Receding horizon control of freeways

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CRIM meeting - November 2007

Content & credits

- Freeway traffic management issues
- Discussions around the LWR model
- Solution of the inhomogeneous LWR (I-LWR)
- Optimal control for the I-LWR
- Finite dimensional approximations

- 11/2006: PhD in automatic control (Grenoble)
- Credits : – C. Canudas de Wit (CNRS – Grenoble)
  – R. Horowitz (UC Berkeley – ME dept.)
- R&D company in modelling, optimization & control

Freeway management problems

- Dimensioning

Freeway management problems

- Dimensioning
- Performance measure

Freeway management problems

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- Performance measure
- Ramp metering

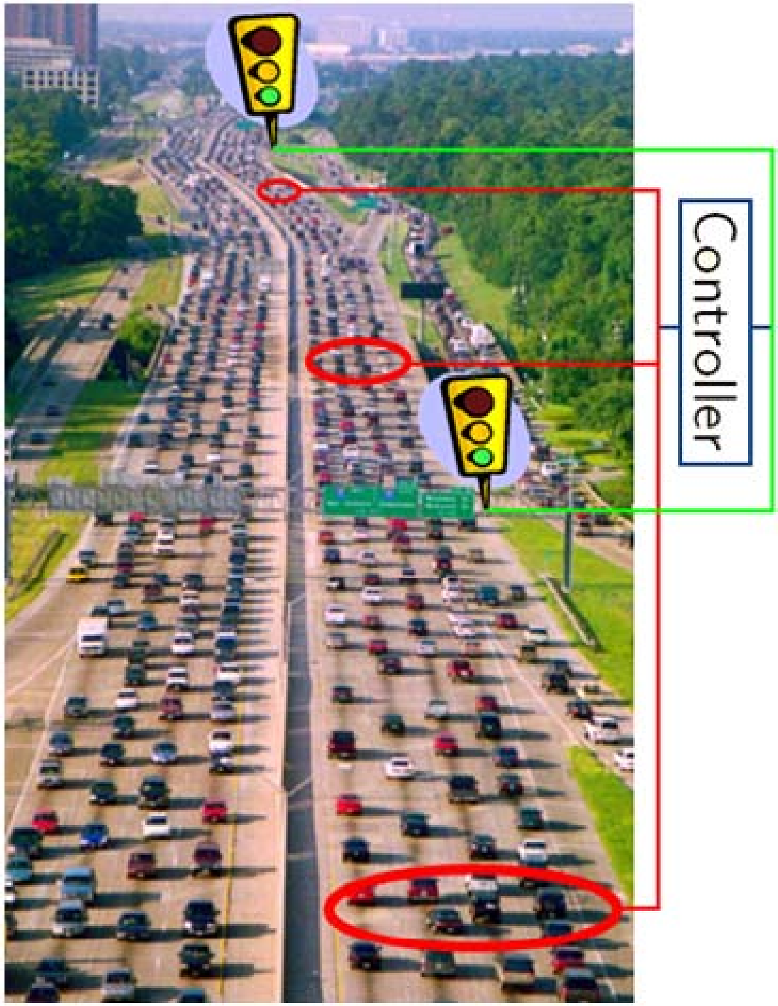
Freeway management problems

- Dimensioning
- Performance measure
- Ramp metering
- Variable speed limits

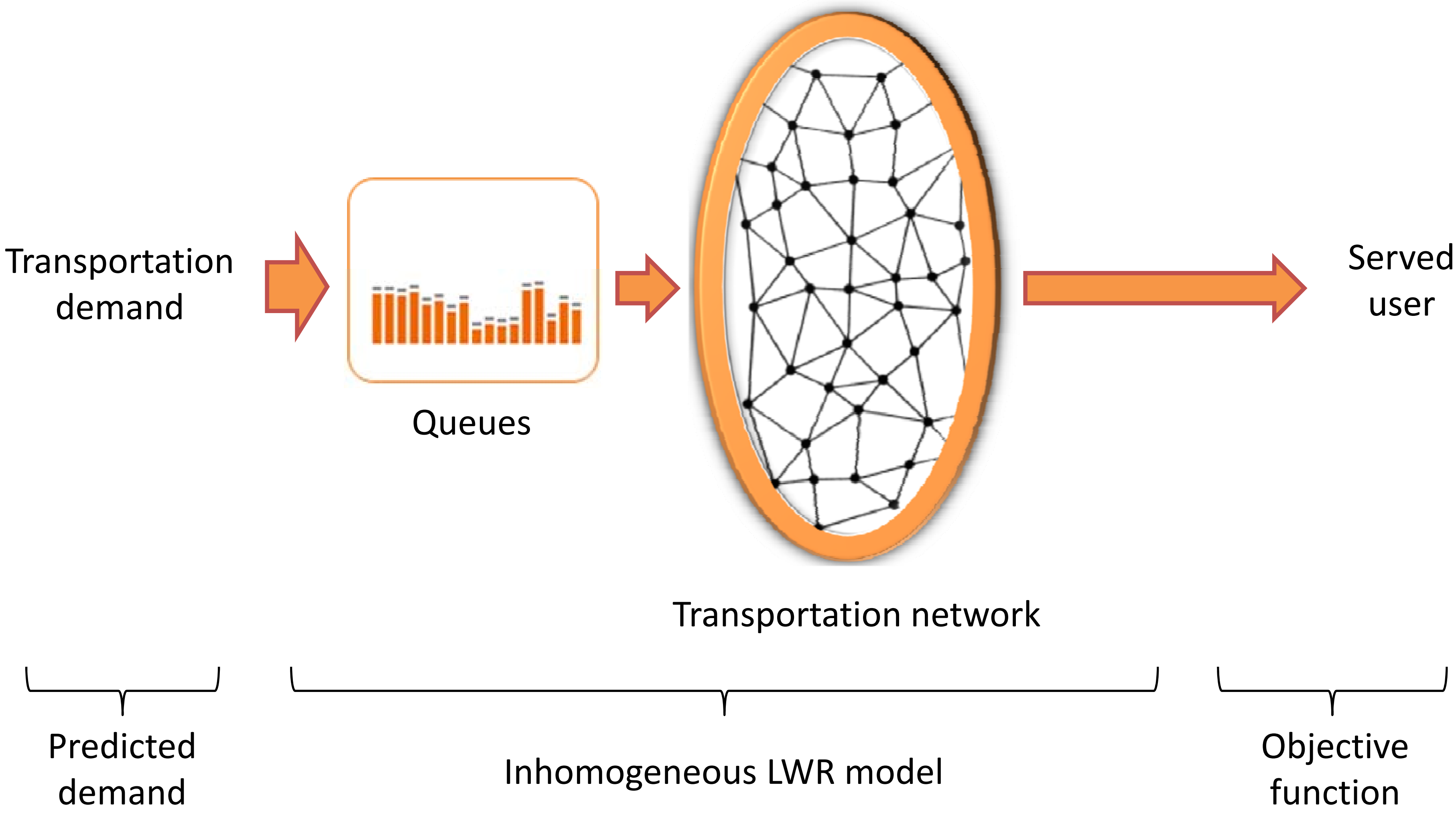
Freeway management problems

- Dimensioning
- Performance measure
- <u>Ramp metering</u>
- Variable speed limits
- Incident detection
- Model parameter estimation
- Traffic state estimation

Ramp metering problem



Ramp metering problem



Classical performance measures

- Vehicle Miles Traveled (VMT)

$$\mathcal{J}_{VMT}(\phi) = \int_0^T \int_0^L \phi(x,t)\ dxdt$$

- Total Travel Time (TTT)

$$\mathcal{J}_{TTT}(\rho) = \int_0^T \int_0^L \rho(x,t)\ dxdt$$

- Total Waiting Time (TWT)

$$\mathcal{J}_{TWT}(q) = \sum_j \int_0^T q_j(t)\ dt$$

- Total Time Spent (TTS = TTT + TWT)
- Total Served Vehicles (TSV)

$$\mathcal{J}_{TSV}(r) = \sum_j \int_0^T r_j(t)\ dt$$

Classically multi-objective

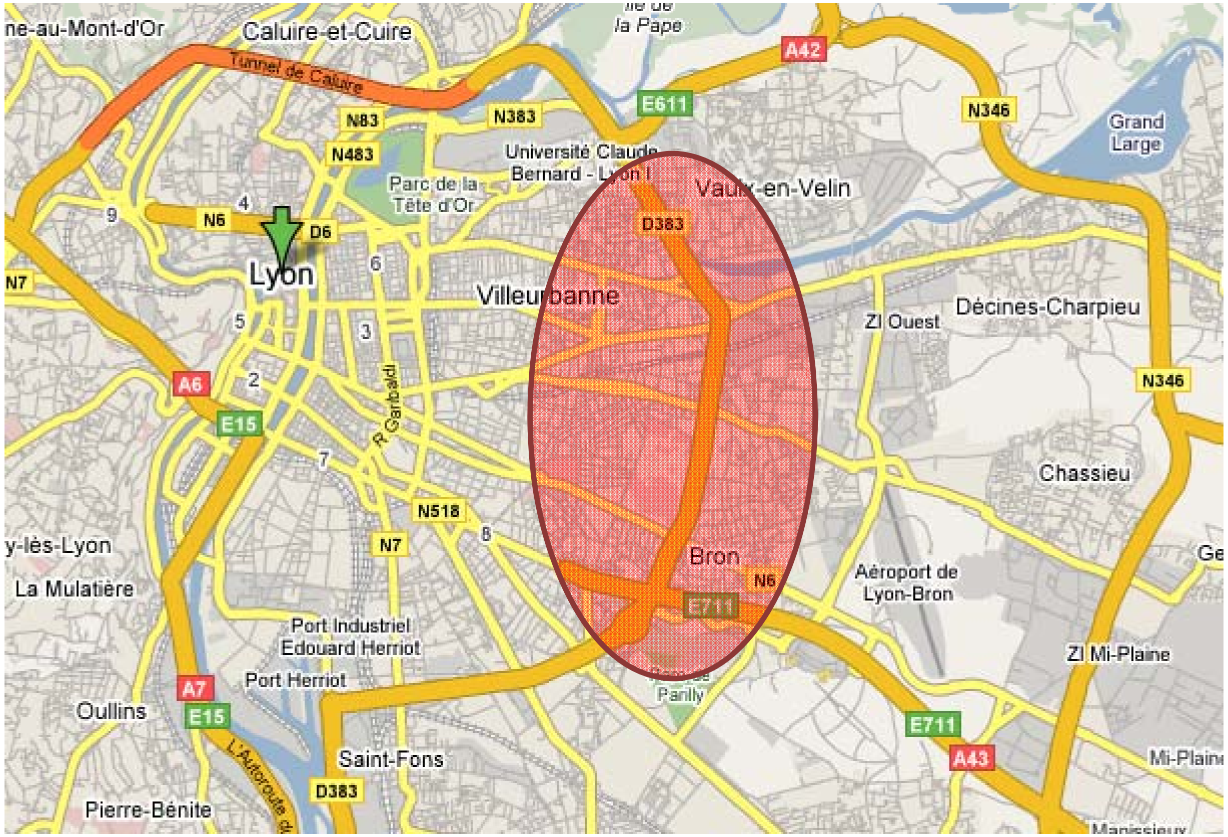
- Travel time & confort for the <u>users</u>
- Safety for the <u>authorities</u>
- Deteriorated condition management for <u>operators</u>
- Optimisation of the investments for the <u>state</u>

$$\mathcal{J}_{WTWT}(q) = \sum_j \int_0^T \omega_j . q_j(t)\ dt$$

$$\mathcal{J}_{WTTS}(\rho, q) = \int_0^T \int_0^L \rho(x,t)\ dx dt\ +\ \kappa \sum_j \int_0^T q_j(t)\ dt$$

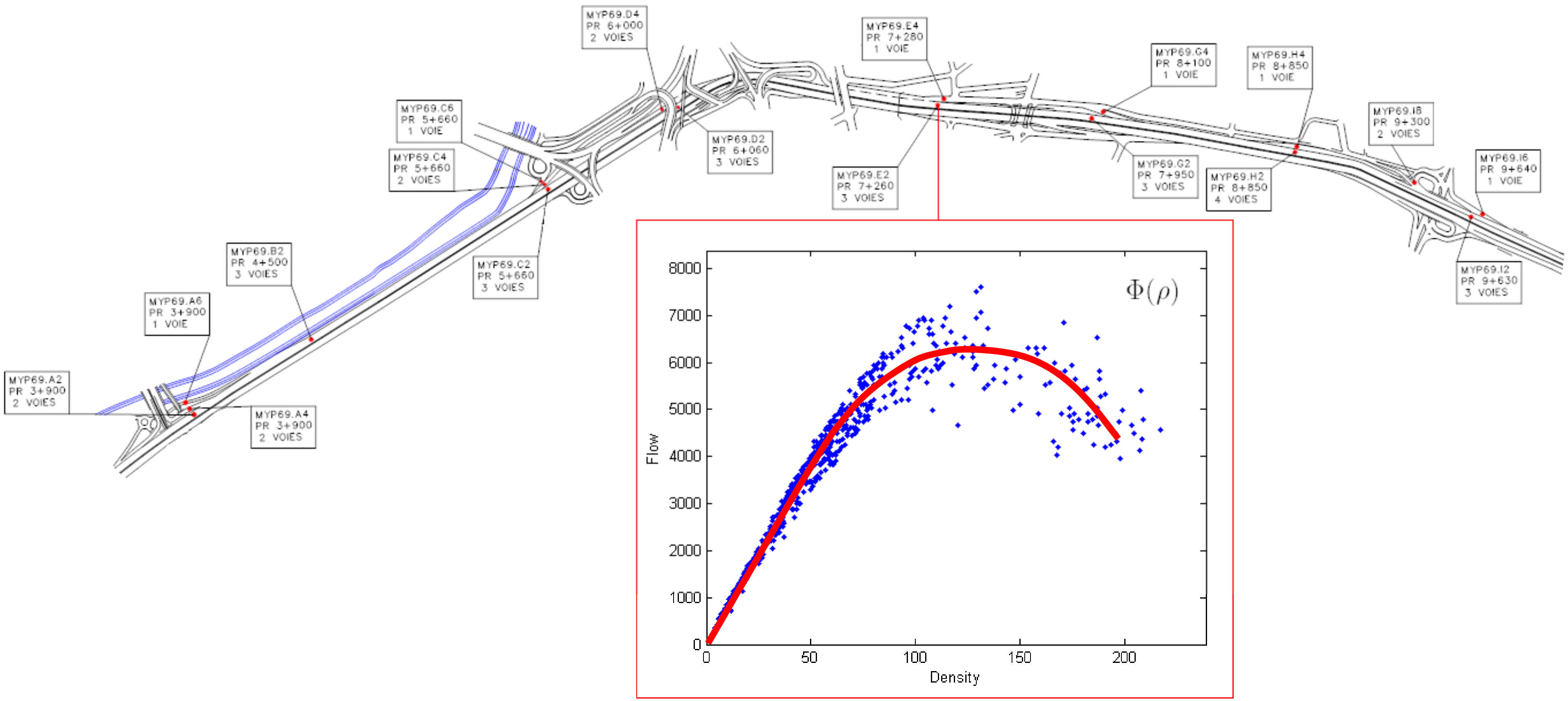
Or discrete versions…

Study case 1



Lyon, France

Study case 1



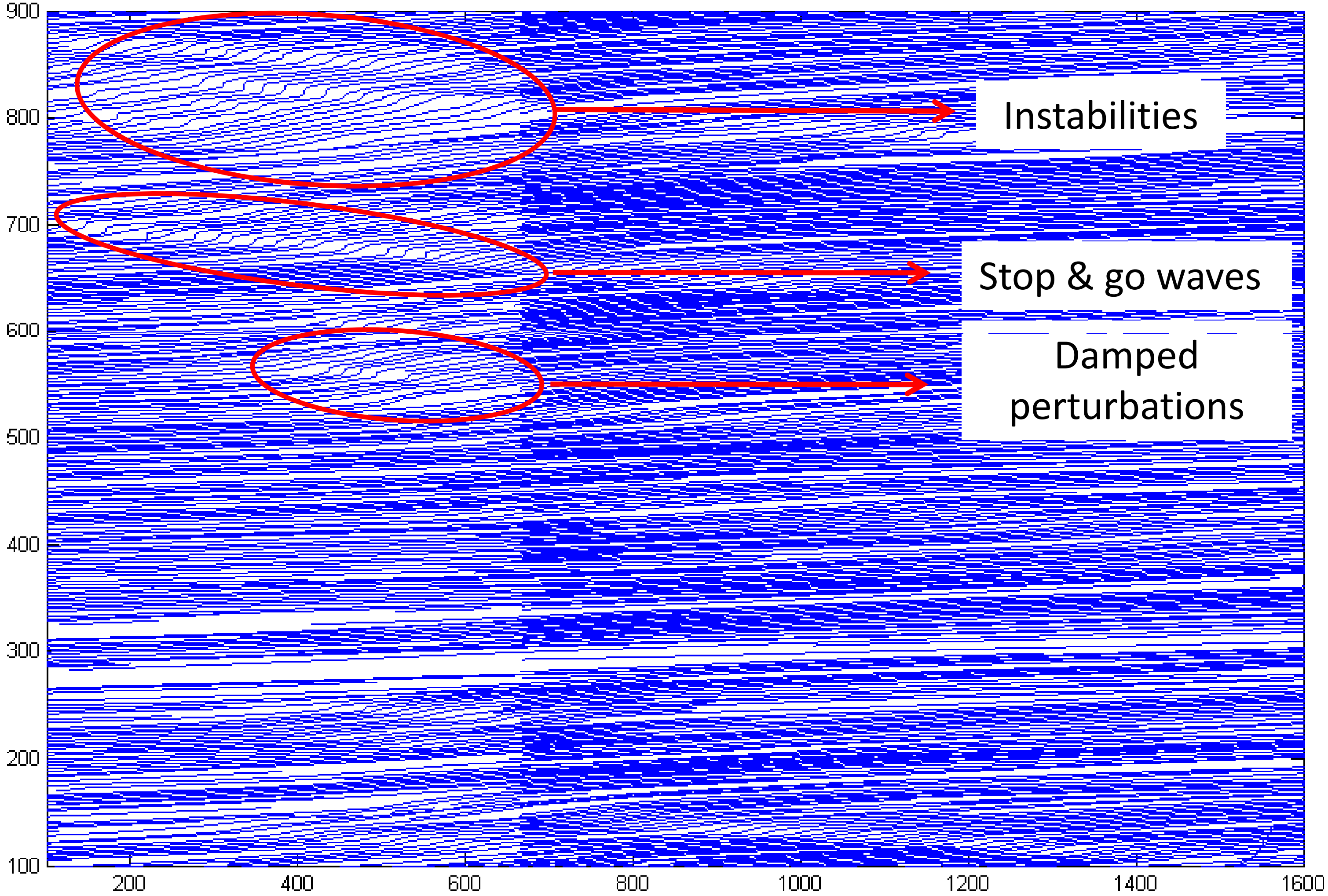
$$\Delta x \Rightarrow (CFL) \Rightarrow \Delta t \qquad \text{versus} \qquad \boxed{\Delta t \Rightarrow (CFL) \Rightarrow \Delta x}$$

Study case 2

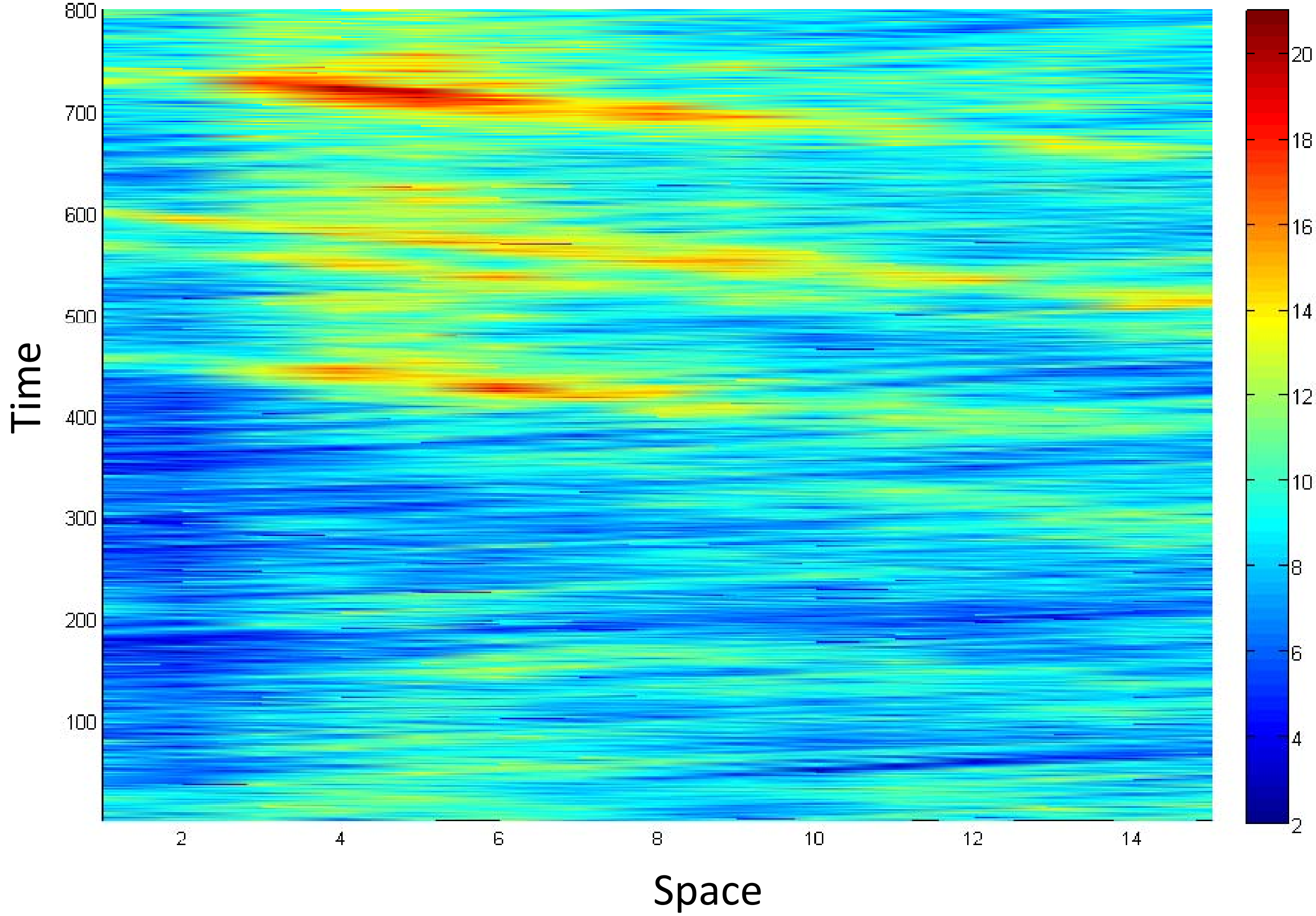
I-80

Bay Area

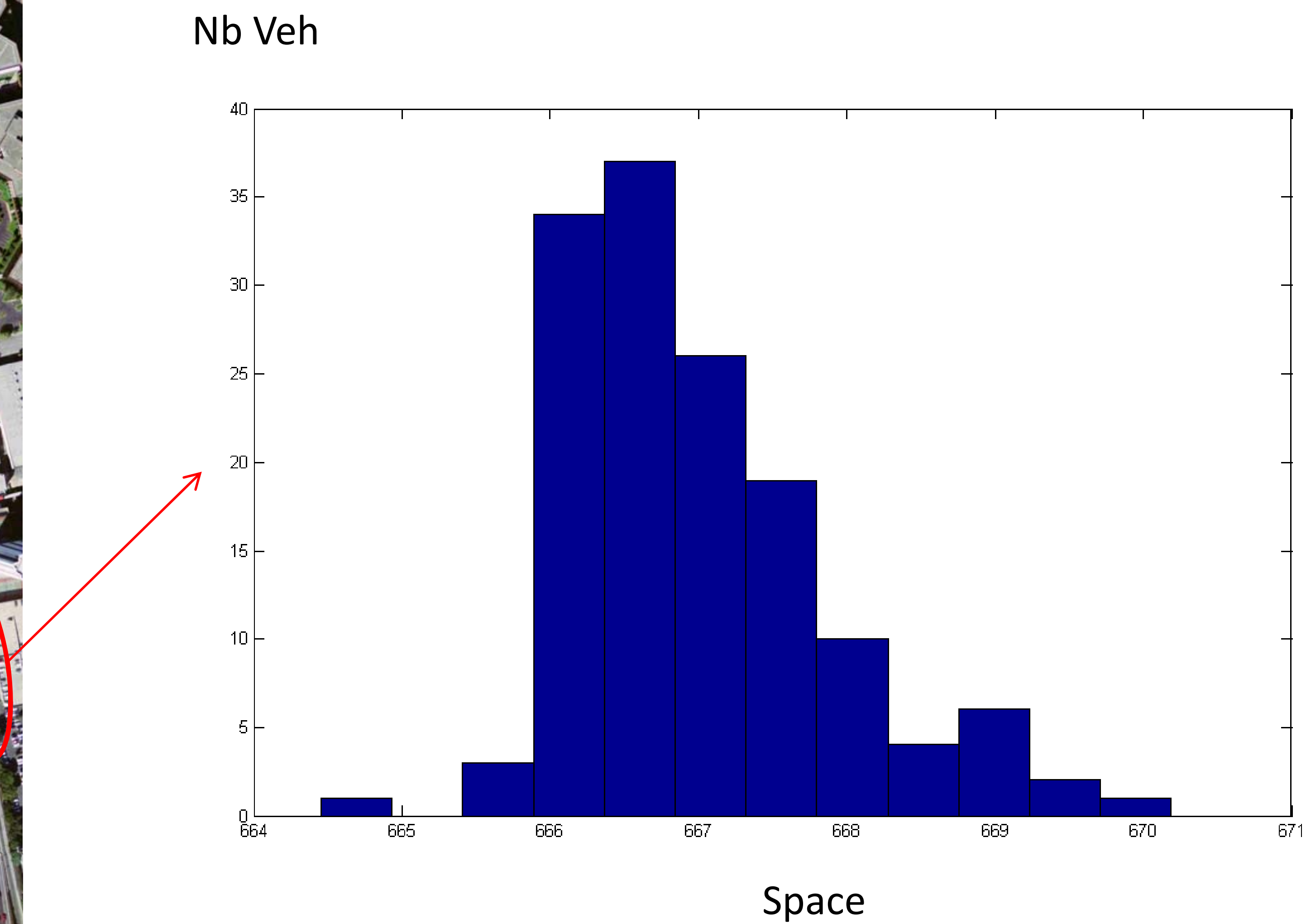
BHL data – courtesy of FHWA



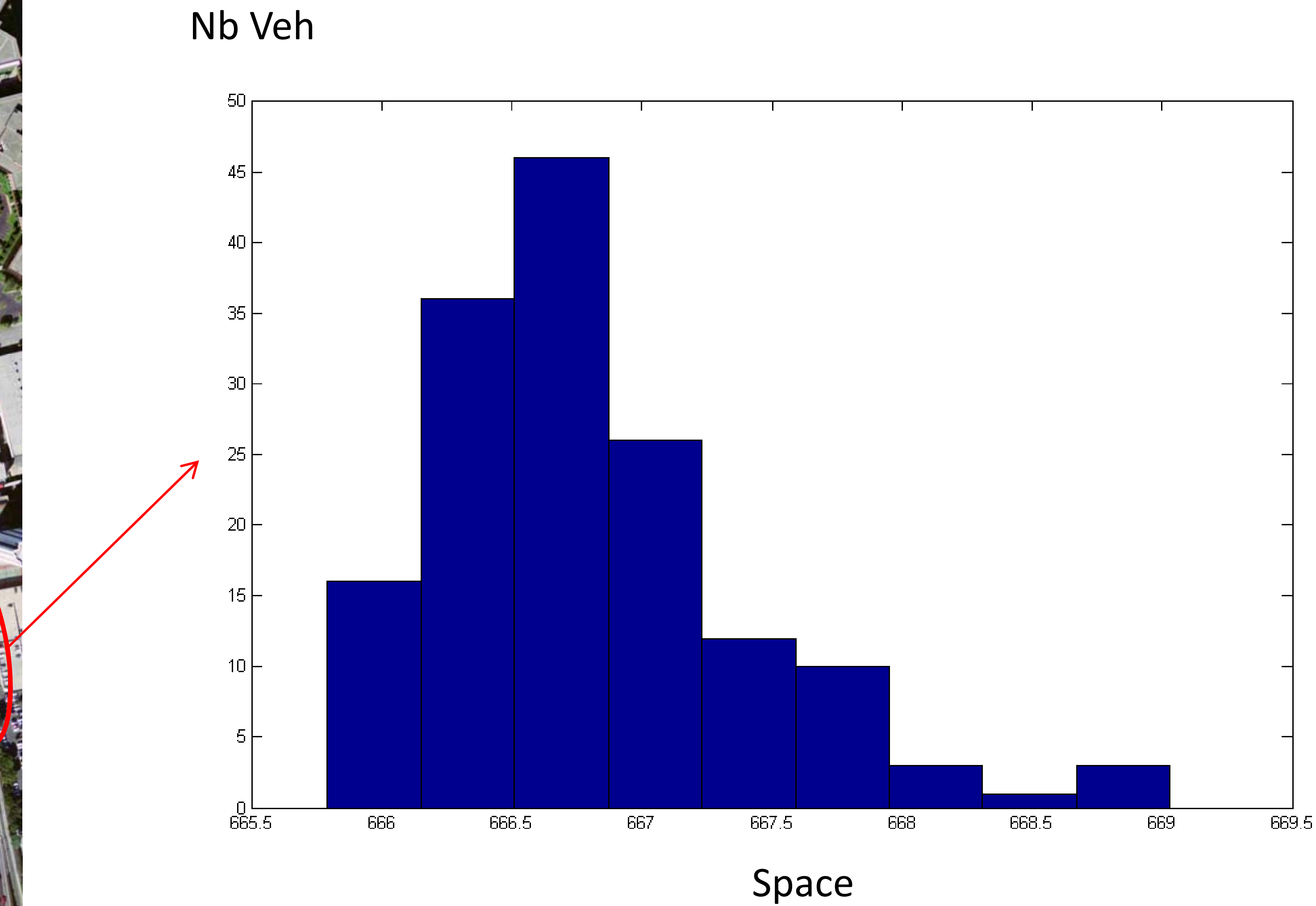
15 cell averages (number of vehicles)



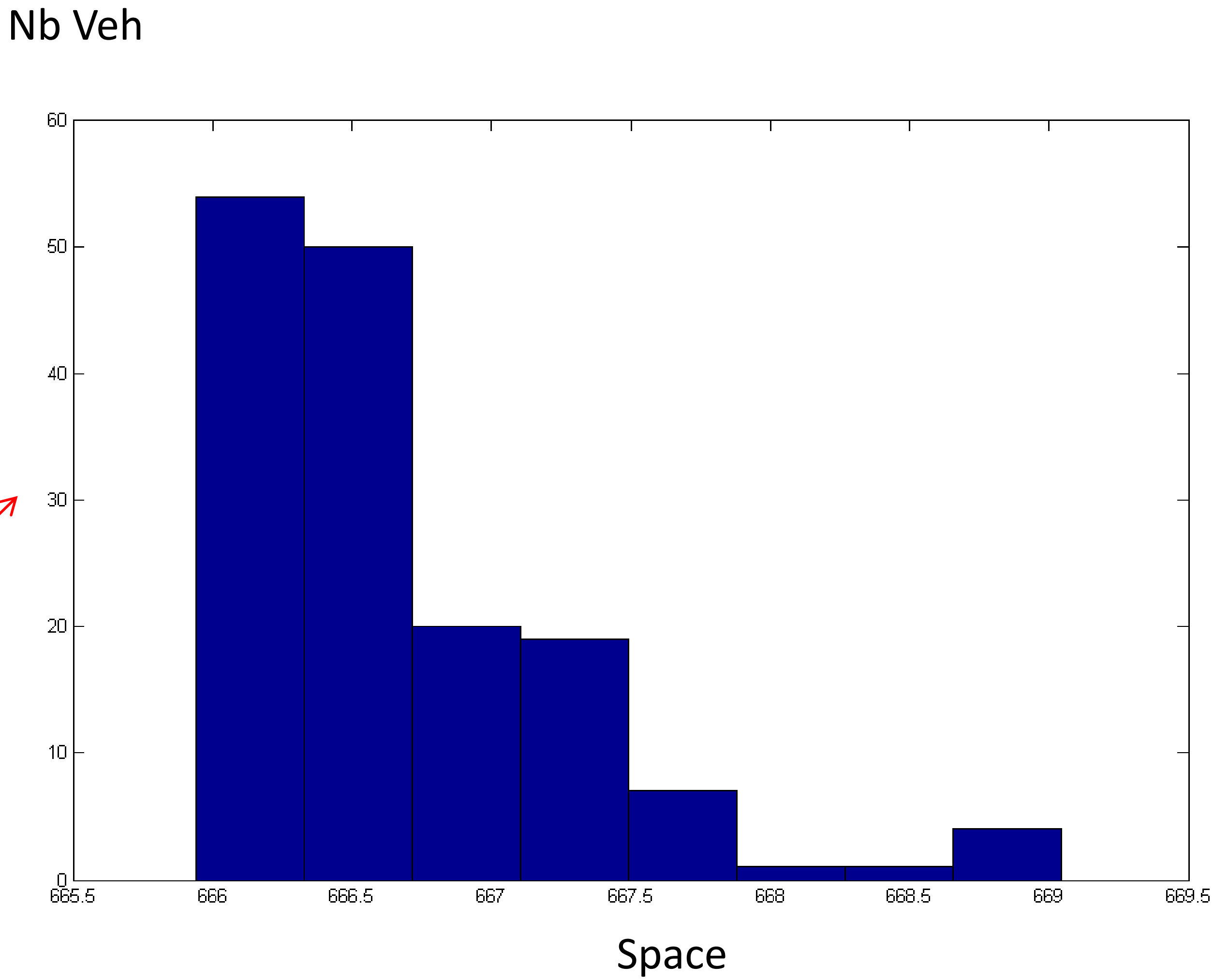
15 min averages



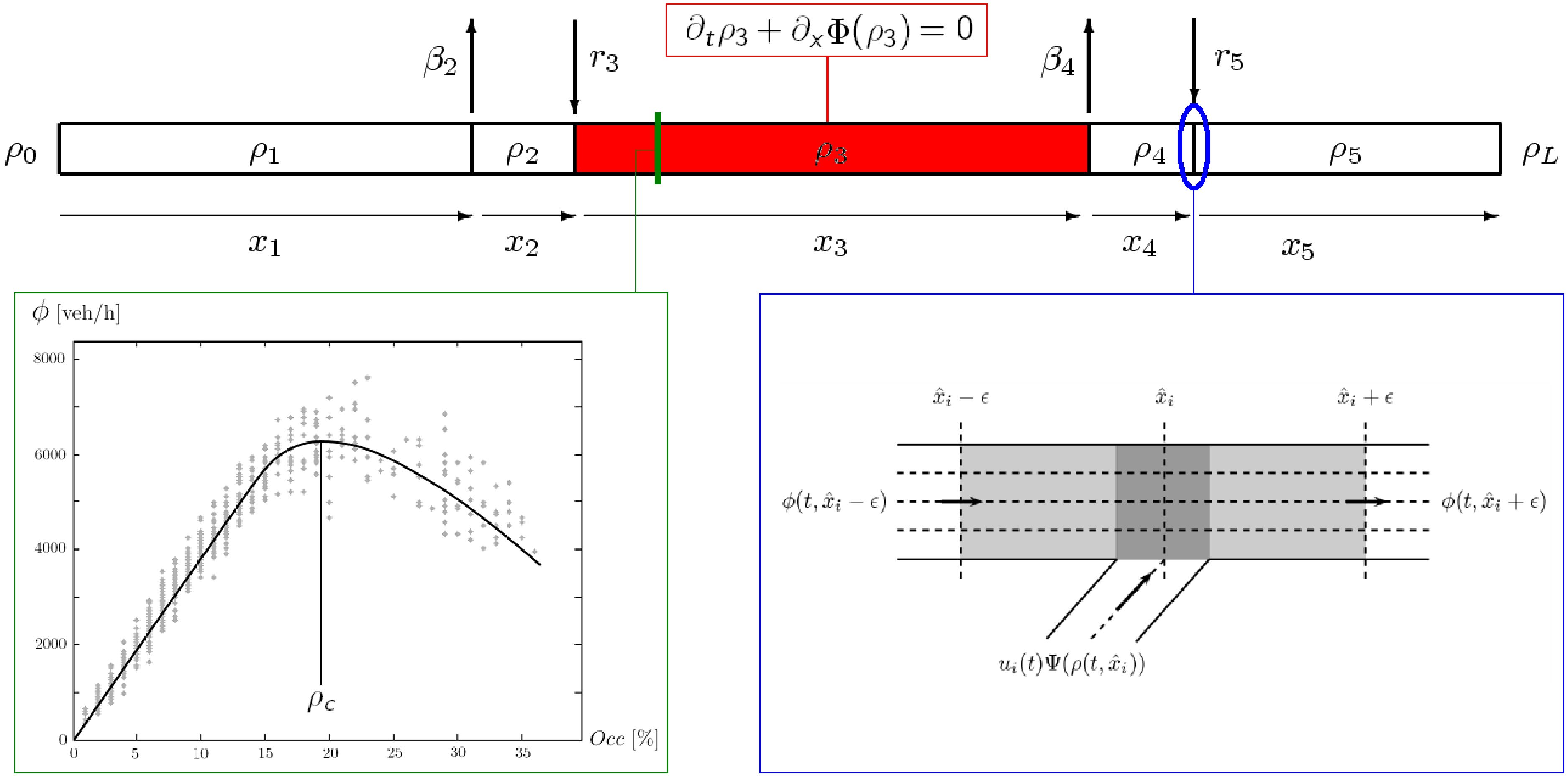
15 min averages



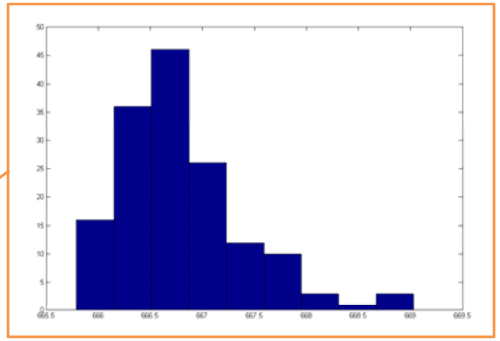
15 min averages



Inhomogeneous LWR



Inhomogeneous LWR



<u>2 ways of thinking:</u>

- Inhomogeneous term

$$\begin{cases} \partial_t \rho + \partial_x \Phi(\rho, x) = g(x, r) \\ \rho(x, 0) = \rho_I(x) \\ \rho(0, t) = \rho_0(t) \text{ and } \rho(L, t) = \rho_L(t) \end{cases}$$

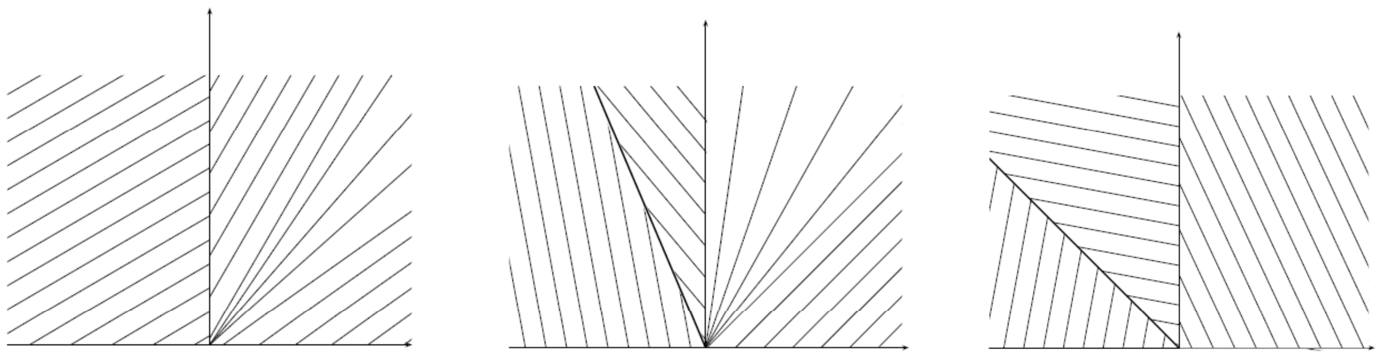
- Homogeneous PDEs + interface conditions

$$\begin{cases} \partial_t \rho + \partial_x \Phi(\rho, x) = 0 \\ Finite\ state\ machine\ (Free,\ Congested,\ Decoupled) \end{cases}$$

Inhomogeneous LWR

- Kruzkov, Bardos-Leroux-Nedelec
- Demand/Supply (Lebacque, Daganzo)
- Discontinuous fluxes (Temple, Towers, Colombo,…)

Riemann solver :



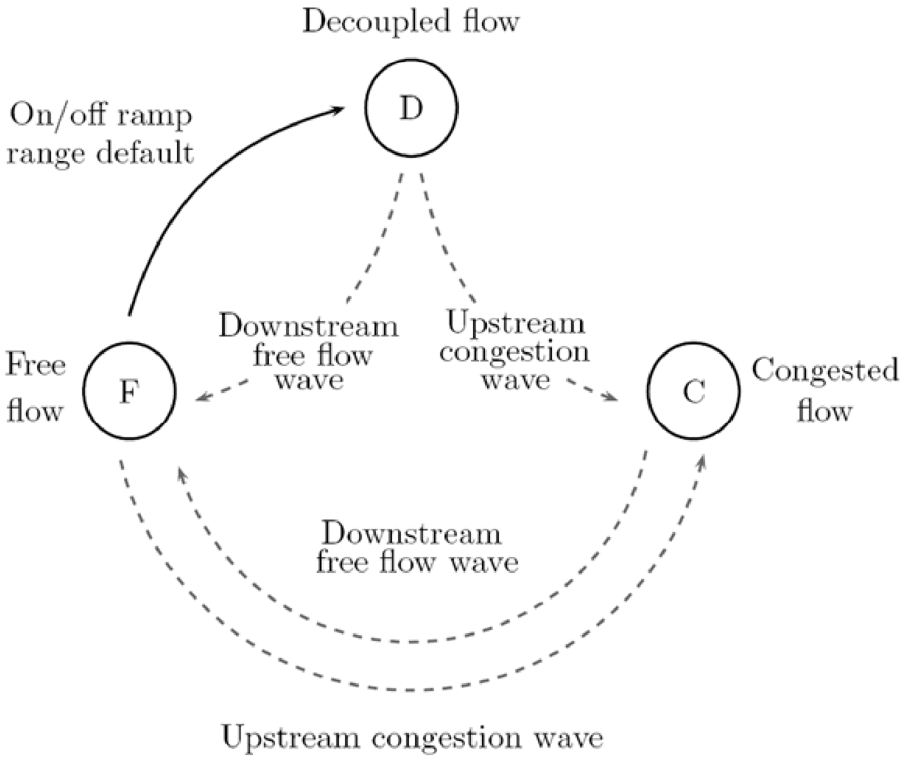
11 cases

…

Classification

$Finite\ state\ machine\ (Free,\ Congested,\ Decoupled)$

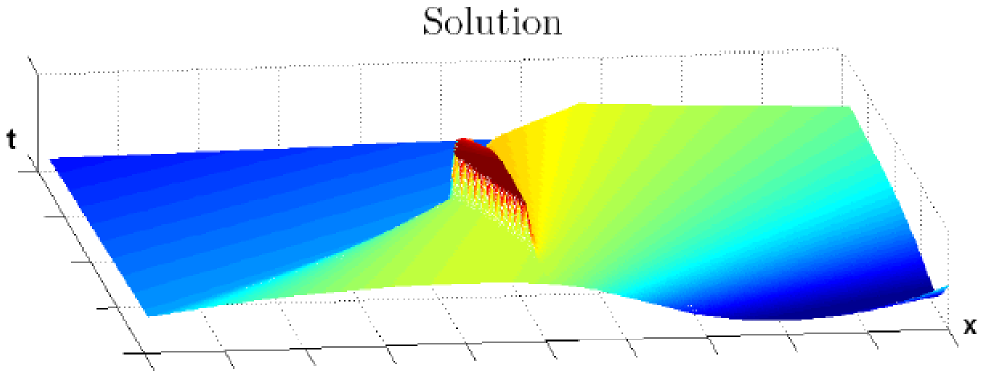
Inhomogeneous LWR

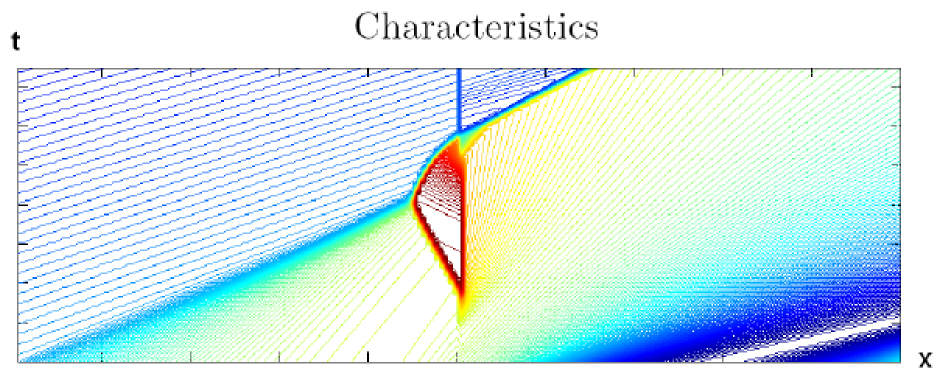


free: $\rho^r = \Phi^{-l}(\Phi(\rho^l) + r)$

congested: $\rho^l = \Phi^{-r}(\Phi(\rho^r) - r)$

decoupled: $\begin{cases} \rho^r = \rho_c \\ \rho^l = \Phi^{-r}(\Phi_m - r) \end{cases}$



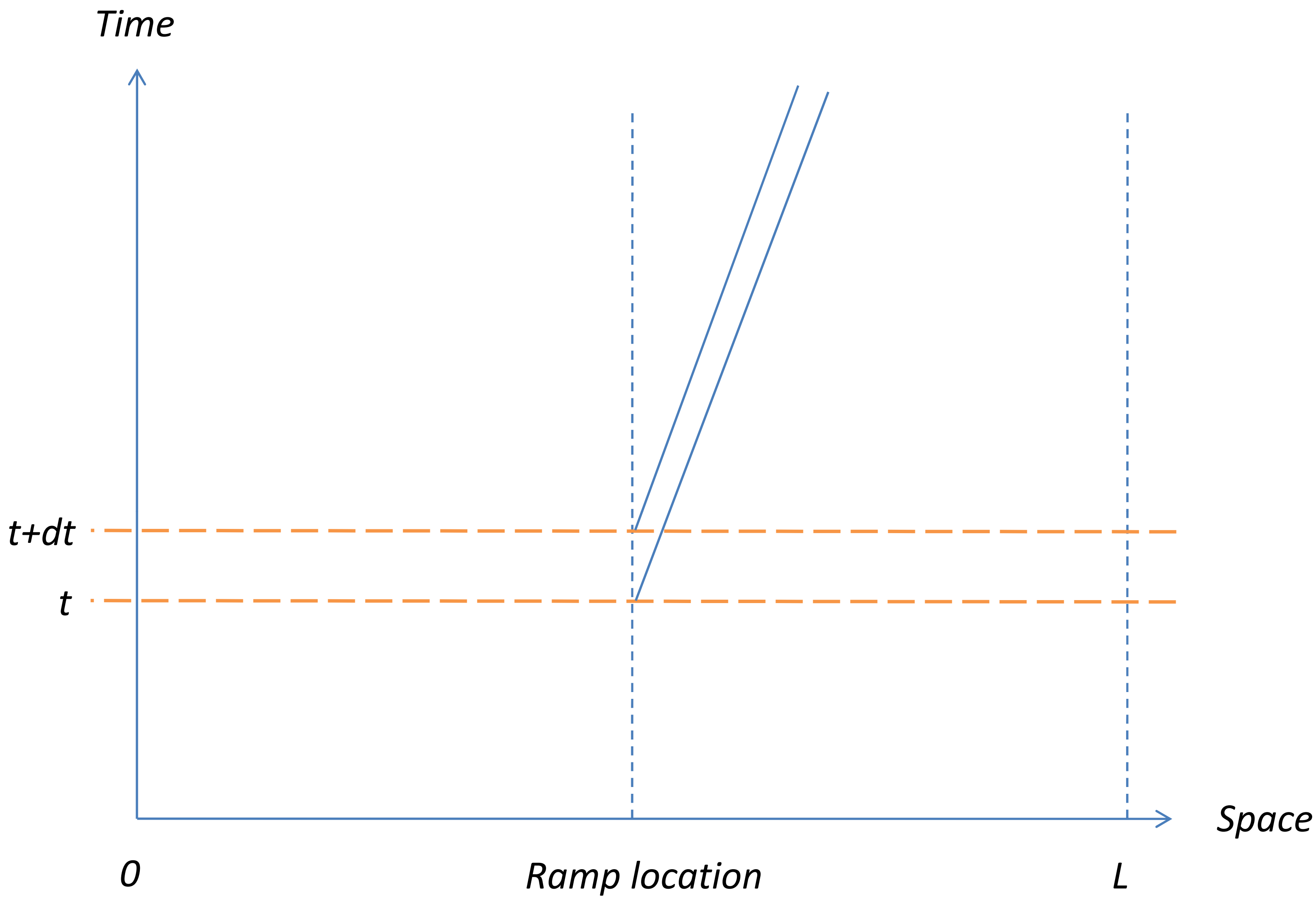


Why receding horizon control ?

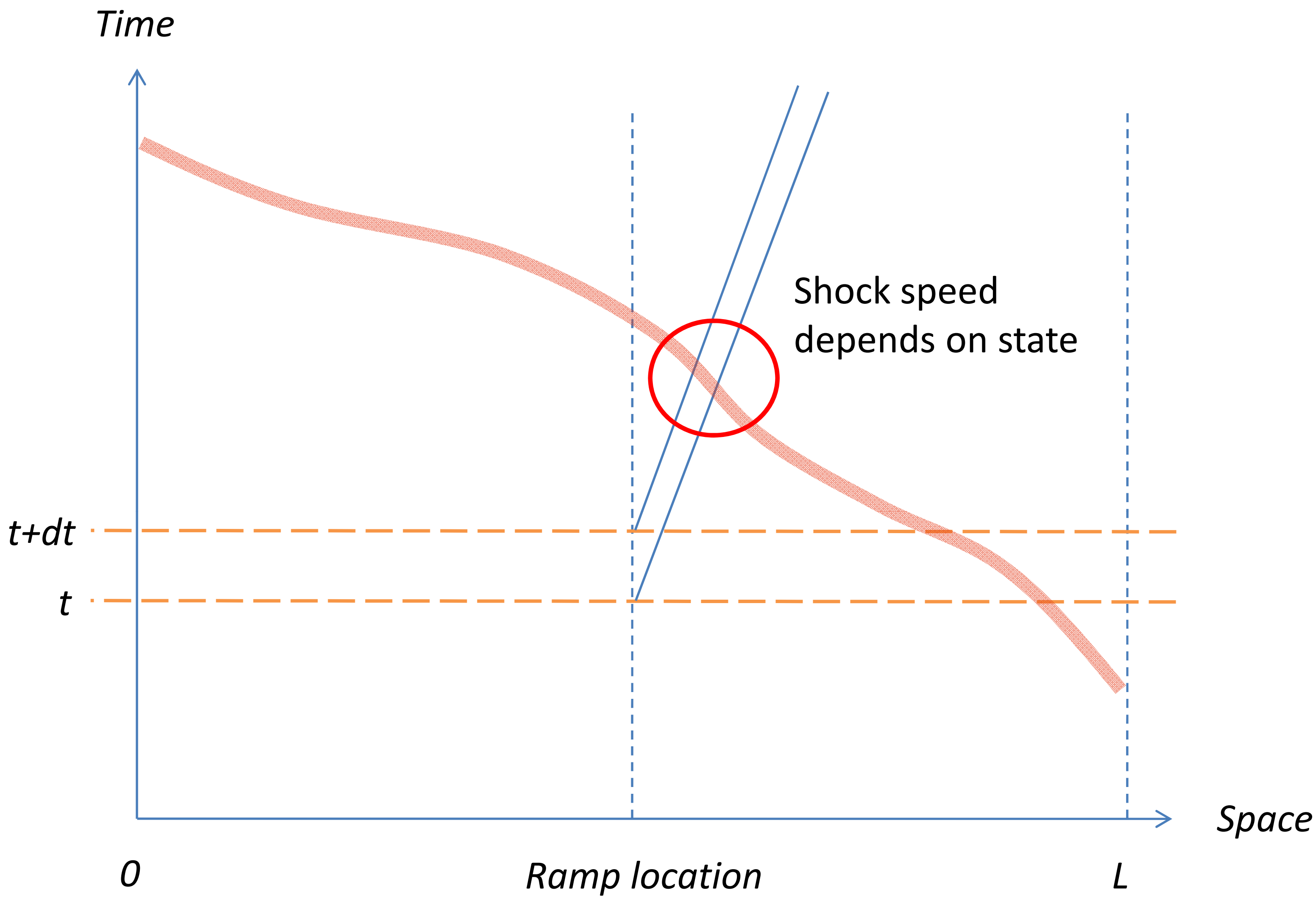
1. Freeway management is mostly an optimal allocation problem, not a tracking problem.

2. Hyperbolicity implies some controllability and observability properties that are not suitable for feedback control.

Why receding horizon control ?



Why receding horizon control ?



Optimal control of I-LWR

Classical optimization loop for PDE:

- Solve the system equation with a candidate
- Solve the adjoint system backwards
- Evaluate the objective gradient and iterate

But some serious issues here:

- What is the linearization of a conservation law ?
- How to solve the adjoint system ?

Optimal control of I-LWR

<u>Linearization:</u>

- Godlewski-Raviart, Bardos-Pironneau
- Bressan-Guerra, Bianchini, Colombo

↳ Shift differentiability

↳ Euler-Lagrange equations

$$L^1(|Du|)$$

Optimal control of I-LWR

Solution structure: well-behaved BV functions

[Di Perna, Dafermos] Solutions are mesure theoretically C^1 with jumps along measure theoretically C^1 surfaces.

Optimal control of I-LWR

<u>Solution structure:</u> well-behaved BV functions

[Di Perna, Dafermos] Solutions are mesure theoretically *C*¹ with jumps along measure theoretically *C*¹ surfaces.

- Decomposition in absolutely continuous and singular parts
- Integration by parts : $\Omega = (0, L) \times (0, T) \subset \mathbb{R}^2$

$$\int_\Omega u \cdot \nabla\phi \, \mathrm{d}\mathcal{L}^2 = -\int_{\Omega \setminus \cup_i \Gamma_i} \phi \, \mathrm{div} u \, \mathrm{d}\mathcal{L}^2 + \int_{\partial\Omega} u \cdot \nu \, \phi \, \mathrm{d}\mathcal{H}^1$$
$$+ \sum_{i=1}^{N_s} \int_{t_i^I}^{t_i^F} \dot{s}_i(t)[u_2\phi]_{|_{x=s_i(t)}} - [u_1\phi]_{|_{x=s_i(t)}} dt$$

Optimal control of I-LWR

$$\min_{y_I,u} \mathcal{J}(y,s,u) = \mathcal{J}_{\mathrm{obs}}(y) + \mathcal{J}_{\mathrm{s}}(s) + \mathcal{J}_{\mathrm{bar}}(u)$$
$$= \int_\Omega \mathcal{P}(y) + \sum_{i=1}^{N_s} \int_{t_i}^{T} \mathcal{Q}_i(s_i) + \int_\Omega \mathcal{R}(u)$$

Subject to
$$\begin{cases} \partial_t y + \partial_x f(y) = g(x,u) \\ y(x,t=0) = y_I(x) \\ y(0,t) = y_0(t) \text{ and } y(L,t) = y_L(t) \end{cases}$$

where

- $\mathcal{J}_{\mathrm{obs}}(y)$ weights the value of the distributed state y
- $\mathcal{J}_{\mathrm{s}}(s)$ weights the N_s shock locations $s(t) = (s_1(t), \ldots, s_{N_s}(t))$
- $\mathcal{J}_{\mathrm{bar}}(u)$ weights the control $u = (u_1, \ldots, u_{N_u}) \in U_{\mathrm{ad}}$

Optimal control of I-LWR

Weak solution of $\partial_t \tilde{y} + \partial_x \left(f'(\bar{y})\tilde{y}\right) = \partial_u g(x, \bar{u})\tilde{u}$

$$\tilde{y} = \tilde{y}_s + \sum_{i=1}^{N_s} \kappa_i \delta_{\Gamma_i}$$

with $\tilde{y}_s$ *the strong solution of the PDE*

$$\begin{cases} \partial_t \tilde{y}_s + \partial_x \left(f'(\bar{y})\tilde{y}_s\right) = \partial_u g(x, \bar{u})\tilde{u} \\ \tilde{y}_s|_{t=0} = \tilde{y}_I \\ \tilde{y}_s|_{x=0} = 0 \ \text{ and } \ \tilde{y}_s|_{x=L} = 0 \ \text{ depending on } sign\left(f'(\bar{y})\right) \end{cases}$$

and $\kappa_i = -\tilde{s}_i[\bar{y}]|_{x=\bar{s}_i(t)}$, *for* $i = \{1, \ldots, N_s\}$, *the solutions of the ODEs*

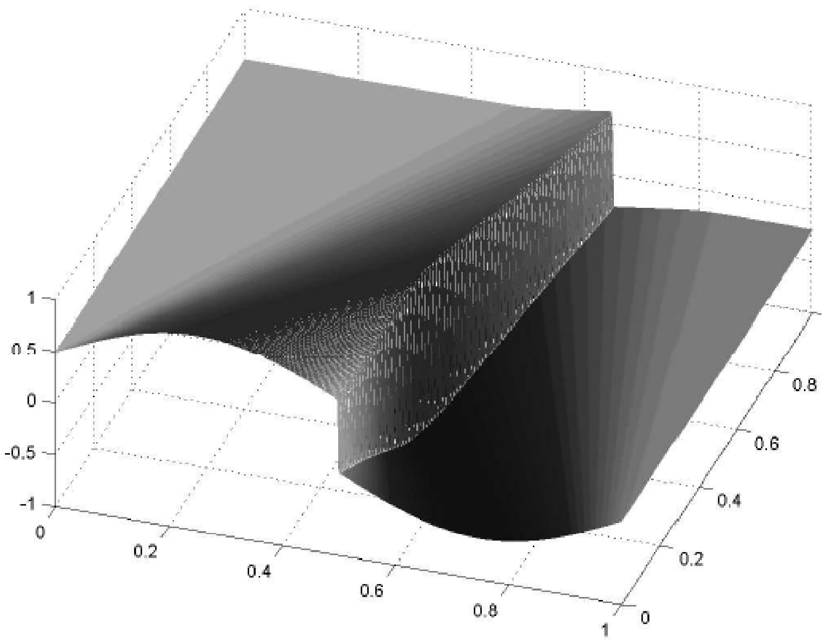
$$\begin{cases} \frac{d\kappa_i}{dt} = -\left[f'(\bar{y})\tilde{y}_s\right]|_{x=\bar{s}_i(t)} + \dot{\bar{s}}_i[\tilde{y}_s]|_{x=\bar{s}_i(t)} \\ \kappa_i(t_i^I) = 0 \end{cases}$$

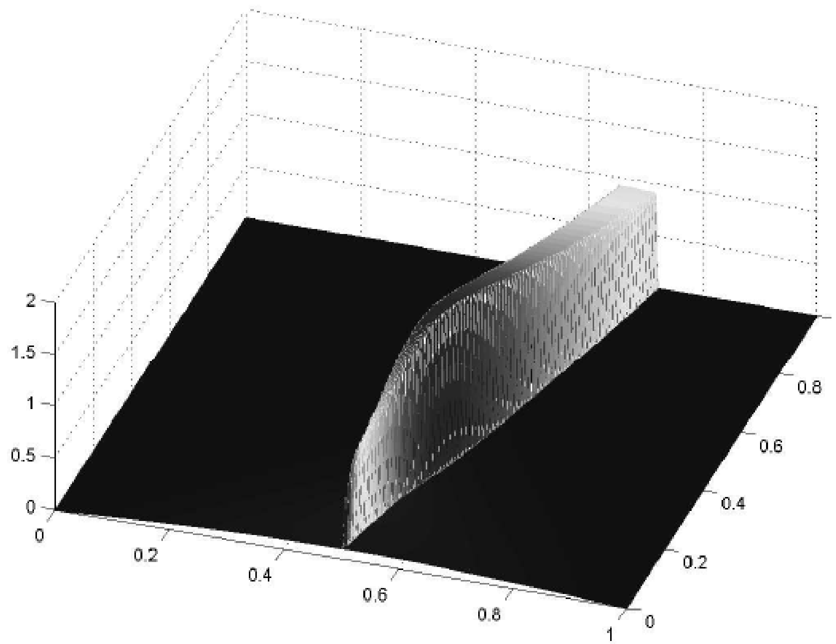
Relationship with $L^1(|Du|)$?

Example

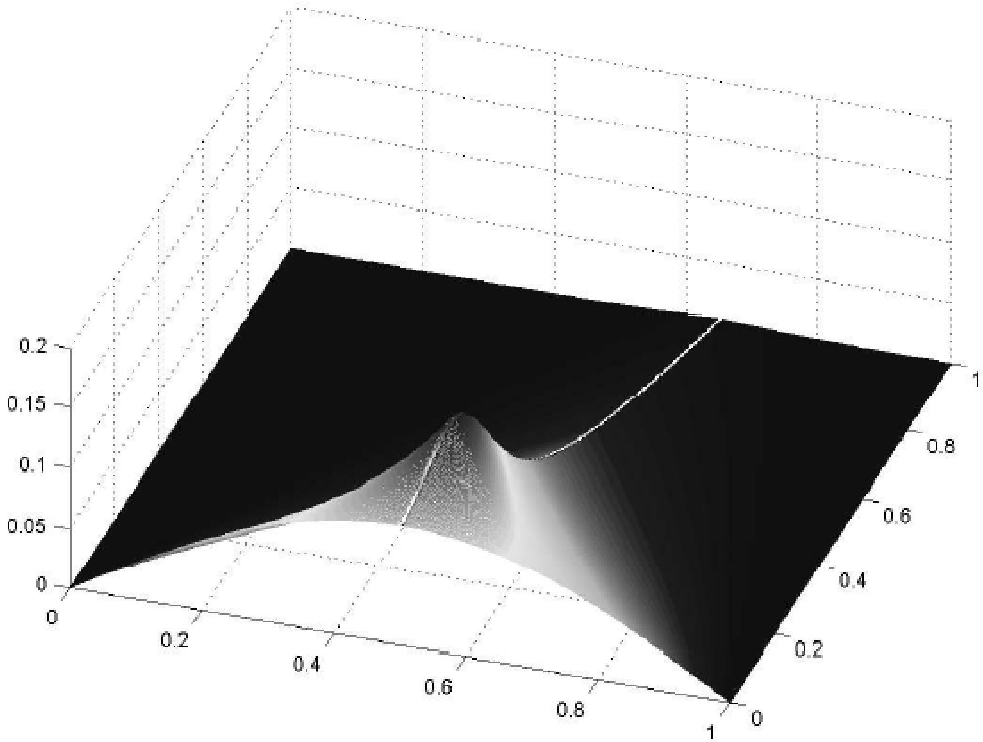
Burgers equation with:

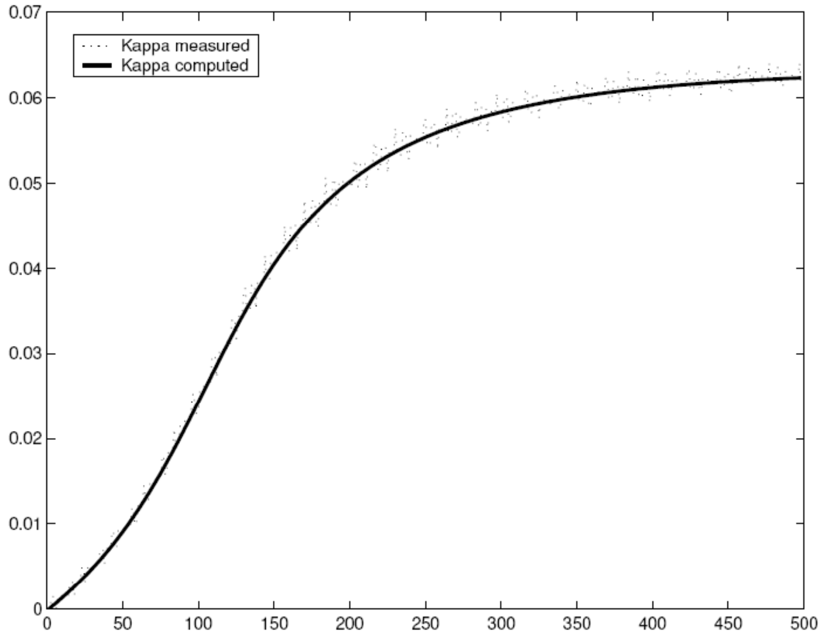
$$\begin{cases} y_I = 0.5 - 0.7\ H(x - 0.5) + 0.4 \sin(2\pi x) \\ y_0(t) = 0.5 \text{ and } y_L(t) = -0.2 \\ \tilde{y}_I = 0.1 \sin(\pi x) \end{cases}$$





PDE & ODE solutions





Absolutely continuous part

Smooth part sensitivity

Singular part

Shock position sensitivity

Optimal control of I-LWR

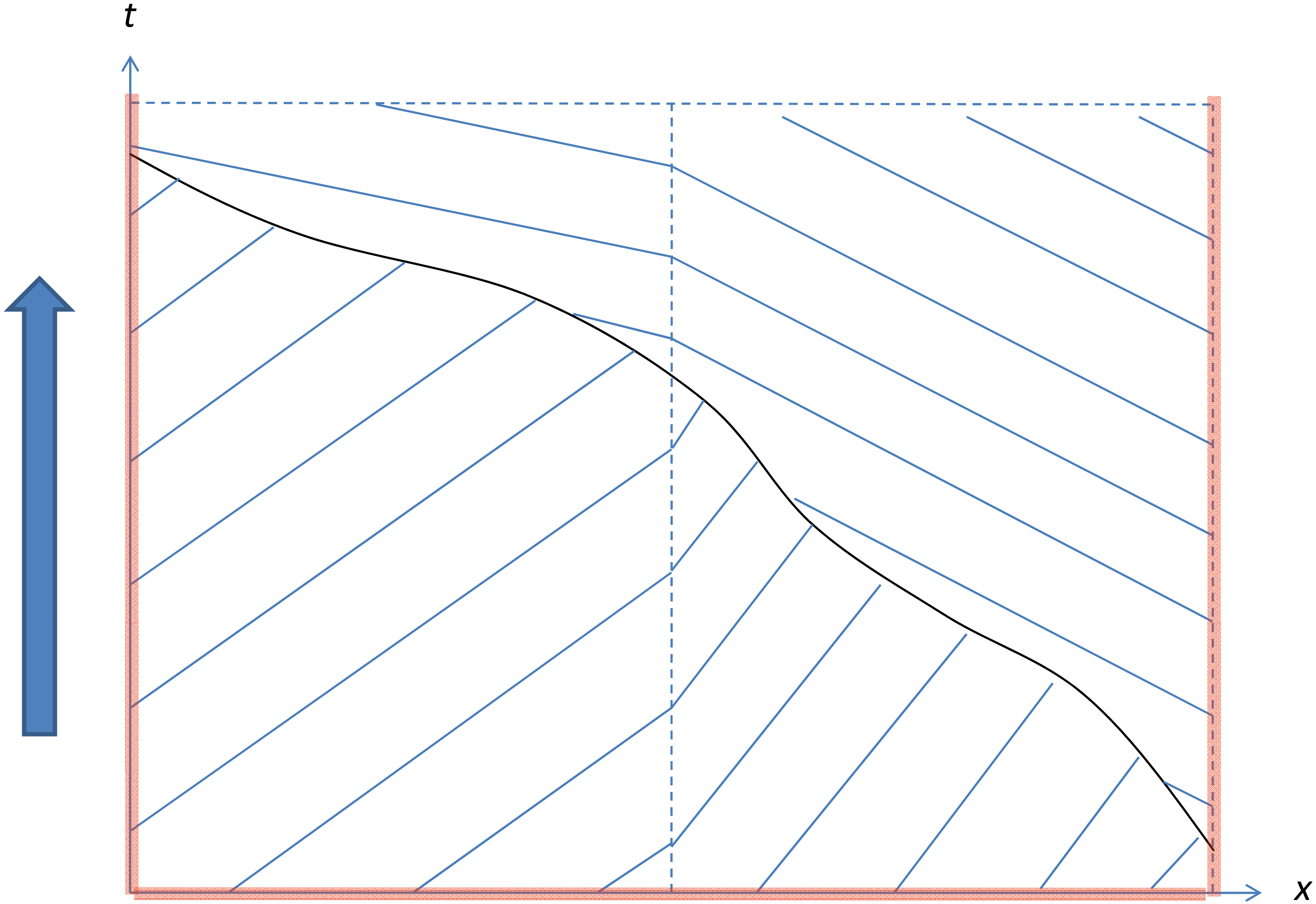
Rewriting
$$\begin{cases} \partial_t \tilde{y}_s + \partial_x \alpha(x,t)\tilde{y}_s = \gamma(x,t)\tilde{u} \\ \dot{\kappa}_i = -[\alpha(\bar{s}_i(t),t)\tilde{y}_s(\bar{s}_i(t),t)] + \dot{\bar{s}}_i(t)[\tilde{y}_s(\bar{s}_i(t),t)] \end{cases}$$

The gradients of the cost $\mathcal{J} = \int_\Omega \mathcal{P}(y) + \sum_{i=1}^{N_s} \int_{t_i}^{T} \mathcal{Q}_i(s_i) + \int_\Omega \mathcal{R}(u)$ *writes*

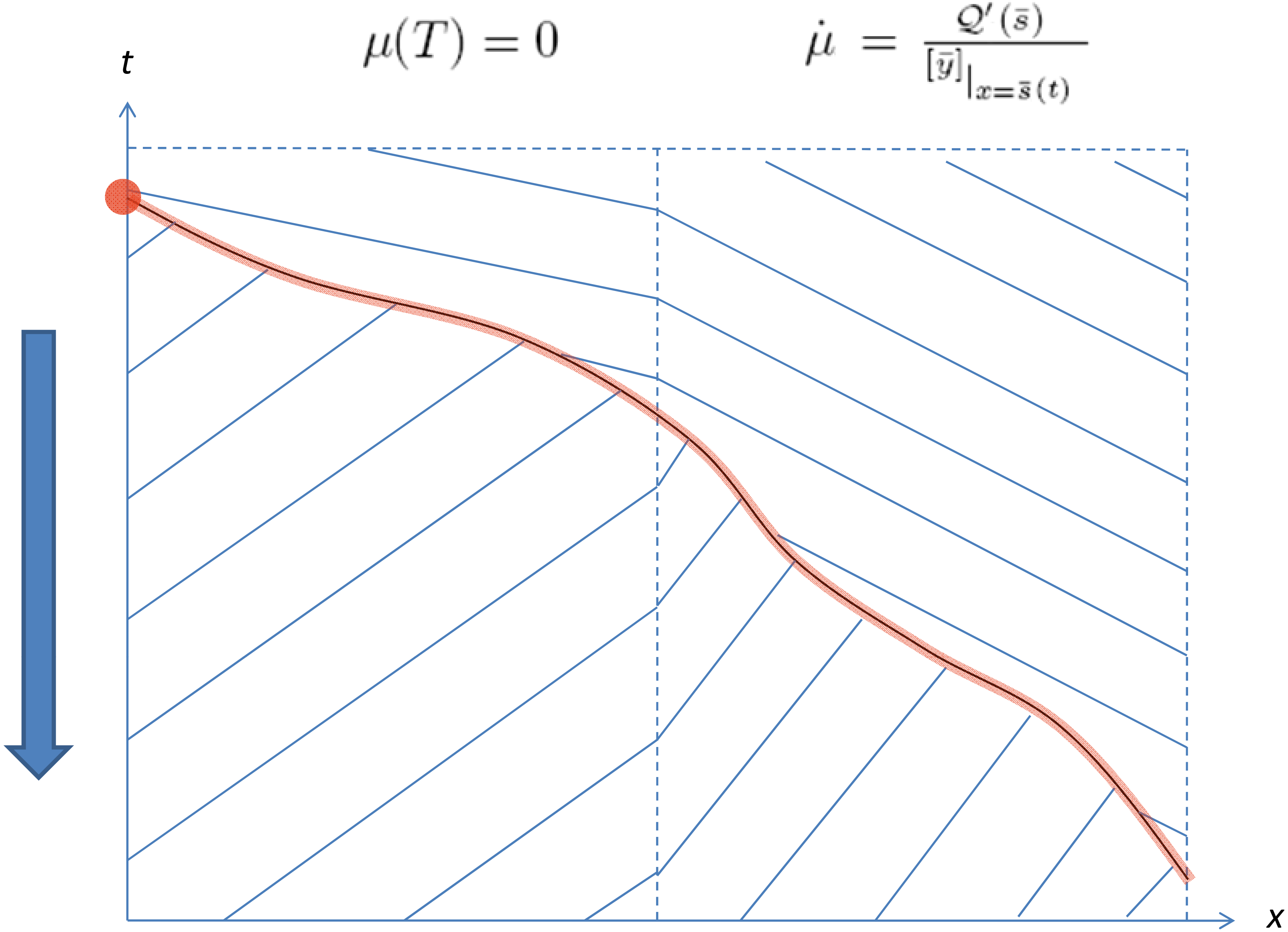
$$\nabla_u \mathcal{J} = \mathcal{R}'(\bar{u}) + \int_0^L \gamma(x,t)\lambda(x,t)\mathrm{d}x \qquad \text{and} \qquad \nabla_{y_I} \mathcal{J} = \lambda(x,0)$$

with λ *and* μ *the adjoint variables, solution of the coupled problem*

$$\begin{array}{l} (\mathrm{ODE-DE}) \\ (\mathrm{ODE-FC}) \\ (\mathrm{PDE-SC}) \\ (\mathrm{PDE-DE}) \\ (\mathrm{PDE-FC}) \\ (\mathrm{PDE-BC}) \end{array} \begin{cases} \dot{\mu}_i = \frac{\mathcal{Q}_i'(\bar{s}_i)}{[\bar{y}]_{|\Gamma_i}} \\ \mu(T) = 0 \\ \lambda^-(\bar{s}_i(t),t) = \lambda^+(\bar{s}_i(t),t) = \mu_i(t) \\ -\partial_t \lambda - \alpha(x,t)\partial_x \lambda = \mathcal{P}'(\bar{y}) \\ \lambda(x,T) = 0 \\ \begin{cases} \lambda(0,t) = 0 & \text{when } \tilde{y}_0 \text{ not active} \\ \lambda(L,t) = 0 & \text{when } \tilde{y}_L \text{ not active} \end{cases} \end{cases}$$



t
x

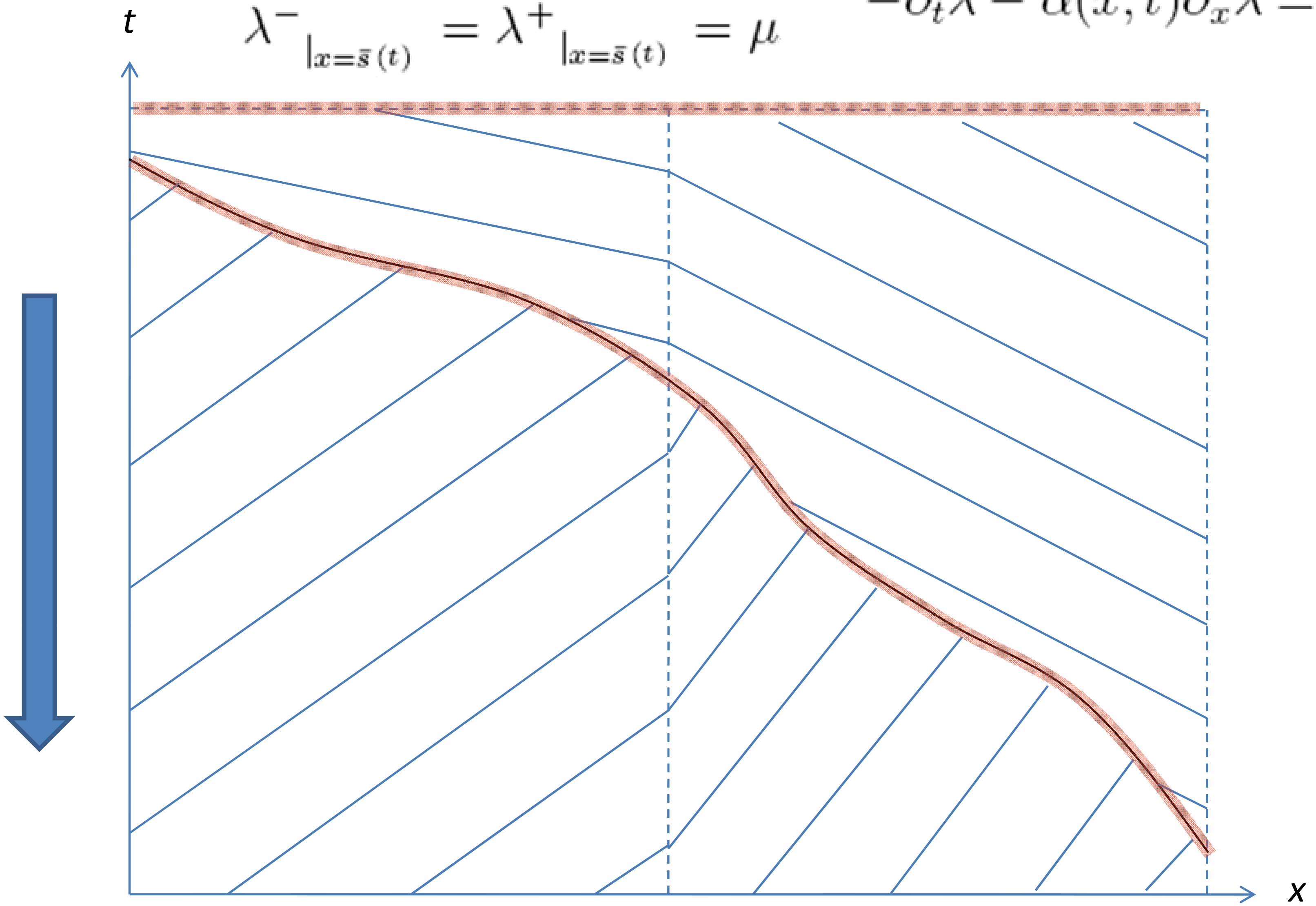


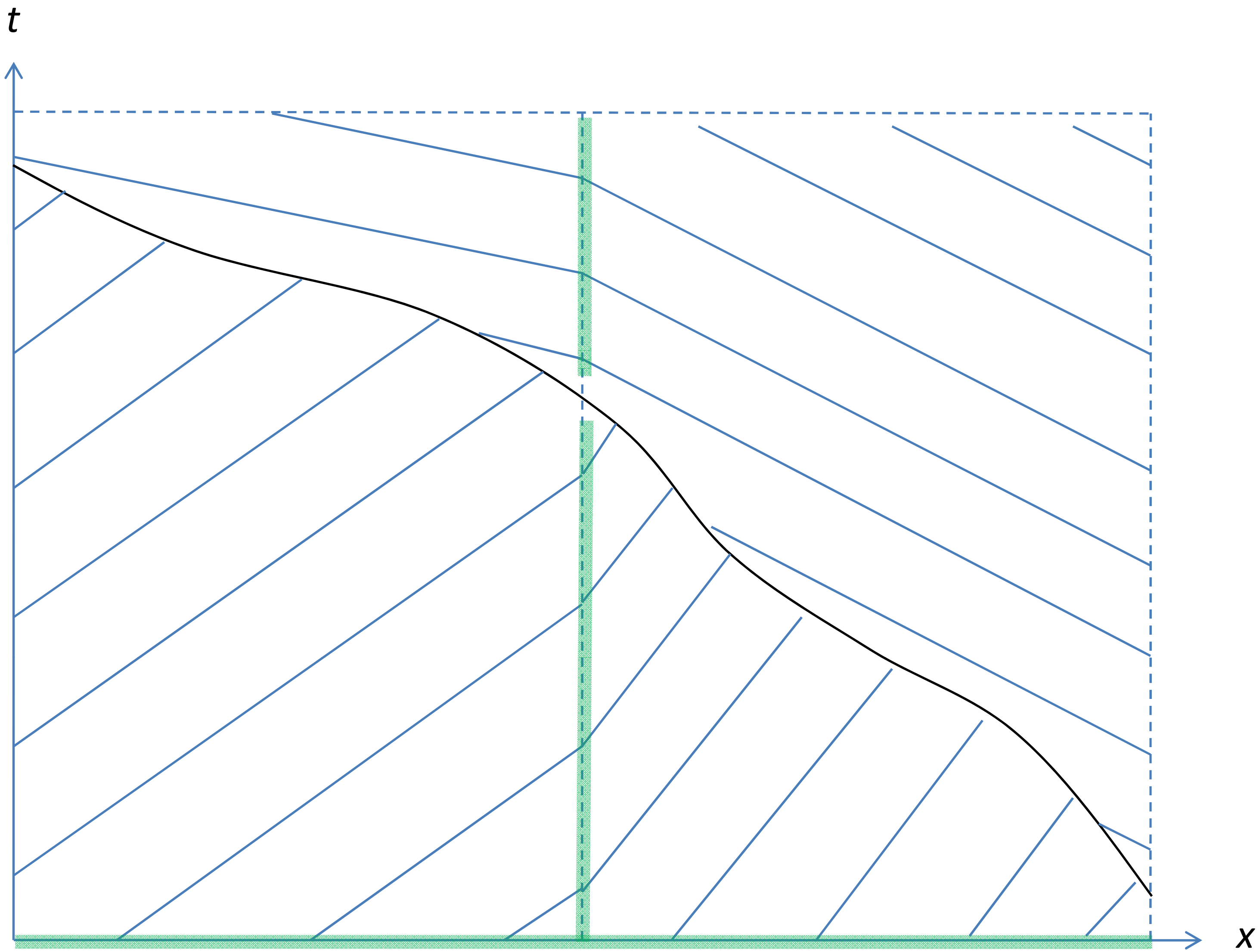
$\mu(T) = 0$
$\dot{\mu} = \frac{Q'(\bar{s})}{[\bar{y}]|_{x=\bar{s}(t)}}$
t
x

$$\lambda(x,T)=0$$

$$\lambda^{-}\big|_{x=\bar{s}(t)} = \lambda^{+}\big|_{x=\bar{s}(t)} = \mu$$

$$-\partial_t\lambda - \alpha(x,t)\partial_x\lambda = \mathcal{P}'(\bar{y})$$





$$\nabla_u \mathcal{J} = \mathcal{R}'(\bar{u}) + \int_0^L \gamma(x,t)\lambda(x,t)\mathrm{d}x \quad \text{and} \quad \nabla_{y_I} \mathcal{J} = \lambda(x,0)$$

Optimal control of I-LWR

Conclusion:

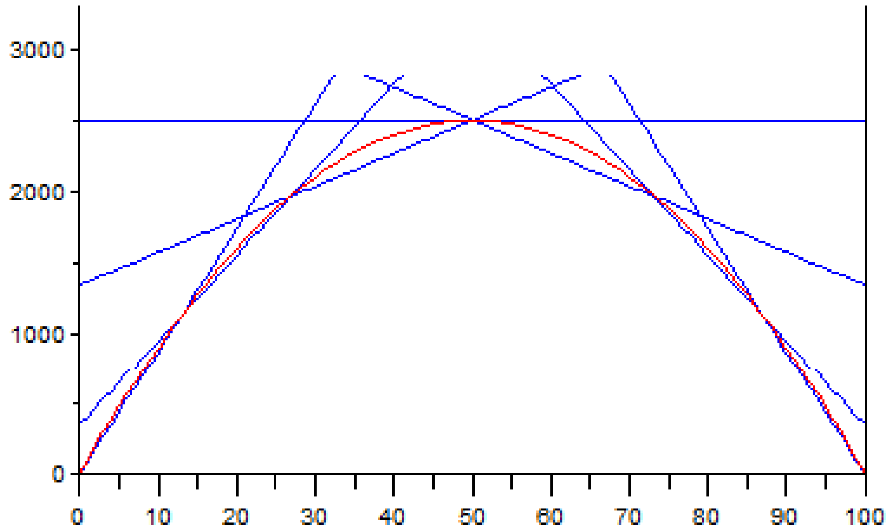
- Formal
- Nice interpretations
- Give some insights on the limitations

Main drawbacks:

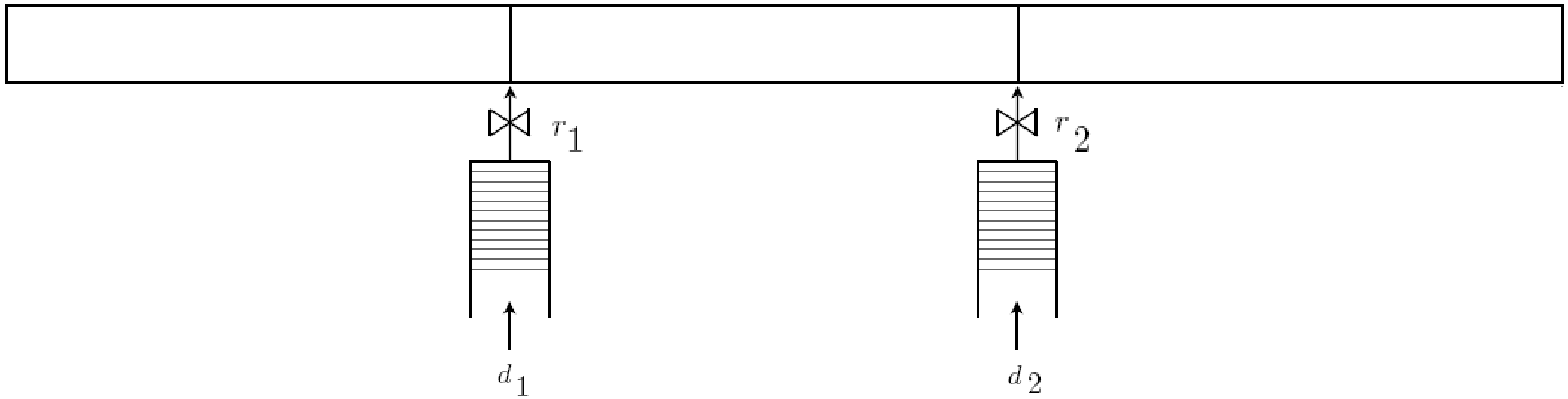
- Computationnaly tricky (shock detection)
- No ramp queue model

Finite dimensional approximation

- Piecewise affine approximation of the fundamental diagram (Front Tracking, CTM)
- « concave » relaxation [Gomes-Horowitz]
- Leads to an LP problem for ramp metering
- Demand/Supply ramp model



Finite dimensional approximation



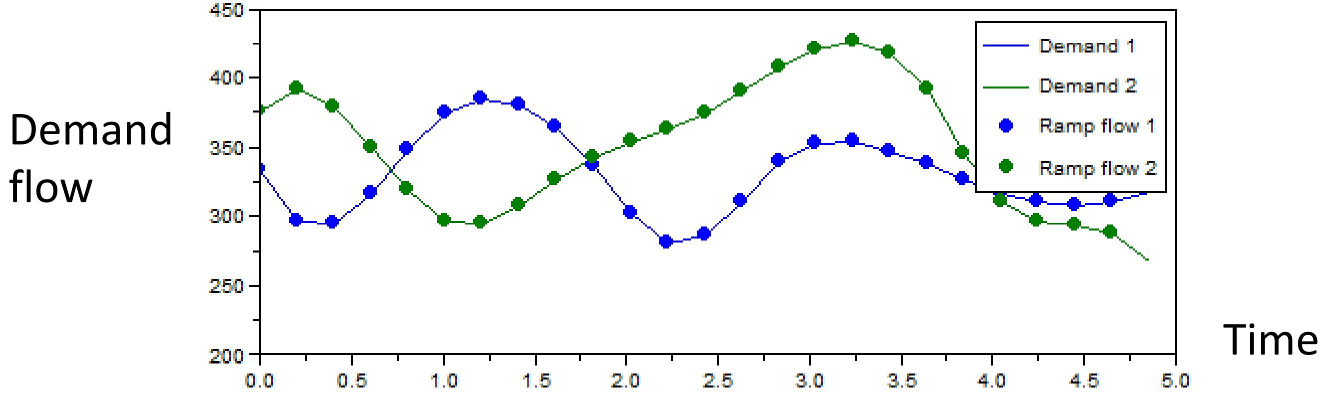
Space: 4.5 km

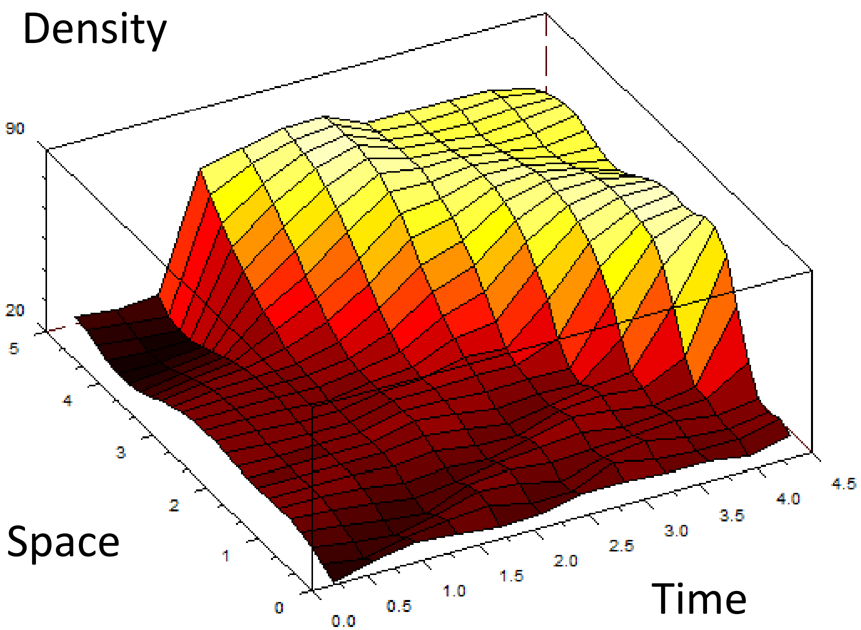
Time: 5 min

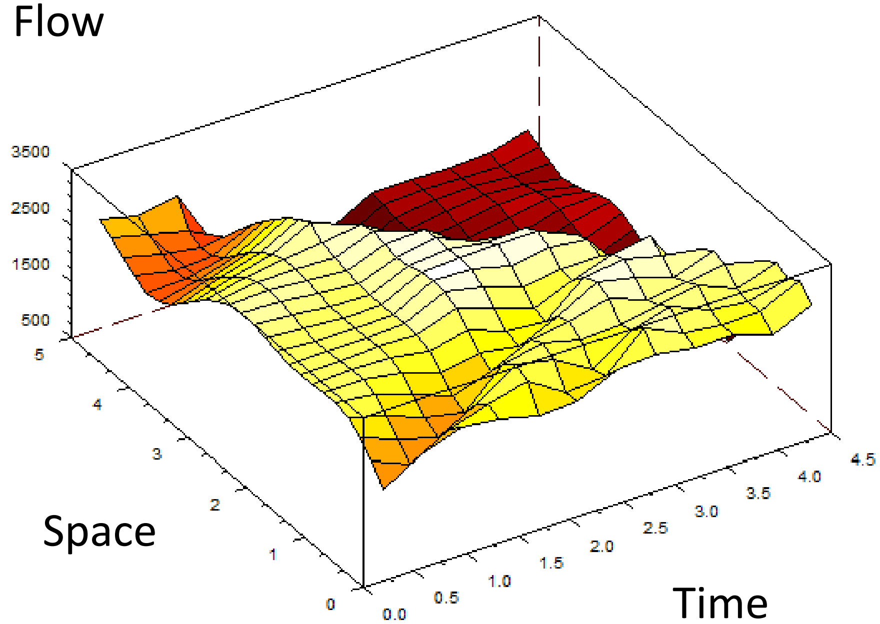
Max ramp flow: 1000 veh/h

$$\mathcal{J} = VMT + \kappa . TSV = \sum_{i,k} \phi_i^k . \Delta x_i . \Delta t + \kappa . \sum_j r_j^k . \Delta t$$

Finite dimensional approximation

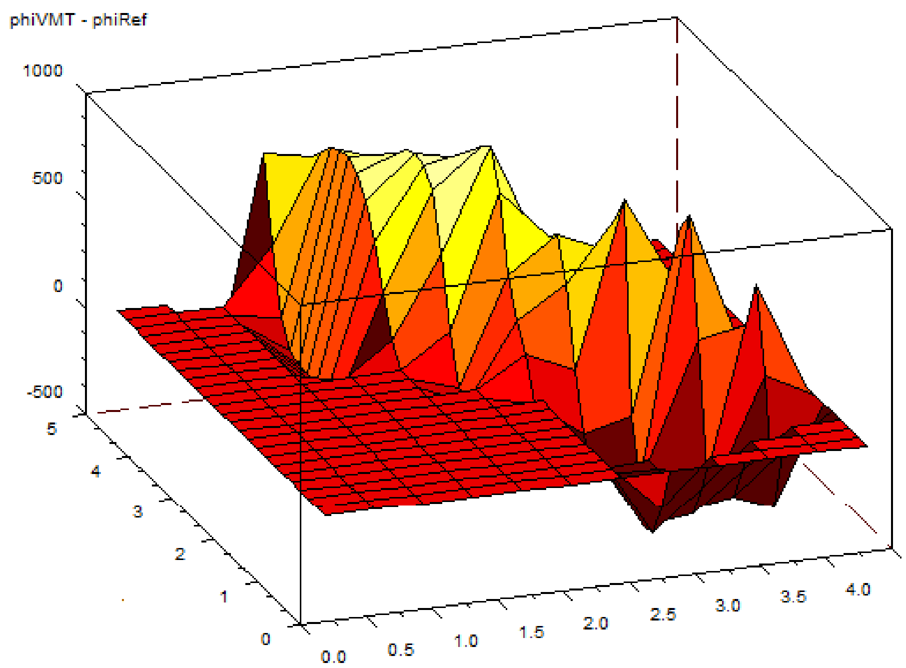




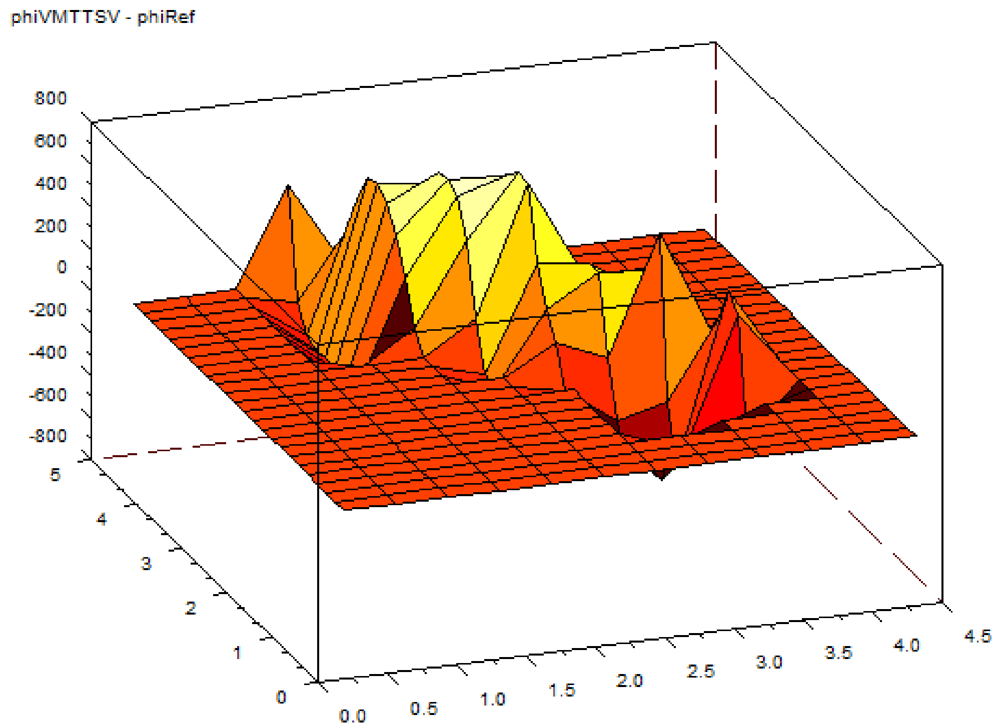


Finite dimensional approximation

VMT



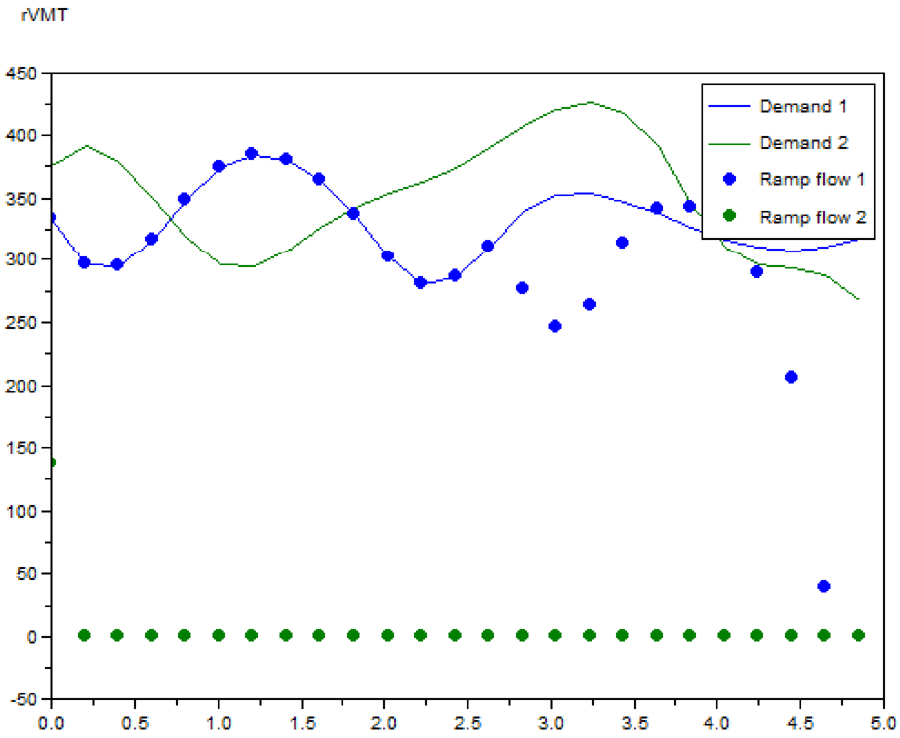
VMT + k . TSV



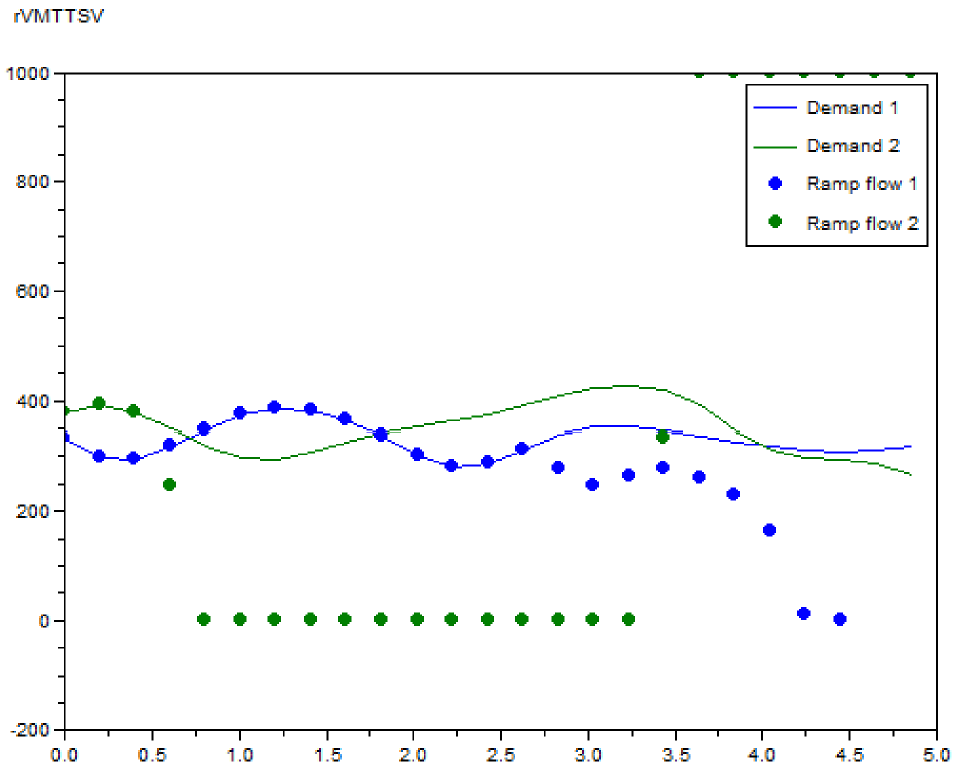
Flow improvement on mainlane

Finite dimensional approximation

VMT



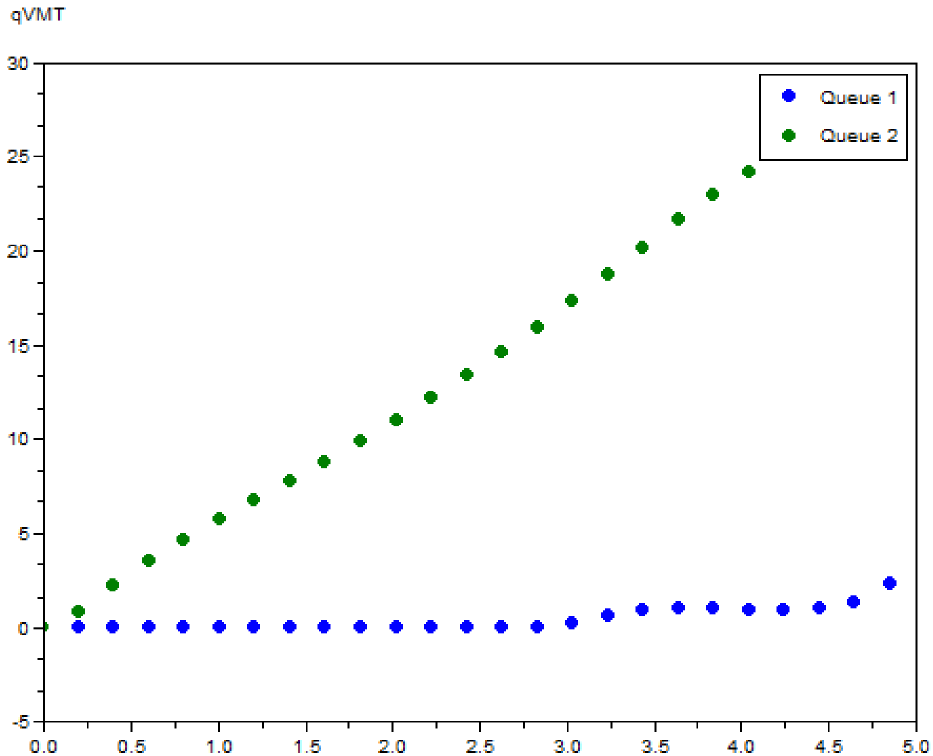
VMT + k . TSV



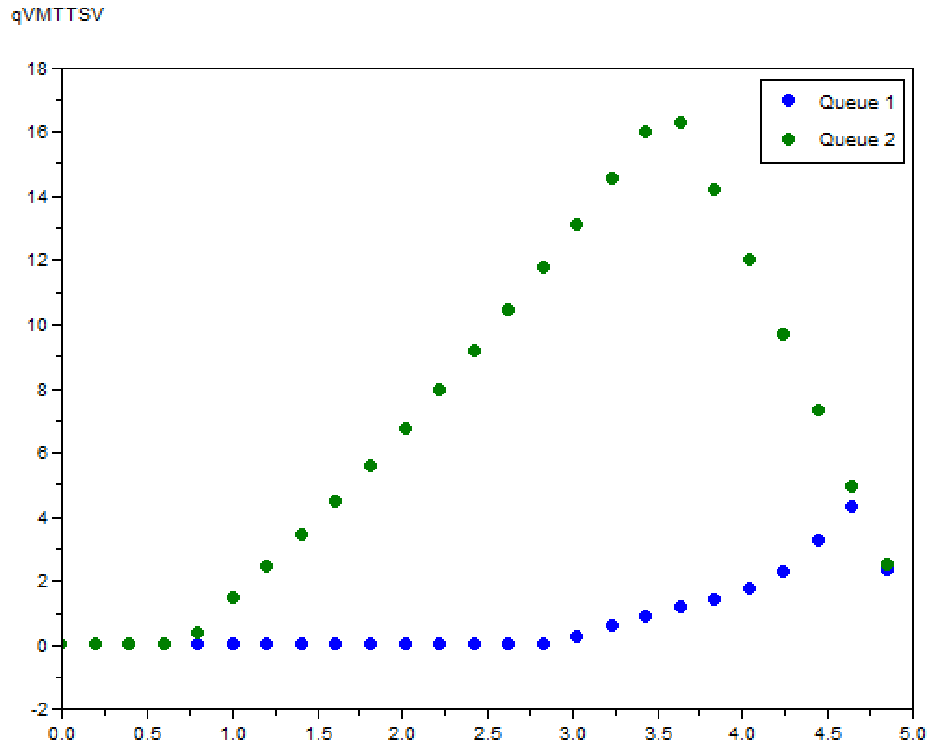
<u>Ramp flow signals</u>

Finite dimensional approximation

VMT



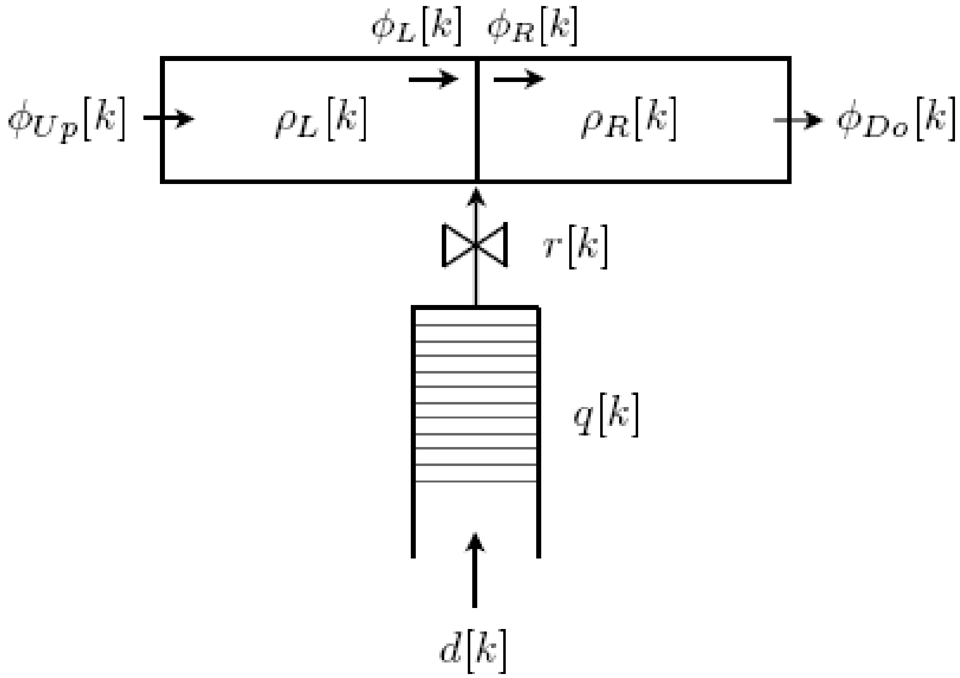
VMT + k . TSV



<u>Ramp queues</u>

Finite dimensional approximation

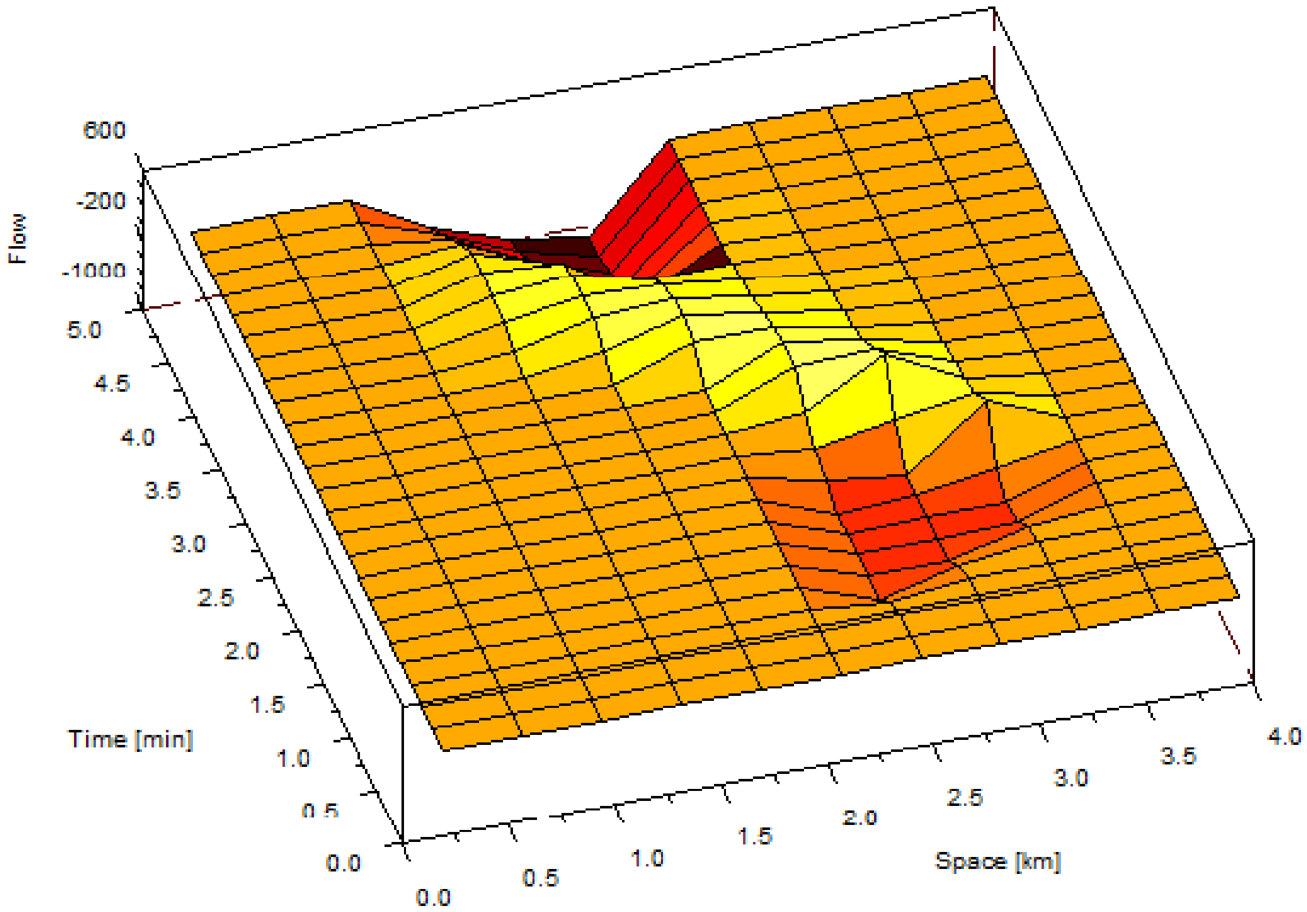
<u>Instantaneous control leads to a local structure</u>



Local Instantaneous Control (LIC)

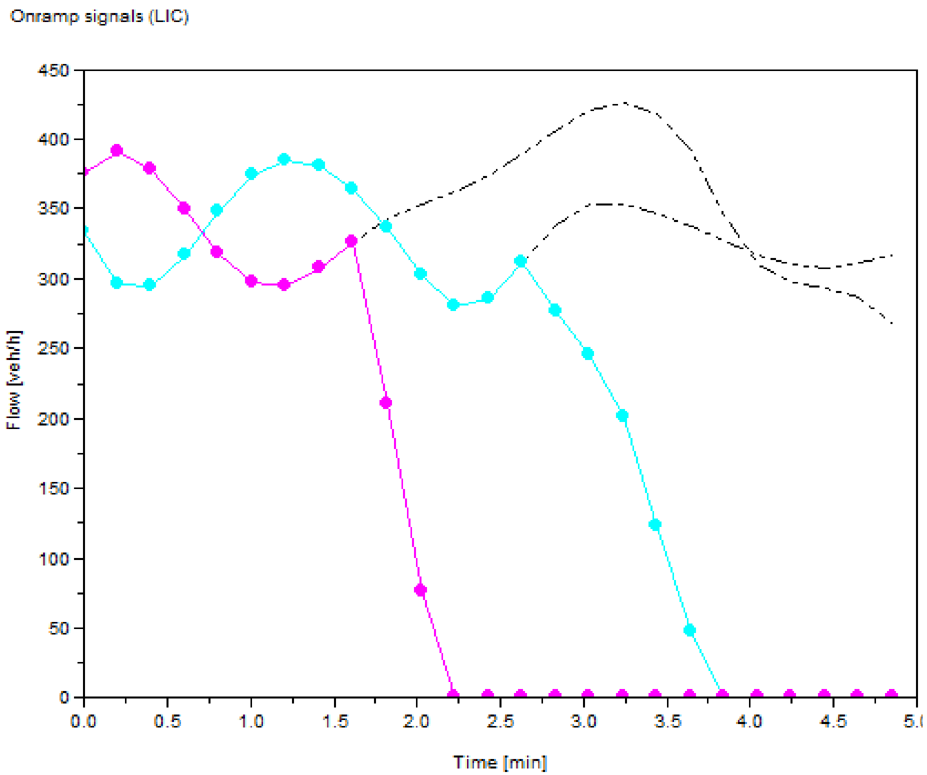
Finite dimensional approximation

Flow improvement = Flow (MPC) - Flow (LIC)



Finite dimensional approximation

Ramp flow



Ramp queues