Phi(\rho, u, \xi, v)|\, d\xi\, dv\, dx \to 0, \text{ lorsque } n \to +\infty.$$

Finalement, sur $E \cup E^C = \mathbf{R}$, on a pour une sous-suite

$$\Phi(\rho_n, u_n, \xi, v) \to \Phi(\rho, u, \xi, v) \quad \text{p.p. } x, \xi, v. \tag{0.9}$$

Ecrivant

$$(\xi + v)(\omega_1 + \omega_2) - 2(\omega_1\omega_2 + \xi v) = (\omega_2 - \xi)(v - \omega_1) + (\xi - \omega_1)(\omega_2 - v)$$

dans la définition (0.7) de $z(\omega_1, \omega_2, \xi, v)$, on voit que

$$1 \le z \le \frac{\omega_2 - \omega_1}{|\xi - v|} \le \frac{\omega_M - \omega_m}{|\xi - v|} \equiv z^0,$$

puisque les bornes $\rho, u \in L^\infty$ entraîne l'existence de deux constantes ω_m et ω_M telles que $\omega_m \le \omega_1 \le \omega_2 \le \omega_M$. On a alors la majoration

$$\begin{aligned} \Phi(\rho_n, u_n, \xi, v) &= k\mathbb{1}_{\omega_1 < \xi, v < \omega_2} |\xi - v|^{2\lambda - 1} \Upsilon_{\lambda - 1}(z) \\ &\le k\mathbb{1}_{\omega_m < \xi, v < \omega_M} |\xi - v|^{2\lambda - 1} \Upsilon_{\lambda - 1}(z^0), \end{aligned} \tag{0.10}$$

qui, avec la convergence (0.9) et le lemme 1 permet d'obtenir (0.8) pour une sous-suite. Comme la limite obtenue ne dépend pas de la sous-suite, on a finalement la convergence voulue.
Le support compact en v permet d'avoir de même $vf \in C_t([0, \infty[, L^1_{(loc\ x), \xi, v}(\mathbf{R}^3))$. La relation (0.10) donne de plus $f, vf \in L^\infty_{t,x}(]0, \infty[\times\mathbf{R}, L^1_{\xi, v}(\mathbf{R}^2))$.
Définissons maintenant une distribution m par (0.3) et prenons des fonctions tests $\varphi(t,x) \ge 0$, $\psi_1(\xi) \ge 0$ et $\psi(v) \ge 0$. En posant $S(v) = \int_{v_0}^v \int_{v_0}^y \psi(z)\, dz\, dy \mathbb{1}_{v \ge v_0}$ avec v_0 la borne inférieure du support de ψ, S est sous-quadratique, $S'' = \psi \ge 0$ et on a, puisque Φ est à support compact en v,

$$\begin{aligned} < m, \varphi(t,x)\psi_1(\xi)\psi(v) > &= \iiiint \Phi(\rho, u, \xi, v)\psi_1(\xi) S(v) \partial_t \varphi(t,x)\, dt\, dx\, d\xi\, dv \\ &\quad + \iiiint \xi \Phi(\rho, u, \xi, v)\psi_1(\xi) S(v) \partial_x \varphi(t,x)\, dt\, dx\, d\xi\, dv. \end{aligned}$$

Mais $\partial_{vv} m$ est à support compact en ξ, donc m l'est aussi et on peut prendre $\psi_1(\xi) = 1$, d'où

$$\begin{aligned} < m, \varphi(t,x)\psi(v) > &= \iint \eta_S(\rho, u) \partial_t \varphi(t,x)\, dt\, dx + \iint G_S(\rho, u) \partial_x \varphi(t,x)\, dt\, dx \\ &\ge 0, \end{aligned}$$

et ainsi $< m, 1 >_\xi$ est une distribution positive.
On note $M =< m, 1 >_\xi$ cette mesure. Puisque $\partial_{vv} m$ est à support compact en v, on peut prendre $S(v) = v^2/2$ dans l'égalité

$$< \partial_{vv} M, \varphi(t,x) S(v) >=< M, \varphi(t,x)\psi(v) >,$$

d'où

$$< M, \varphi(t,x) >= \iint \eta(\rho,u)\partial_t\varphi(t,x)\, dt\, dx + \iint G(\rho,u)\partial_x\varphi(t,x)\, dt\, dx.$$

Prenant une fonction test $\varphi(t,x) = \varphi_1(t)\varphi_R(x)$ avec φ_R paire valant 1 sur $[-R, R]$ et décroissante sur $[R, +\infty[$, puis faisant $R \to \infty$, on obtient

$$< M, \varphi_1(t) >= \iint \eta(\rho,u)\varphi_1'(t)\, dt\, dx.$$

On montre, comme pour $f \in C_t(L^1_{(loc\ x),\xi,v})$, que $\eta(\rho,u) \in C_t(L^1_{loc\ x})$ et ainsi faisant $R' \to \infty$ avec les fonctions tests positives $\varphi_1^{R'}$ valant 1 sur $[1/R', T - 1/R']$, croissante sur $[0, 1/R']$, décroissante sur $[T - 1/R', T]$ et nulle en 0 et T, on trouve

$$\int_0^T \iint M(\, dt,\, dx,\, dv) = \int_{\mathbf{R}} (\eta(\rho,u)(T) - \eta(\rho,u)(0))\, dx,$$

et donc pour une solution d'énergie finie, $< m, 1 >_\xi$ est bornée.
Il reste à voir que $< m, 1 >_\xi$ est à support compact en v, ceci provient du résultat du lemme 2 ci-après.
Preuve du 2). On a $(\rho, \rho u) = \int_{\mathbf{R}} \int_{\mathbf{R}} (1, v) f(t,x,\xi,v)\, d\xi\, dv$, d'où par exemple

$$\int_K |\rho_n(x) - \rho(x)|\, dx \le \int_K \iint_{\mathbf{R}^2} |\Phi(\rho_n(x), u_n(x), \xi, v) - \Phi(\rho(x), u(x), \xi, v)|\, d\xi\, dv\, dx,$$

ce qui permet d'avoir les régularités.
Maintenant, pour des fonctions tests $\varphi(t,x,\xi)$ et $\psi(v)$, on a

$$\begin{aligned} < m, \varphi(t,x,\xi)\psi''(v) > \;&=\; \iiiint \Phi(\rho,u,\xi,v)\psi(v)\partial_t\varphi(t,x,\xi)\, dt\, dx\, d\xi\, dv \\ &\quad + \iiiint \xi\Phi(\rho,u,\xi,v)\psi(v)\partial_x\varphi(t,x,\xi)\, dt\, dx\, d\xi\, dv. \end{aligned}$$

Grâce aux supports compacts en v, on peut prendre $\psi(v) = 1$ et v pour obtenir (0.1). M bornée permet d'avoir une solution d'énergie finie. Prenant ψ convexe, sous-quadratique, on obtient (0.2) pour $\psi \in C^\infty$ puis par approximation pour $\psi \in C^1$. $\square$

Il reste à prouver le lemme sur les supports, à savoir :

Lemme 2 *Si $M(dt, dx, dv)$ est une mesure positive bornée telle que $\partial_{vv} M$ soit à support compact en v, alors M est à support compact en v.*

Preuve. Il existe $R_0 > 0$ tel que si $\psi \in \mathcal{D}(]-\infty, -R_0[\cup]R_0, +\infty[)$, alors on a $< \partial_{vv}M, \varphi(t,x)\psi(v) >= 0$. La distribution $\partial_{vv}M$ est donc nulle sur l'ouvert $]0,\infty[_t \times \mathbf{R}_x \times]R_0, +\infty[_v$ et on peut appliquer localement sur cet ouvert le lemme suivant, ce qui est possible d'après le choix des ψ_R de la preuve de ce lemme. On obtient alors $M = 0$ sur $]0,\infty[_t \times \mathbf{R}_x \times]R_0, +\infty[_v$. On procède de même sur $]0,\infty[_t \times \mathbf{R}_x \times]-\infty, -R_0[_v$. □

Lemme 3 *Si $M(dt, dx, dv)$ est une mesure positive bornée telle que $\partial_{vv}M = 0$, alors $M = 0$.*

Preuve. Comme $\partial_v(\partial_v M) = 0$, il existe une distribution T indépendante de v telle que $\partial_v T = M$, c'est-à-dire qu'il existe $\tilde{T}$ telle que pour toute fonction test $\varphi(t,x,v)$, on ait

$$< T, \varphi >=< \tilde{T}, \int_{\mathbf{R}} \varphi \, dv > .$$

On a de plus

$$\begin{aligned} < \partial_v(vT), \varphi > &= - < T, v\partial_v\varphi >= - < \tilde{T}, \int_{\mathbf{R}} v\partial_v\varphi \, dv > \\ &= < \tilde{T}, \int_{\mathbf{R}} \varphi \, dv >=< T, \varphi >, \end{aligned}$$

et donc $\partial_v(vT) = T$. Les solutions de l'équation $\partial_v M = T$ d'inconnue M sont distinctes d'une distribution indépendante de v, donc $M = vT + S$ où S est indépendante de v.
Soit $\varphi(t,x)$ et $\psi(v)$ des fonctions tests positives et plus petites que 1. On a

$$\begin{aligned} < M, \varphi\psi > &= < T, v\varphi\psi > + < S, \varphi\psi > \\ &= < \tilde{T}, \varphi > \int_{\mathbf{R}} v\psi(v) \, dv + < \tilde{S}, \varphi > \int_{\mathbf{R}} \psi(v) \, dv, \end{aligned}$$

d'où

$$0 \leq \int_{\mathbf{R}} v\psi(v) \, dv < \tilde{T}, \varphi > + \int_{\mathbf{R}} \psi(v) \, dv < \tilde{S}, \varphi > \leq \iiint M(\,dt,\, dx,\, dv).$$

On fixe φ et on prend ψ_R nulle pour $v \leq R$ et $v \geq 2R$, valant 1 sur $[5R/4, 7R/4]$ et monotone sur $[R, 5R/4]$ et $[7R/4, 2R]$. On obtient alors

$$R/2 \leq \int_{\mathbf{R}} \psi_R(v) \, dv \leq R, \qquad R^2/2 \leq \int_{\mathbf{R}} v\psi_R(v) \, dv \leq 2R^2.$$

Si $< \tilde{T}, \varphi >$ et $< \tilde{S}, \varphi >$ sont tous deux positifs alors $R^2/2 < \tilde{T}, \varphi > +R/2 < \tilde{S}, \varphi >$ est bornée pour tout R ce qui ne se peut que si $< \tilde{T}, \varphi >=< \tilde{S}, \varphi >= 0$. Si maintenant $< \tilde{T}, \varphi > \geq 0$ et $< \tilde{S}, \varphi > \leq 0$, alors $R^2/2 < \tilde{T}, \varphi > +R < \tilde{S}, \varphi >$ est bornée pour tout R et donc $< \tilde{T}, \varphi >= 0$, mais alors $0 \leq \int_{\mathbf{R}} \psi(v) \, dv < \tilde{S}, \varphi >$ entraîne $< \tilde{S}, \varphi >= 0$.
Finalement, si $< \tilde{T}, \varphi > \leq 0$ et $< \tilde{S}, \varphi > \geq 0$ alors $R^2/2(- < \tilde{T}, \varphi >) \leq R < \tilde{S}, \varphi >$ ce qui devient impossible pour R assez grand lorsque $< \tilde{T}, \varphi >< 0$ et donc $< \tilde{T}, \varphi >= 0$. On est alors revenu au premier cas.

Le dernier cas de figure est quant à lui évident. Ainsi dans tous les cas, on trouve $< \tilde{T}, \varphi > = < \tilde{S}, \varphi > = 0$ et on peut conclure que $M = 0$. $\square$

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Annexe B

Relaxation des solutions non bornées

Dans cette Annexe, on prouve la relaxation $f_\varepsilon - M[f_\varepsilon] \to 0$ sans supposer que les données initiales sont dans un domaine invariant et sans avoir recours à un principe du maximum.
Soit f_ε la solution du Théorème d'existence globale pour le modèle BGK de la dynamique des gaz isentropiques de [1] avec la même donnée initiale $f^0(x,\xi)$. Les seules hypothèses sur la donnée initiales sont donc : $f^0 \in L^1(\mathbf{R}_x \times \mathbf{R}_\xi)$, $f^0(x,\xi) \in D$ p.p. et $\int_{\mathbf{R}^2} H(f^0(x,\xi),\xi)\, dx\, d\xi = C_H^0 < \infty$.
Nous prouvons le résultat suivant :

Théorème 0.2 *Après extraction d'une sous-suite, $f_\varepsilon - M[f_\varepsilon] \underset{\varepsilon\to 0}{\to} 0$ dans $L^1_{(loc\ t,x),\xi}$.*

Preuve Le plan de la preuve est le même que celui de la partie 2, chapitre 2 ([2]) mais la technique est plus délicate.
Nous utilisons tout d'abord le résultat de dissipation d'entropie prouvé à la référence ci-dessus :

$$\iiint_{]0,T[\times\mathbf{R}\times\mathbf{R}} |(H'(f_\varepsilon,\xi) - T(\rho_\varepsilon,u_\varepsilon)) \cdot (f_\varepsilon - M[f_\varepsilon])|\, dtdxd\xi \le C_H^0 \varepsilon, \tag{0.1}$$

ce qui entraîne pour une sous-suite

$$(H'(f_\varepsilon,\xi) - T(\rho_\varepsilon,u_\varepsilon)) \cdot (f_\varepsilon - M[f_\varepsilon]) \underset{\varepsilon\to 0}{\to} 0 \quad a.e.\ t,x,\xi. \tag{0.2}$$

On remarque que pour $(M[f_\varepsilon])_0 \neq 0$, $T(\rho_\varepsilon,u_\varepsilon) = H'(M[f_\varepsilon],\xi)$.
Première étape : pour $\lambda \le 1$, $(f_\varepsilon)_0 - (M[f_\varepsilon])_0 \underset{\varepsilon\to 0}{\to} 0$ dans $L^1_{(loc\ t,x),\xi}$.
On a les majorations suivantes qui permettent de contrôler le comportement à l'infini :

$$\iiint_{|\xi|\ge R} (f_\varepsilon)_0\, dtdxd\xi \le \frac{1}{R^2} \iiint_{|\xi|\ge R} |\xi|^2 (f_\varepsilon)_0\, dtdxd\xi \le \frac{1}{R^2}\frac{4}{\theta} T C_H^0,$$

$$F^{1/\lambda} \iiint_{(f_\varepsilon)_0 \ge F} (f_\varepsilon)_0\, dtdxd\xi \le \iiint_{(f_\varepsilon)_0 \ge F} (f_\varepsilon)_0^{1/\lambda+1}\, dtdxd\xi \le (1+1/\lambda)\frac{2c_{\gamma,\kappa}^{1/\lambda}}{\theta} T C_H^0,$$

et des relations semblables pour $(M[f_\varepsilon])_0$.
Soit $\mu > 0$. Il existe donc $F > 0$ et $R > 0$ tels que

$$\iiint_{(f_\varepsilon)_0 \geq F} (f_\varepsilon)_0 \, dtdxd\xi \leq \mu, \qquad \iiint_{(M[f_\varepsilon])_0 \geq F} (M[f_\varepsilon])_0 \, dtdxd\xi \leq \mu,$$

$$\iiint_{|\xi| \geq R} (f_\varepsilon)_0 \, dtdxd\xi \leq \mu, \qquad \iiint_{|\xi| \geq R} (M[f_\varepsilon])_0 \, dtdxd\xi \leq \mu.$$

Notons

$$B_R = \{(t,x,\xi) \; ; \; (t,x) \in B, \; |\xi| \leq R\},$$
$$P_1 = \{(t,x,\xi) \; ; \; (M[f_\varepsilon])_0 \neq 0, \; (f_\varepsilon)_0 \leq F, \; (M[f_\varepsilon])_0 \leq F\},$$
$$P_2 = \{(t,x,\xi) \; ; \; (M[f_\varepsilon])_0 \neq 0 \text{ et } ((f_\varepsilon)_0 > F \text{ ou } (M[f_\varepsilon])_0 > F)\}.$$

La relation de conservation $\int_{\mathbf{R}} (f_\varepsilon)_0 \, d\xi = \int_{\mathbf{R}} (M[f_\varepsilon])_0 \, d\xi$ p.p. t, x entraîne, pour tout domaine borné B en (t,x),

$$\iiint_{B_R \cap \{(M[f_\varepsilon])_0 = 0, (f_\varepsilon)_0 \leq F\}} (f_\varepsilon)_0 \, dtdxd\xi \tag{0.3}$$
$$= \iiint_{\{(t,x) \in B, |\xi| \geq R\}} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0) \, dtdxd\xi + \iiint_{B_R \cap P_1} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0) \, dtdxd\xi$$
$$+ \iiint_{B_R \cap P_2} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0) \, dtdxd\xi + \iiint_{B_R \cap \{(M[f_\varepsilon])_0 = 0, (f_\varepsilon)_0 > F\}} -(f_\varepsilon)_0 \, dtdxd\xi.$$

Le premier terme du membre de droite est plus petit que 2μ et le dernier est plus petit que μ.
Sur l'ensemble P_1, $(f_\varepsilon)_0$ et $(M[f_\varepsilon])_0$ sont bornées par F et $(M[f_\varepsilon])_0$ est non nul, donc (voir partie 2, chapitre 2 ([2]))

$$(H'(M[f_\varepsilon], \xi) - H'(f_\varepsilon, \xi)) \cdot (M[f_\varepsilon] - f_\varepsilon) \geq K_1 F^{1/\lambda - 1} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2,$$

ce qui avec (0.2) donne

$$\iiint_{B_R \cap P_1} |(M[f_\varepsilon])_0 - (f_\varepsilon)_0| \, dtdxd\xi \underset{\varepsilon \to 0}{\to} 0,$$

et donc pour ε petit, ce terme est inférieur à μ.
Notant

$$P_2^f = \{(t,x,\xi) \; ; \; (M[f_\varepsilon])_0 \neq 0, \; (f_\varepsilon)_0 > F, \; (M[f_\varepsilon])_0 \leq F\},$$

on a

$$\iiint_{B_R \cap P_2^f} (M[f_\varepsilon])_0 \, dtdxd\xi \leq \iiint_{B_R \cap P_2^f} F \, dtdxd\xi \leq \iiint_{B_R \cap P_2^f} (f_\varepsilon)_0 \, dtdxd\xi \leq \mu, \tag{0.4}$$

et de même sur $\{(t,x,\xi)\ ;\ (M[f_\varepsilon])_0 \neq 0,\ (f_\varepsilon)_0 \leq F,\ (M[f_\varepsilon])_0 > F\}$. Le terme avec l'ensemble $\{(t,x,\xi)\ ;\ (M[f_\varepsilon])_0 \neq 0,\ (f_\varepsilon)_0 > F,\ (M[f_\varepsilon])_0 > F\}$ est encore plus clair, d'où

$$\iiint_{B_R \cap P_2} |(M[f_\varepsilon])_0 - (f_\varepsilon)_0|\ dtdxd\xi \leq 6\mu.$$

On obtient finalement, grâce à (0.3), que pour ε assez petit

$$\iiint_{B_R \cap \{(M[f_\varepsilon])_0 = 0, (f_\varepsilon)_0 \leq F\}} (f_\varepsilon)_0\, dtdxd\xi \leq 10\mu.$$

Maintenant, on a la décomposition

$$\begin{aligned}
&\iiint_{\{(t,x)\in B\}} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\, dtdxd\xi \\
\leq\ & \iiint_{B_R \cap \{(M[f_\varepsilon])_0 = 0, (f_\varepsilon)_0 \leq F\}} (f_\varepsilon)_0\, dtdxd\xi + \iiint_{B_R \cap \{(M[f_\varepsilon])_0 = 0, (f_\varepsilon)_0 > F\}} (f_\varepsilon)_0\, dtdxd\xi \\
&+ \iiint_{B_R \cap P_1} |(M[f_\varepsilon])_0 - (f_\varepsilon)_0|\, dtdxd\xi + \iiint_{B_R \cap P_2} |(M[f_\varepsilon])_0 - (f_\varepsilon)_0|\, dtdxd\xi \\
&+ \iiint_{\{(t,x)\in B, |\xi| \geq R\}} |(M[f_\varepsilon])_0 - (f_\varepsilon)_0|\, dtdxd\xi,
\end{aligned}$$

ce qui donne pour ε assez petit,

$$\iiint_{\{(t,x)\in B\}} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\, dtdxd\xi \leq 20\mu.$$

Deuxième étape : pour $\lambda < 1$, $(f_\varepsilon)_0 - (M[f_\varepsilon])_0 \underset{\varepsilon\to 0}{\to} 0$ dans $L^1_{(loc\ t,x),\xi}$.
La relation de conservation $\int_{\mathbf{R}} (f_\varepsilon)_0\, d\xi = \int_{\mathbf{R}} (M[f_\varepsilon])_0\, d\xi$ p.p. t,x entraîne pour tout domaine borné B en (t,x),

$$\begin{aligned}
&\iiint_{B_R \cap \{(M[f_\varepsilon])_0 \leq \alpha, (f_\varepsilon)_0 \leq F\}} (f_\varepsilon)_0\, dtdxd\xi \qquad (0.5)\\
=\ & \iiint_{\{(t,x)\in B, |\xi| \geq R\}} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0)\, dtdxd\xi + \iiint_{B_R \cap P_1} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0)\, dtdxd\xi \\
&+ \iiint_{B_R \cap P_2} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0)\, dtdxd\xi + \iiint_{B_R \cap \{(M[f_\varepsilon])_0 \leq \alpha, (f_\varepsilon)_0 > F\}} -(f_\varepsilon)_0\, dtdxd\xi \\
&+ \iiint_{B_R \cap \{(M[f_\varepsilon])_0 \leq \alpha\}} (M[f_\varepsilon])_0\, dtdxd\xi,
\end{aligned}$$

où cette fois

$$P_1 = \{(t,x,\xi)\ ;\ (M[f_\varepsilon])_0 > \alpha,\ (f_\varepsilon)_0 \leq F,\ (M[f_\varepsilon])_0 \leq F\},$$

$$P_2 = \{(t,x,\xi)\ ;\ (M[f_\varepsilon])_0 > \alpha \text{ et } ((f_\varepsilon)_0 > F \text{ ou } (M[f_\varepsilon])_0 > F)\}.$$

On peut fixer F et R tels que la somme du premier, du troisième et du quatrième terme du membre de droite de cette décompostion est plus petit que 9μ selon la même démarche que pour le cas $\lambda \geq 1$. Le dernier terme se majore par $\alpha|B_R|$; on fixe α de manière à ce que ce terme soit plus petit que μ. On utilise ensuite le résultat de la partie 2, chapitre 2 ([2]) qui donne pour $(f_\varepsilon)_0$ et $(M[f_\varepsilon])_0$ bornés,

$$\begin{aligned} & \mathbb{I}_{(M[f_\varepsilon])_0>\alpha}(H'(M[f_\varepsilon],\xi) - H'(f_\varepsilon,\xi))\cdot(M[f_\varepsilon]-f_\varepsilon) \\ \geq\ & K_1(1/2)^{1/\lambda}\alpha^{1/\lambda-1}((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2\mathbb{I}_{(M[f_\varepsilon])_0>\alpha} \quad \text{a.e.}, \end{aligned}$$

soit avec (0.2),

$$\iiint_{B_R\cap P_1} |(M[f_\varepsilon])_0 - (f_\varepsilon)_0| \underset{\varepsilon\to 0}{\to} 0 \quad a.e.\ t,x,\xi.$$

La décomposition (0.5) permet alors d'obtenir, pour ε assez petit,

$$\iiint_{B_R\cap\{(M[f_\varepsilon])_0\leq\alpha,(f_\varepsilon)_0\leq F\}} (f_\varepsilon)_0\, dtdxd\xi \leq 11\mu.$$

On utilise maintenant la décomposition

$$\begin{aligned} & \iiint_{(t,x)\in B} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\ dtdxd\xi \\ \leq\ & \iiint_{B_R\cap\{(M[f_\varepsilon])_0\leq\alpha,(f_\varepsilon)_0\leq F\}} (f_\varepsilon)_0\, dtdxd\xi + \iiint_{\{(t,x)\in B,|\xi|\geq R\}} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\, dtdxd\xi \\ & + \iiint_{B_R\cap P_1} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\, dtdxd\xi + \iiint_{B_R\cap P_2} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\, dtdxd\xi \\ & + \iiint_{B_R\cap\{(M[f_\varepsilon])_0\leq\alpha,(f_\varepsilon)_0> F\}} (f_\varepsilon)_0\, dtdxd\xi + \iiint_{B_R\cap\{(M[f_\varepsilon])_0\leq\alpha\}} (M[f_\varepsilon])_0\, dtdxd\xi \\ \leq\ & 22\mu, \end{aligned}$$

pour ε assez petit.
Troisième étape : on prouve que $(f_\varepsilon)_1 - (M[f_\varepsilon])_1 \underset{\varepsilon\to 0}{\to} 0$ dans $L^1_{(loc\ t,x),\xi}$.
Soit $\mu > 0$. Utilisant les bornes de la partie 2, chapitre 1 ([1]) pour $|\xi|(f_\varepsilon)_1$ et $|(f_\varepsilon)_1|^{p_1}$, on obtient l'existence de $F > 0$ et $R > 0$ tels que

$$\iiint_{|(f_\varepsilon)_1|\geq F} |(f_\varepsilon)_1|\, dtdxd\xi \leq \mu, \qquad \iiint_{|(M[f_\varepsilon])_1|\geq F} |(M[f_\varepsilon])_1|\, dtdxd\xi \leq \mu,$$

$$\iiint_{|\xi|\geq R} |(f_\varepsilon)_1|\, dtdxd\xi \leq \mu, \qquad \iiint_{|\xi|\geq R} |(M[f_\varepsilon])_1|\, dtdxd\xi \leq \mu,$$

et on peut supposer de plus que $\frac{4T}{\mu}\int_{\mathbf{R}^2} f_0^0(x,\xi)\,dxd\xi \le F$.
On a la décomposition suivante pour B un domaine borné en (t,x),

$$\begin{aligned}
&\iiint\limits_{(t,x)\in B} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1|\,dtdxd\xi \\
\le\ & \iiint\limits_{\{(t,x)\in B, |\xi|\ge R\}} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1|\,dtdxd\xi + \iiint\limits_{B_R\cap P_3} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1|\,dtdxd\xi \\
&+ \iiint\limits_{B_R\cap P_4} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1|\,dtdxd\xi + \iiint\limits_{B_R\cap P_5} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1|\,dtdxd\xi
\end{aligned}$$

où

$$P_3 = \{(t,x,\xi)\ ;\ |(f_\varepsilon)_0| > F^2 \text{ ou } |(M[f_\varepsilon])_0| > F^2\},$$

$$P_4 = \{(t,x,\xi)\ ;\ 0 \ne (f_\varepsilon)_0 \le F^2, 0 \ne (M[f_\varepsilon])_0 \le F^2\},$$

et

$$P_5 = \{(t,x,\xi)\ ;\ (f_\varepsilon)_0 \le F^2,\ (M[f_\varepsilon])_0 \le F^2, ((f_\varepsilon)_0 = 0 \text{ ou } (M[f_\varepsilon])_0 = 0)\}.$$

Le terme $\int_{\{(t,x)\in B, |\xi|\ge R\}} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1|\,dtdxd\xi$ est inférieur à 2μ.
On commence par traiter le terme avec P_4 où tout devient régulier et borné. On utilise pour cela la décomposition suivante prouvée dans la partie 2, au chapitre 2 ([2]),

$$\begin{aligned}
&\mathbb{1}_{(M[f_\varepsilon])_0\ne 0; (f_\varepsilon)_0\ne 0}\big((f_\varepsilon)_1 - (M[f_\varepsilon])_1\big) \\
=\ & \mathbb{1}_{(M[f_\varepsilon])_0\ne 0;\ (f_\varepsilon)_0\ne 0;\ (f_\varepsilon)_0\le (M[f_\varepsilon])_0} \frac{(f_\varepsilon)_1(M[f_\varepsilon])_0 - (f_\varepsilon)_0(M[f_\varepsilon])_1}{(M[f_\varepsilon])_0} \\
&+ \mathbb{1}_{(M[f_\varepsilon])_0\ne 0;\ (f_\varepsilon)_0\ne 0;\ (M[f_\varepsilon])_0 < (f_\varepsilon)_0} \frac{(f_\varepsilon)_1(M[f_\varepsilon])_0 - (f_\varepsilon)_0(M[f_\varepsilon])_1}{(f_\varepsilon)_0} \\
&- \mathbb{1}_{(M[f_\varepsilon])_0\ne 0;\ (f_\varepsilon)_0\ne 0;\ (f_\varepsilon)_0\le (M[f_\varepsilon])_0} \big((M[f_\varepsilon])_0 - (f_\varepsilon)_0\big)\frac{(M[f_\varepsilon])_1}{(M[f_\varepsilon])_0} \\
&- \mathbb{1}_{(M[f_\varepsilon])_0\ne 0;\ (f_\varepsilon)_0\ne 0;\ (M[f_\varepsilon])_0 < (f_\varepsilon)_0} \big((M[f_\varepsilon])_0 - (f_\varepsilon)_0\big)\frac{(f_\varepsilon)_1}{(f_\varepsilon)_0}.
\end{aligned} \tag{0.6}$$

Pour $(f_\varepsilon)_0$ et $(M[f_\varepsilon])_0$ bornées par F^2 et non nuls, on utilise la majoration montrée au chapitre 2 de la partie 2 ([2]),

$$\begin{aligned}
&\mathbb{1}_{(M[f_\varepsilon])_0\ne 0; (f_\varepsilon)_0\ne 0}\big(H'(M[f_\varepsilon],\xi) - H'(f_\varepsilon,\xi)\big)\cdot\big(M[f_\varepsilon] - f_\varepsilon\big) \\
\ge\ & \frac{K_2}{F^2}\left(\frac{(f_\varepsilon)_1(M[f_\varepsilon])_0 - (f_\varepsilon)_0(M[f_\varepsilon])_1}{(f_\varepsilon)_0 + (M[f_\varepsilon])_0}\right)^2 \mathbb{1}_{(M[f_\varepsilon])_0\ne 0; (f_\varepsilon)_0\ne 0},
\end{aligned}$$

ce qui avec (0.2) permet d'obtenir

$$\mathbb{1}_{P_4}\frac{(f_\varepsilon)_1(M[f_\varepsilon])_0 - (f_\varepsilon)_0(M[f_\varepsilon])_1}{(f_\varepsilon)_0 + (M[f_\varepsilon])_0} \underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ p.p. } t,x,\xi,$$

et ainsi les deux premiers termes du membre de droite de (0.6) tendent vers 0 sur P_4. Pour $|(f_\varepsilon)_1/(f_\varepsilon)_0| > A$, le dernier terme de cette décomposition se majore par

$$\begin{aligned}
&\iiint\limits_{\substack{\{(M[f_\varepsilon])_0 < (f_\varepsilon)_0, \\ |(f_\varepsilon)_1/(f_\varepsilon)_0| > A\}}} |((M[f_\varepsilon])_0 - (f_\varepsilon)_0)\frac{(f_\varepsilon)_1}{(f_\varepsilon)_0}|\, dtdxd\xi \\
\leq\ & \iiint\limits_{\substack{\{(M[f_\varepsilon])_0 < (f_\varepsilon)_0, \\ |(f_\varepsilon)_1/(f_\varepsilon)_0| > A\}}} (1 - \frac{(M[f_\varepsilon])_0}{(f_\varepsilon)_0})|(f_\varepsilon)_1|\, dtdxd\xi \\
\leq\ & \iiint\limits_{\substack{\{(M[f_\varepsilon])_0 < (f_\varepsilon)_0, \\ |(f_\varepsilon)_1/(f_\varepsilon)_0| > A\}}} |(f_\varepsilon)_1|\, dtdxd\xi \\
\leq\ & \frac{1}{A} \iiint\limits_{\substack{\{(M[f_\varepsilon])_0 < (f_\varepsilon)_0, \\ |(f_\varepsilon)_1/(f_\varepsilon)_0| > A\}}} \frac{(f_\varepsilon)_1^2}{(f_\varepsilon)_0}\, dtdxd\xi \\
\leq\ & \frac{1}{A} 2C_H^0 T,
\end{aligned}$$

ce terme peut donc être rendu arbitrairement petit. Fixant A, on a alors sur $|(f_\varepsilon)_1/(f_\varepsilon)_0| \leq A$, $((M[f_\varepsilon])_0 - (f_\varepsilon)_0)\frac{(f_\varepsilon)_1}{(f_\varepsilon)_0}\mathbb{1}_{(M[f_\varepsilon])_0 < (f_\varepsilon)_0}$ tend vers 0 lorsque $\varepsilon \to 0$, et on obtient

$$\iiint\limits_{\{(M[f_\varepsilon])_0 \neq 0;\ (f_\varepsilon)_0 \neq 0;\ (M[f_\varepsilon])_0 < (f_\varepsilon)_0\}} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0)\frac{(f_\varepsilon)_1}{(f_\varepsilon)_0}\, dtdxd\xi \underset{\varepsilon \to 0}{\to} 0.$$

On procède de même avec le troisième terme de (0.6) et on obtient finalement

$$\iiint\limits_{B_R \cap P_4} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1|\, dtdxd\xi \underset{\varepsilon \to 0}{\to} 0.$$

Le terme qui autorise la prise en compte du vide, à savoir celui avec P_5, va se traiter grâce à l'inégalité de Cauchy-Schwarz

$$\begin{aligned}
&\left(\iiint\limits_{B_R \cap \{(M[f_\varepsilon])_0 = 0\}} |(f_\varepsilon)_1|\, dtdxd\xi \right)^2 \\
\leq\ & \left(\iiint\limits_{B_R} \frac{((f_\varepsilon)_1)^2}{(f_\varepsilon)_0}\, dtdxd\xi \right) \left(\iiint\limits_{B_R \cap \{(M[f_\varepsilon])_0 = 0\}} (f_\varepsilon)_0\, dtdxd\xi \right),
\end{aligned}$$

soit, comme $(M[f_\varepsilon])_0 = 0 \Rightarrow (M[f_\varepsilon])_1 = 0$,

$$\left(\iiint\limits_{B_R \cap \{(M[f_\varepsilon])_0 = 0\}} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1|\, dtdxd\xi \right)^2$$

$$\leq \quad 2C_H^0 T \left(\iiint\limits_{B_R \cap \{(M[f_\varepsilon])_0 = 0\}} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0| \, dt dx d\xi \right),$$

terme qui tend vers 0 lorsque $\varepsilon \to 0$. On procède de même pour $(f_\varepsilon)_0 = 0$ et on obtient

$$\iiint\limits_{B_R \cap P_5} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1| \, dt dx d\xi \underset{\varepsilon \to 0}{\to} 0.$$

Il reste maintenant à traiter le terme avec P_3 qui correspond au cas non borné. On a la majoration

$$\begin{aligned} \iiint\limits_{B_R \cap P_3} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1| \, dt dx d\xi \quad \leq & \iiint\limits_{\substack{B_R \cap P_3 \cap \{|(f_\varepsilon)_1| \leq F, \\ |(M[f_\varepsilon])_1| \leq F\}}} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1| \, dt dx d\xi \\ & + \iiint\limits_{\substack{\{|(f_\varepsilon)_1| > F, \\ \text{ou } |(M[f_\varepsilon])_1| > F\}}} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1| \, dt dx d\xi. \end{aligned}$$

Le second terme du membre de droite est majoré par 6μ grâce à un raisonnement semblable à celui de la première étape au niveau de (0.4). Pour le premier terme, on utilise que pour $(f_\varepsilon)_0 > F^2$ ou $(M[f_\varepsilon])_0 > F^2$, on a $1 < \frac{(f_\varepsilon)_0 + (M[f_\varepsilon])_0}{F^2}$, d'où

$$\begin{aligned} & \iiint\limits_{B_R \cap P_3 \cap \{|(f_\varepsilon)_1| \leq F, |(M[f_\varepsilon])_1| \leq F\}} |(f_\varepsilon)_1 - (M[f_\varepsilon])_1| \, dt dx d\xi \\ \leq \quad & 2F \iiint\limits_{B_R \cap P_3 \cap \{|(f_\varepsilon)_1| \leq F, |(M[f_\varepsilon])_1| \leq F\}} dt dx d\xi \\ \leq \quad & \frac{2}{F} \iiint (f_\varepsilon)_0 + (M[f_\varepsilon])_0 \, dt dx d\xi \\ \leq \quad & \frac{4T}{F} \iint\limits_{\mathbf{R}^2} f_0^0(x,\xi) \, dx d\xi \leq \mu. \end{aligned}$$

Ceci termine la preuve.

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