Some Remarks on Second Order macroscopic Flow

Michel Rascle

Laboratoire JA Dieudonné,
Université de Nice Sophia-Antipolis
Parc Valrose 06108 Nice Cedex 02, France
http://math.unice.fr/~rascle/

26 octobre 2007

Intro

- Several years ago, we have introduced with Aw a class of "second order" models of traffic flow, for (very good) heuristic reasons, and later on, with Klar et al, made rigorous links with microscopic models.
- I will first recall the basic arguments
- If time allows, I will mention a few recent extensions of this class of models. In any case, I will discuss its (dis)ability of describing more realistic phenomena, such as oscillating solutions ...

Outline

Introduction : At least 3 classes of models :

- Microscopic : Follow the Leader ...
- Kinetic
- Fluid : First Order : Lighthill-Whitham-Richards

$$\partial_t \rho + \partial_x(\rho v) = 0 \ , \ v = V(\rho) \ , \ V'(\rho) < 0$$

Second Order : Payne-Whitham (cf Gas Dynamics)

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t v + v\, \partial_x v = -\rho^{-1} p'(\rho) \partial_x \rho + ... := -\tilde{p}'(\rho) \partial_x \rho + ... \end{cases}$$

- Daganzo (Requiem, 95) PW is a terrible model ! ! [Diffusion still worse !]
  Paradoxes : 1 : $v < 0$ **and 2 :** $\lambda_2 = v + c > v$ **! !**
- Aw-Rascle (Resurrection ?, 2000), Zhang(2002). Fixing : $\partial_x p \rightarrow \partial_t p + v \partial_x p$
- Second equation in (PW) becomes :

$$\partial_t v + v\, \partial_x v = -\tilde{p}'(\rho)(\partial_t + v \partial_x)(\rho)$$

The Fluid model. Eulerian System

- Therefore, setting (new) $p(\rho) := \tilde{p}(\rho)$,

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t w + v\,\partial_x w = 0, \end{cases}$$

where here w : Lagrangian marker ("color") has an influence on v, and $w := v + p(\rho) := v + v_{max} - V_{eq}(\rho)$, but could be much more general

- In conservative form, the system becomes (E) :

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w v) = 0 \end{cases} \tag{1.1}$$

- here $p(\rho) = v_{max} - V(\rho)$ is a known function, which satisfies : $p'(\rho) > 0$ and (strict concavity, again can be extended !), λ_1 is GNL :

$$\forall \rho, \rho p''(\rho) + 2p'(\rho) \neq 0 \ (> 0 \text{ here}) : \tag{1.2}$$

Riemann Problem. Quick version

- Strictly hyperbolic system, (except for $\rho = 0$ ...)
- Eigenvalues of 2x2 matrix :

$$\lambda_1(U) = v - \rho p'(\rho) < \lambda_2(U) = v$$

- λ_1 : genuinely nonlinear shock (braking) or rarefaction (acceleration),whose curves coincide here, since $[\rho w(v-\sigma)] = ((\rho(v-\sigma)_{\pm}) \cdot [w] = 0$
- λ_2 is linearly degenerate : 2-contact discontinuity.
- Diagonalization : Riemann invariants (say on road i) :

$$w(U) := w_i(U) = v + p_i(\rho) \text{ and } v(U) = v$$
$$\partial_t w + v\,\partial_x w = 0, \quad \partial_t v + \lambda_{1,i}(U)\partial_x v \approx 0$$

Riemann Problem with initial data U^- and U^+

- $1-$ waves between U^- and U :
  - Rarefaction : $w(U) := v + p(\rho) = w(U^-)$, if $v > v^-$
  - or shock : $w(U) := v + p(\rho) = w(U^-)$, if $v > v^-$ (coinciding)
- $2-$ waves between U and U^+ : contact discontinuity : $v = v^+$

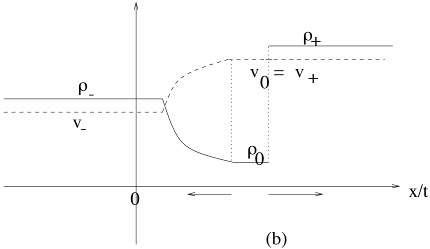
Geometric interpretation : In $(\rho, \rho v)$ plane, $\lambda_{1,2}$ = slope of tangent (or secant : RH relations) to curve $\{w = C\}$ (or to $\{v = C\}$.

Solution of Riemann Pb : Define : $U := U^0 := (w^-, v^+)$, then connect U^- with U through a $1-$wave, finally connect U^0 with U^+ through a $2-$wave, see Figures.

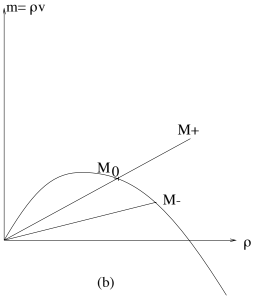
In all cases,if $d(U^1, U^2)) := |v_1 - v_2| + |w_1 - w_2|$, then (BV estimates)

$$d(U^-, U^+) = d(U^-, U^0) + d(U^0, U^+),$$

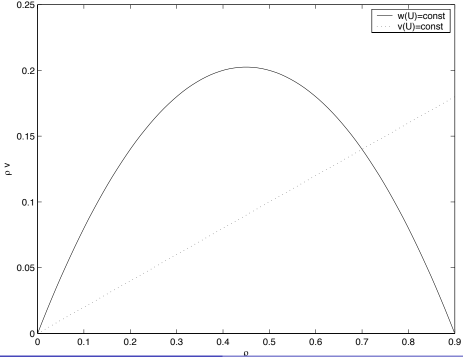
and (bounded) rectangles in (v, w) plane are invariant regions : L^∞ estimates. No more paradox 1 ($v < 0$) or 2 ($\lambda_2 > v$). Compare with PW !

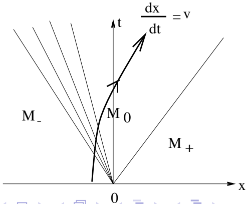

ρ+
v0 = v+
ρ-
v-
ρ0
0
x/t

(b)

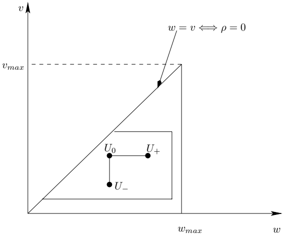

m= ρv
M+
M0
M-
ρ

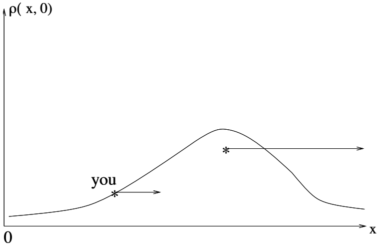
(b)


w(U)=const
v(U)=const
ρ v
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


dx/dt = v
t
M-
M0
M+
x
0

Riemann Pb in (v, w) **plane** BV estimate :
$d(U^-, U^+) = d(U^-, U^0) + d(U^0, U^+)$. No oscillation ...

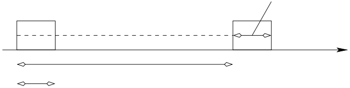




First motivation : x **or** t **dependence ?** Do we react to flow variations in x or t : if the "wave" is faster than you, should you brake (cf gas dynamics), or accelerate (cf our model) ? ? Compare :
$\partial_t v + v\, \partial_x v = \partial_x\, \tilde{p}(\rho)$ or $= -(\partial_t + v\, \partial_x)(\tilde{p}(\rho))$

Motivations. Lagrangian version

- Lagrangian mass coordinates (Courant-Friedrichs)



- From mass conservation,

$$\partial_t \partial_x X = \partial_x \partial_t X \; ; \quad X(x,t) = \int_{-\infty}^{x} \rho(y,t)dy$$

(Essentially, $X = -N$, N : cumulated flow).

- $$\tau_j := \frac{1}{\rho_j} = \frac{x_{j+1} - x_j}{\Delta X} : \qquad \text{adimensional}$$

- τ is additive (on a single lane), not ρ ! ! ... Homogenization : average τ, not ρ ! ! ! ...

Link with microscopic Models (FLM)

- Follow The Leader Model (FLM)

$$\begin{cases} \dot{x}_j = v_j \Longrightarrow \dot{\tau}_j = \frac{v_j+1-v_j}{\Delta X} \\ \\ \dot{v}_j = -P'\left(\frac{x_j+1-x_j}{\Delta X}\right)\frac{v_j+1-v_j}{\Delta X} = -P'(\tau_j)\dot{\tau}_j \end{cases} \tag{1.3}$$

- Equivalent form (FLM') :

$$\begin{cases} \dot{\tau}_j = \frac{v_j+1-v_j}{\Delta X} \\ \\ \dot{w}_j = 0 \quad ; \quad w_j := v_j + P(\tau_j) \end{cases} \tag{1.4}$$

- When $\Delta X \to 0$, (FLM') formally CV to Lagrangian System (L) :

$$\begin{cases} \partial_t \tau - \partial_X v = 0\,,\ \tau := \rho^{-1}\,, \\ \partial_t w = 0\,,\ w = v + P(\tau) := v + p(\rho). \end{cases} \tag{1.5}$$

- Even for weak solutions (Wagner, 87) (L) is equivalent to system (E)

- Now, the Euler first order explicit discretization of (FLM') is (God) :

$$\begin{cases} \tau_j^{n+1} = \tau_j^n + \frac{\Delta t}{\Delta X}\left(v_{j+1}^n - v_j^n\right) \\ w_j^{n+1} = w_j^n = ... = w_j \end{cases} \tag{1.6}$$

- Its numerical solution of CV to the solution of (FLM') when $\Delta t \to 0$, with ΔX fixed, which inherits same stability properties.
- Therefore (FLM') CV **rigorously** to the same system (L) when $\Delta X \to 0$.
- Now, make a **hyperbolic scaling :** let a zoom parameter $\epsilon \to 0$ and $(x', t', X', \Delta t', \Delta X') := \epsilon(x, t, X, \Delta t, \Delta X)$
- ρ, τ, v, system (L) and (God) are unchanged in this scaling, but not the **initial data**

$$U_0(X, \epsilon X) := U_0(\frac{X'}{\epsilon}, X')$$

- Now, (God) is also the exact Godunov approximation of Lagrangian system and (**exceptional**) has same BV stability as the Riemann Pb (monotonicity argument : $w = C$ in Lagrangian cells).

- Therefore, **if there is no small scale** $\frac{X'}{\epsilon}$ in the initial data the solution of (God) converges to the **(unique)** solution of (L) when $\epsilon \to 0$: withAw-Klar-Materne-Rascle, SIAP 2002)
- Independent, formal M. Zhang (2002)
- First $\exists$ result (no scaling) : J. Greenberg (SIAP 2001), with Relax, (sub)"characteristic" case ; Aw, PhD
- If small scales in initial data (oscillations in w and τ), **homogenize** : with P. Bagnerini, SIMA 2003. Corrector ... cf Hamilton-Jacobi aspects
- Oscillations in w (mixture) on outgoing roads in junctions : with Herty, Moutari
- Summary : start from (FLM'). Between two cars $j, j+1$, make $\rho = \rho_j(t)$ in Eulerian coord or $\tau = \tau_j(t)$ in Lagrangian coordinates, then choose between ODE an PDE approach, e.g. for hybrid schemes.

Lagrangian Godunov Scheme

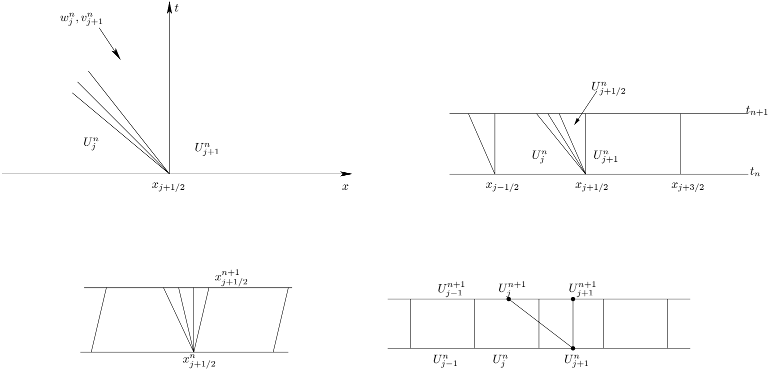


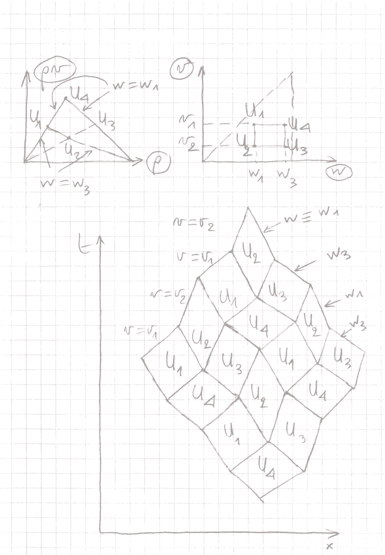
FIG.: In Eulerian moving coordinates, $x_{j+1/2}^{n+1} = x_{j+1/2}^n + \Delta t\, v_{j+1}^n$. Therefore $\tau_j^{n+1} = \frac{x_{j+1/2}^{n+1} - x_{j-1/2}^{n+1}}{\Delta X} = \tau_j^n + \Delta t\, \frac{v_{j+1}^n - v_j^n}{\Delta X}$, same formulas (God), and the trajectories !

Question : how to propagate oscillations ? I

First case : LD case :

only contact discont.

no shock, no rarefact



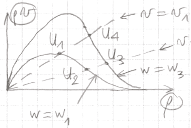
Question : how to propagate oscillations ? II

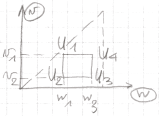
Convex-concave case, cf Greenberg -Rascle (below), Colombo-Tzavaras. Here,

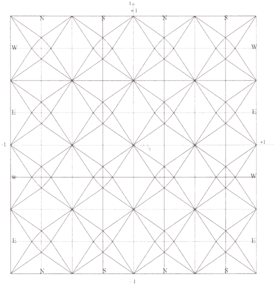
$U_1 \to U_2,\ U_3 \to U_4$: shock (CC)

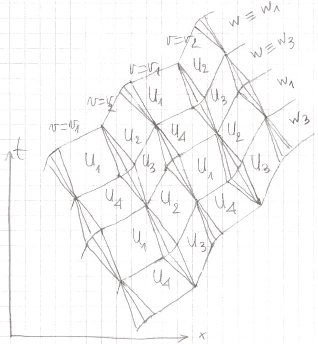
$U_2 \to U_1,\ U_4 \to U_3$: rarefaction.

A 1-rarefaction $\leftrightarrow$ a CD produces centered compression wave (CC) .







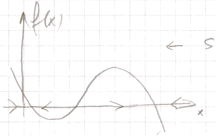


Question : how to initiate oscillations ? I

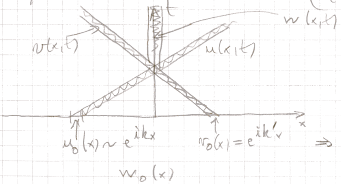
. HOW TO <u>INITIATE</u> OSCILLATIONS ?

. ADD A SOURCE-TERM, VIOLATING SUBCHARACTERISTIC CONDITION (+ DELAY ? → BANDO ...), WITHOUT DELAY: J. GREENBERG et al, F. SIEBEL et al ...

← SOURCE TERM, cf <u>ODE</u> $\dot{x} = f(x)$



. ADD A SOURCE TERM (POSSIBLY STABLE), PRODUCING <u>RESONANCE</u> BETWEEN ≠ WAVES: ARCHETYPE: $(\partial_t + \partial_x)(u) = 0$; $(\partial_t - \partial_x)(v) = 0$; $\partial_t w = uv$



⇒ <u>If</u> <u>$k + k' = 0$</u> : <u>resonance</u> : $\partial_t w = e^{ikx} \cdot e^{-ik'x} \equiv 1$.

Open question : how to initiate oscillations ? II

Combine the two waves in the model (even in GNL case ? ?) with a third wave, produced at a junction, even 1-1 junction (bottleneck) ? ? cf demand -supply discussion : a small change in the supply produces a big shock on incoming roads, see other talks ...