Modeling, simulation and analysis for traffic intersections

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Modéles mathématiques du trafic – 2007

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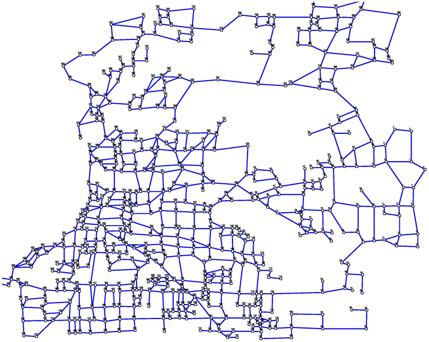
Optimal routing problems



Measure load on the network

$$\sum_j \int_0^T \int_a^b \mathcal{F}_j(\rho_j)dxt$$

Reduce load by redistribution of cars at intersections ?

- I Ignore desired destination of drivers
- II Suboptimal control strategies
- III Include desired destinations

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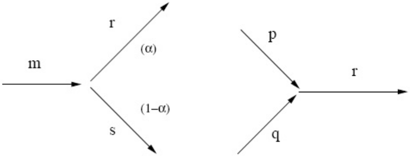
Optimal control problem for the LWR equation

Present derivation and results for LWR equation.
Mathematical problem:

$$\sum_j \int_0^T \int_a^b \mathcal{F}_j(\rho_j) dxt \rightarrow \min \quad \text{subject} \quad \partial_t \rho_j + \partial_x(\rho_j u(\rho_j)) = 0$$



Coupling conditions for intersections:
Type A: Distribution possible

$$\rho_r u_r = \alpha_m \rho_m u_m$$

$$\rho_s u_s = (1 - \alpha_m) \rho_m u_m$$

Type B: No distribution

$$\rho_r u_r = \rho_p u_p + \rho_q u_q$$

Existence of a solution (ρ_j, u_j) proven by B. Piccoli/ M. Garavello/C. Coclite

Formal optimality system (KKT system on cont. level)

First–order optimality conditions α_m distributes from m to r

$$\begin{aligned} \partial_t\rho_j + \partial_x \rho_j u_j &= 0 & \partial_t \mu_j + (\rho_j u_j)' \partial_x \mu_j &= \mathcal{F}'(\rho_j) \\ \rho_j(x,0) &= \rho_{j,0} & \mu_j(x,T) &= 0 \\ \rho_r u_r &= \alpha_m \rho_m u_m & \mu_r &= \alpha_m \mu_r + (1-\alpha_m)\mu_s \\ \rho_r u_r &= \rho_p u_p + \rho_q u_q & \mu_p = \mu_q &= \mu_r \end{aligned}$$

$$\begin{aligned} 0 &= \partial_{\alpha_m} \sum_i \int_0^T \int_a^b \mathcal{F}(\rho_j) dxt \\ &= \int_0^T (\mu_s - \mu_r)\, \rho_m u_m dt \end{aligned}$$

Properties of the functional

Assume $\mathcal{F}_j$ convex, $\rho \to (\rho u(\rho))$ concave and for fixed control $\vec{\alpha}$ the solution to adjoint equations satisfy $\partial_x \mu_j \geq 0$, then

$$\vec{\alpha} \to F(\vec{\alpha}) := \sum_j \int_0^T \int_a^b \mathcal{F}_j(\rho_j) dxt$$

is convex in each component α_j.

- If $x \to \rho_j(x,t)$ is decreasing for all times t (i.e., no shock), then $\partial_x \mu_j$ is non–negative.
- Theorem covers the case $\mathcal{F}_j(\rho) = \rho$.
- Result is strict, in the sense that there is an examples for two controls where F is not convex (but componentwise convex).

Additional remarks on the optimization problem

- Under the assumption $\partial_x \mu_j \geq 0$ we can use a simple upwind discretization. Then, the discrete KKT system coincides with discretized optimality system up to order $O(\Delta x)$. Further, the discretized cost functional F^Δ is componentwise convex.
- Similar calculus applies for optimal inflow profile problems:

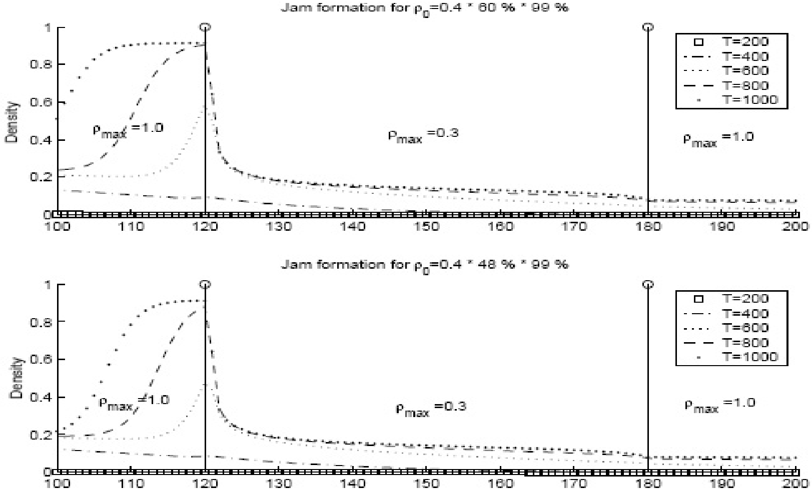
$$\min \sum_j \int_0^T \int_a^b \mathcal{F}_j(\rho_j) dxt \quad \text{subject to pde and}$$

$$const = \int_0^T \rho_0(t) dt$$

- Numerical optimization by e.g. quasi–Newton methods on the reduced cost functional. Problem is computational time.

Model	Solver	Gradient	Parameter N	CPU Time
Macroscopic	Godunov Scheme	DD	100	135.655s
Macroscopic	Front-Tracking	DD	25	45.172s
Macroscopic	Godunov Scheme	DD	50	45.172s

Numerical results without and with optimization



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Towards instantaneous controls

Problem: $\min \sum_j \int_0^T \int_a^b \mathcal{F}_j(\rho_j) dxt$ subject to pde + cplg for a fixed (possible) large time horizon T

- KKT–system with time– and space–dependent multipliers and right–hand side depending on $\rho_j(x,t) \implies$ need to store the solution $\rho_j(t,x)$ for each point in space and each arc
- Possible solution: Instantaneous/receding horizon control as suboptimal control strategy
  - Implicit Euler discretization (in time) of the LWR equation
  - Successively determine optimal control $\vec{\alpha}$ on (t_i, t_{i+1}) by minimizing

$$H_i := \sum_j \int_{t_i}^{t_{i+1}} \int_a^b \mathcal{F}_j(\rho_j) dxt \text{ subject to pde + cplg } \to \min$$

Properties of the optimization problem

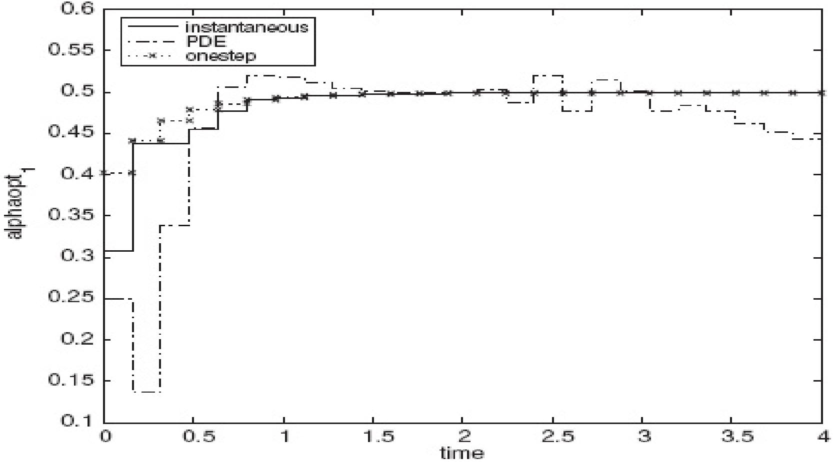
- H_i is componentwise convex
- Only suboptimal controls expected
- Sequence of optimization problems has to be solved. Each problem is independent of time and hence cheaper in terms of storage and computational time
- Different strategies exist for prolongation of a control and to prevent oscillations in the final control $\vec{\alpha}(t)$
- If KKT system for $\min H_i$ is solved by projected gradient typically the first gradient step yields the strongest decrease in the value of the cost functional.

Numerical results – computational times

Table I. CPU-times in seconds.

# Arcs		$N = 10$	$N = 20$	$N = 30$	$N = 40$	$N = 50$
	Full OCP	66.5	146.2	344.3	378.3	565.1
	Gradient steps	20	20	26	22	22
	Av. Armijo tests	1	1.2	2.15	1.59	2.36
$\|E\| = 11$	Instantaneous OCP	75.7	135.1	154.9	172.4	228.9
	Av. gradient steps	21.9	16.05	9.57	7.03	6.48
	Av. Armijo tests	1.08	1.27	1.56	1.47	1.82
	Onestep OCP	3.6	6.7	11.1	17.1	20.8
	Av. Armijo tests	1	1	1	1	1.02
	Full OCP	108.6	292.6	546.8	615.3	829.1
	Gradient steps	22	27	25	23	24
	Av. Armijo tests	1	2	2.64	1.82	1.75
$\|E\| = 17$	Instantaneous OCP	123.0	273.2	262.8	227.2	275.3
	Av. gradient steps	24.8	25.8	14.0	7.1	5.8
	Av. Armijo tests	1.3	1.2	1.3	1.63	1.82
	Onestep OCP	4.6	9.8	15.5	23.1	32.3
	Av. Armijo tests	1	1	1	1.08	1.22
	Full OCP	356.3	427.2	1081.3	1061.3	1549.4
	Gradient steps	36	23	31	25	28
	Av. Armijo tests	1.89	1.74	2.48	1.24	1.28
$\|E\| = 32$	Instantaneous OCP	228.4	388.4	505.7	363.2	366.8
	Av. gradient steps	26.6	21.4	17.8	7.85	5.16
	Av. Armijo tests	1.09	1.14	1.15	1.55	2.40
	Onestep OCP	8.6	21.8	29.7	42.5	60.2
	Av. Armijo tests	1	2	1.37	1.3	1.42

Numerical results – qualitative behavior α_1



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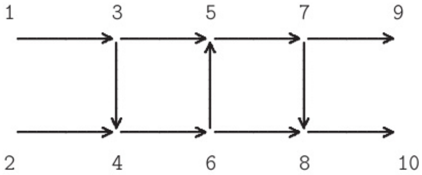
Necessary extensions for optimal load of networks

- More than one in– and outflow arc
- Optimization of distribution rates not desirable(!)



- Need for introducing commodities, e.g., commodity one – cars with destination (9) and commodity two – cars with destination (10)

Governing equations and coupling conditions

- Commodities $\vec{a}$ travel with cars and are Lagrangian variables (like w but without influence on the velocity)

$$\partial_t \vec{a} + u_j \partial_x \vec{a} = 0, \; \sum_k a_k = 1$$

- + LWR model

$$\partial_t \rho_j + \partial_x(\rho_j u_j) = 0, \; u_j = u(\rho_j)$$

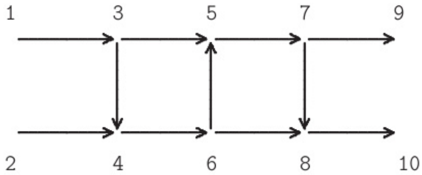
or + ARZ model with $w_j = u_j + p(\rho_j)$

$$\partial_t \rho_j + \partial_x(\rho_j u_j) = 0, \partial_t \rho_j w_j + \partial_x(\rho_j u_j w_j) = 0$$

- $\Longrightarrow$ Discussion of coupling conditions at intersections

[0] see e.g. Piccoli/Garavello or H./Kirchner/Moutari/Rascle (2007)

Modeling aspects



- Recall: E.g. commodity a_1 are drivers traveling from (1) to (9)
- There are several possible paths they can take, e.g., 1-3-5-7-9 or 1-3-4-6-5-7-9
- For optimal load they have to be distributed only(!) among the admissible paths

Modeling of possible controls for optimization

- We introduce $\gamma^i_{jk} \in (0,1)$ as the proportion of flow of commodity i on the incoming road k that take the outgoing road j (replacing α_{jk})
- Some values of γ^i_{jk} are fixed a–priori, since some destination of commodity i might not be reached from road k
- For each commodity i (fixed origin) we introduce the set of all possible paths to (fixed) destination $\mathcal{P}_i$
- If and only if $k \in P^i$ for at least one $P^i \in \mathcal{P}_i$ and if $j \in P^i$ for at least one $P^i \in \mathcal{P}_i$, then $\gamma^i_{jk} \in [0,1]$ and else $\gamma^i_{jk} = 0$ (NP–hard problem if graph contains no closed loops)

$$\implies \sum_{j \in \delta^+} \gamma^i_{jk} = 1 \forall k \in \delta^-$$

We can define commodity dependent distribution rates

Coupling conditions for the ARZ-system

Distribution rates for commodities imply

A1 Distribution rates for the car flux

$$\alpha_{jk} = \sum_i \gamma^i_{jk} a^i_k, \quad (\rho_j u_j) = \sum_k \alpha_{jk} \rho_j u_j$$

A2 Distribution of commodities a^i_j

$$a^i_j = \frac{\sum_{k \in \delta^-} \gamma^i_{jk} a^i_k}{\sum_i \sum_{k \in \delta^-} \gamma^i_{jk} a^i_k}$$

Coupling conditions for the ARZ-system (cont'd)

A3 Behavioral rule for the mixture of car flow on outgoing roads

$$\beta_{jk} = \frac{\sum_i \gamma^i_{jk} a^i_k}{\sum_i \sum_{j \in \delta^-} \gamma^i_{jk} a^i_j}, \quad w_j = \sum_{k \in \delta^-} \beta_{jk} w_k$$

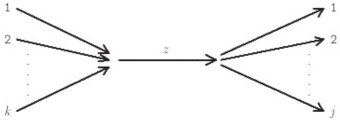
E.g., two incoming, one outgoing road yields $\beta_{jk} = \frac{1}{2}$ (equal priority)

Conditions conserve all moments $(\rho_j u_j), (\rho_j u_j w_j)$ and $(\rho_j u_j a_j)$

Conditions (A1), (A2) necessary for LWR and are similar to conditions proposed by B. Piccoli, M. Garavello

Riemann problem at the intersection

- Case of LWR, c.f. B. Piccoli/ M. Garavello – here, with a choice of the optimal path vs. fixed paths
- Case of ARZ: Homogenization on the outgoing roads necessary, c.f. presentation of S. Moutari or F. Siebel.
  now two possibilities
  - *First* distribute the incoming flux to the outgoing roads and *then* homogenize the outgoing total flow
    $\Longrightarrow$ mix the colors of only those who take the same outgoing road
  - *First* homogenize the total flow and *then* distribute with rate γ^i_{jk}
    $\Longrightarrow$ mix the colors of all cars and then distribute colored cars



Note: The commodities remain not *homogenized* – this would be fatal!

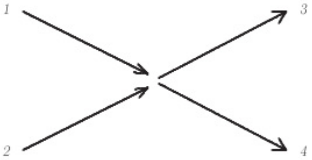
Riemann problem at the intersection (cont'd)

- Previous discussion and results seen in many talks before $\Longrightarrow$
- For given constant initial densities and constant commodities we can solve the Riemann problem at the intersection for the LWR and the ARZ(!)
- Numerical results different depending on model or homogenization procedure
- Junctions with at most three connected arcs: No difference between both procedures

Results on a single junction

Initial data: $a_1^1 = 20\%, a_2^1 = 10\%$, constant initial densities ρ_i define velocities by fundamental diagram $q = \rho(1-\rho)$ and then $w = q/\rho + \rho$

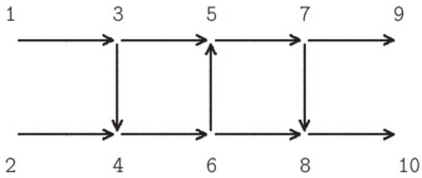
$$\gamma_{13}^1 = 1, \quad \gamma_{13}^2 = 0, \quad \gamma_{23}^1 = 1, \quad \gamma_{23}^2 = 0.$$



	LWR	HD	DH
ρ_1^+	0.5	0.6	0.6
ρ_2^+	0.6	0.8656	0.8
ρ_3^-	0.5	0.318	0.35
ρ_3^*	–	0.318	0.35
ρ_4^-	0.3	0.448	0.497
ρ_4^*	–	0.18	0.15

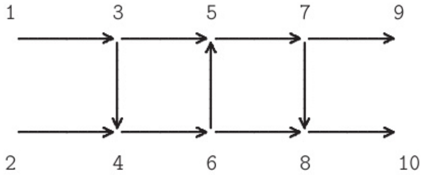
Different absolute values depending on the model, trends similar, qualitative behavior similar

Results on a test network



- Commodity one (two) traveling from 1(2) to 9(10)
- Assume 1-3-5-7-9 and 2-4-6-5-7-9 (one) or 1-3-4-6-8-10 and 2-4-6-8-10 (two) as possible paths only
- Initial distribution $a^1_{(1,3)} = 70\%$, $a^1_{(2,4)} = 40\%$
- No optimization on γ^i_{jk} since no freedom in choice of paths

Results on a test network (cont'd)



Densities and commodities on each road

Arc (e)	ρ_e		a_e^1		a_e^2	
	LWR	AR	LWR	AR	LWR	AR
(1,3)	0.2	0.2	0.7	0.7	0.3	0.3
(2,4)	0.25	0.25	0.4	0.4	0.6	0.6
(4,6)	0.38	0.1198	0.32	0.2	0.68	0.8
(3,5)	0.1285	0.1373	1	1	0	0
(6,5)	0.08167	0.0233	1	1	0	0

Conclusion

Optimization for load of networks possible

- Componentwise convexity of cost functional
- Properties can be inherited for suitable discretizations
- Commodity models: guarantee that drivers arrive at desired locations
- Predicted optimal road load strongly depending on modeling of intersection
- No efficient numerical optimization algorithm for large–scale problems, but suboptimal control exist

Thank you for your attention.