

# Car-Following Models as Dynamical Systems and the Mechanisms for Macroscopic Pattern Formation

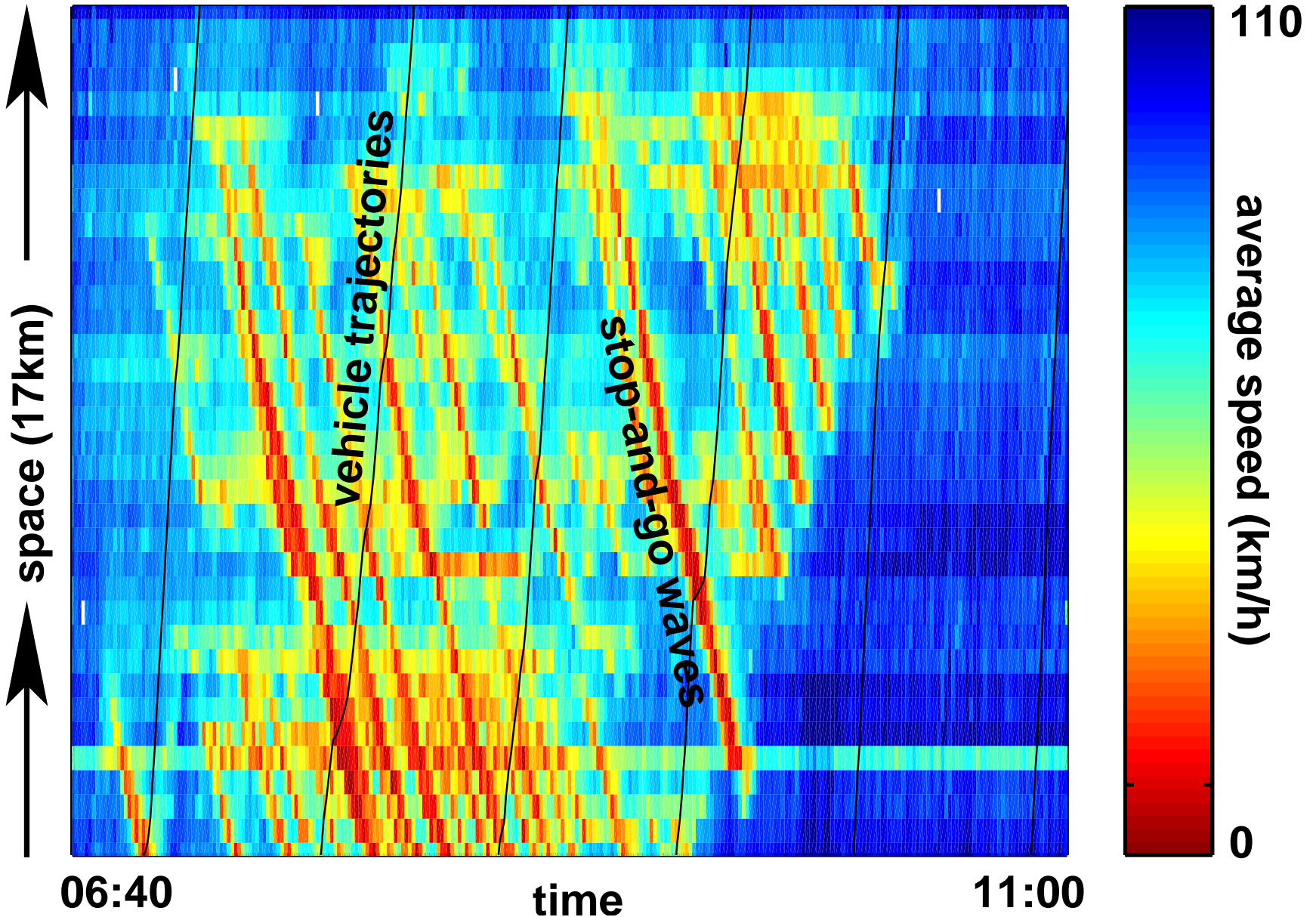
R. Eddie Wilson, University of Bristol

EPSRC Advanced Research Fellowship EP/E055567/1

<http://www.enm.bris.ac.uk/staff/rew>

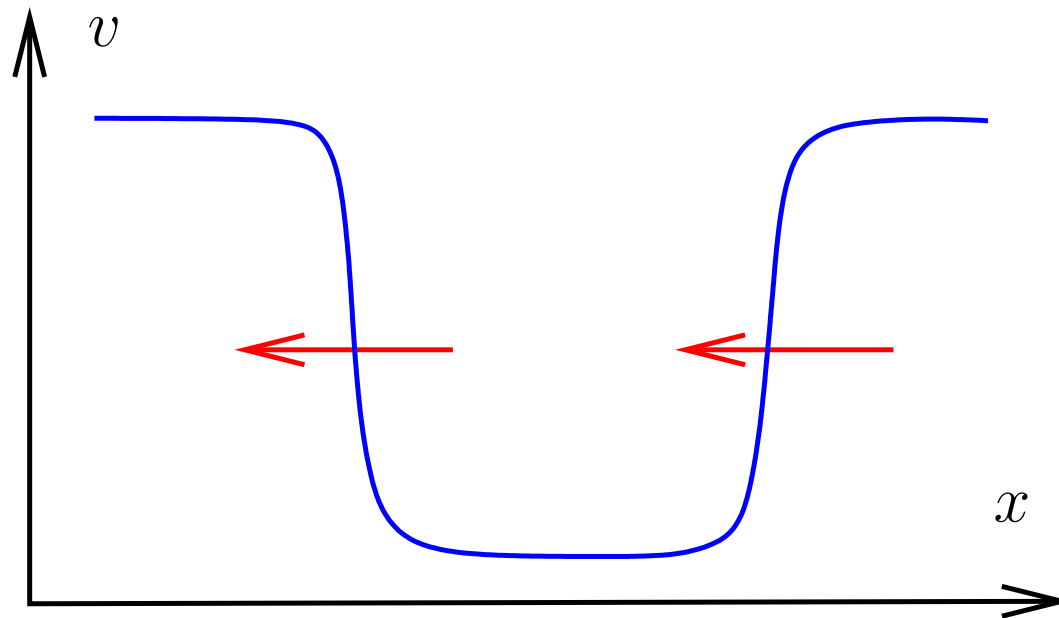
# Macroscopic Traffic Data

M25 anticlockwise carriageway 1/4/2000



# Some facts and conclusions (I)

- Propagation of stop-and-go is (fairly) regular
  - so can be captured by macroscopic deterministic models?



- Downstream interface does not spread (Kerner 90s) — problem for LWR and I believe ARZ / Lebacque framework

# Some facts and conclusions (II)

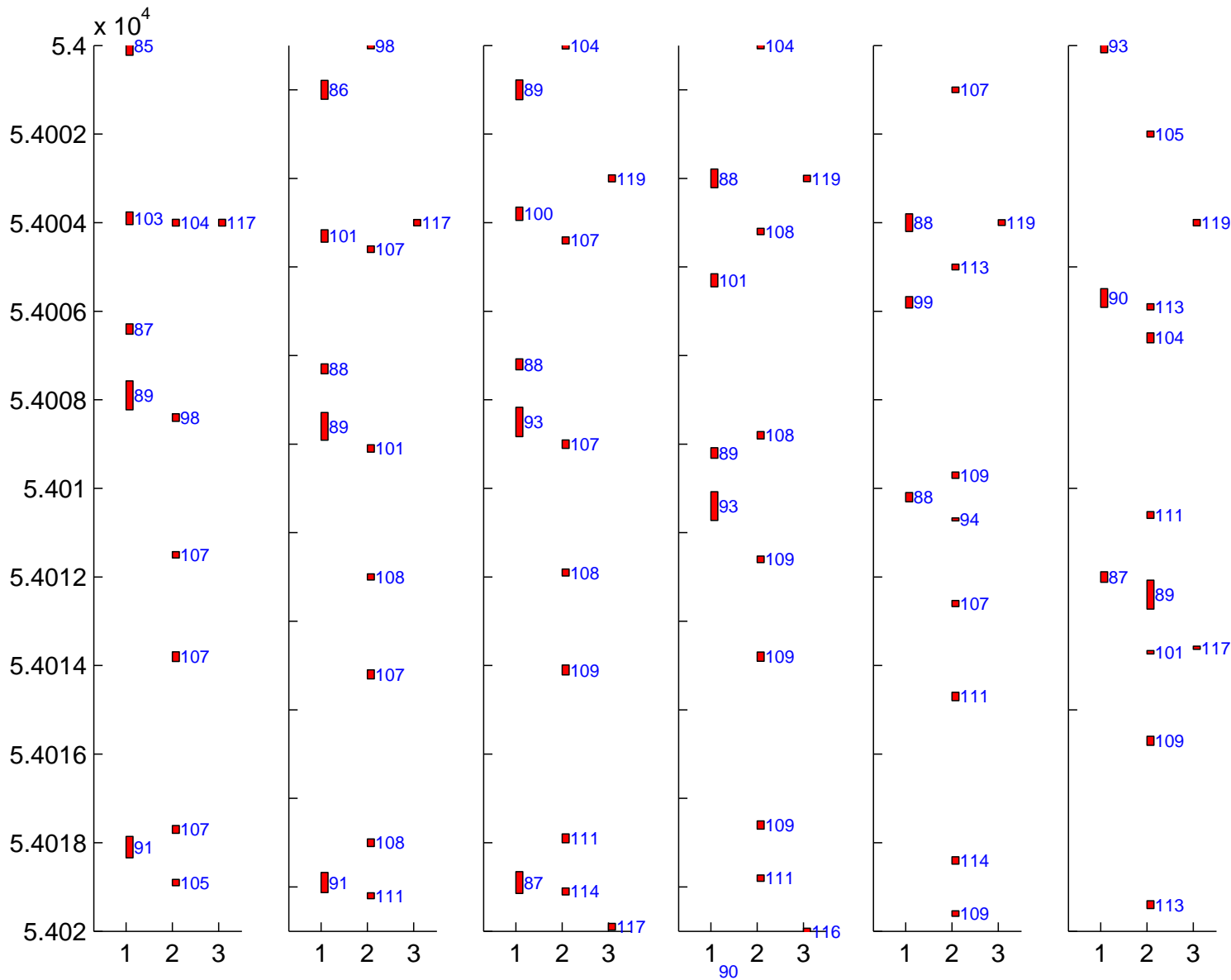
- Ignition of stop-and-go waves is irregular
  - needs full noisiness of microscopic description (but predictions can only be probabilistic)
- Wavelength is much longer than vehicle separation
  - how to capture the upscaling effect?
- General idea: identify families of models which are qualitatively ok and throw away models which are qualitatively inadequate
- IN FUTURE
  - Fit models to microscopic data
  - Use emergent macroscopic dynamics for predictions

# Active Traffic Management system

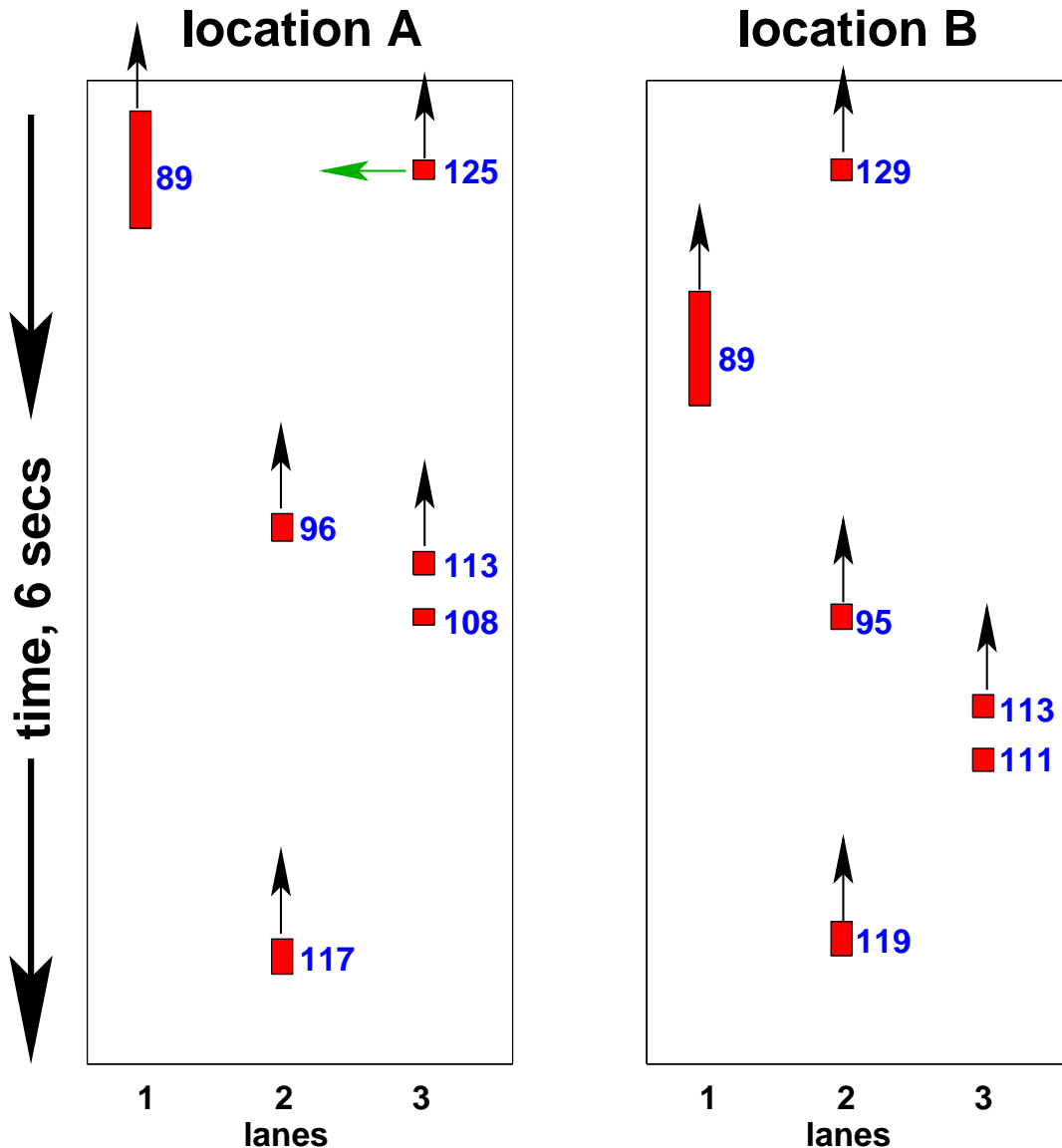


- Aim, reduce: accidents, (variance of) journey times
  - *Queue Ahead* warning systems
  - Temporary speed limits
  - Lane management
- Spacing of inductance loop pairs is in range 30m to 100m

# Individual Vehicle Data from ATM system



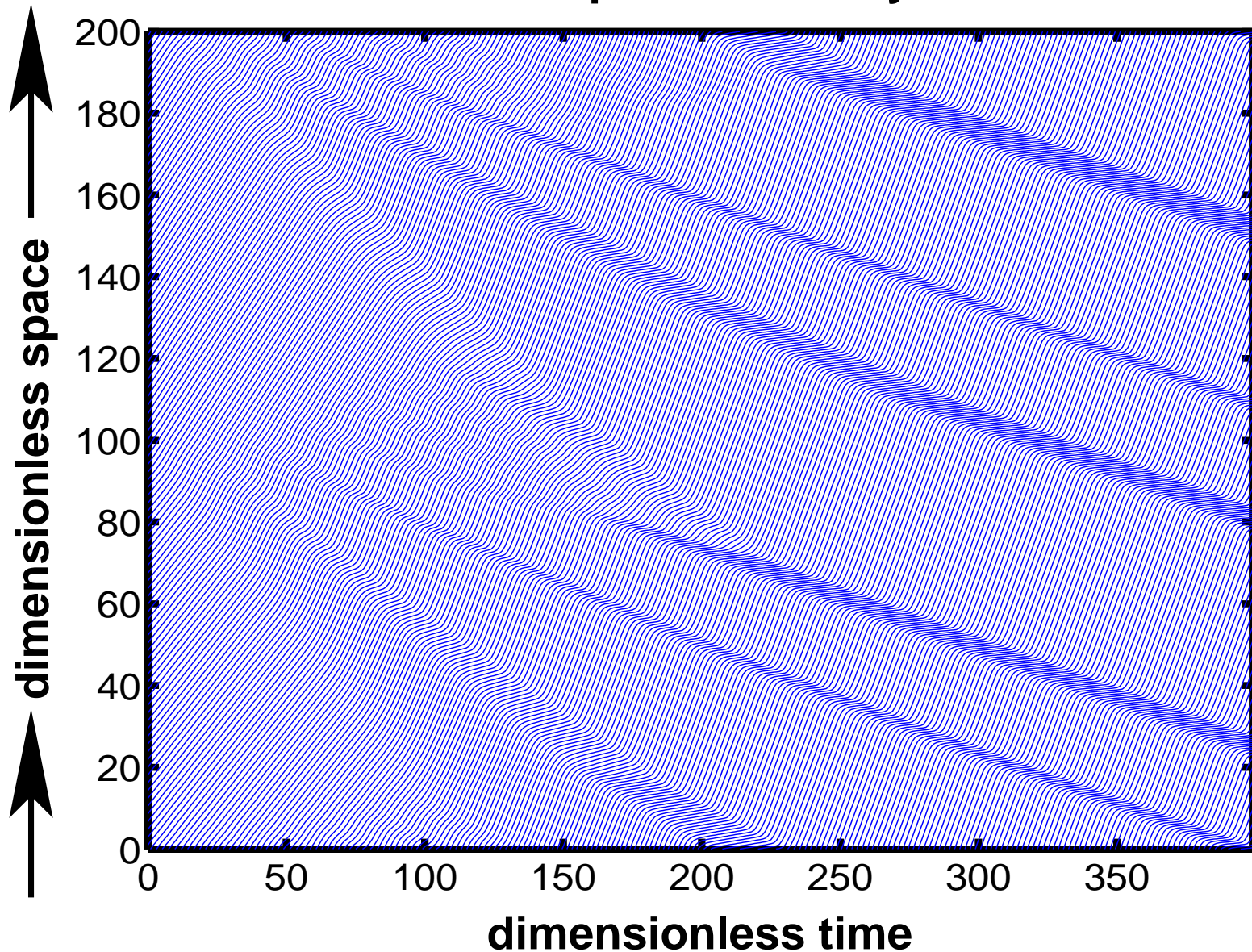
# Zoom-view and future scope



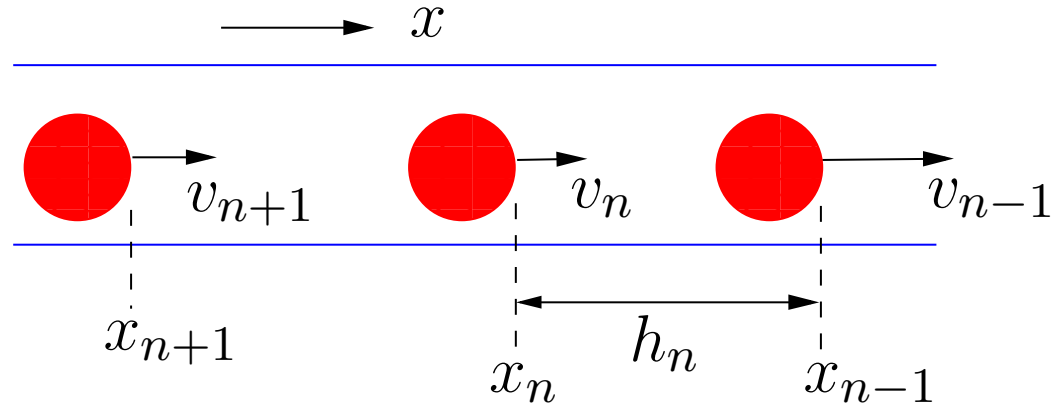
- Individual vehicle data gives 'helicopter view' (speeds km/h)
- Location B is 100m downstream of location A: note lane change
- Propose to *reconstruct* vehicle trajectories over  $55 \times 100\text{m} \times 1$  week

# Jam formation in simulations

simulation of Optimal Velocity model



# Car-following models



- Typical form

$$\dot{x}_n = v_n,$$

$$\dot{v}_n = f(h_n, \dot{h}_n, v_n) \quad \text{and generalisations}$$

- E.g. *Bando* model (1995)

$$f = \alpha \{V(h_n) - v_n\}, \quad \alpha > 0$$

$V$  is *Optimal Velocity* or *Speed-Headway* function

# Linear stability framework

- General car-following model

$$\dot{v}_n = f(h_n, \dot{h}_n, v_n),$$

- Equilibrium condition, there exists  $V(h)$  so that

$$f(h_*, 0, V(h_*)) = 0 \quad \text{for all } h_* > 0.$$

- Linearisation yields

$$\dot{\tilde{v}}_n = (D_h f) \tilde{h}_n + (D_{\dot{h}} f) \dot{\tilde{h}}_n + (D_v f) \tilde{v}_n,$$

with sensible sign constraints

$$D_h f, D_{\dot{h}} f \geq 0 \quad \text{and} \quad D_v f \leq 0.$$

# Linear stability, part 2

- Re-arrangement  $\dot{h}_n = v_{n-1} - v_n$  gives

$$\ddot{\tilde{h}}_n = (D_h f)(\tilde{h}_{n-1} - \tilde{h}_n) + (D_{\dot{h}} f)(\dot{\tilde{h}}_{n-1} - \dot{\tilde{h}}_n) + (D_v f)\dot{\tilde{h}}_n.$$

- Then try exponential ansatz  $\tilde{h}_n = \text{real}(ce^{in\theta}e^{\lambda t})$

- $\theta$  is perturbation's discrete wavenumber
- $\text{real}(\lambda)$  is growth rate

to obtain quadratic

$$\lambda^2 + \left\{ (D_{\dot{h}} f)(1 - e^{-i\theta}) - (D_v f) \right\} \lambda + (D_h f)((1 - e^{-i\theta})) = 0.$$

- Then derive results for  $\lambda(\theta)$  in quite general terms (proofs omitted)

# Technical details

- Short wavelength analysis,  $\theta = \pi$

$$\lambda^2 + \{2(D_h f) - (D_v f)\} \lambda + 2D_h f = 0$$

All coeffs positive, therefore stable roots

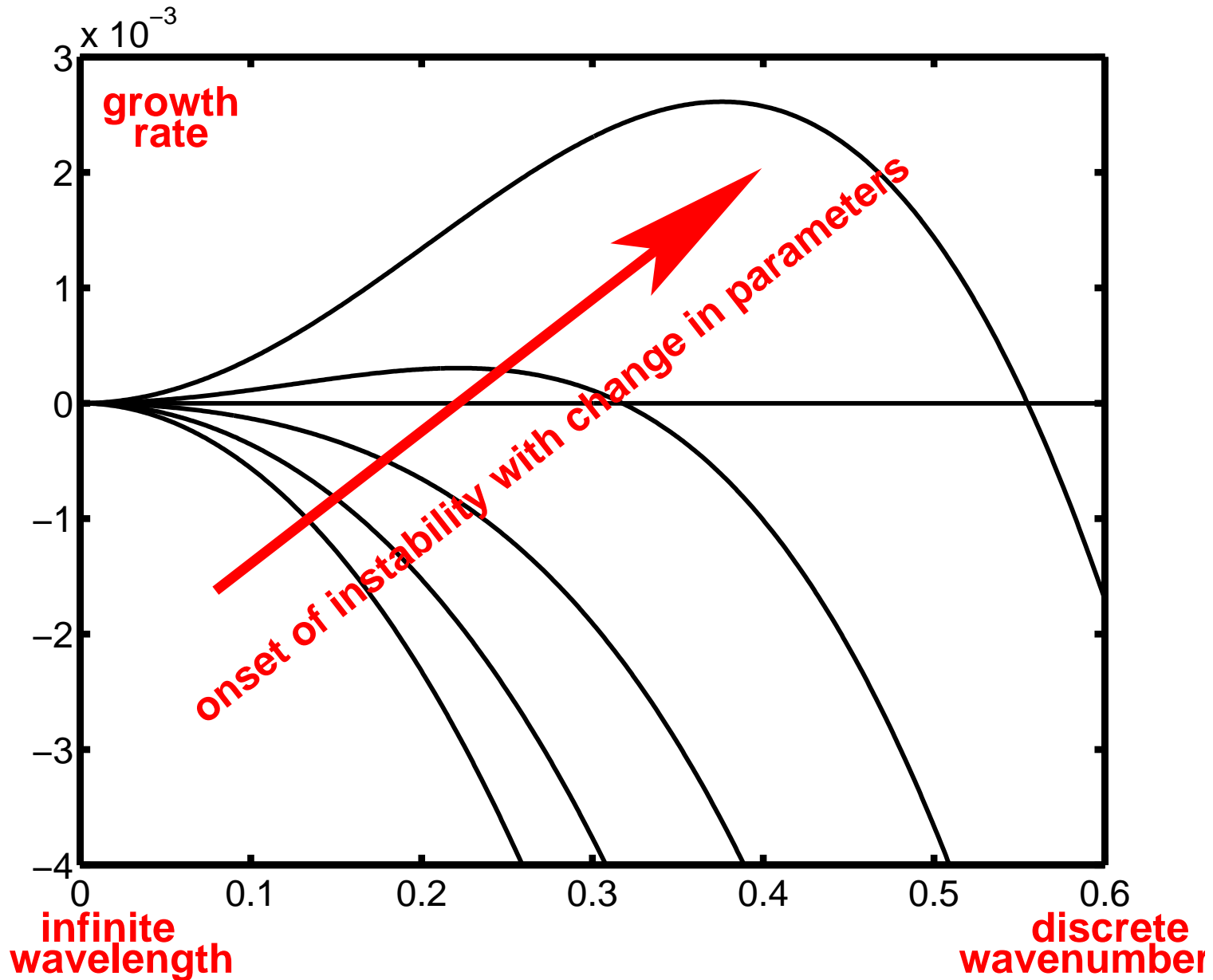
- Long wavelength analysis,  $\theta > 0$  small,  $\lambda = \lambda_1 \theta + \lambda_2 \theta^2$  gives  $\lambda_1 = i(D_h f)/(D_v f)$  and

$$\lambda_2 = \frac{(D_h f)}{(D_v f)^3} \left\{ \frac{1}{2}(D_v f)^2 - (D_h f) - (D_h f)(D_v f) \right\}$$

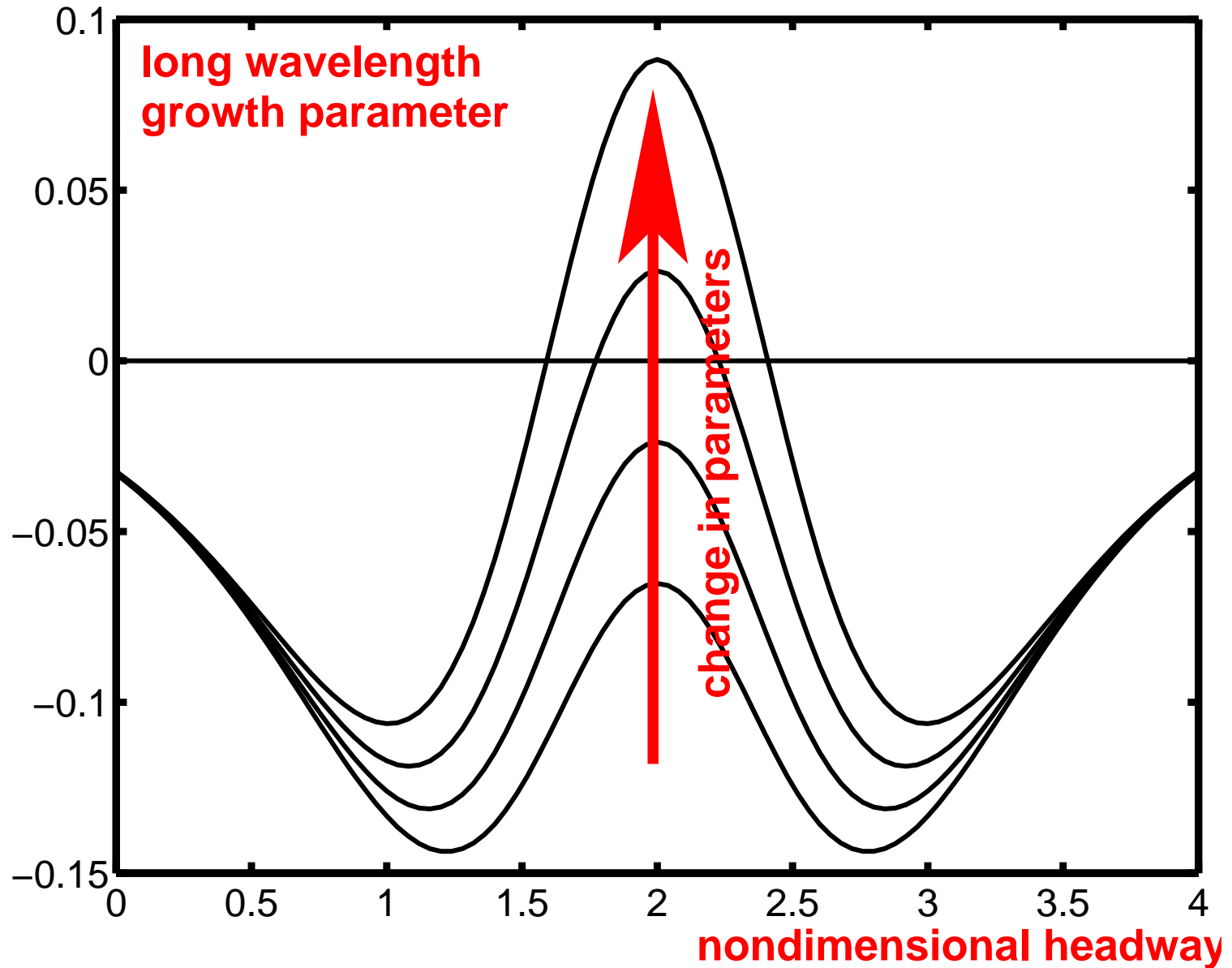
- Can show neutral stability  $\lambda = i\omega$  for general  $\theta$  is equivalent to  $\lambda_2 = 0$ .

Therefore: need only analyse  $\lambda_2$

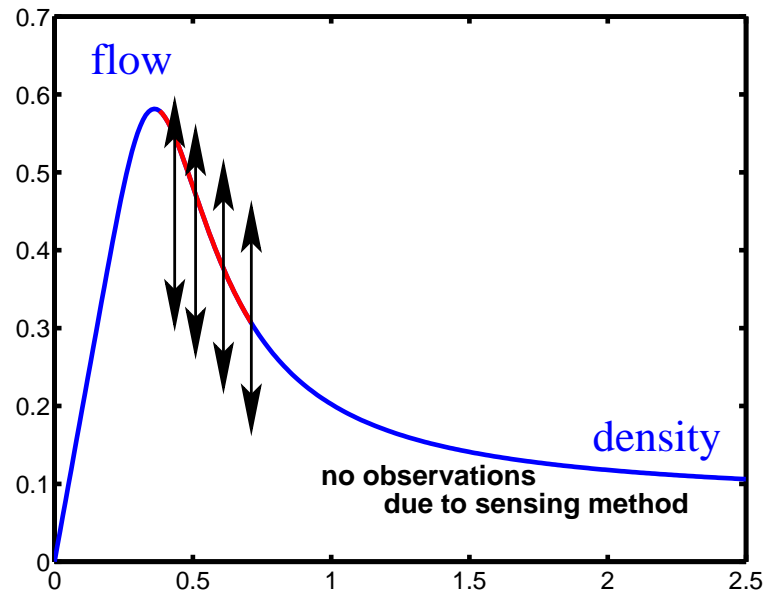
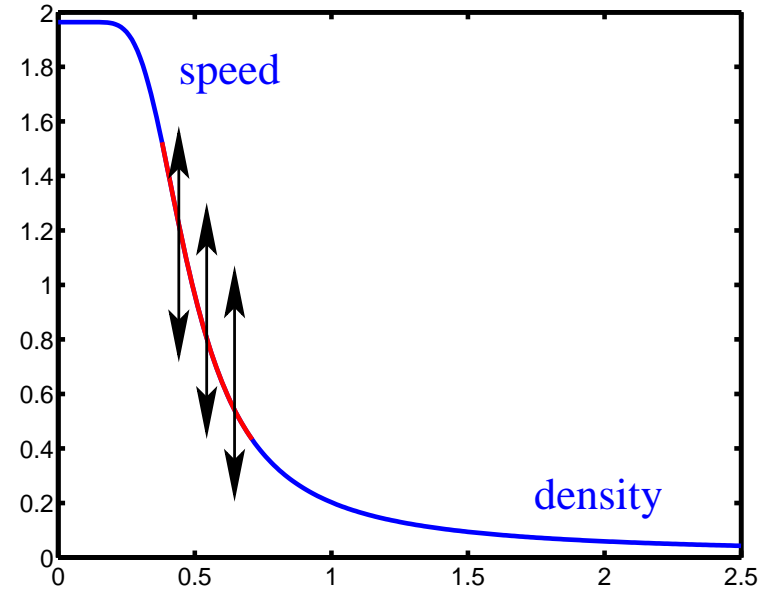
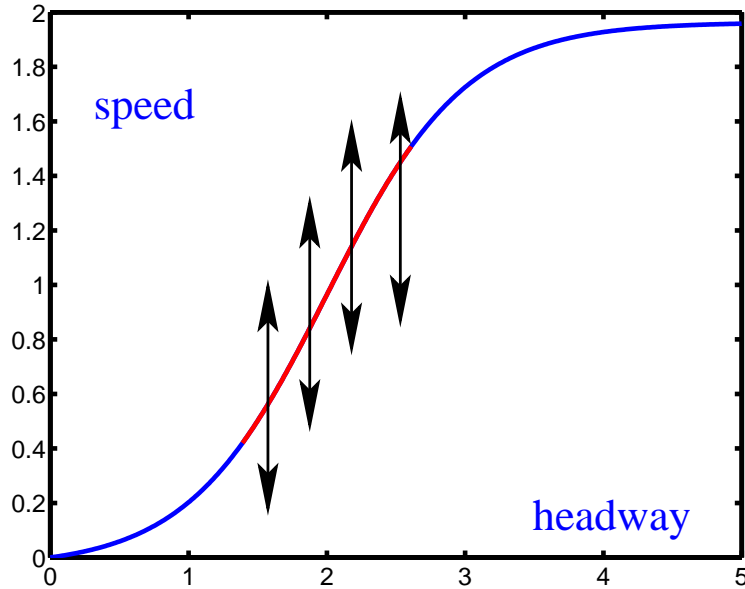
# Onset from infinite wavelength



# Onset at medium densities



# Equilibrium curves



# Other types of linear (in)stability

Notional experiment in semi-infinite column of vehicles where second vehicle is instantaneously perturbed out of equilibrium

- Linearised dynamics of  $n$ th vehicle

$$\ddot{\tilde{h}}_n + [(\mathbf{D}_{\dot{h}} f) - (\mathbf{D}_v f)] \dot{\tilde{h}}_n + (\mathbf{D}_h f) \tilde{h}_n = (\mathbf{D}_h f) \tilde{h}_{n-1} + (\mathbf{D}_{\dot{h}} f) \dot{\tilde{h}}_{n-1}$$

- Solve resonant oscillators inductively, large  $t$

$$\tilde{h}_n(t) \sim \frac{t^{n-1}}{(n-1)!} \left[ \frac{\lambda(\mathbf{D}_{\dot{h}} f) + (\mathbf{D}_h f)}{2\lambda + (\mathbf{D}_{\dot{h}} f) - (\mathbf{D}_v f)} \right]^{n-1} e^{\lambda t}$$

where  $\lambda$  is stable ‘platoon’ eigenvalue

- Use moving absolute space frame  $t = nh^*/(c + v^*)$  and Stirling’s formula to define growth ‘wedge’

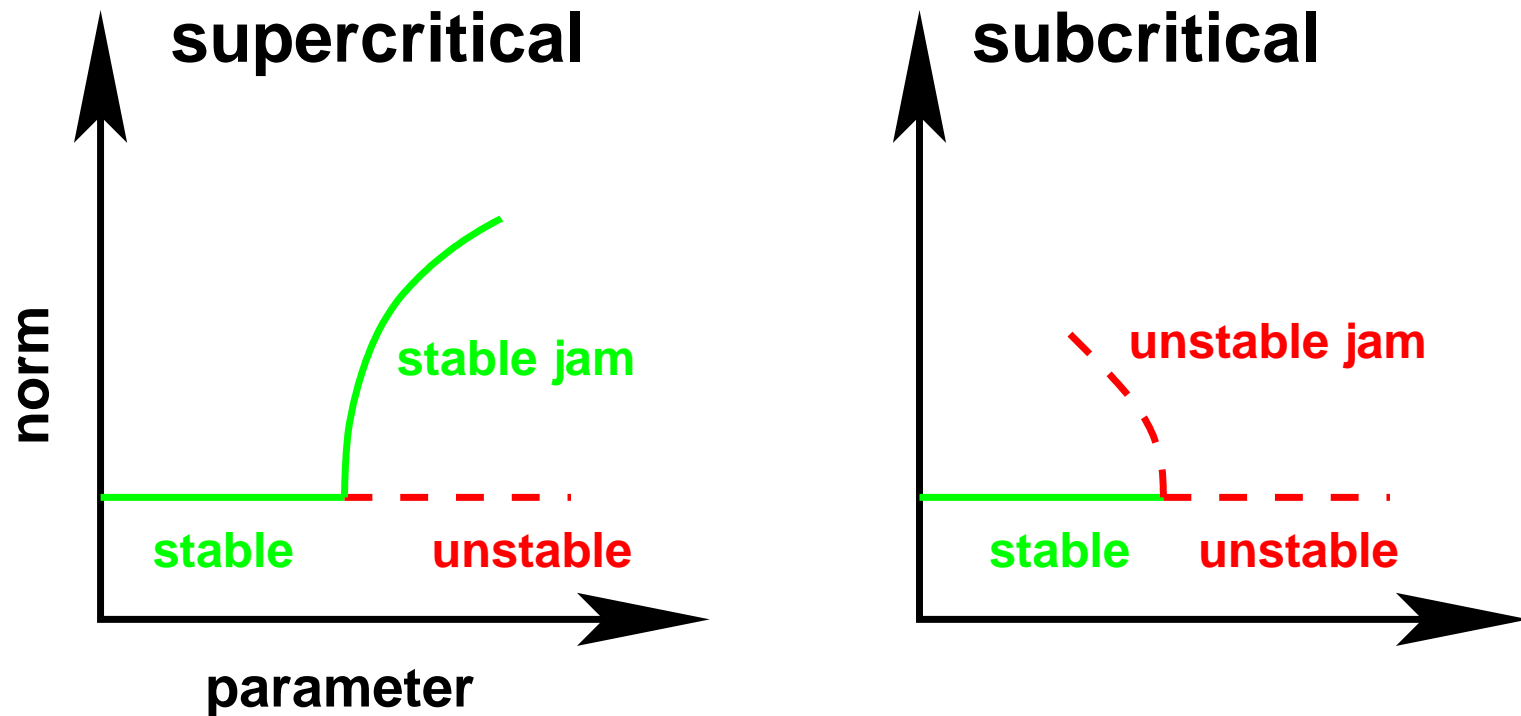
# Problems (?) with linear instability

- Setting a reduced speed limit to induce mid-range density and increase flow does not induce flow breakdown
- Stop-and-go waves almost always ignite at merges or other large amplitude ‘externalities’

These problems may explain the continuing adherence to one-phase PDE models, be they first order like LWR or second order like ARZ/Lebacque

# Introduction to bifurcation theory

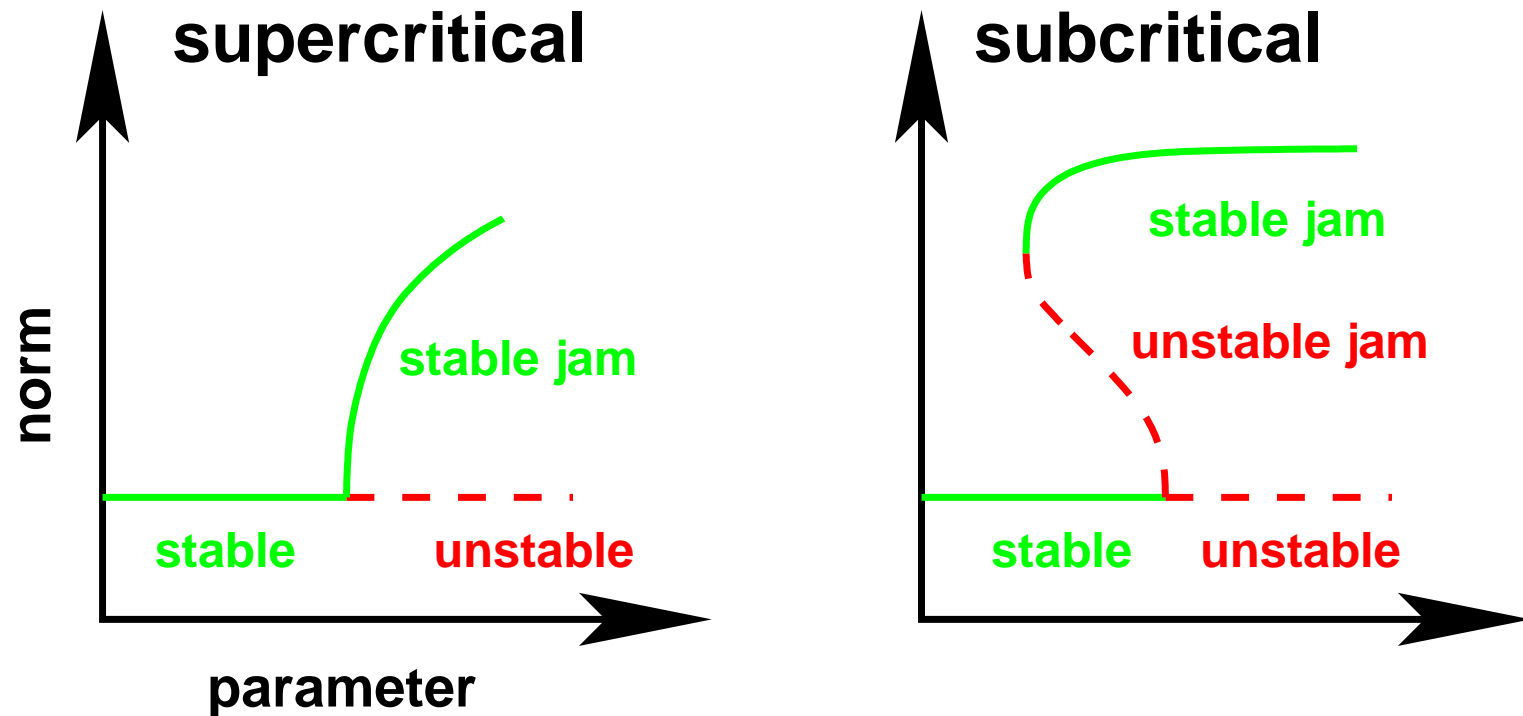
- Loss of stability of uniform flow is via a Hopf bifurcation, of which there are two types:



- supercritical: stable periodic solutions are born
- subcritical: unstable periodic solutions are born, branch bends back — so what is dynamics?

# Introduction to bifurcation theory

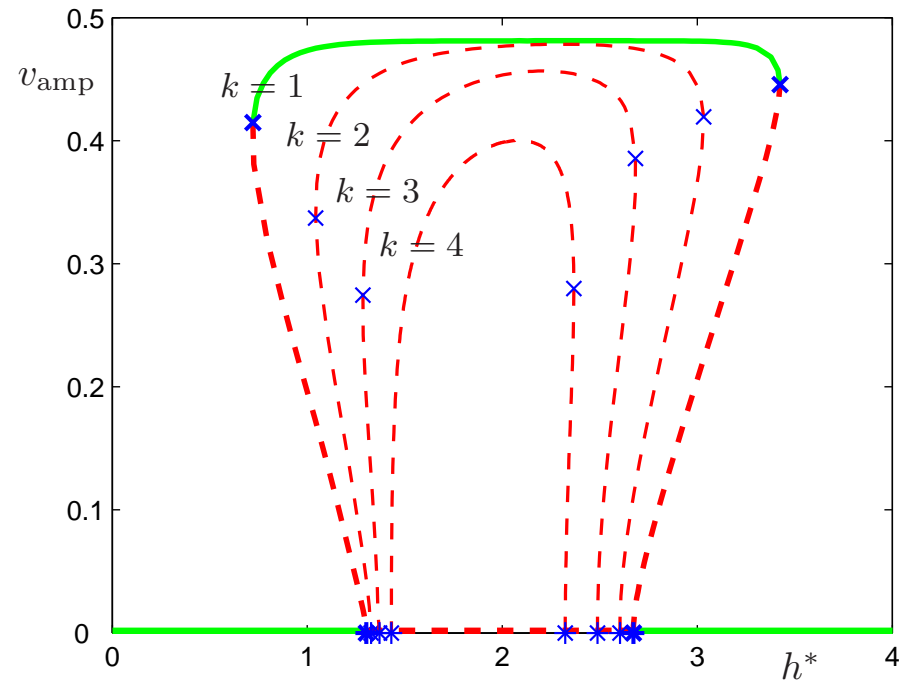
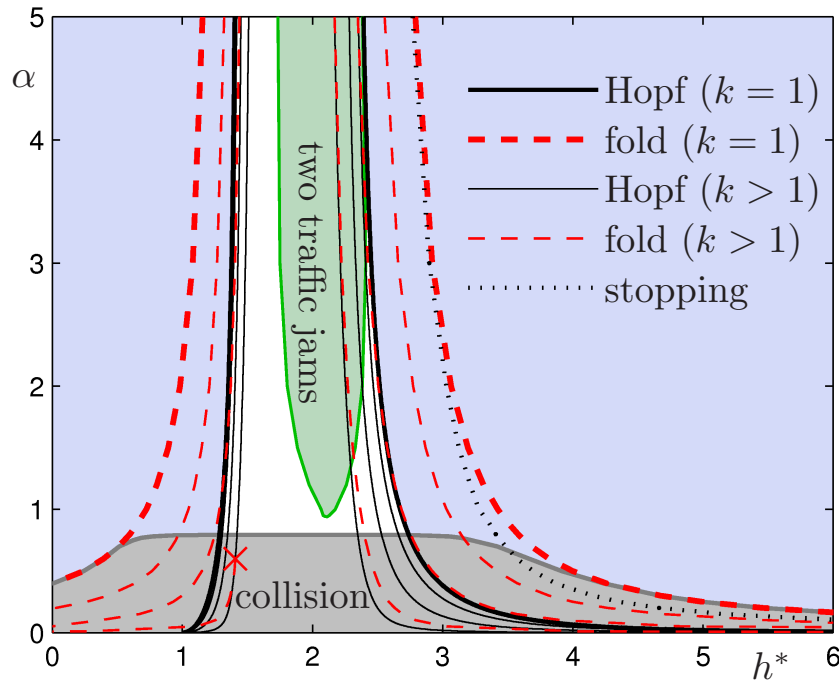
- Loss of stability of uniform flow is via a Hopf bifurcation, of which there are two types:



- Subcritical bifurcation with cyclic fold gives jump to large amplitude traffic jam solution plus region of bistability

# Computational results

- Application of numerical parameter continuation tools to analyse stop-and-go waves on the ring road



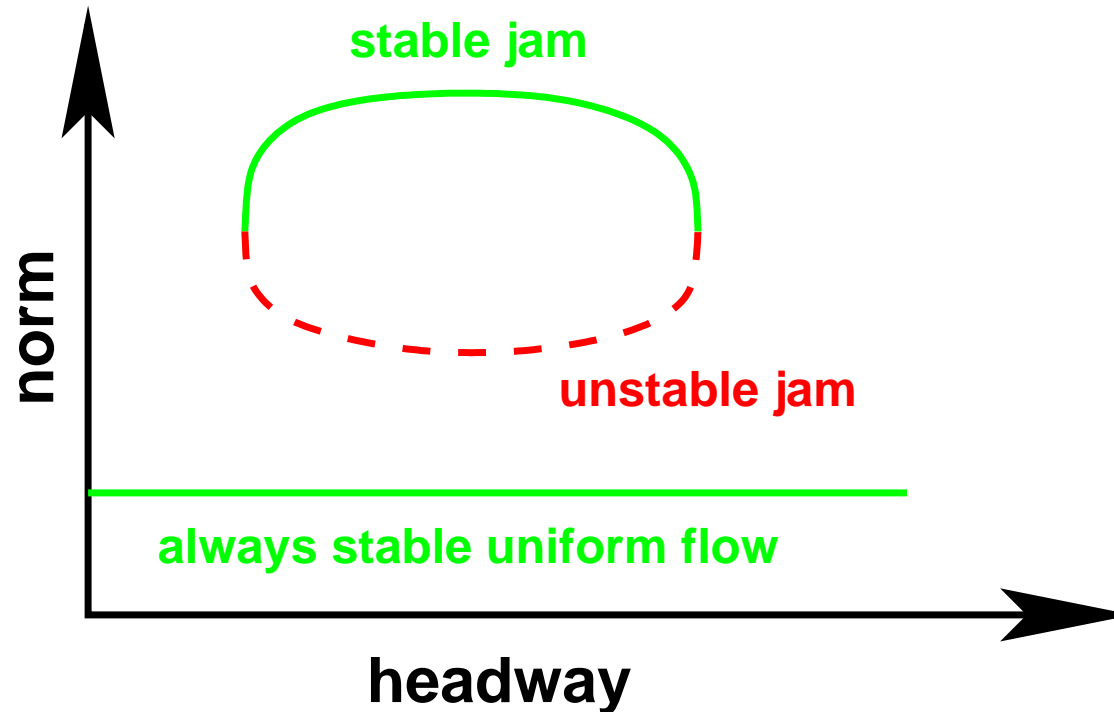
- REW, Krauskopf and Orosz, also group of Gasser
- Large perturbations (lane changes at merges?) cause jump to jammed state

# Search for new dynamics

- This explanation still requires uniform flow to be unstable in some parameter regime. Is a fix possible?

# Search for new dynamics

- This explanation still requires uniform flow to be unstable in some parameter regime. Is a fix possible?
- ‘Design’ bifurcation diagram:



- Ongoing work  $\dot{v}_n = \alpha(\dot{h}_n)F(V(h_n) - v_n)$

# Alternative: travelling wave analysis

- Computationally wasteful (and perhaps inappropriate) to analyse wave structures via bifurcations of periodic orbits of large systems of ODEs/DDEs
- Instead: travelling wave analysis. Two methods:
  - Weakly nonlinear continuum limit (Kim, Lee, Lee):

$$\rho_t + (\rho v)_x = 0, \quad \text{see TGF '01}$$

$$v_t + vv_x = \alpha \left\{ \hat{V}(\rho) - v \right\} + \alpha \left[ \hat{V}'(\rho) \frac{\rho_x}{2\rho} + \frac{v_{xx}}{6\rho^2} \right],$$

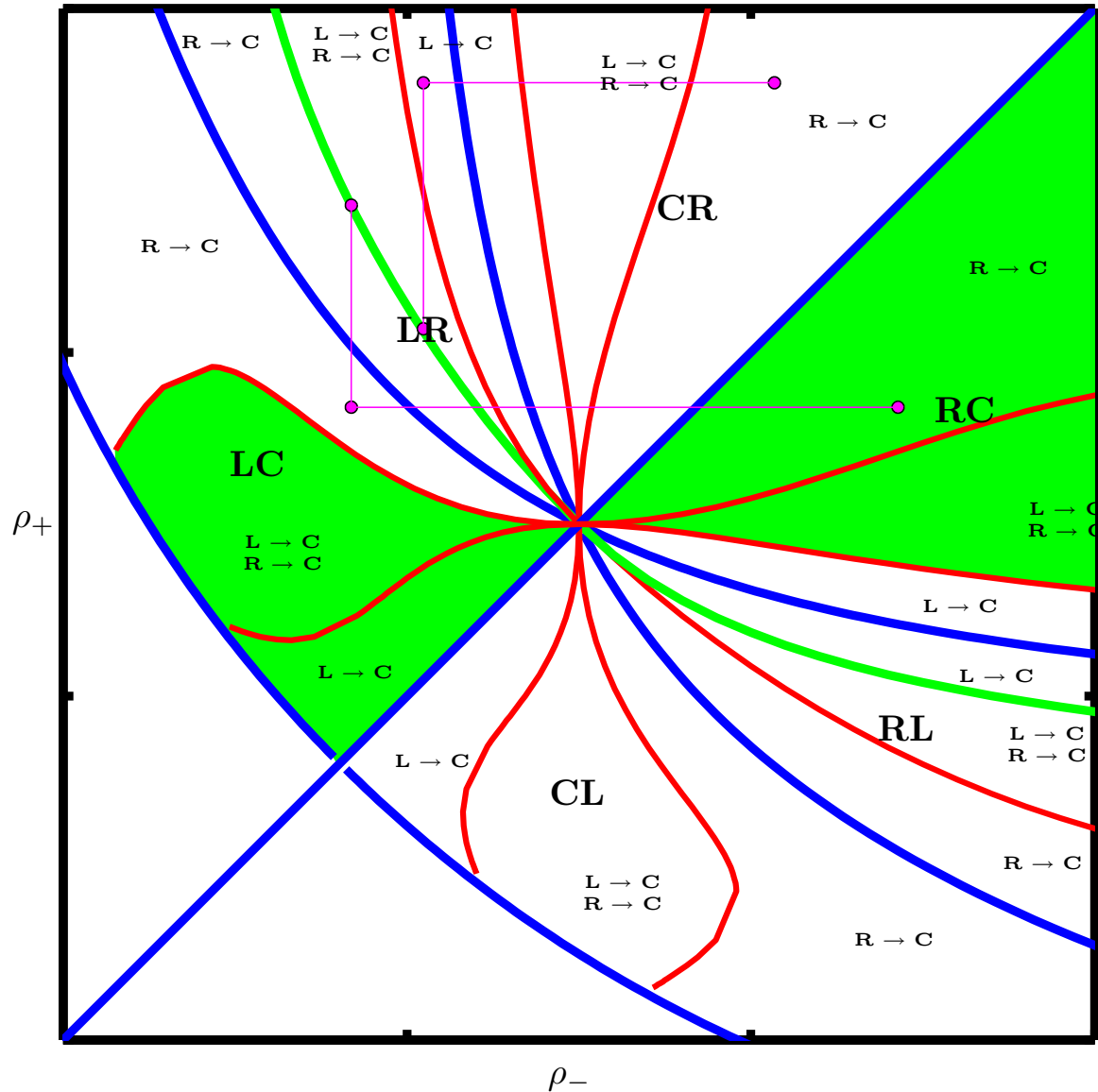
- Single advance/delay equation, derived from

$$h_{n-1}(t) = h_n(t + \tau), \quad v_{n-1}(t) = v_n(t + \tau)$$

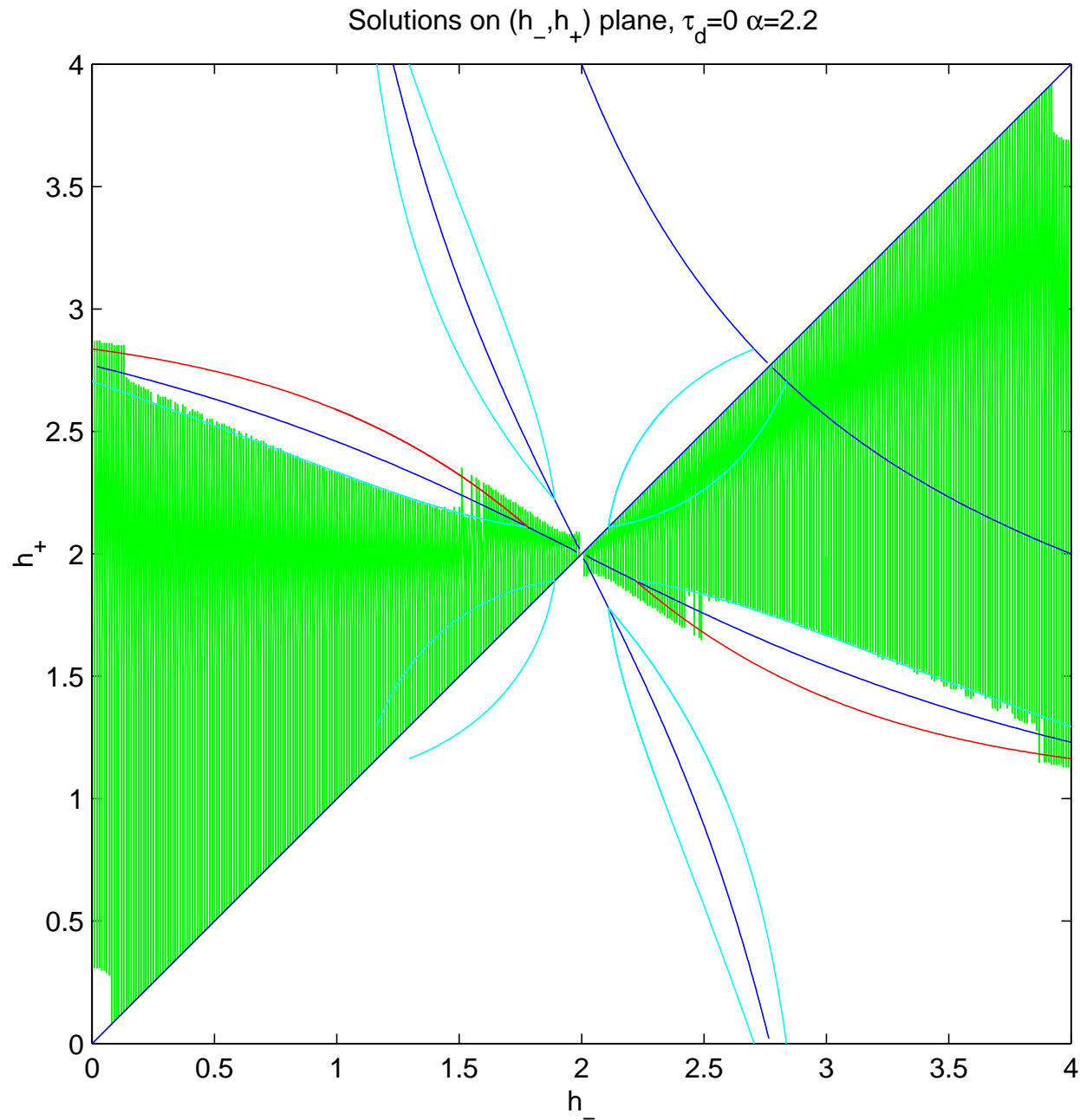
substitution in car-following model (ongoing work with Tony Humphries, McGill)

# Travelling wave phase diagram

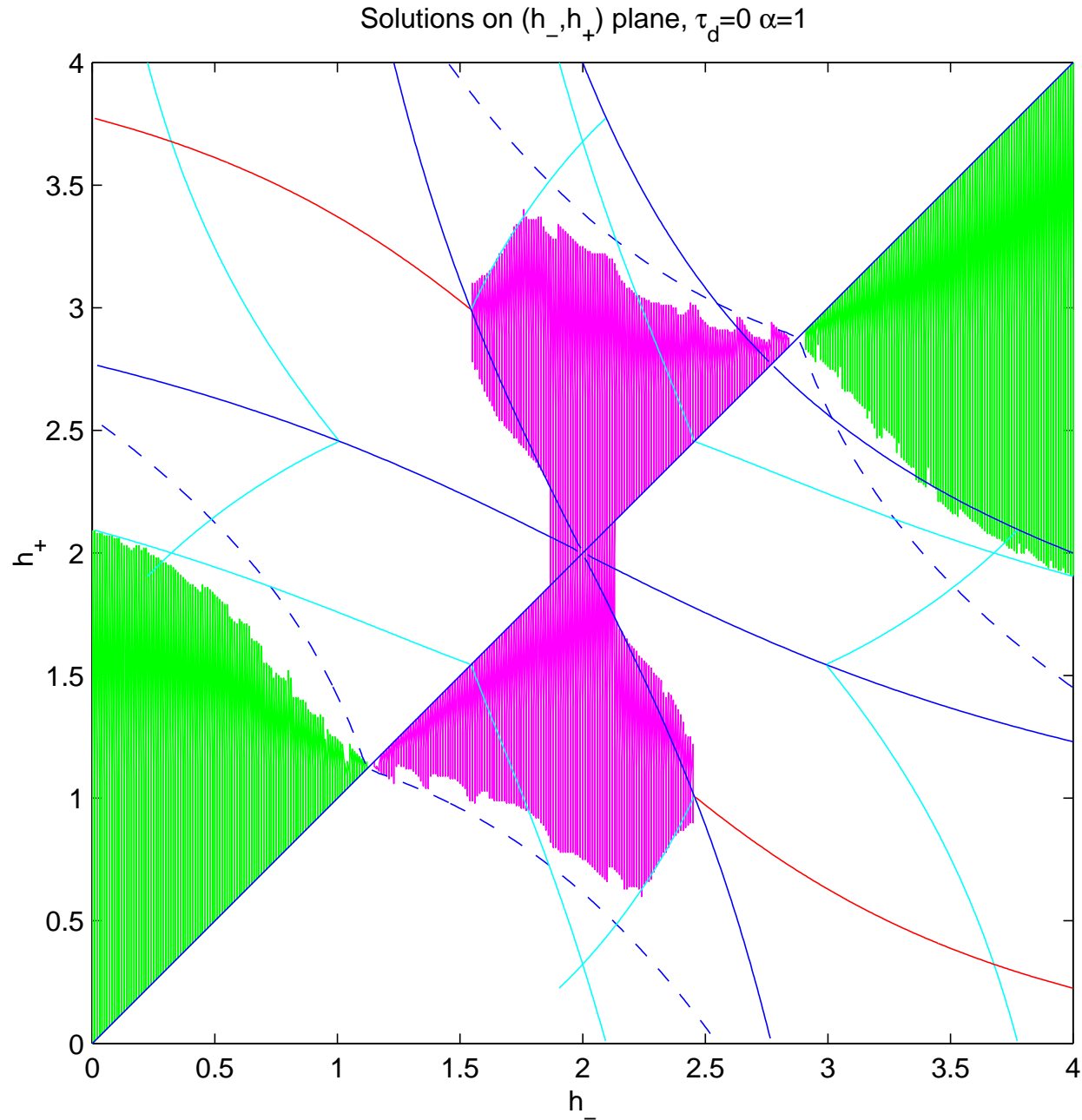
See TGF '01



# Recent discrete computation (stable)



# Recent discrete computation (unstable)



# Broad conclusions

- For the car-following community:
  - Still some work to do in understanding fully pattern mechanisms at the nonlinear level and on the infinite line. Fitting models to new sources of microscopic data.
- For the PDE community:
  - Vanilla versions of LWR/ARZ/Lebacque do not qualitatively replicate data or what car-following models do generically (even at the linear level). This needs a fix — NB global existence results will become ugly / difficult.