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Generic second order traffic flow modeling: recent progress

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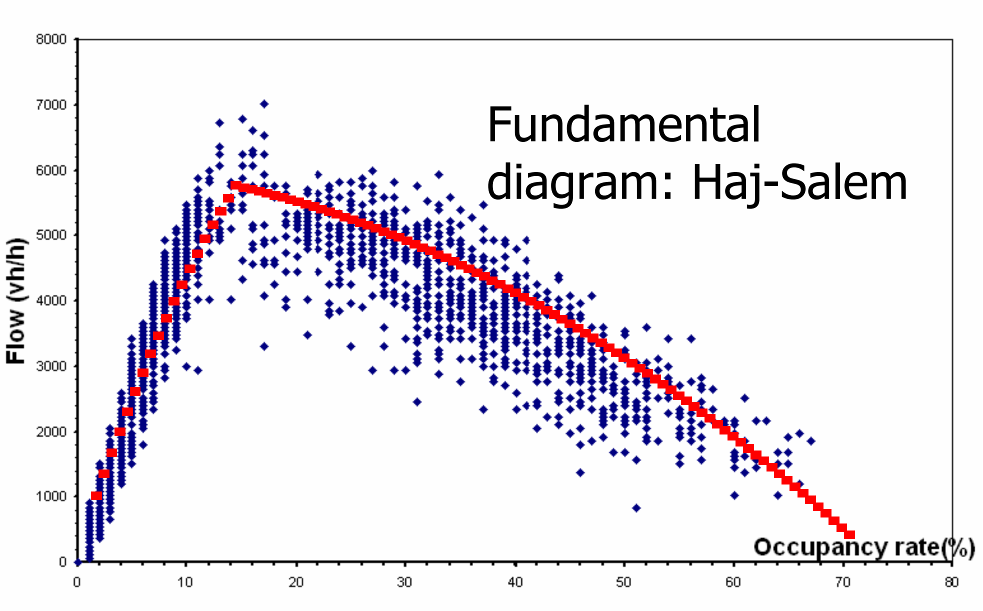
Outline of the presentation

1. Introduction: generic second order traffic modelling
2. Some examples (LWR, ARZ, 1-phase Colombo, Cremer-Papageorgiou)
3. Common features
4. Further extensions: driver specific parameters, , multiclass, stochastic
5. Conclusion

Macroscopic traffic description: notations

- Continuum hypothesis: traffic state can be described by functions of location x and time t
- Variables:
  - Density $\rho(x,t)$
  - Flow $q(x,t)$
  - Velocity $v(x,t)$



Fundamental diagram: Haj-Salem

Introduction: definition of the GSOM family

- **Ideas:**
  - Conservation of vehicles
  - Dynamics of some driver-specific attributes, induced by a **variable fundamental diagram**
- **Variability:**
  - Driver-specific attributes (none: LWR, FD parameters, relative speed, etc)
  - Dynamics of the driver specific attributes (conservation, relaxation, stochastic)

GSOM family: basic description

(Lebacque, Mammar, Haj-Salem 2005-2006, 2007)

- Conservation of vehicles (density)

$$\partial_t \rho + \partial_x \left(\rho v\right) = 0$$

- Variable fundamental diagram, depending on a (possibly vector of) user attribute(s) I

$$v = \Im\left(\rho, I\right)$$

- Evolution equation for I along vehicle trajectories (example: relaxation)

$$\dot{I} = \partial_t I + v \partial_x I = -f\left(I\right)$$

GSOM: basic equations

Conservation of vehicles

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0$$

Dynamics of I along trajectories

$$\frac{\partial \rho I}{\partial t} + \frac{\partial \rho I v}{\partial x} = -\rho f(I)$$

$$\Leftrightarrow \quad \dot{I} = -f(I)$$

Variable fundamental diagram

$$v = \Im(\rho, I) \quad \Rightarrow \quad \rho v \overset{def}{=} \Re(\rho, I)$$

Example 0: the LWR model (Lighthill, Whitham, Richards 1955, 1956)

- No driver-specific attribute

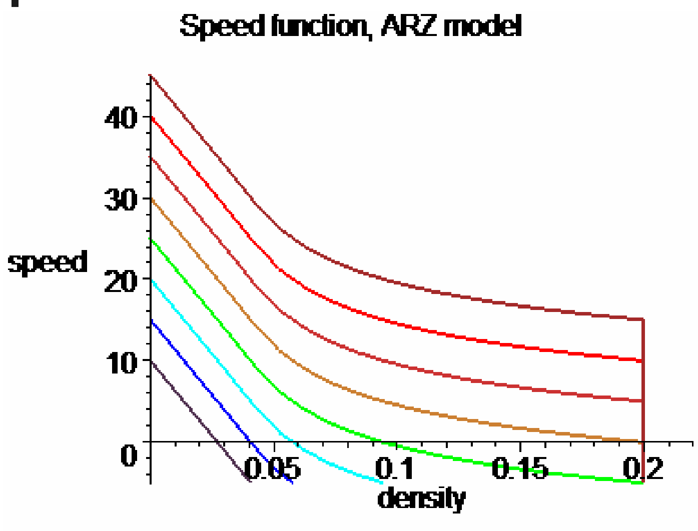
$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho v}{\partial t}=0 \quad \text{Conservation equation}$$

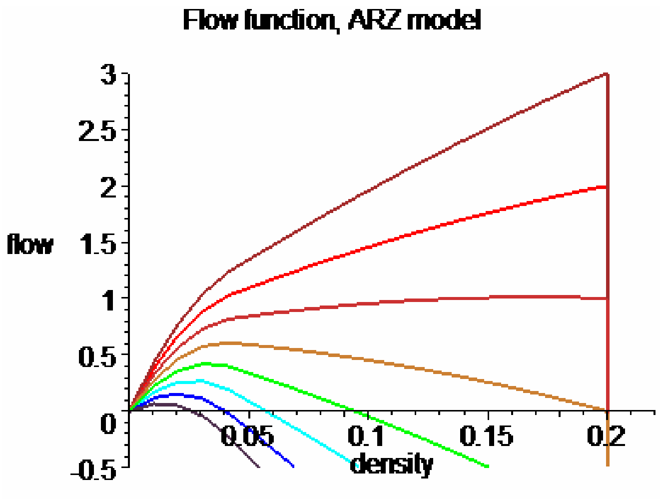
$$v=V_e(\rho, x) \quad \text{Fundamental diagram}$$

- One conservation equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial t}Q_e(\rho, x)=0$$

Example 1: The ARZ model (Aw,Rascle 2000, Zhang,2002)


Speed function, ARZ model


Flow function, ARZ model

- The attribute ***I*** is the relative speed

$$I = v - V_e(\rho, x) \quad \Leftrightarrow \quad v = \Im(\rho, I) = I + V_e(\rho)$$

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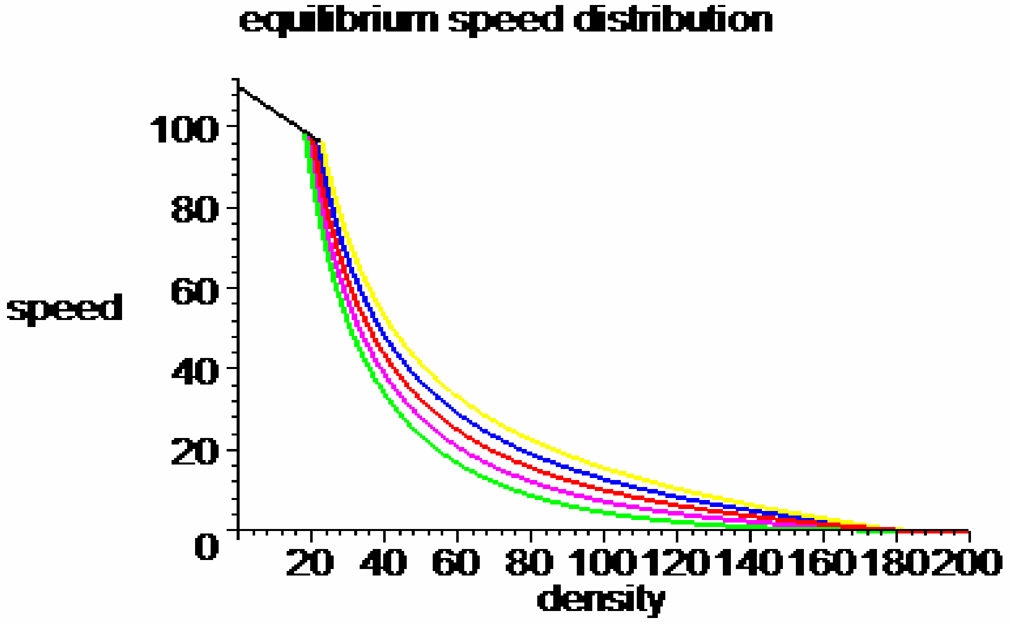
Example 2: the 1-phase Colombo model (Colombo 2002, Lebacque Mammar Haj-Salem 2007)

- **Variable FD** (speed-density fundamental diagram, congested domain)

$$\Im(\rho, I) = \left(I + \frac{q_*}{\rho} \right) v_0(\rho)$$

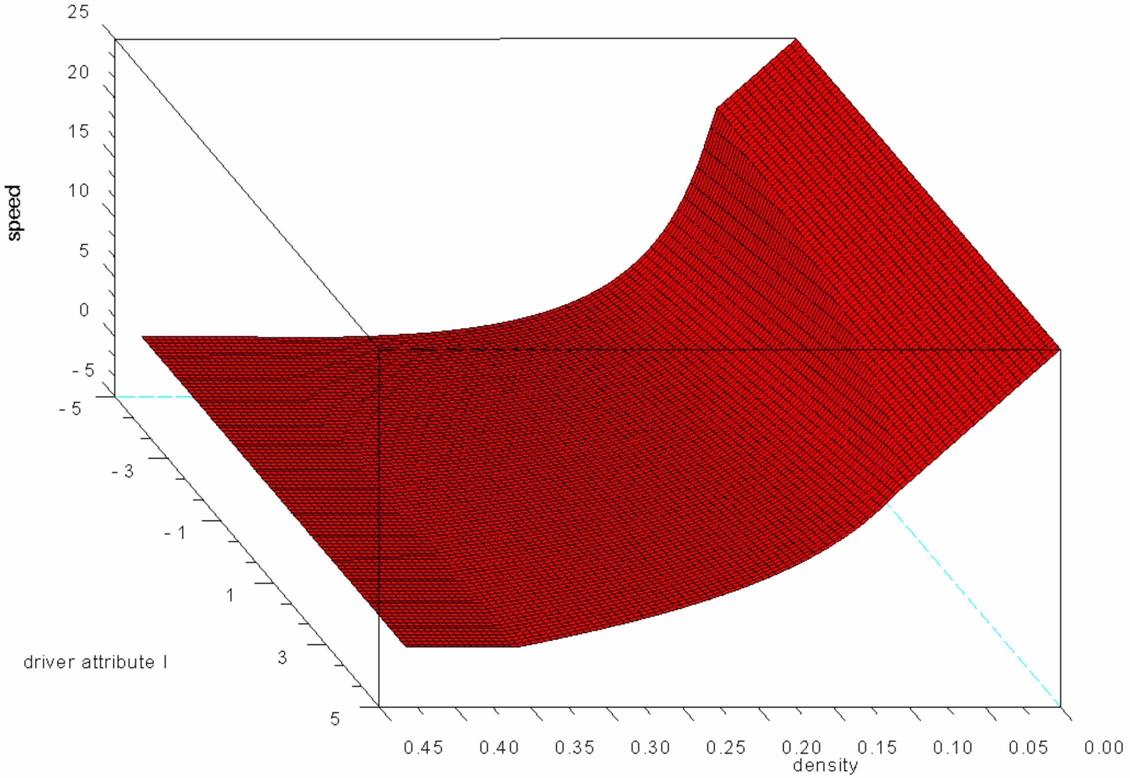
$$v_0(\rho) = 1 - \frac{\rho}{\rho_{max}}$$



- The attribute *I* is the parameter of the FD family

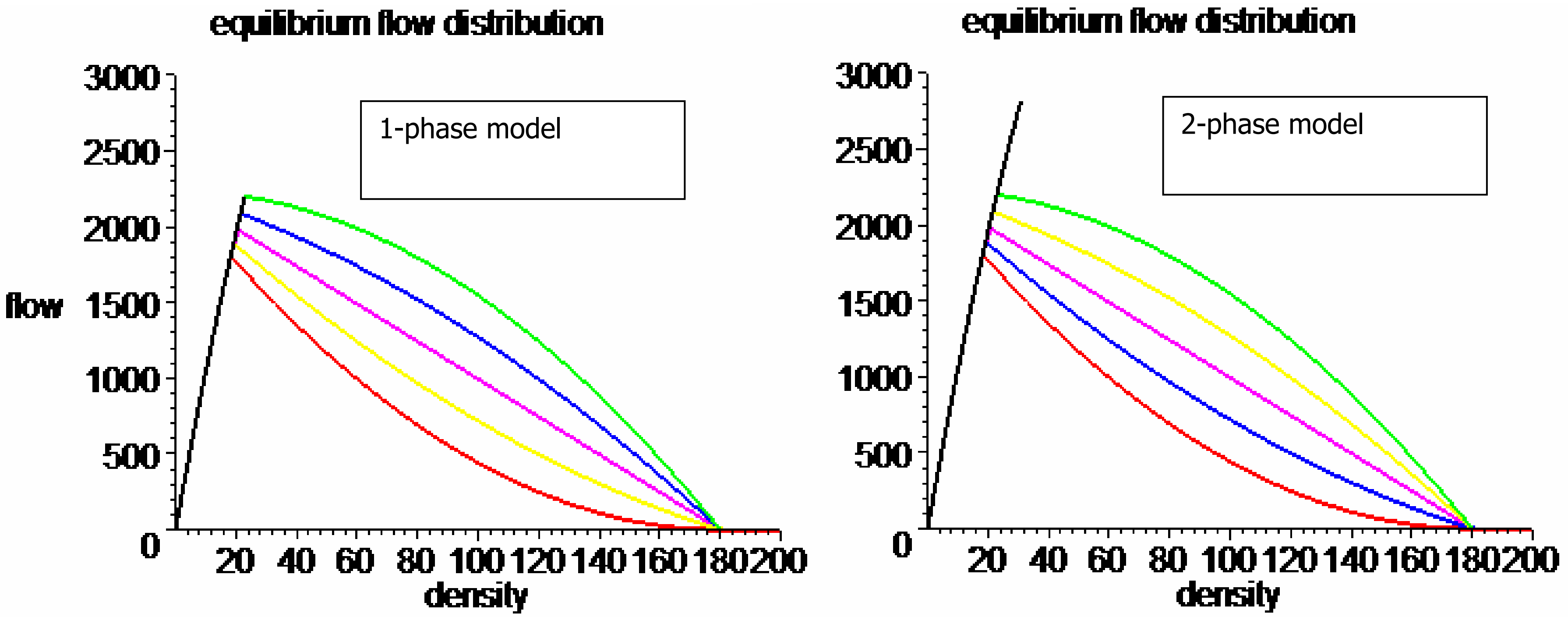
$$v = \Im(\rho, I) \overset{def}{=} \begin{cases} V_{max} - \dfrac{V_{max} - V_{crit}}{\rho_{crit}(I)}\rho & \text{if} \quad \rho \le \rho_{crit}(I) \\ \left(I + \dfrac{q_*}{\rho}\right)\left(1 - \dfrac{\rho}{\rho_{max}}\right) & \text{if} \quad \rho \ge \rho_{crit}(I) \end{cases}$$

- Fundamental diagram (speed-density)



Example 2 continued (1-phase Colombo model)

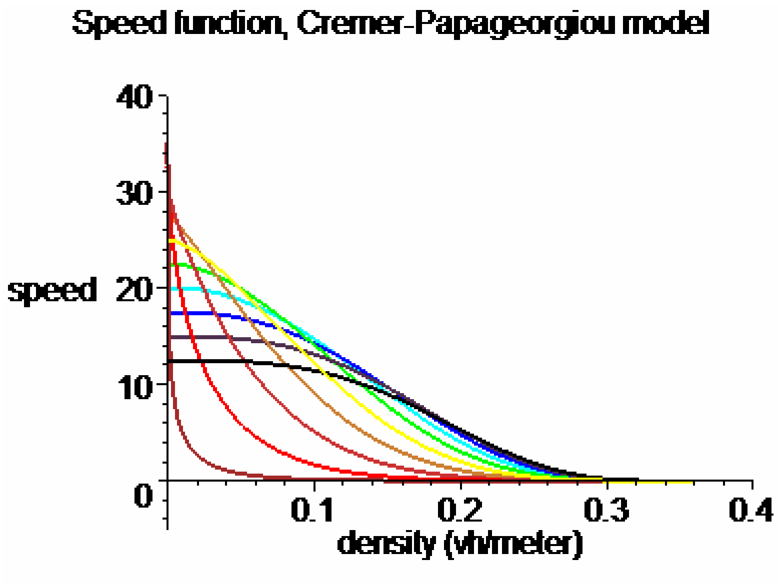
- **1-phase vs 2-phase**: flow-density FD

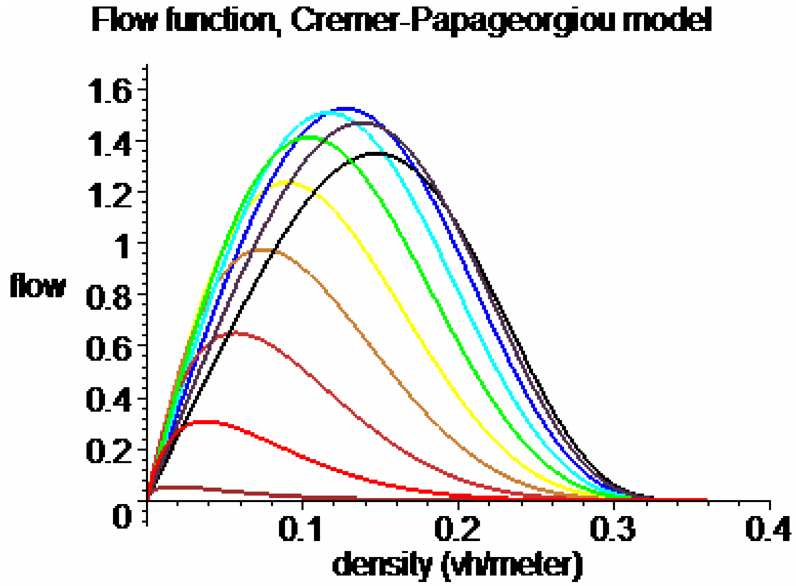


Example 3: Cremer-Papageorgiou

- Based on the Cremer-Papageorgiou fundamental diagram (Haj-Salem 2007)

$$\Im(\rho, I) = V_f I \left(1 - \left(\frac{\rho}{\rho_{max}}\right)^{l(3-2I)}\right)^m$$





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Two fundamental properties of the GSOM family

- **1.** The discontinuities of I propagate at the speed v of the traffic flow
- **2.** If the invariant I is initially piecewise continuous, it stays so at all times $t > 0$
- $\Rightarrow$ On a domain on which I is uniform the GSOM model reduces to a shifted LWR model (piece-wise LWR models)

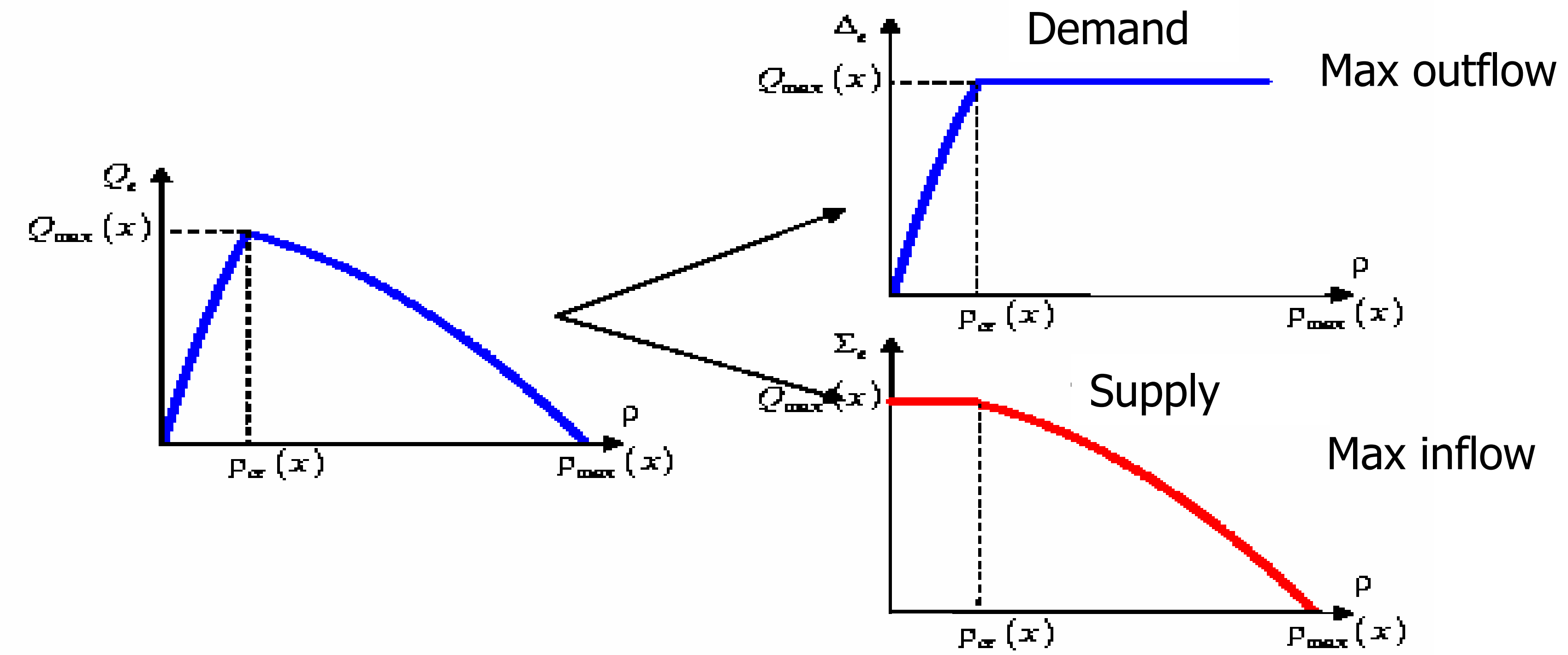
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \Re(\rho, I) = 0$$

Consequence: most properties of LWR translate to the GSOM family with few changes

- Inhomogeneous Riemann problem solved as in LWR
- Shifted supply and demand
- Boundary conditions
- Godunov scheme
- Particle (lagrangian) discretizations
- Wave tracking
- Hybridization (Mammar, Moutari)

The LWR model: supply / demand

- the *equilibrium supply* Σ_e and *demand* Δ_e functions (Lebacque, 1996)



Demand

Max outflow

Supply

Max inflow

The LWR model: the min formula

- The local supply and demand:

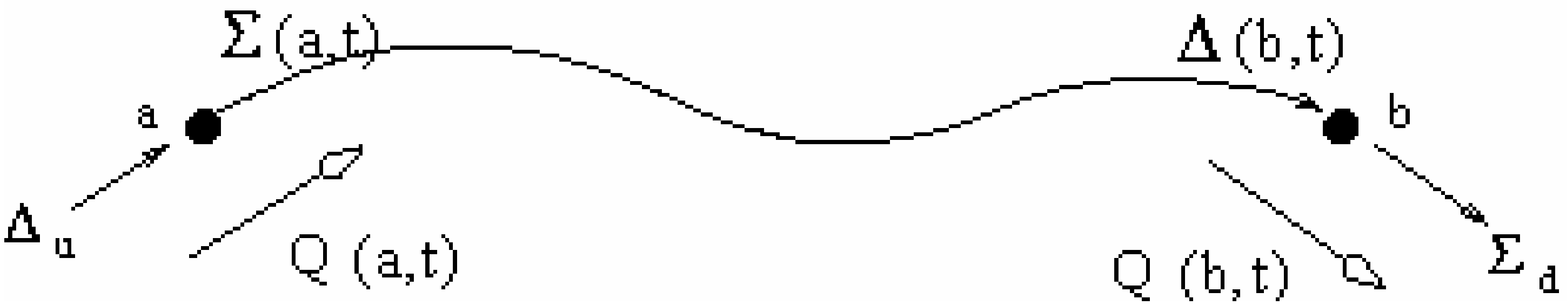
$$\Sigma(x,t) = \Sigma_e\left(\rho\left(x+,t\right), x+\right)$$
$$\Delta(x,t) = \Delta_e\left(\rho\left(x-,t\right), x-\right)$$

- The min formula

$$Q\left(x,t\right) = \mathrm{Min}\left[\Sigma\left(x,t\right), \Delta\left(x,t\right)\right]$$

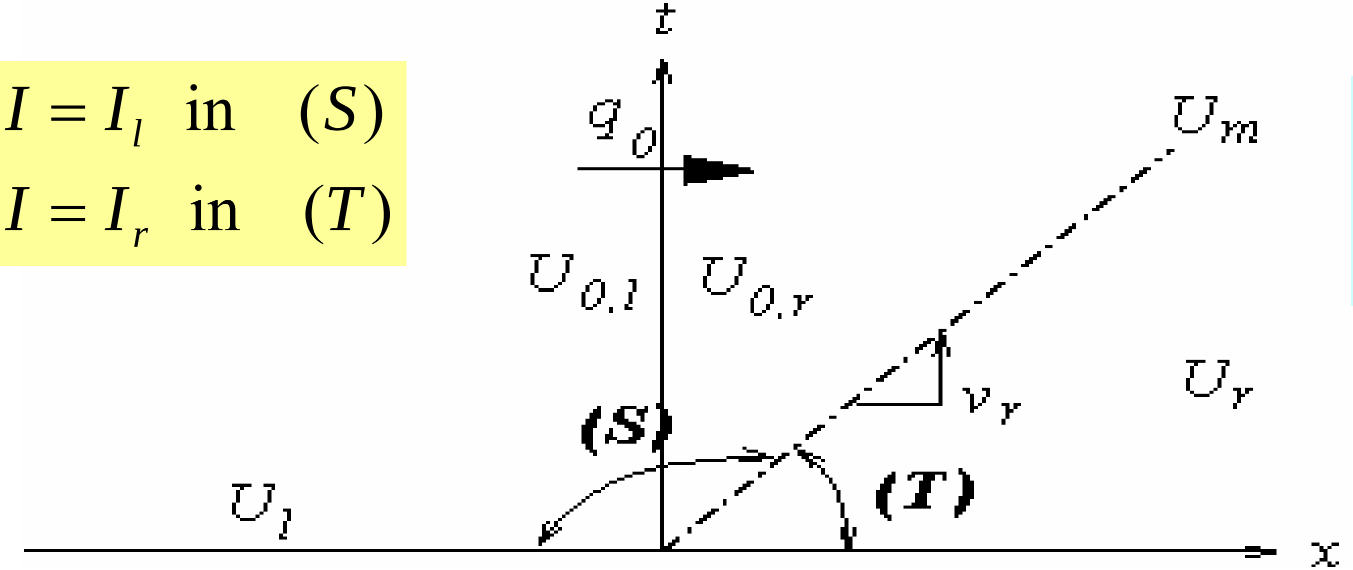
Supply-demand boundary conditions (equivalent to Bardos-LeRoux-Nédélec)

- Link supply : $\Sigma(a,t) = \Sigma_e\big(K(a+,t),a\big)$
- Link demand : $\Delta(b,t) = \Delta_e\big(K(b-,t),b\big)$
- Min formula : $$Q(a,t) = \mathrm{Min}\left[\Delta_u(t),\Sigma(a,t)\right]$$ $$Q(b,t) = \mathrm{Min}\left[\Delta(b,t),\Sigma_d(t)\right]$$



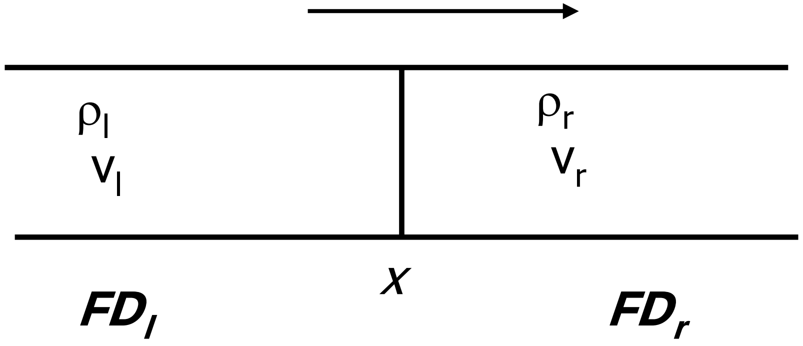
Inhomogeneous Riemann problem

$$I = I_l \quad \text{in} \quad (S)$$
$$I = I_r \quad \text{in} \quad (T)$$

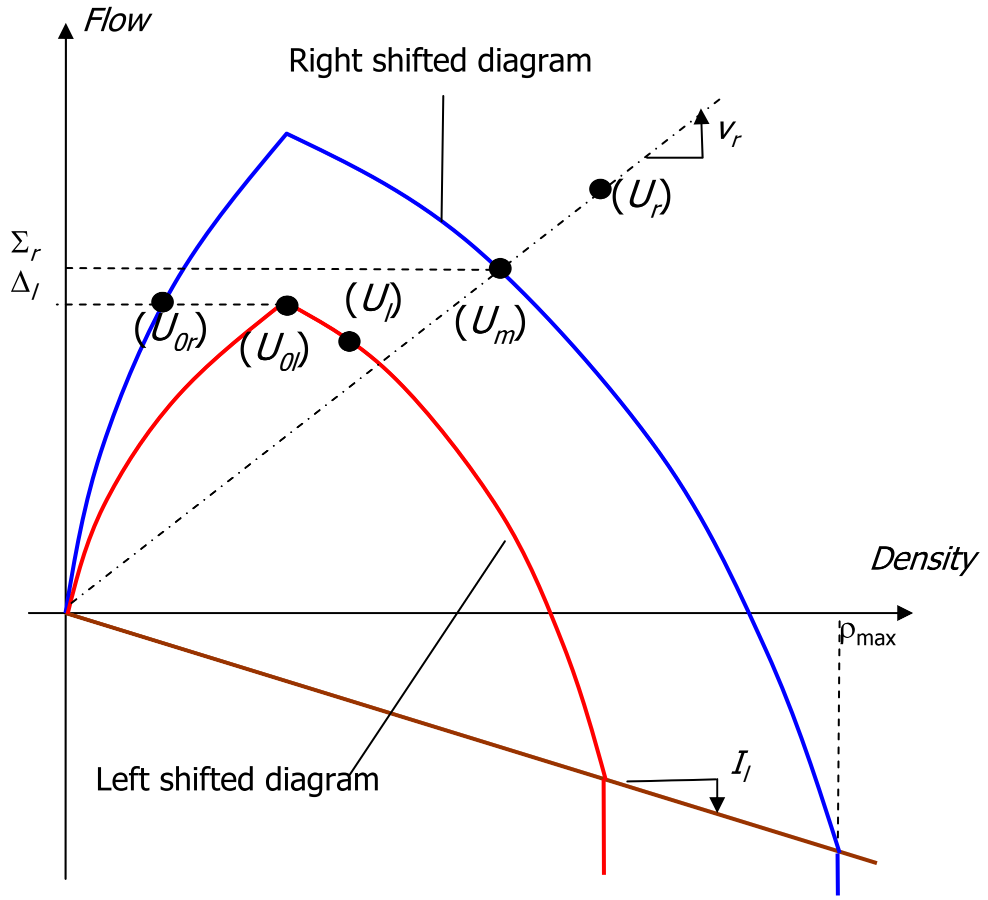


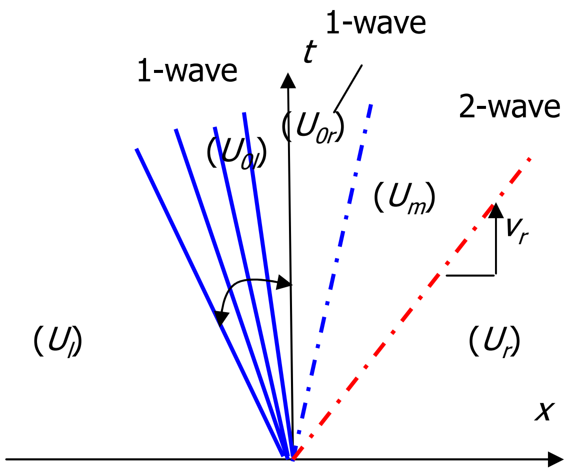
$$v_m = \Im(\rho_m, I_l) = v_r$$
$$\Rightarrow \rho_m = \Im_\rho^{-1}(v_r, I_l)$$

$$U_l \rightarrow U_m : \quad \text{shifted LWR wave}$$
$$U_m \rightarrow U_l : \quad I \text{ discontinuity}$$



Complete resolution example (GSOM - ARZ)

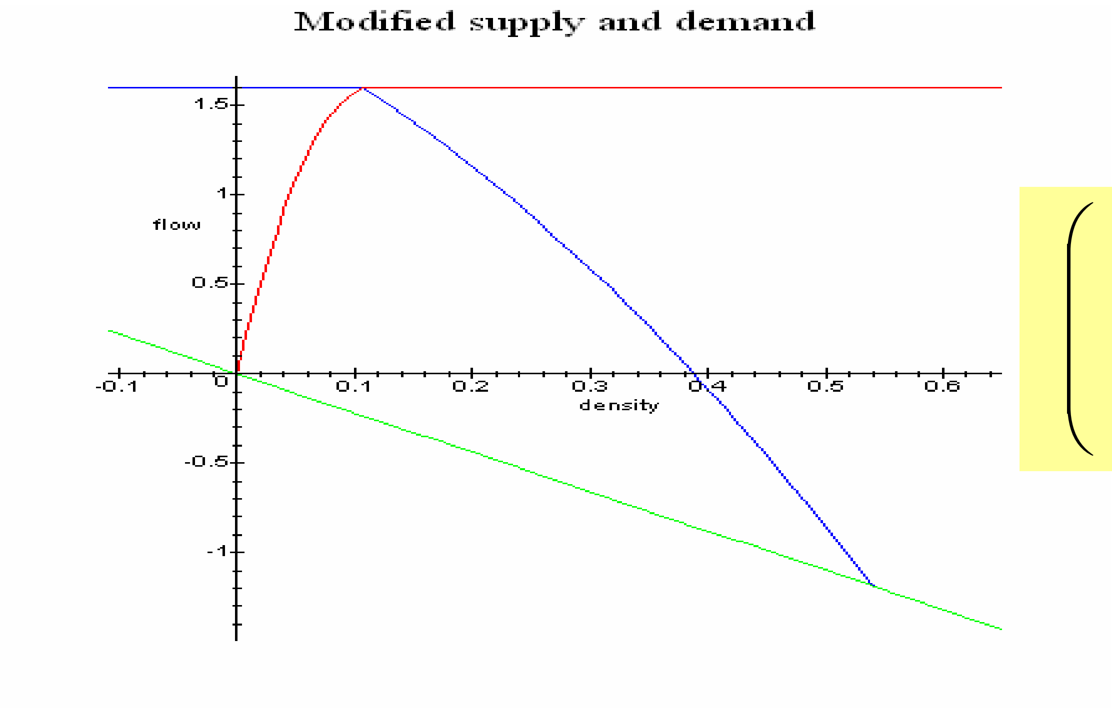




Shifted supply and demand

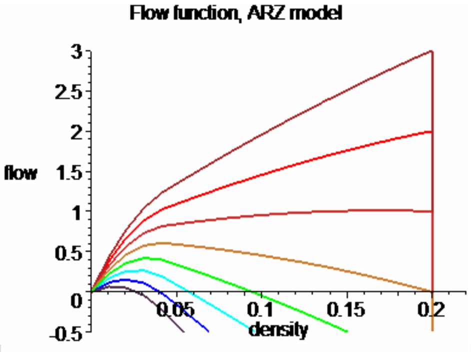
(Lebacque Mammar Haj-Salem 2005-2007)

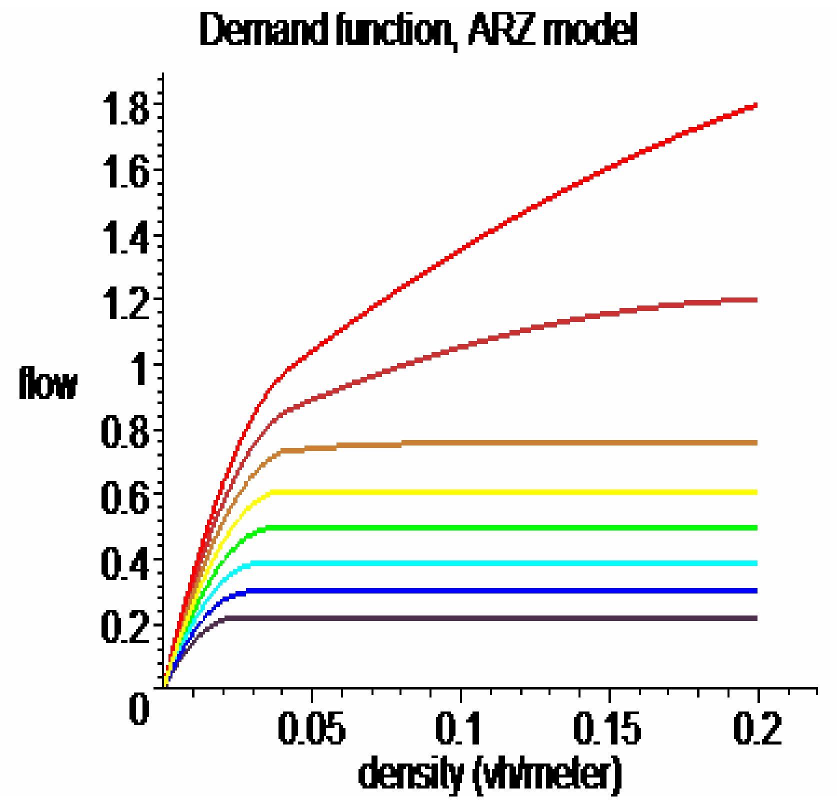
- Shifted supply and demand = supply and demand for the shifted FD

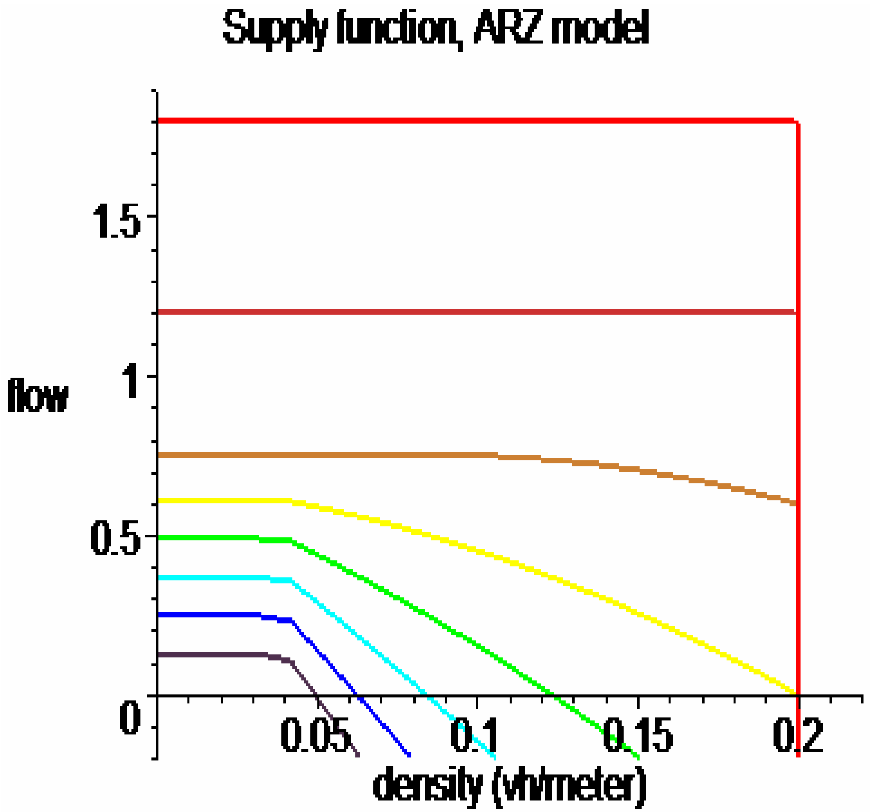


$$\begin{cases} \Delta(\rho, I) \overset{def}{=} \mathrm{Max}_{0\leq\varsigma\leq\rho}\, \Re(\varsigma, I) \\ \Sigma(\rho, I) \overset{def}{=} \mathrm{Max}_{\varsigma\geq\rho}\, \Re(\varsigma, I) \end{cases}$$

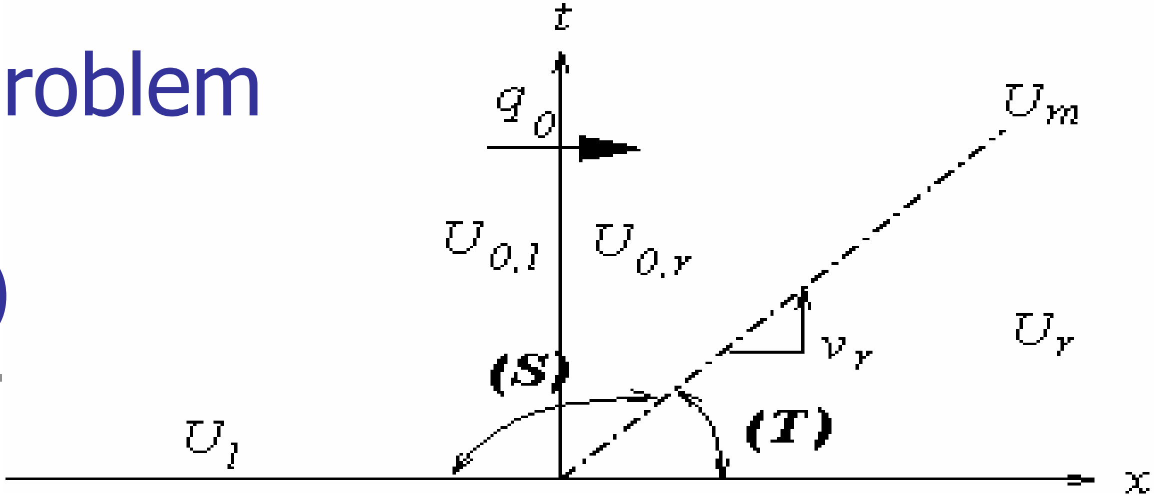
Shifted supply and demand: ARZ family







Riemann problem solution (summary)



- Define upstream demand and downstream supply (both depend on I_l):

$$\delta_l \overset{def}{=} \Delta_{e,l}(\rho_l, I_l), \quad \sigma_r \overset{def}{=} \Sigma_{e,r}(\rho_m, I_l)$$

- The state $\boldsymbol{U_m}$ is given by

$$\begin{aligned} &I_m = I_l \\ &v_m = v_r \quad i.e. \quad \mathfrak{I}_r(\rho_m, I_l) = v_r = \mathfrak{I}_r(\rho_r, I_r) \end{aligned}$$

- The upstream supply and downstream demand (as functions of initial conditions) are expressed as:

$$\left(\begin{aligned} &\delta_l \overset{def}{=} \Delta_l(\rho_l, I_l) \\ &\sigma_r \overset{def}{=} \Sigma_r(\rho_m, I_l) = \Sigma_r\left(\mathfrak{I}_{r,\rho}^{-1}(v_r, I_l), I_l\right) = \Sigma_r\left(\mathfrak{I}_{r,\rho}^{-1}(\mathfrak{I}_r(\rho_r, I_r), I_l), I_l\right) \end{aligned} \right.$$

Riemann problem: fluxes at the origin

- Through flow q_0 = Min [upstream demand, downstream supply]

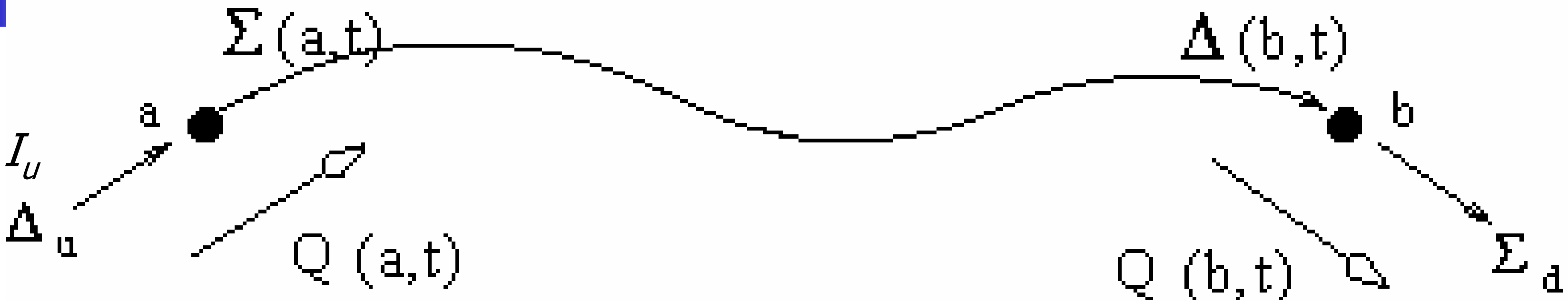
$$q_0 = \mathrm{Min}\left[\delta_l, \sigma_r\right]$$

- Through relative pressure $p_0 = q_0 \times I_l$

  (flux of density x I)

- Basis of all **Godunov schemes** (Eulerian + lagrangian)

Boundary conditions (revisited for GSOM)



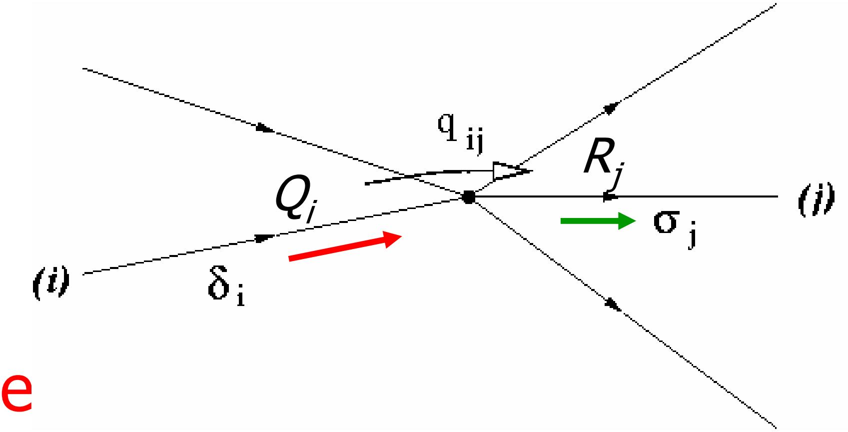
- Note the complex dependance with respect to the driver attributes

$$\Sigma(a,t) = \Sigma_e(\rho(a,t), I(a,t), I_u(t); a)$$
$$Q(a,t) = \text{Min}\left[\Delta_u(t), \Sigma(a,t)\right]$$

$$\Delta(b,t) = \Delta_e(\rho(b,t), I(b,t); b)$$
$$\Sigma_d(t) = \Sigma_e(\rho(b,t), I_d(t), I(b,t); b)$$
$$Q(b,t) = \text{Min}\left[\Delta(b,t), \Sigma(a,t)\right]$$

LWR intersection modeling (point wise)

$$\left|\begin{array}{ll} 0 \le Q_i \le \delta_i & \forall i \\ 0 \le R_j \le \sigma_j & \forall j \\ R_j - \sum_i \gamma_{ij} Q_i = 0 & \forall j \end{array}\right.$$



- Flows are constrained by upstre downstream supplies (+ flow conservation)

- Node models $(Q,R) = f(\delta,\sigma)$ must satisfy the invariance principle (Lebacque-Khoshyaran 2005)

$$\left[\begin{array}{lll} \delta_i \to Q_{i,max} & \text{if} & Q_i < \delta_i \\ \sigma_j \to R_{j,max} & \text{if} & R_j < \sigma_j \end{array}\right.$$

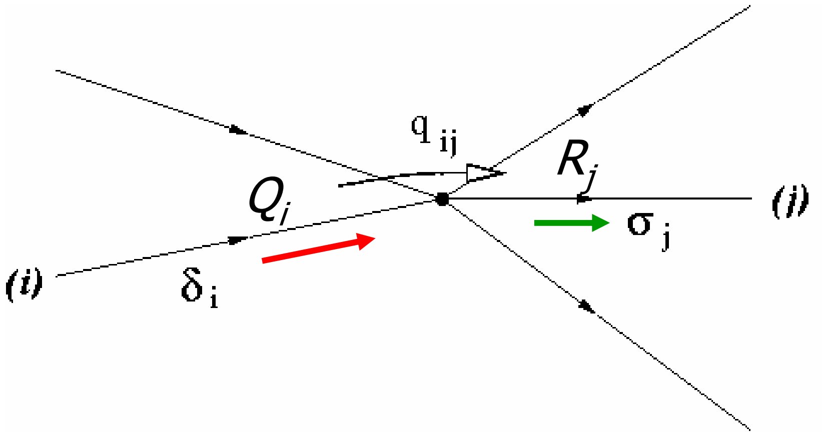
Meaning of these structural constraints

- Generalized Riemann problem for the intersection
  ⇒ supply and demand constraints
- Self-similarity + feasibility of solution ⇒ invariance principle

- The invariance principle guaranties consistent and convergent numerical schemes

Intersection modeling in the GSOM models: what changes?

- The upstream demands depend on upstream driver attributes
- The downstream supplies depend on downstream driver attributes and on the mix of upstream driver attributes



- The shifted supply/demand functions are increasing with respect to driver attributes
- Treatment in mathematical literature : Rascle Herty, Haut Bastin Chitour, Piccoli Garavello, etc… following different but equivalent rationale

$$\left|\begin{array}{ll} 0 \leq Q_i \leq \delta_i(\rho_i, I_i) & \forall i \\ 0 \leq R_j \leq \sigma_j(\rho_j, I_j, J_j) & \forall j \\ R_j - \sum_i \gamma_{ij} Q_i = 0 & \forall j \\ R_j J_j - \sum_i \gamma_{ij} Q_i I_i = 0 & \forall j \end{array}\right.$$

Solutions:

- Homogenization (Herty Rascle)
- Optimization model: computationally costly (non-linear constraints) therefore unsuitable for applications
- Internal state node model (LWR: Lebacque Khoshyaran 1998-2002-2005, ARZ: Mammar Lebacque Haj-Salem 2005, GSOM: Lebacque Mammar Khoshyaran Haj-Salem 2005-2008,…)

Adaptation of the internal state node model to the GSOM family

- The internal node state comprises the variables: N, N_j and I
- The dynamics of these variables are fast, but smooth: conservation ODE for node variables

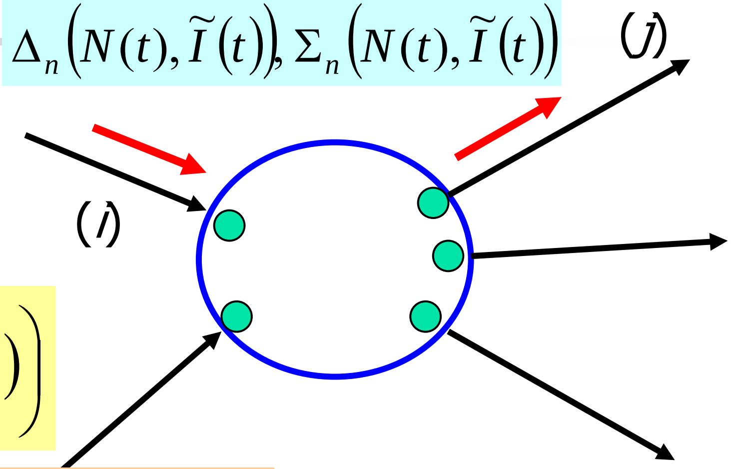
- The shifted supply and demand concept applies
- The node shifted supply and demand are split between upstream and downstream links

- The node dynamics are smooth: they **provide proper boundary conditions** for adjacent links

Internal state node model

$N(t), N_j(t), \tilde{I}(t)$

$\Delta_n\left(N(t), \tilde{I}(t)\right), \Sigma_n\left(N(t), \tilde{I}(t)\right)$



- The node supply is split

$$\Omega_i = \beta_i \Omega\left(N(t), \tilde{I}\right)$$

- The node demand is split

$$\Delta_j(t) = \frac{N_j(t)}{N(t)} \Delta\left(N(t), I(\tilde{t})\right)$$

- Flows = Min(demand, supply)
- All conservative quantities are conserved

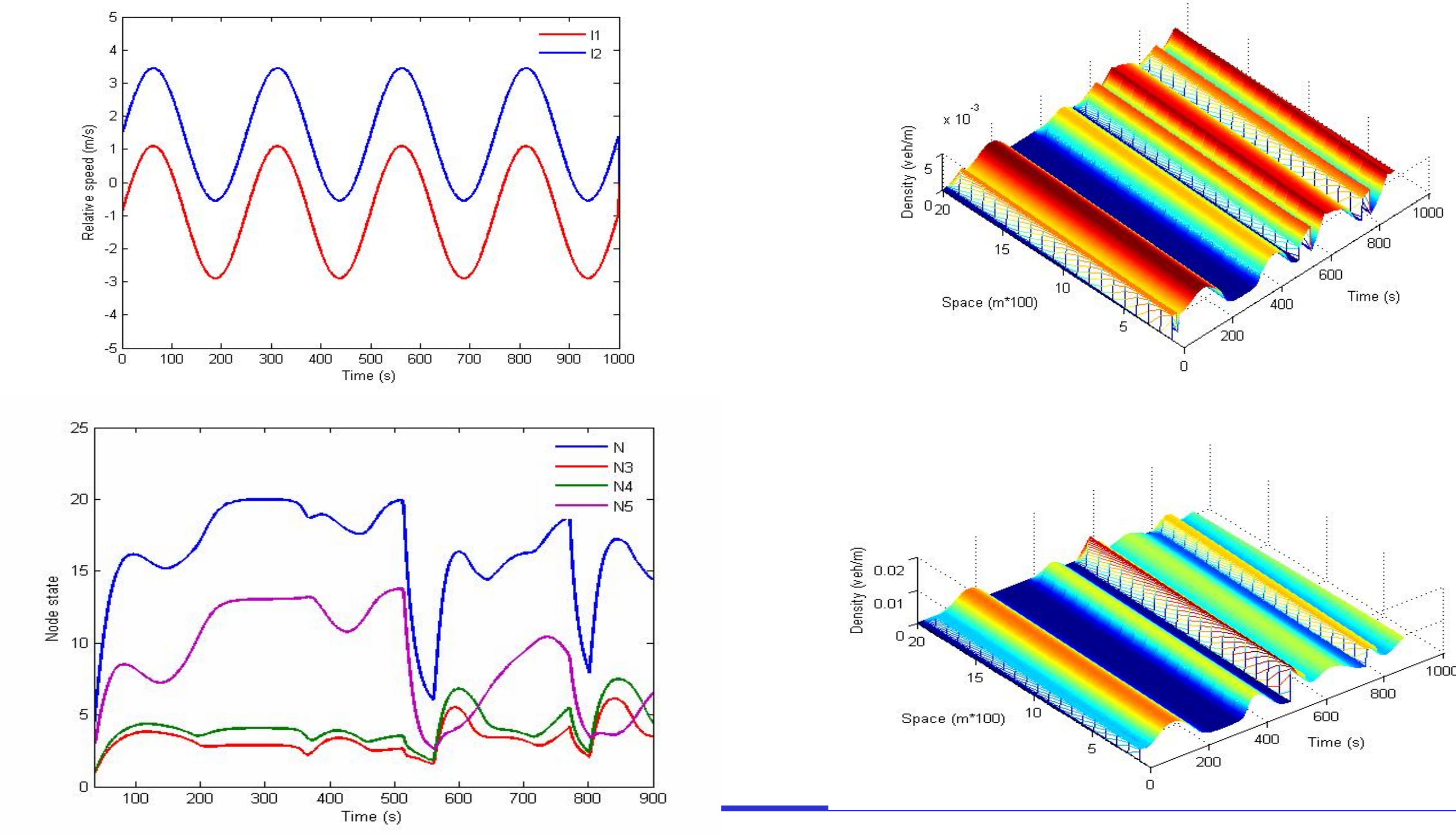
$$q_i(t) = \mathrm{Min}\left[\delta_i(t), \Omega_i(t)\right]$$

$$r_j(t) = \mathrm{Min}\left[\Delta_j(t), \sigma_j(t)\right]$$

$$\dot{N}_j(t) = -r_j(t) + \sum_i \gamma_{ij} q_i(t)$$

$$(\dot{NI}) = \sum_i q_i I_i - \sum_j r_j I$$

Example: 2 upstream links, 3 downstream links



Other extensions of the GSOM concept

- Driver dependent FD: I := set of FD parameters. (applicability: model identification and traffic control)

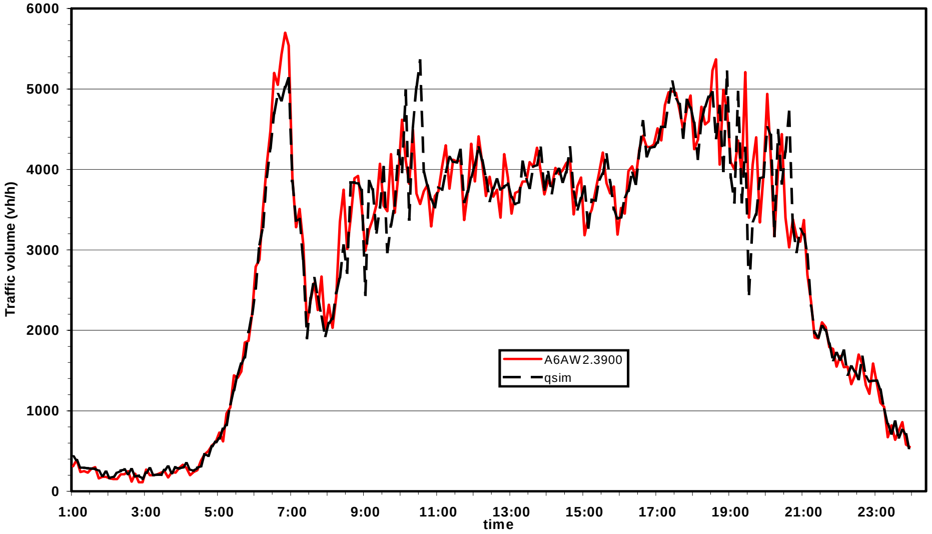
- I := set of driver classes (ex: drivers classes = destinations). We recapture the FIFO multi-class model (all drivers have the same speed)(Moutari Herty Rascle) and extend LWR FIFO multiclass (Jin Zhang, Khoshyaran Lebacque)

- Stochastic macroscopic model (applicability: model identification and traffic control)

Why stochastic macroscopic traffic modeling

- Traffic flow exhibits stochastic (like) features: scatter of FD data, fractal aspect of measurements
- Stochastic aspects of human behavior and interaction
- The system of traffic cannot be known and described comprehensively: imperfect information

Why stochastic traffic modeling: illustrations

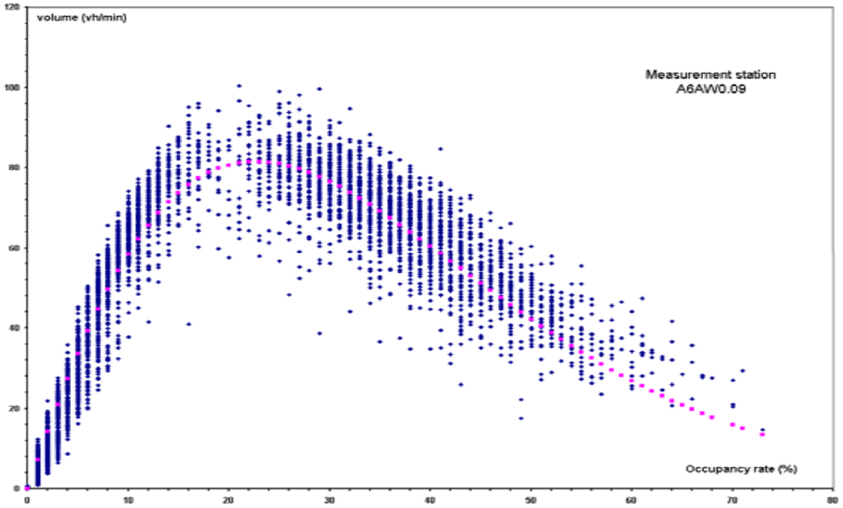


Fractal aspect of flow data

(aggregated over 6 mn periods)

Scatter of FD data

(flow density measurement)



Stochastic GSOM model (Khoshyaran Lebacque 2007-2008)

- Idea:
  - Conservation of vehicles
  - Fundamental diagram depends on *I*, related to cars
  - *I* is subject to stochastic perturbations (other vehicles, traffic conditions, environment)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0$$

$$v = \Im(\rho, I) \quad \Rightarrow \quad \rho v \overset{def}{=} \Re(\rho, I)$$

$$\dot{I} = \Phi\left(I, \frac{dB_t}{dt} \right)$$

$$\Rightarrow I = \Xi(N, t; \omega)$$

Stochastic GSOM model: lagrangian expression

- Necessary: I dynamics are lagrangian
- Lagrangian coordinates: N (vehicle index) and t (time)
- Lagrangian unknowns: $r = 1/\rho$, and I
- Lagrangian ARZ: Aw, Klar, Materne, Rascle, 2002
- **No diffusion** (the perturbation of speed affects the behavioral I variable

$$\frac{\partial r}{\partial t} + \frac{\partial}{\partial N}\Im\left(\frac{1}{r}, I\right) = 0$$

$$\frac{\partial I}{\partial t} = \Phi\left(I, \frac{dB_t}{dt}\right)$$

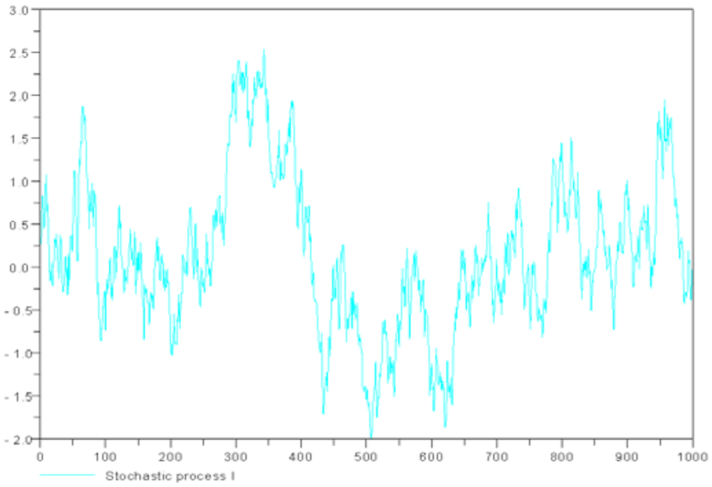
$$v = \Im\left(\frac{1}{r(N,t)}, I(N,t;\omega)\right)$$

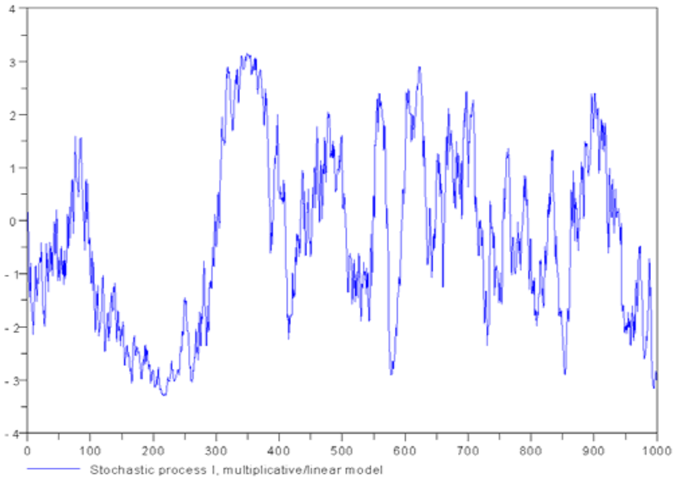
$$\frac{\partial I}{\partial t}(N,t;\omega) = \Phi\left(I(N,t;w), \frac{dB_t}{dt}(N,\omega)\right)$$

Solution method:

- Integrate I (stochastic ode) for each particle
- Apply Godunov to each particle

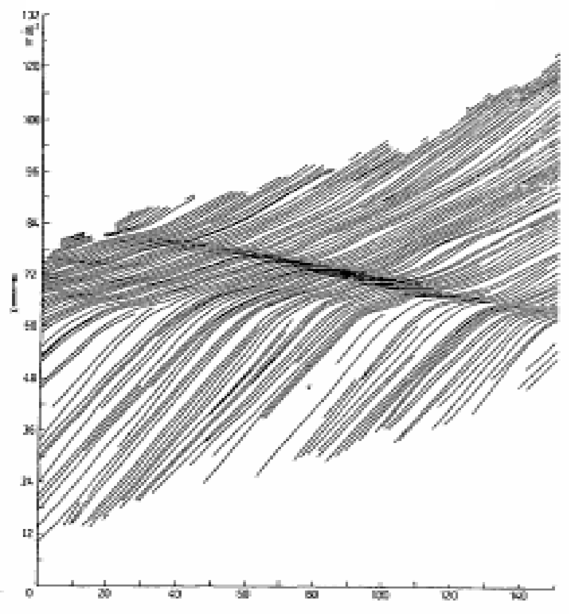
- Examples of processes for I

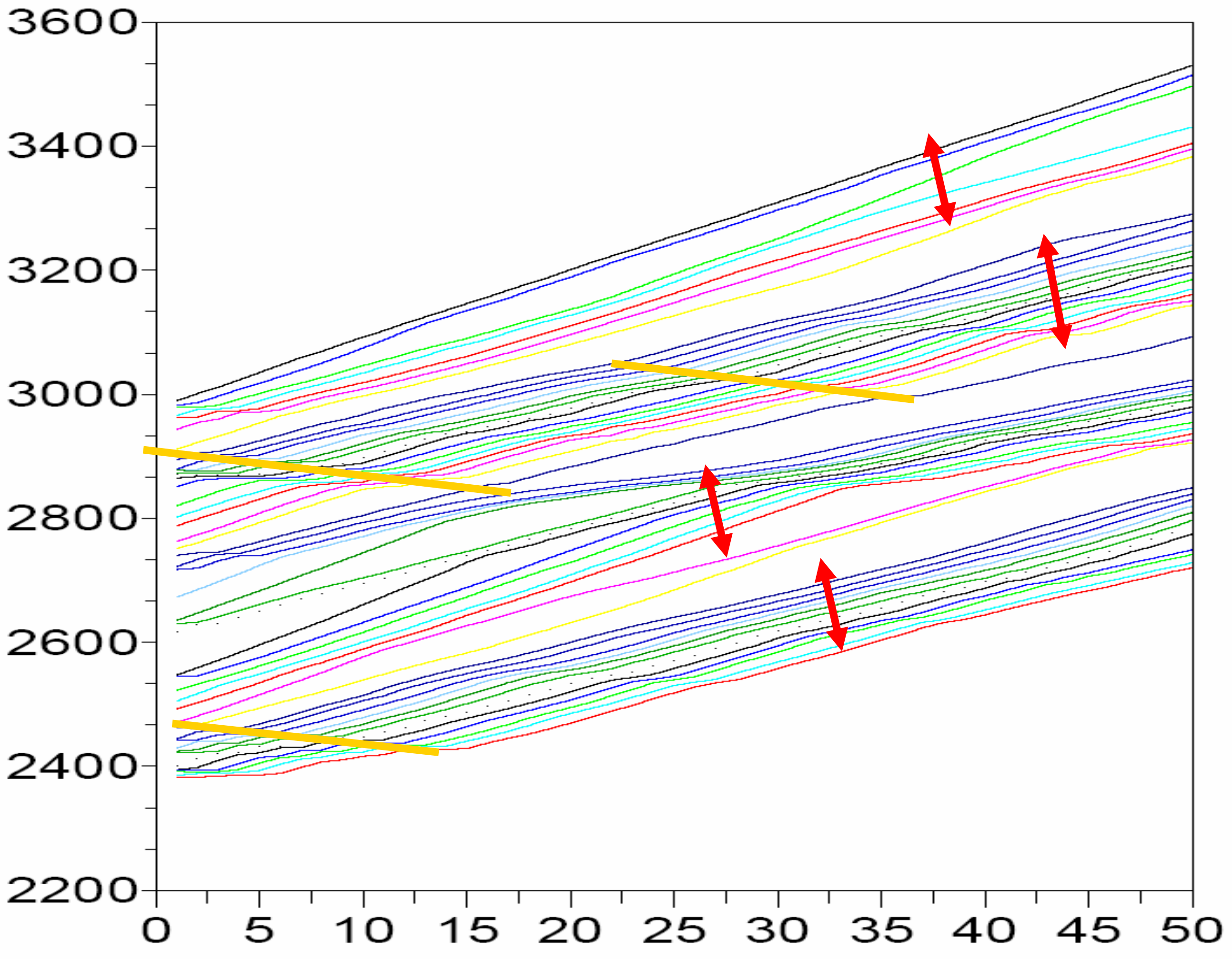




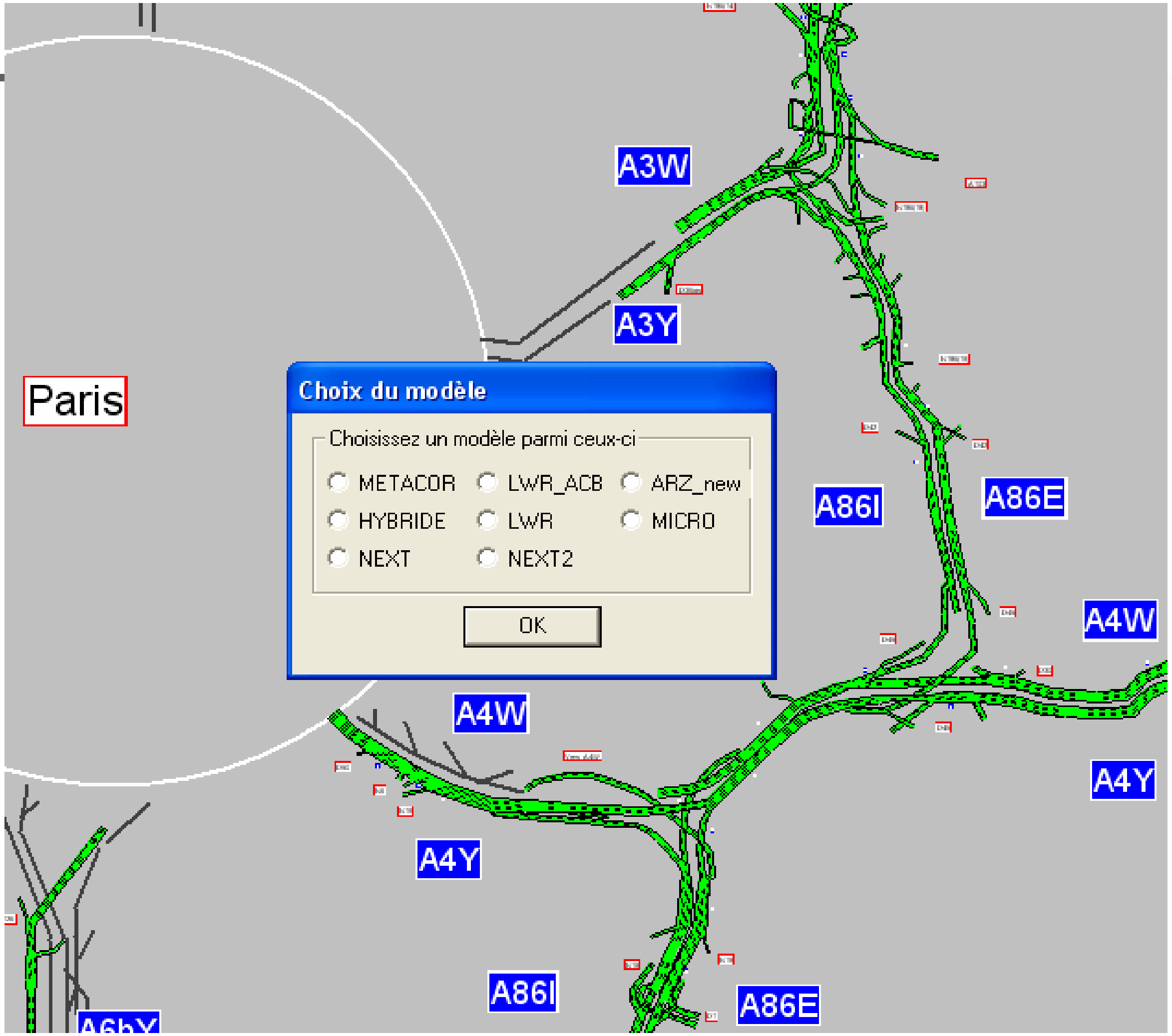
Example: simulation of a motorway section

- Perturbation patterns
- Shockwaves
- Platooning
- Compare with Treiterer and Myers 1978





Application: MAGISTER multi-model platform (2006-2008)



Choix du modèle

Choisissez un modèle parmi ceux-ci

○ METACOR ○ LWR_ACB ○ ARZ_new
○ HYBRIDE ○ LWR ○ MICRO
○ NEXT ○ NEXT2

Conclusion

- The GSOM family includes all models which combine:
    - Kinematic waves
    - Dynamics of driver attributes
- Benefit: common framework and set of properties + great variety of problems are modelled