The LWR Model in Lagrangian coordinates

ACI-NIM: Math Models on Traffic Flow

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Outline

- Lagrangian resolution of the homogeneous LWR model
- Extension to heterogeneous flow
- Numerical examples

The Homogeneous case

The LWR model

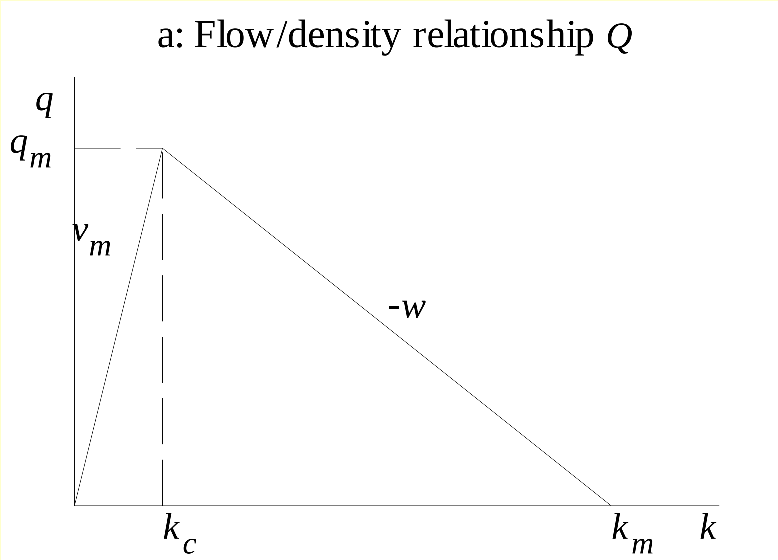
Variables: k, density; v, speed; $q=kv$, flow

Conservation equation:

$$\partial_t k + \partial_x kv = 0$$

Fundamental diagram (FD):

$$v = V(k) \quad \text{or} \quad q = kV(k) = Q(k)$$



The model can be synthesized as a scalar hyperbolic equation:

$$\partial_t k + \partial_x Q(k) = 0$$

The Godunov scheme is classically used for numerical resolution

The cumulative count function $N(x,t)$

- $N(x,t)$ represents the cumulative number of vehicles that cross location *x* by time *t*
- $k=-\partial_x N$ and $q=\partial_t N$
- The conservation equation reduces to: $\partial_{x,t}N=\partial_{t,x}N$ (existence of N)
- The LWR model can then be expressed as:

$$\partial_t N = Q(-\partial_x N)$$

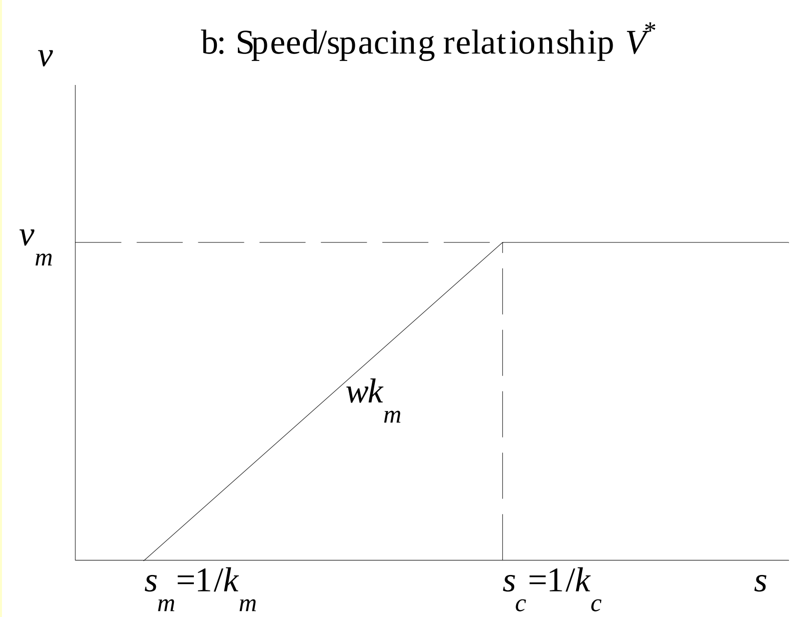
Hamilton-Jacobi equation

(x,t) Eulerian coordinates → (N,t) Lagrangian coordinates

The LWR Model in (N,t) coordinates as a conservation law

Variables: $s=1/k$, spacing; $N(x,t)$, cumulative count function

The conservation equation becomes : $\partial_t s + \partial_N v = 0$



FD can be expressed as :

$v = V(1/s) = V^*(s)$

The model reduces to a scalar hyperbolic equation:

$$\partial_t s + \partial_x V^*(s) = 0$$

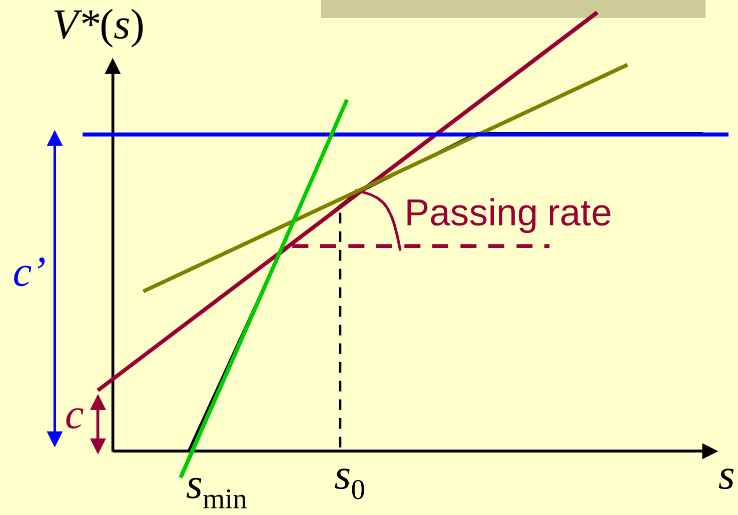
The LWR model in (N,t) coordinates as a variational principle

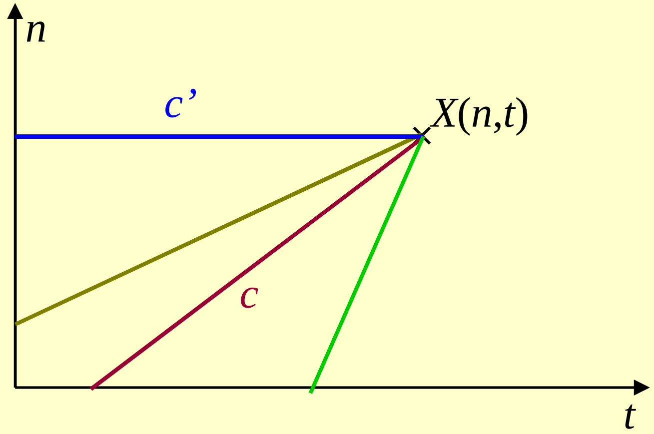
- Let $X(n,t)$ be the inverse of $N(x,t)$
  (X is obtained by solving for x in $n=N(x,t)$)

- $X(n,t)$ represents the trajectory of vehicle n
- X verifies: $\frac{\partial X}{\partial t} = V^*\left(-\frac{\partial X}{\partial N}\right)$ (Hamilton-Jacobi equation)
- The model solutions in X also satisfy a least-cost path problem

Definition of the least-cost path problem in (*N*,*t*)

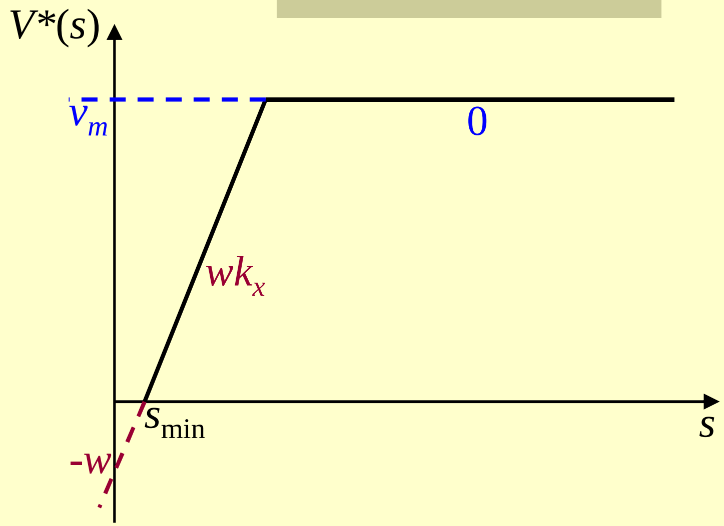
- Valid Paths are locations in (*N*,*t*) where *s* is constant (⇔ to characteristics in (*x*,*t*))
- Path Slopes ⇔ passing rates
- Path Costs ⇔ intercept

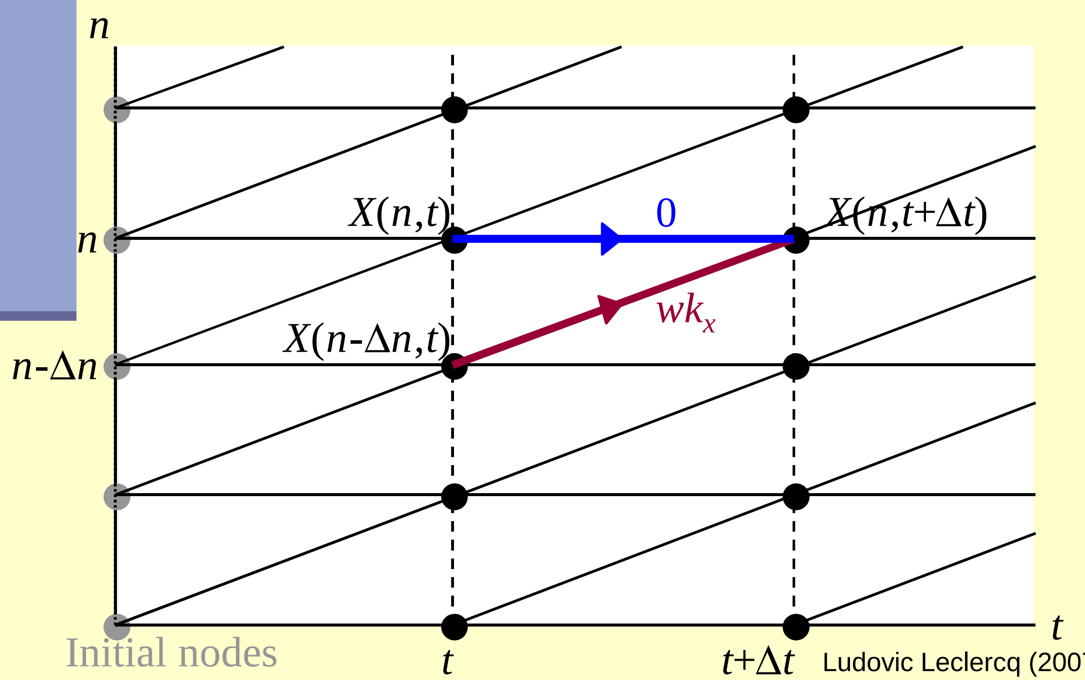




Numerical resolution using the variational principle

- When the FD is triangular, only two paths have to be considered:
  - free-flow path (slope: 0 ; cost: v_m)
  - congested path (slope: wk_x ; cost: $-w$)





$$X(n,t+\Delta t) = \min\begin{pmatrix} X(n,t) + v_m \Delta t \\ X(n-\Delta n, t) - w\Delta t \end{pmatrix}$$

with $\Delta t = \Delta n / wk_x$

This scheme is exact as each node is linked by a valid path

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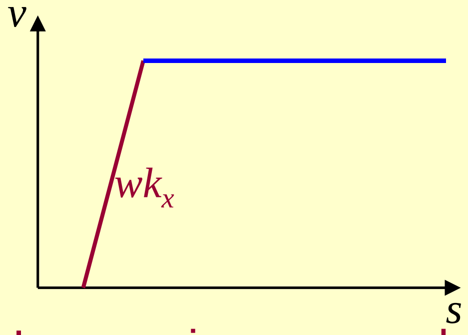
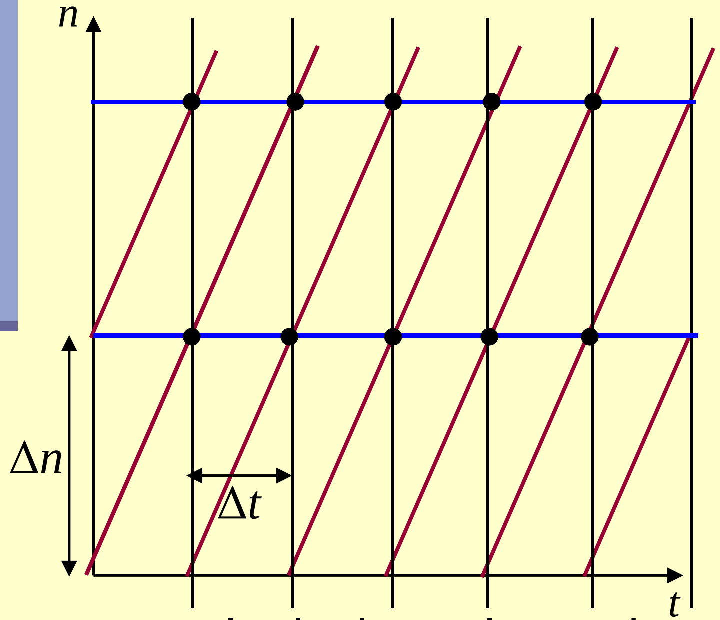
Advantages of (N,t) coordinates (1)

- when $\Delta n=1$
  - the outputs are vehicle trajectories (car-following model)
  - vehicle characteristics can then be easily incorporated:
    - destination
    - own characteristics (desired speed, size, reaction time)
- when $\Delta n>1$ ⇔ mesoscopic models
- $\Delta n<1$ ⇔ moving boundary conditions

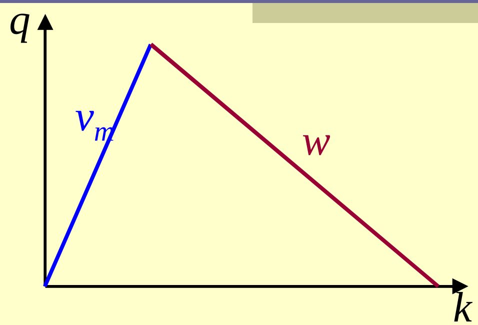
Advantages of (N,t) coordinates (2)



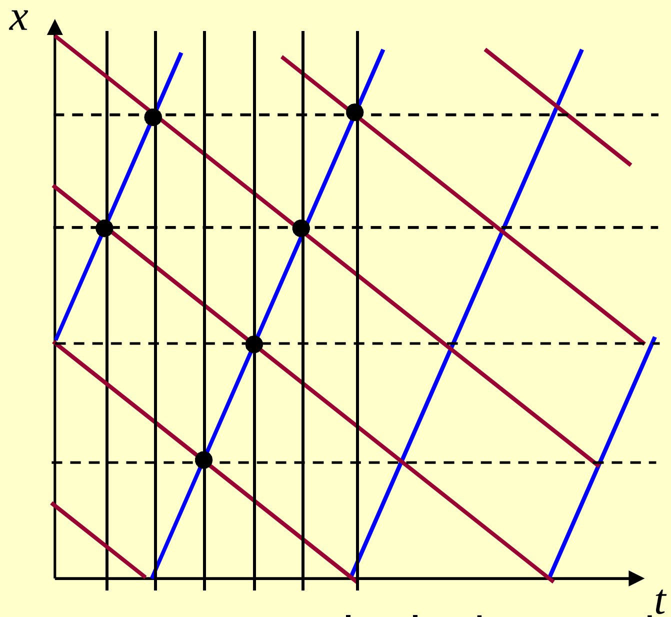
Lagrangian approach



A rectangular lattice only requires
$\Delta t = \Delta n / wk_x$



Eulerian approach

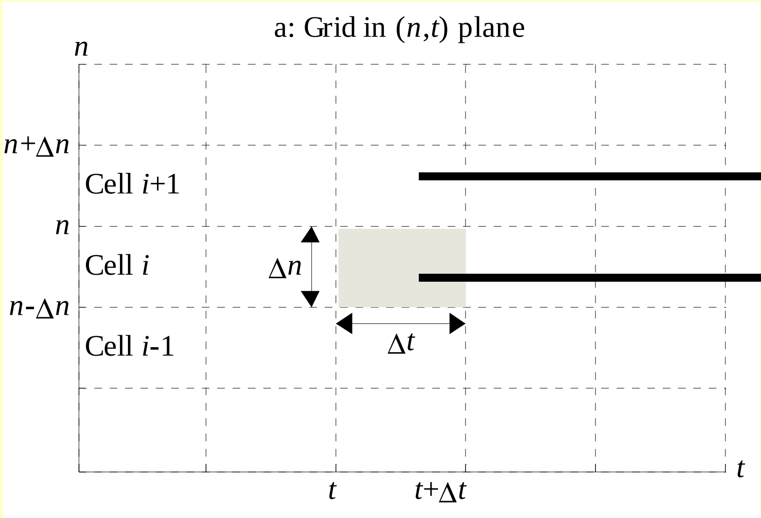


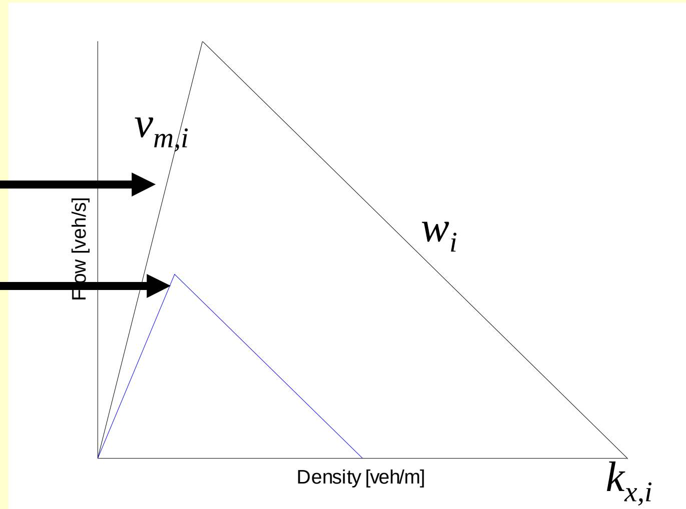
A rectangular lattice requires
v_m/w to be an integer

Multiclasse in Lagrangian Framework

Principle

- Define a specific FD for each Lagrangian cell (Δn=1)
- Each vehicle type *i* is defined by three parameters:
  - The free-flow speed $v_{m,i}$
  - The jam density $k_{x,i}$ (≈inverse of vehicle size)
  - The wave-speed w_i





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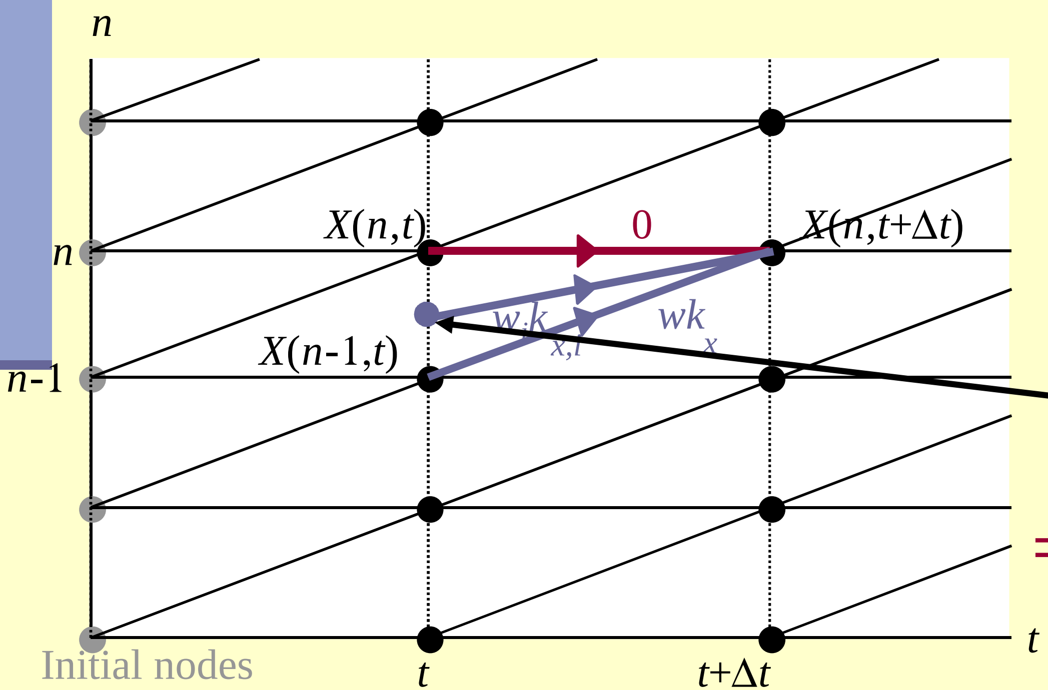
Numerical resolution using the variational principle

- Free-flow path: only the cost ($v_{m,i}$) is modified
- Congested path: the slope ($w_i k_{x,i}$) and the cost ($-w_i$) are modified



Slope modifications change the structure of the network

This point is not on the network grid
=> the value of *X* is unknown

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X value estimation – solution 1

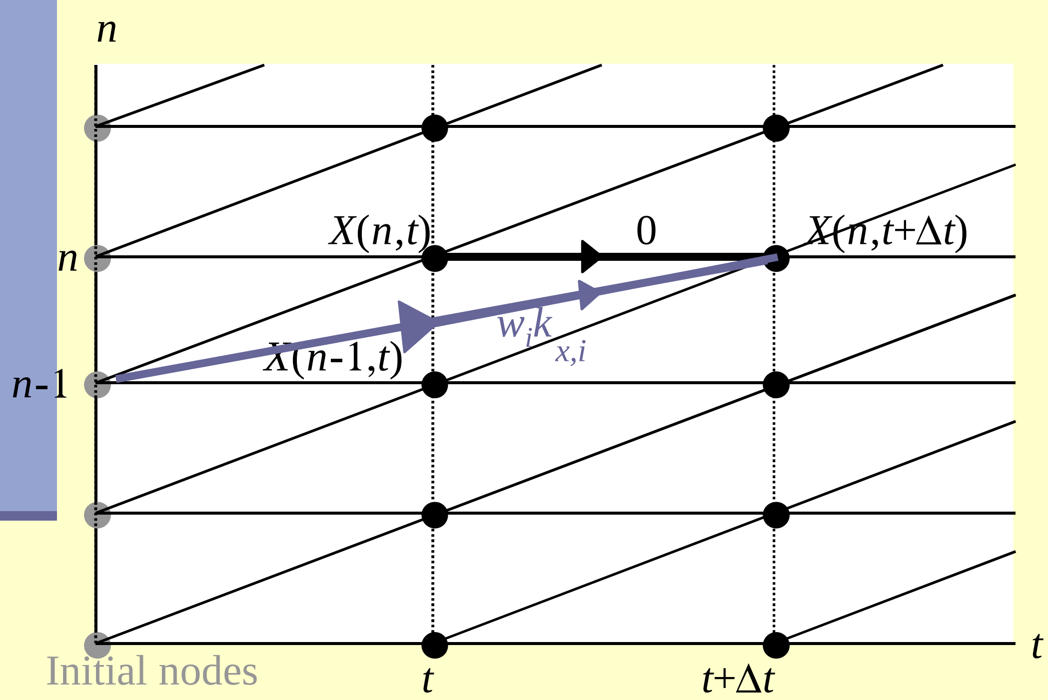
- Assume that the spacing is uniform between *n* and *n*-1 at time *t* and estimate the *X* value:

$$X(t) = (1 - \alpha_i) X(n,t) + \alpha_i X(n-1,t) \quad \text{with} \quad \alpha_i = w_i k_{m,i} \Delta t$$

- The numerical scheme does not remain exact

X value estimation – solution 2

- Store the X values at previous time steps and look for the time when the congested path join a network node



This is possible if for each vehicle:

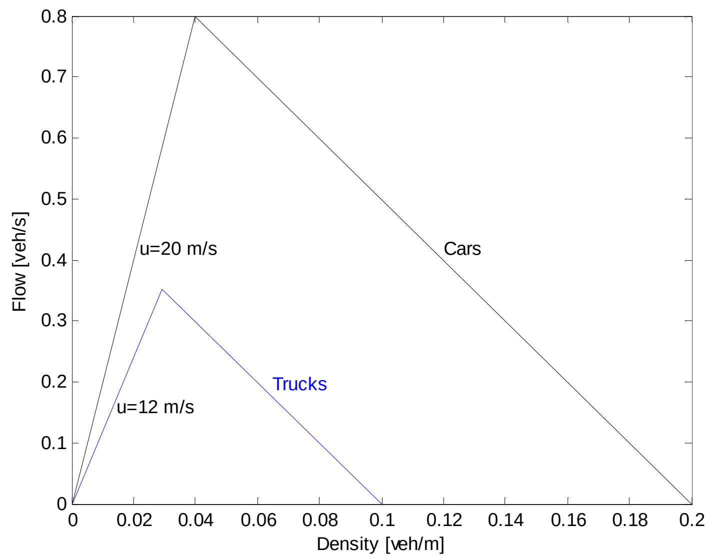
- w_i is the same
- the ratio $k_{x,i}/\max(k_{x,i})$ is an integer

Under these assumptions, the numerical scheme is exact

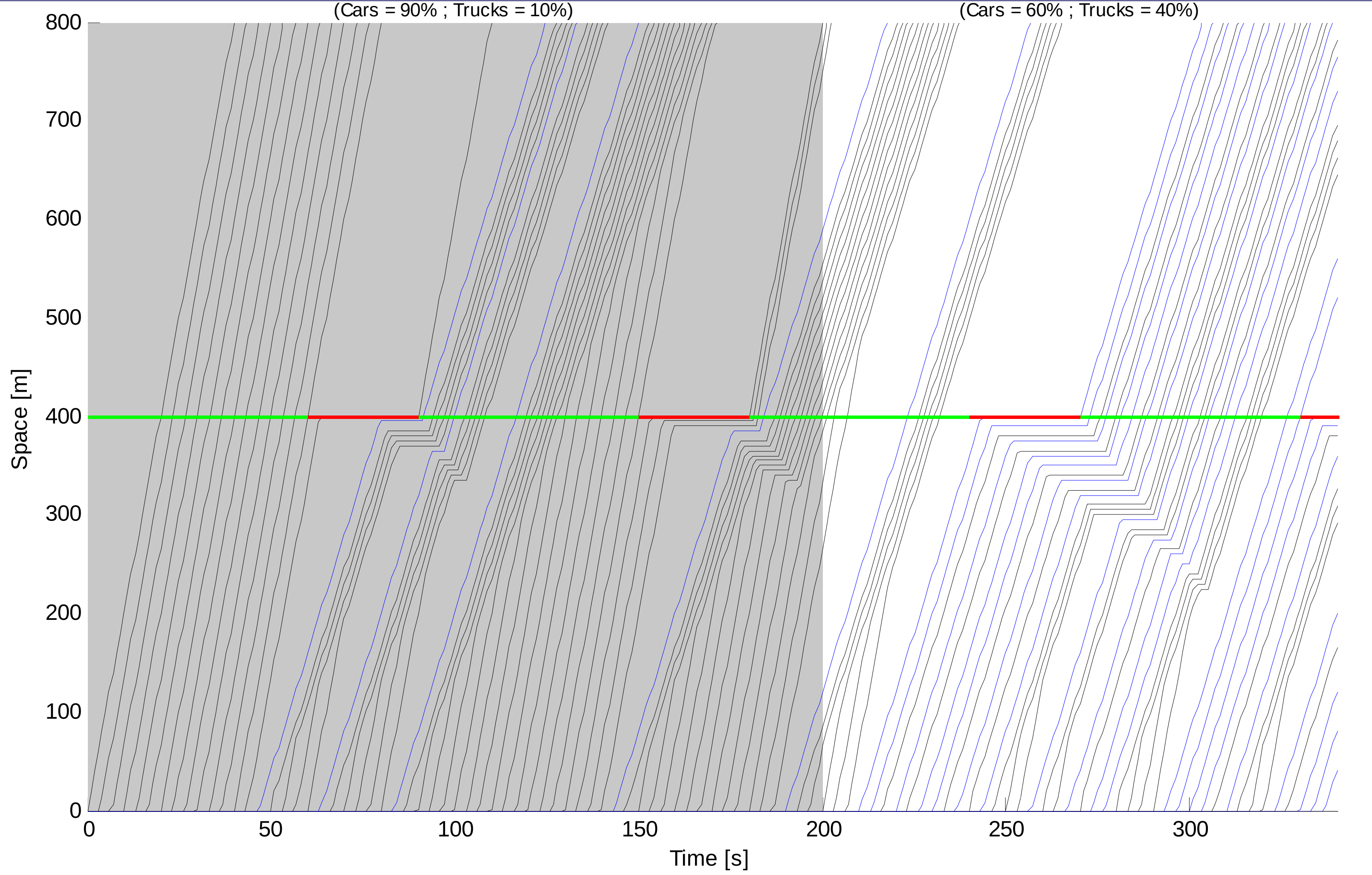
Numerical examples

Case of study

- One-lane road
- Cars and trucks with specific FDs
- A constant flow rate at the entry (1080 veh/h)
- Composition:
  - 0-150s: 90% cars, 10% trucks
  - 150s-350s: 60% cars, 40% trucks
- A traffic signal (cycle=90s, green time=60s)



Variational scheme



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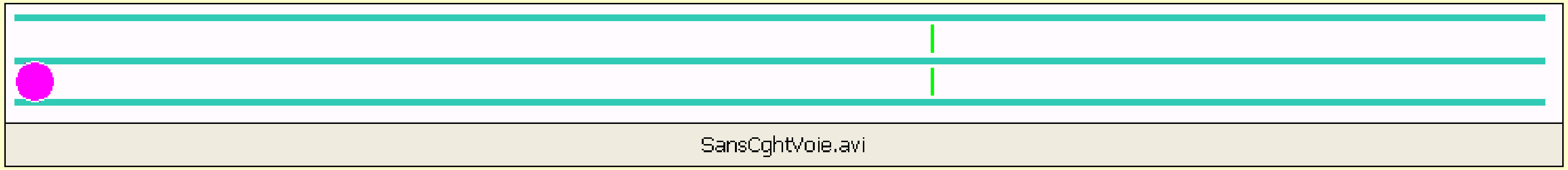
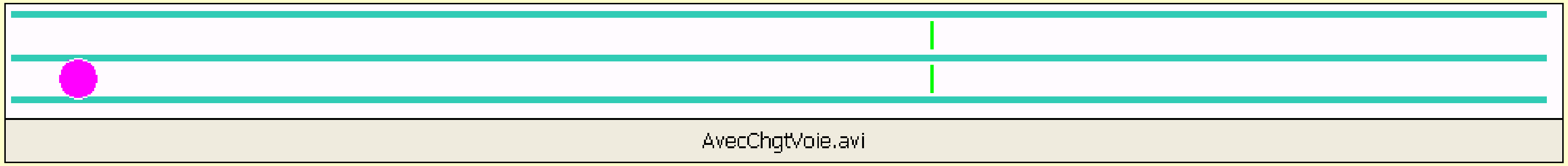
Simulation

Without lane-changing:



Coupled with a lane-changing model:



Conclusion on Lagrangian approach

- output are vehicle trajectories. This makes it easier to incorporate extensions such as:
  - origin/destination
  - different vehicle types
  - moving bottlenecks
- Allow a direct extension to heterogenous flow:
  - Parsimonious and easy to calibrate (FD by vehicle type)
  - Exact under little restrictive assumptions (triangular FD, constant *w*, integer ratio between jam densities)
  - Fully compatible with existing extension of the LWR model and especially lane-changing one (Laval and Leclercq, 2007)

Thank you for your attention