## 1 Introduction

In the last several years there has emerged an extensive literature [1-11] on "higher order" traffic models. These models were developed in an attempt to explain the strong permanent waves which appear in congested traffic. At the continuum level all of these models are of the form:

$$\frac{\partial s}{\partial t} - \frac{\partial u}{\partial m} = 0 \tag{1.1}$$

and

$$\epsilon \left( \frac{\partial u}{\partial t} - P'(s) \frac{\partial u}{\partial m} \right) = V(s) - u. \tag{1.2}$$

Here, $t \geq 0$ is time, $m$ is a "continuous" car index, and $\epsilon > 0$ has the interpretation of a relaxation time. The velocity of the $m^{th}$ car at time $t$ is $u(m, t)$ and the trajectory of the $m^{th}$ car, $t \longrightarrow x(m, t)$, is given as the solution of

$$\frac{\partial x}{\partial t} = u \quad \text{and} \quad x(m, 0) = x_0(m) \tag{1.3}$$

where $x_0(m)$ is the position of the $m^{th}$ car at $t = 0$. $s(m, t)$ is related to $x(m, t)$ by

$$s(m, t) = \frac{\partial x}{\partial m}(m, t) \tag{1.4}$$

and measures the spacing between successive cars.

The function $s \to V(s)$ in (1.2) represents an "equilibrium" velocity and typically it is assumed that $V(\cdot)$ is defined on $s \geq L$ and satisfies

$$V(L) = 0, \quad 0 < V'(s) \text{ for } L < s, \text{ and } \lim_{s\to\infty} V(s) = v_\infty < \infty. \tag{1.5}$$

The parameter $L > 0$ has the interpretation of the length of a car on the roadway. Finally, the term $P'(s)\frac{\partial u}{\partial m}$ appearing in (1.2) is referred to as the "anticipatory" acceleration and all modelers assume that $P'(s) \geq 0$ on $s \geq L$.

An equivalent system to (1.1) and (1.2) is

$$\frac{\partial s}{\partial t} - \frac{\partial}{\partial m}(P(s) + \alpha) = 0 \tag{1.6}$$

and

$$\epsilon\frac{\partial \alpha}{\partial t} + \alpha = (V(s) - P(s)). \tag{1.7}$$

Here, $P(s)$ is an indefinite integral of $P'(\cdot)$ normalized so that $P(L) = 0$ and, of course,

$$u(m,t) = P(s(m,t)) + \alpha(m,t). \tag{1.8}$$

The hypothesis $P'(s) \geq 0$ implies that the system (1.1) and (1.2) or equivalently (1.6) and (1.7) is hyperbolic with wave speeds $c = -P'(s) \leq 0$ and $c = 0$ and thus information propogates from right to left. This observation implies that when constructing finite difference schemes the appropriate spatial differences should be downwind, i.e., that

$$s(m,t) \doteq \frac{x(m + \Delta m, t) - x(m,t)}{\Delta m} \tag{1.9}$$

and

$$\begin{aligned} \frac{\partial u}{\partial m}(m,t) = & \frac{\partial}{\partial m}(P(s) + \alpha)(m,t) \doteq \\ & \frac{(P(s(m+\Delta m,t)) + \alpha(m+\Delta m,t) - P(s(m,t)) - \alpha(m,t))}{\Delta m}. \end{aligned} \tag{1.10}$$

If one chooses to discretize (1.1)-(1.4) spatially, keeps time continuous, and, moreover, chooses $\Delta m = 1$ (recalling that cars are really discrete entities) one is led to the classic follow-the-leader system

$$\frac{dx_m}{dt} = u_m \tag{1.11}$$

and

$$\epsilon \frac{du_m}{dt} = \epsilon P'(x_{m+1} - x_m)(u_{m+1} - u_m) + V(x_{m+1} - x_m) - u_m \tag{1.12}$$

studied by traffic engineers. Moreover, if we let

$$s_m = x_{m+1} - x_m \tag{1.13}$$

we see that solving (1.11) and (1.12) is equivalent to solving

$$\frac{ds_m}{dt} = (P(s_{m+1}) + \alpha_{m+1} - P(s_m) - \alpha_m) \tag{1.14}$$

$$\epsilon \frac{d\alpha_m}{dt} + \alpha_m = (V(s_m) - P(s_m)) \tag{1.15}$$

and

$$u_m = P(s_m) + \alpha_m \tag{1.16}$$

and this latter system is nothing more than the appropriate spatial discretization of (1.6) and (1.7) with $\Delta m = 1$.

On the other hand, if one lets

$$\rho(x,t) = \frac{1}{s(m,t)} \quad \text{and} \quad v(x,t) = u(m,t) \tag{1.17}$$

when

$$x = x(m,t), \tag{1.18}$$

one finds that as functions of $x$ and $t$ the functions $\rho$ and $v$ satisfy

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \tag{1.19}$$

and

$$\epsilon\left(\frac{\partial v}{\partial t} + (v + \rho\mathcal{R},_{\rho}(\rho))\frac{\partial v}{\partial x}\right) = W(\rho) - v, \tag{1.20}$$

where

$$\mathcal{R}(\rho) \stackrel{def}{=} P(1/\rho) \quad \text{and} \quad W(\rho) \stackrel{def}{=} V(1/\rho). \tag{1.21}$$

Of course

$$\rho^2\mathcal{R},_{\rho}(\rho) = -P'(s = 1/\rho) \leq 0 \quad \text{and} \quad \rho^2 W,_{\rho}(\rho) = -V'(s = 1/\rho) \leq 0. \tag{1.22}$$

In the parlance of continuum mechanic the system (1.1) and (1.2) is the Lagrangian description of the traffic and (1.19) and (1.20) the Eulerian description. Here, we shall work with the Lagrangian description.

We start with a number of observations about solutions of (1.1) and (1.2). The first is that for any number $s_{eq} > L$ that the functions

$$(s(m,t), u(m,t)) \equiv (s_{eq},\ V(s_{eq})),\ -\infty < m < \infty \tag{1.23}$$

are constant solutions of (1.1) and (1.2).

**One of the issues before us is under what conditions are these solutions stable and more importantly what happens if they are unstable.** For definiteness we shall confine our attention to the ring-road scenario; that is we shall assume that all solutions satisfy the periodic boundary conditions:

$$(s,u)(m+M,t) = (s,u)(m,t),\ -\infty < m < \infty. \tag{1.24}$$

Here, $M > 0$ represents the number of cars on our ring-road. The conservation structure of (1.1) also guarantees that for all $t > 0$

$$\int_0^M s(m,t)dm \equiv \int_0^M s(m,0)dm \stackrel{def}{=} \ell \tag{1.25}$$

and $\ell$ represents the length of the ring-road.

For the discrete system (1.11) and (1.12) (or equivalently (1.14) and (1.15)) we impose the discrete analog of (1.24) and (1.25), namely the conditions that

$$(s_{m+M}, u_{m+M})(t) = (s_m, u_m)(t) \tag{1.26}$$

$$\sum_{j=0}^{M} s_j(t) = \sum_{j=0}^{M} s_j(0) = \ell. \tag{1.27}$$

The latter condition implies that $x_{m+M}(t) = x_m(t) + \ell$.

In an early paper on this subject I proved the following Theorem for solutions of (1.1) - (1.4) satisfying the boundary conditions (1.24) and (1.25) which assume the initial conditions

$$s(m, 0^+) = s_0(m) \text{ and } u(m, 0^+) = u_0(m), -\infty < m < \infty. \tag{1.28}$$

**Theorem 1.** Suppose $s \to P(s)$ satisfies

$$P(L) = 0 \text{ and } 0 < P'(s),\ \ P''(s) < 0,\ \text{ and } P(s) > V(s) \text{ for } s > L. \tag{1.29}$$

Suppose further that at $t = 0$ the initial data satisfies

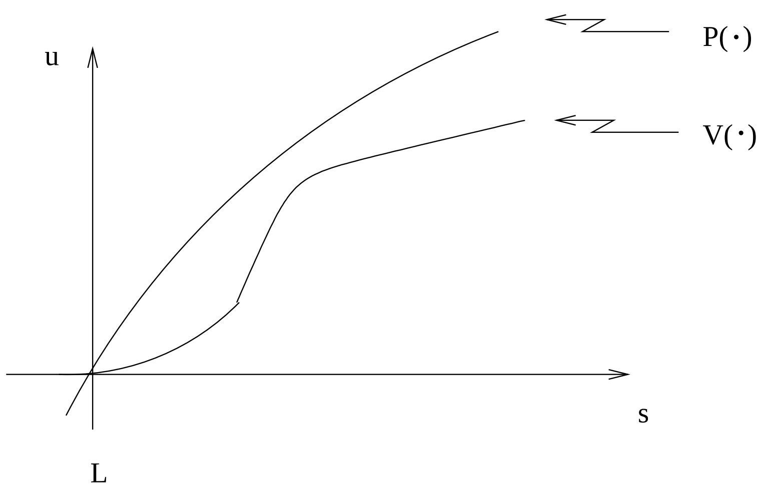
$$L \leq s_0(m) \text{ and } 0 \leq u_0(m) \leq P(s_0(m)) \tag{1.30}$$

for all $m$. Then, for any $t > 0$ the same inequalities hold; that is for all $m$

$$L \leq s(m,t) \text{ and } 0 \leq u(m,t) \leq P(s(m,t)). \tag{1.31}$$

Moreover, the analogus result is true for solutions of the discrete system (1.11) and (1.12). □

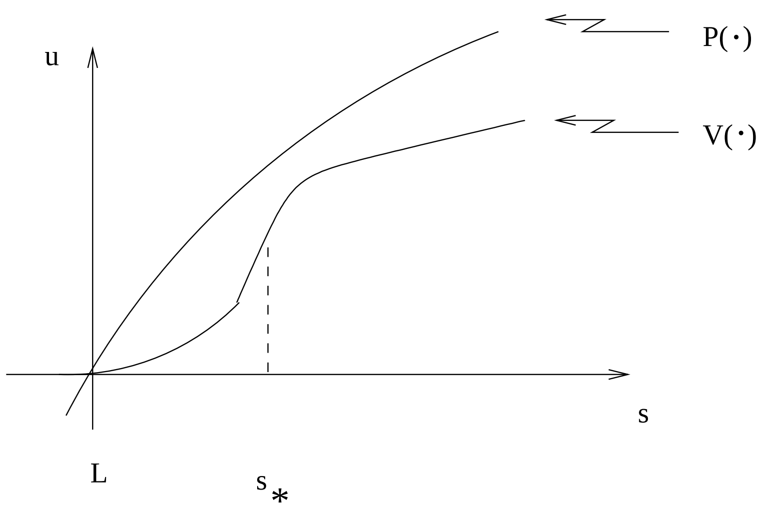
These a-priori estimates form the basis for establishing an existence theorem for the system (1.1) and (1.2).

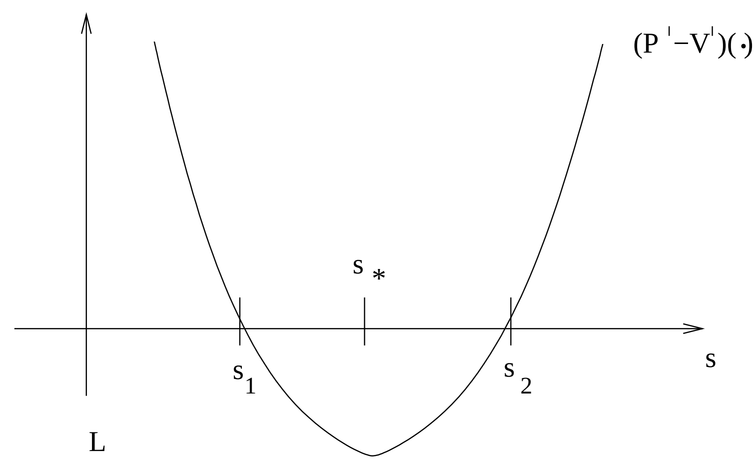


In the remainder of this talk I shall limit myself to the analysis of (1.1)-(1.2) when $P(\cdot)$ and $V(\cdot)$ satisfy (1.29) and (1.5). I shall also assume that $V'(\cdot)$ has an isolated maximum at $s_* > L$, that

$$V''(s) > 0, \quad L \leq s < s_* \quad \text{and } V''(s) < 0, \quad s_* < s < \infty, \tag{1.32}$$

that the difference $(P' - V')(\cdot)$ has two isolated zeros at points $s_1$ and $s_2$ satisfying $L < s_1 < s_* < s_2 < \infty$, and finally that $(P' - V')(\cdot) > 0$ on $(L, s_1) \cup (s_2, \infty)$.



I shall give a simple argument that shows that if the equilibrium solution for $s_{eq}$ is in $(s_1, s_2)$, the constant solution is unstable. This latter result will be established by using a Chapman-Enskog expansion of the solutions of (1.1) and (1.2). I'll also show some numerical simulations. I shall limit myself to the case where

$$P(s) = \lambda(1 - L/s) \quad , \quad L \leq s \tag{1.33}$$

and $V(\cdot)$ given by

$$V(s) = \frac{v_\infty \left(\tanh\left(\frac{s-rL}{\delta}\right) + \tanh\left(\frac{(r-1)L}{\delta}\right)\right)}{\left(1 + \tanh\left(\frac{(r-1)L}{\delta}\right)\right)} \tag{1.34}$$

when

$$r > 1,\ v_\infty > 0,\ \text{and } \delta > 0. \tag{1.35}$$

I shall demonstrate that for nonconstant initial data taking on values in the unstable interval $(s_1, s_2)$ solutions converge to traveling waves. These simulations will be run on the follow-the-leader model (1.11)-(1.12). Time permitting I shall show how to construct the large amplitude periodic traveling wave solutions to (1.1)-(1.2) reminiscent of the waves seen in congested traffic.

## 2 Stability of the Equilibrium Solution (1.23)

Here we seek a simple criterion indicating whether the equilibrium solution defined in (1.23) is stable or not. For simplicity we assume that the relaxation time $0 < \epsilon$ is small. We focus on obtaining a solution of (1.6) and (1.7) where the $\alpha$ component of the solution is of the form

$$\alpha \sim \alpha_0 + \epsilon\alpha_1 + \ldots + \epsilon^n\alpha_n + \ldots \tag{2.1}$$

and each of the $\alpha_i$'s is independent of $\epsilon$ and, moreover, is a functional $s$ and its spatial derivatives.

Instead of examining the stability of the solution $(s_{eq}, V(s_{eq}))$ of the original system we shall examine whether $s_{eq}$ is a stable solution of

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial m}\left(P(s) + \sum_{i=0}^{\infty} \epsilon^i \alpha_i\right). \tag{2.2}$$

Insertion of the ansatz (2.1) into (1.7) yields

$$\alpha_0 = V(s) - P(s) \text{ and } \alpha_i = -\frac{\partial \alpha_{i-1}}{\partial t} \tag{2.3}$$

which is not exactly a solution of the desired type. But, if we exploit $(2.3)_1$ and (2.2) we find that

$$\frac{\partial s}{\partial t} = V'(s)\frac{\partial s}{\partial m} + 0(\epsilon) \tag{2.4}$$

and from this latter relation we readily obtain

$$\alpha_1 = V'(s)(P'(s) - V'(s))\frac{\partial s}{\partial m}. \tag{2.5}$$

If desired, the remaining $\alpha_i$'s may be obtained from (2.3) by exploiting the fact that (2.4) implies

$$\frac{\partial^{p+1} s}{\partial t (\partial m)^p} = \frac{\partial^p}{(\partial m)^p} \left( V'(s)(P'(s) - V'(s)) \frac{\partial s}{\partial m} \right) + 0(\epsilon). \tag{2.6}$$

Rather than carry through the infinite process of determining all of the $\alpha_i$'s we shall truncate the series at order 1, that is insist that

$$\alpha = V(s) - P(s) + \epsilon V'(s)(P'(s) - V'(s)) \frac{\partial s}{\partial m} \tag{2.7}$$

and examine whether or not $s \equiv s_{eq}$ is a stable solution of

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial m} \left( V(s) + \epsilon V'(s) \left( P'(s) - V'(s) \right) \frac{\partial s}{\partial m} \right). \tag{2.8}$$

This latter equation has a strong maximum principle so long as the initial data for $s$ satisfies either

$$L \leq s(m,0) < s_1 \quad \text{for all } m \tag{2.9}$$

or

$$s_2 \leq s(m,0) < \infty \quad \text{for all } m \tag{2.10}$$

because in either of these cases the diffusion coefficient, $V'(s)(P'(s) - V'(s))$, is positive. On the other hand, when $s_1 < s < s_2$, the diffusion coefficient is negative and this yields explosive growth of the solution. Thus, if $s_{eq} \in (s_1, s_2)$, the interval where $(P' - V')(s) < 0$, the constant solution $s \equiv s_{eq}$ is an unstable solution of (2.8). It is easily checked that this same conclusion is valid as regards the linear instability of (1.23) for the full system (1.6) and (1.7).

## 3 Simulations

In this section we present some numerical simulations for the follow-the-leader system (1.11) and (1.12). We choose non-constant initial data which lies wholly within the unstable interval. Throughout we shall work with

$$P(s) = \lambda\left(1 - \frac{L}{s}\right) \quad , \quad L \leq s \tag{3.1}$$

and

$$V(s) = v_\infty \frac{\left(\tanh\left(\frac{s-rL}{\delta}\right) + \tanh\left(\frac{(r-1)L}{\delta}\right)\right)}{\left(1 + \tanh\left(\frac{(r-1)L}{\delta}\right)\right)}. \tag{3.2}$$

The specific parameters used were

$$L = 15 \text{ feet} \tag{3.3}$$

$$\lambda = 150 \text{ feet/sec} \;= 102.2727\ldots \;\text{ mph}, \tag{3.4}$$

$$v_\infty = 100 \text{ feet/sec} \;= 68.1818\ldots \;\text{ mph}, \tag{3.5}$$

$$\delta = 15 \text{ feet} \tag{3.6}$$

and

$$r = 3. \tag{3.7}$$

For initial data, we choose three sets of data

$$x_m^{(k)}(0) = 45m + 4\sum_{j=0}^{m-1} \sin\left(\frac{kj\pi}{200}\right) \tag{3.8}$$

and

$$u_m^{(k)}(0) = 35 \text{ feet/sec} \tag{3.9}$$

for $m = 0, \pm 1, \pm 2, \ldots$ and $k = 1, 2$, and 3. The observation that

$$x_{m+400}^{(k)}(0) = x_m^{(k)}(0) + 18000 \tag{3.10}$$

implies that we may interpret the data as initial data for a ring-road with 400 cars which is of length 18000 feet.

For our choice of parameter values the unstable region for $(P' - V')(\cdot)$ is the interval $33.59625\ldots < s < 69.8215$ and our data has initial car spacings

$$s_m^{(k)}(0) = x_{m+1}^{(k)}(0) - x_m^{(k)}(0) \tag{3.11}$$

which lie in that interval. A graph of $s \rightarrow (P' - V')(s)$ is shown in the fourth panel of Figures 1-3. Simulations were run with relaxation times

$$\epsilon = 1,\ 5,\ \text{ and } 10. \tag{3.12}$$

We show the spatially periodic solutions at time $t = 1$ hour when $\epsilon = 10$ seconds. Figures 1, 2 and 3 correspond to the initial data indexed by $k = 1, 2$, and 3 respectively. The solution indexed by each particular $k$ has $k$ discontinuities per period after one hour. Run over a longer period, they all revert to a solution with one discontinuity per period.

The first two frames in each figure are self-explanatory. In the third frame of each figure we plot the curve $m \longrightarrow (s_m = x_{m+1} - x_m, u_m)$. This curve is shown in green. The blue curve is the equilibrium curve $s \longrightarrow (s, V(s))$ and the black curve is a suitably normalized image of $P(\cdot)$. The black dot -o- is the image of $(s_1, u_1)$.

## 4 Large amplitude periodic traveling waves

Based on the computational evidence presented in the last section we are led to look for traveling wave solutions to the full system (1.6) and (1.7). These will be solutions which are functions of

$$\xi = m + ct \quad , \quad c > 0 \tag{4.1}$$

which are periodic in $\xi$ with period $M$, the number of cars on the ring-road. The conversation structure of (1.6) implies that the $s(\cdot)$ component of the solution satisfies

$$\int_0^M s(\xi)d\xi = l \tag{4.2}$$

where $l$ is the length of the ring-road.

Insertion of the ansatz (4.1) into (1.6) implies that $u(\cdot)$ and $s(\cdot)$ satisfy

$$P(s) + \alpha = u(\xi) = u_{\#} + c(s(\xi) - s_{\#}) \tag{4.3}$$

and we insist that

$$u_{\#} = V(s_{\#}) \quad \text{and} \quad s(0) = s_{\#} \in (s_1, s_2). \tag{4.4}$$

The relations (4.3) and (4.4) further imply (1.7) reduces to that

$$\epsilon c \left(c - P'(s)\right) \frac{ds}{d\xi} = \left(V(s) - V(s_{\#}) - c(s - s_{\#})\right). \tag{4.5}$$

We seek a solution to (4.4) and (4.5) which is increasing on $-m_a < \xi < M_a$ where $-m_a < 0 < M_a$. For speeds $0 < c < V'(s_{\#})$, we see that the right hand side of (4.5) satisfies

$$\text{sign } \left(V(s) - V(s_{\#}) - c(s - s_{\#})\right) = \text{sign } (s - s_{\#}) \tag{4.6}$$

for $|s - s_{\#}|$ small enough and thus to obtain an increasing solution to (4.4) and (4.5) on some interval containing $\xi = 0$ in its interior we are compelled to choose

$$c = P'(s_{\#}). \tag{4.7}$$

This choice of $c$, together with the hypothesis that $P''(\cdot) < 0$, guarantees that

$$\text{sign } \left(P'(s_{\#}) - P'(s)\right) = \text{ sign } (s - s_{\#}) \tag{4.8}$$

and thus, with this choice of $c$, we are guaranteed a solution of (4.4) and (4.5) defined in some interval $-\tilde{m}_a < \xi < \tilde{M}_a$ where $-\tilde{m}_a < 0 < \tilde{M}_a$. Moreover, this solution satisfies

$$\frac{ds}{d\xi}(0) = \frac{-\left(V'(s_{\#}) - P'(s_{\#})\right)}{\epsilon P'(s_{\#}) P''(s_{\#})} > 0 \tag{4.9}$$

for $s_1 < s_{\#} < s_2$.

We shall now refine the observations of the preceding paragraphs. If

$$V(L) - V(s_2) - P'(s_2)(L - s_2) > 0 \tag{4.10}$$

we let $\bar{s}$ in $(s_1, s_2)$ be the unique solution of

$$V(L) - V(\bar{s}) - P'(\bar{s})(L - \bar{s}) = 0 \tag{4.11}$$

whereas, if

$$V(L) - V(s_2) - P'(s_2)(L - s_2) \leq 0 \tag{4.12}$$

we let

$$\bar{s} = s_2. \tag{4.13}$$

In either case, for any $s_\#$ in $(s_1, \bar{s})$ we let $L < s_-(s_\#) < s_\# < s_+(s_\#)$ be the other two solutions of

$$V(s_\pm) - V(s_\#) - P'(s_*)(s_\pm - s_\#) = 0. \tag{4.14}$$

We of course have

$$V(s) - V(s_\#) - P'(s_\#)(s - s_\#) < 0 \qquad s_-(s_\#) < s < s_\# \tag{4.15}$$

and

$$V(s) - V(s_\#) - P'(s_\#)(s - s_\#) > 0 \quad , \quad s_\# < s < s_+(s_\#). \tag{4.16}$$

For any $s_a$ in $(s_-(s_\#), s_\#)$ we now let $S(s_a) > s_\#$ be the unique solution of

$$\frac{P(S(s_a)) - P(s_a)}{S(s_a) - s_a} = P'(s_\#) \tag{4.17}$$

and note that

$$\frac{dS(s_a)}{ds_a} = \frac{(P'(s_\#) - P'(s_a))}{(P'(s_\#) - P'\left(S(s_a)\right))} < 0. \tag{4.18}$$

We also let $\underline{s}(s_\#)$ be the smallest value of $s_a \geq s_-(s_\#)$ such that $S(s_a) \leq s_+(s_\#)$ and for any $s_a$ in $(\underline{s}(s_\#), s_\#)$ we let

$$-m_a = \epsilon P'(s_\#) \int_{s_a}^{s_\#} \frac{(P'(r) - P'(s_\#))\, dr}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))} < 0 \tag{4.19}$$

and

$$M_a = \epsilon P'(s_\#) \int_{s_\#}^{S(s_a)} \frac{(P'(s_\#) - P'(r))dr}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))} > 0. \tag{4.20}$$

I note that one of the integrals (4.19) or (4.20) or both diverge as $s_a \to \underline{s}(s_\#)^+$. For any $\xi$ in $(-m_a, M_a)$, the solution to (4.4) and (4.5) is given by the quadrature formula

$$\epsilon P'(s_\#) \int_{s_\#}^{s(\xi)} \frac{(P'(s_\#) - P'(r))\, dr}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))} = \xi \tag{4.21}$$

and the solution is extended to $(-\infty, \infty)$ by insisting that the periodicity condition

$$s\left(\xi \pm n(m_a + M_a)\right) = s(\xi), \quad n = 0, 1, \ldots \tag{4.22}$$

holds. As constructed, the solution has jump discontinuities as the points

$M_a \pm n(m_a + M_a)$, $n = 0, 1, \ldots$, and (4.17), (4.19), and (4.20) guarantee that the Rankine Hugoinot condition for (1.6) and (1.7) holds across these discontinuities. The Lax entropy condition that $s^- (M_a \pm n(m_a + M_a)) > s^+ (M_a \pm n(m_a + M_a))$ is also guaranteed since

$$s^-(M_a \pm n(m_a + M_a)) = S(s_a) > s_a = s^+(M_a \pm n(m_a + M_a)). \tag{4.23}$$

What remains to be shown is that for integers $k = 1, 2, \ldots$ we can choose $s_a$ in $(\underline{s}(s_\#), s_\#)$ and $s_\#$ in $(s_1, \overline{s})$ so that

$$k(m_a + M_a) = M \tag{4.24}$$

and

$$\int_0^M s(\xi) d\xi = l. \tag{4.25}$$

The integer $k$ represents the number of increasing segments per period.

In [10] the author gave an exhaustive analysis of the system (4.24) and (4.25). These equations are equivalent to showing the existence of a pair

$$s_a \in (\underline{s}(s_\#), s_\#) \text{ and } s_\# \in (s_1, \overline{s})$$

satisfying

$$k\epsilon P'(s_\#) \int_{s_a}^{S(s_a)} \frac{(P'(s_\#) - P'(r))\, dr}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))} = M. \tag{4.26}$$

$$k\epsilon P'(s_\#) \int_{s_a}^{S(s_a)} \frac{(P'(s_\#) - P'(r))\, r dr}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))} = l. \tag{4.27}$$

For the simulations run in section 3 we had $M = 400$ and $l = 18,000$ and $s_\#$ was $0(40)$. The interested reader may consult [10] for the details.
Once again

$$\frac{P(S(s_a)) - P(s_a)}{S(s_a) - s_a} = P'(s_\#).$$

## 5 Concluding Remarks

We note in passing that many of the results obtained for the traffic system (1.1) and (1.2) obtain for any one-dimensional continuum system of the form:

$$\frac{\partial s}{\partial t} - \frac{\partial u}{\partial m} = 0 \tag{5.1}$$

and

$$\frac{\partial u}{\partial t} - \frac{\partial \sigma(s)}{\partial m} = (V(s) - u)/\epsilon \tag{5.2}$$

provided $s \to \sigma(s)$ and $s \to V(s)$ satisfy

$$\sigma'(s) > 0 \text{ and } \sigma''(s) < 0 \quad , \quad 0 < s, \tag{5.3}$$

$$V(s) > 0 \text{ and } V'(s) < 0 \quad , \quad 0 < s, \tag{5.4}$$

and there is a number $0 < s_2$ so that

$$\text{sign } (\sigma'(s) - (V')^2(s)) = \text{ sign } (s - s_2). \tag{5.5}$$

One typically further assumes that

$$\lim_{s \to 0^+} \sigma(s) = -\infty \text{ and } \lim_{s \to \infty} V(s) = \infty. \tag{5.6}$$

In particular, one finds that the equilibrium solutions

$$(s,u)(m,t) \equiv (s_{eq}, V(s_{eq})), \ -\infty < m < \infty$$

are unstable if $0 < s_{eq} < s_2$ and linearly stable if $s_2 < s_{eq} < \infty$. These systems also support large amplitude periodic traveling waves of the type seen in sections 3 and 4. These waves are functions of $\xi = m + ct$ and satisfy

$$V'(s_\#) < c = -(\sigma'(s_\#))^{1/2} < 0, \tag{5.7}$$

$$(c^2 - \sigma'(s))\frac{ds}{d\xi} = V(s) - V(s_\#) - c(s - s_\#), \tag{5.8}$$

and

$$s(0) = s_\#. \tag{5.9}$$

The smooth portions of these solutions are monotone decreasing on $a < \xi < b$ and the numbers

$$s_a = \lim_{\xi \to a^+ < 0} s(\xi) \text{ and } s_b = \lim_{\xi \to b^- > 0} s(\xi)$$

satisfy

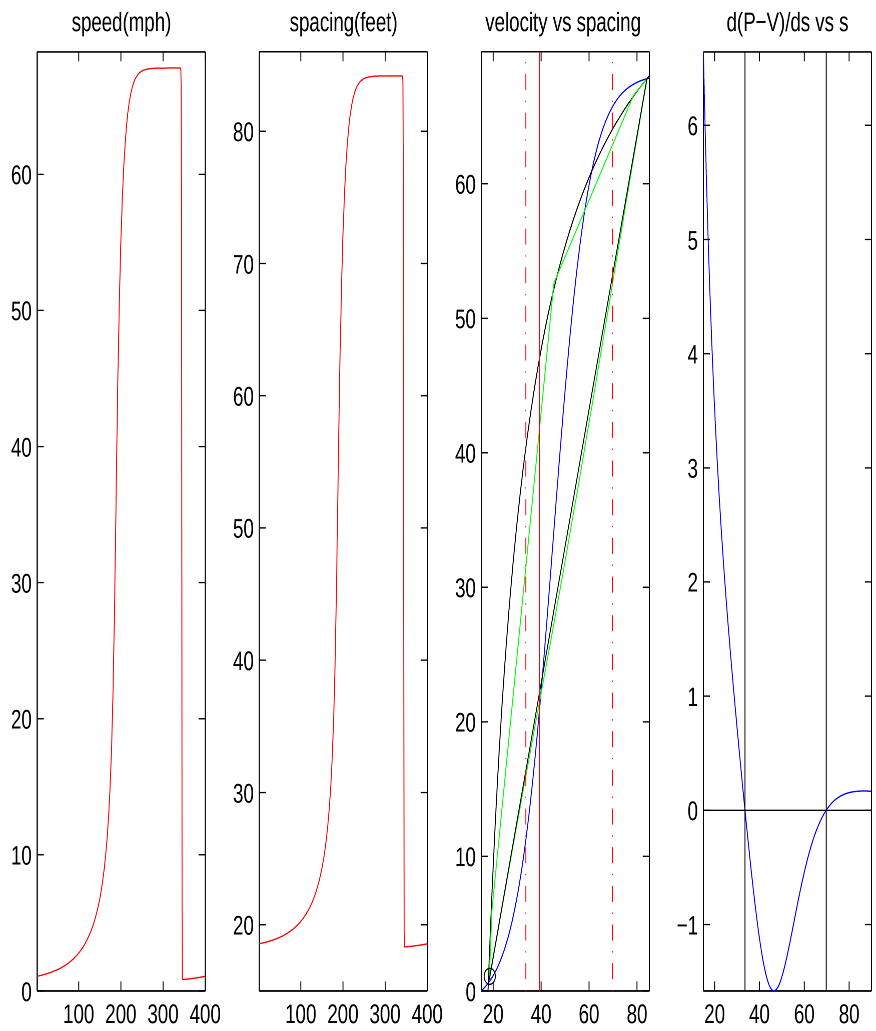
$$c^2 = \sigma'(s_\#) = \frac{\sigma(s_a) - \sigma(s_b)}{s_a - s_b}$$

which is the Rankine-Hugoniot condition for the system. On the interval $a < \xi < b$, $u$ is given by

At each of the jump points these solutions satisfy the Lax entropy condition for the system, namely the condition that

$$\lim_{\xi \to (b \pm n(b-a))^-} u(\xi) \quad > \quad \lim_{\xi \to (b \pm n(b-a))^+} u(\xi) \quad .$$

Roll waves are an interesting example of such waves; for details see [12] and [13].


speed(mph)
spacing(feet)
velocity vs spacing
d(P−V)/ds vs s
0
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20
30
40
50
60
100
200
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400
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70
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−1
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40
60
80

**Figure 1**

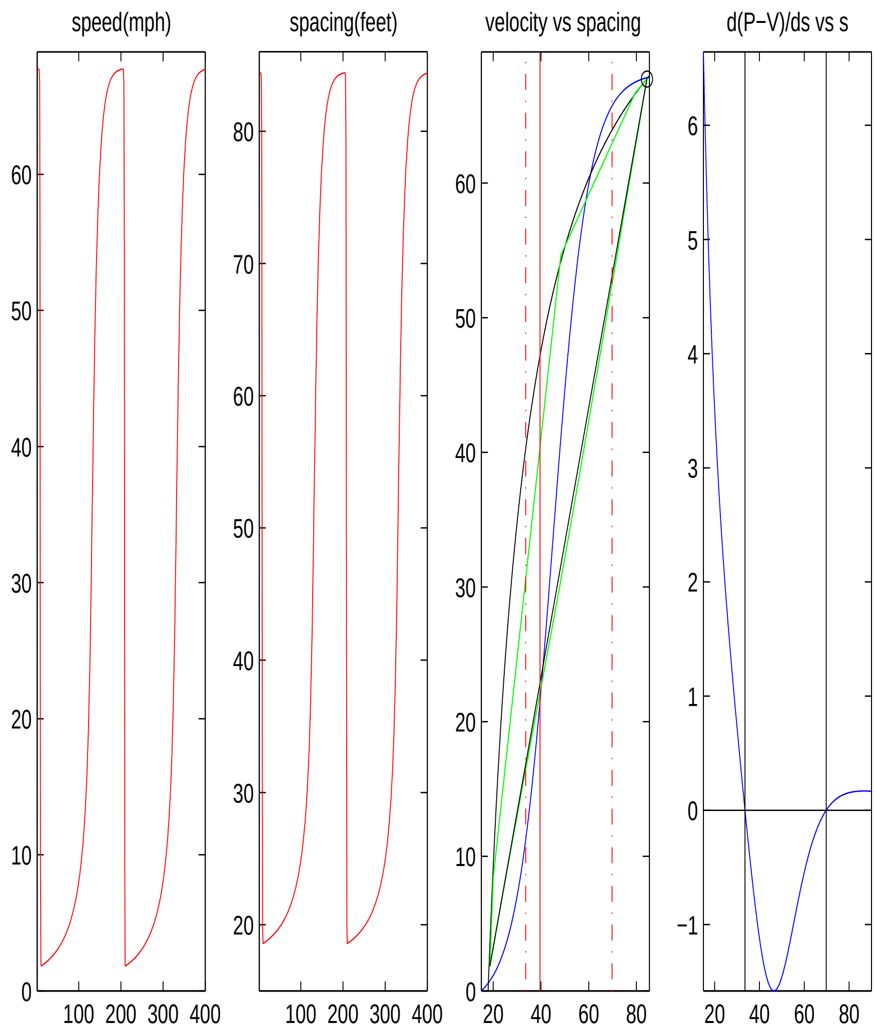
speed(mph)
spacing(feet)
velocity vs spacing
d(P−V)/ds vs s
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**Figure 2**

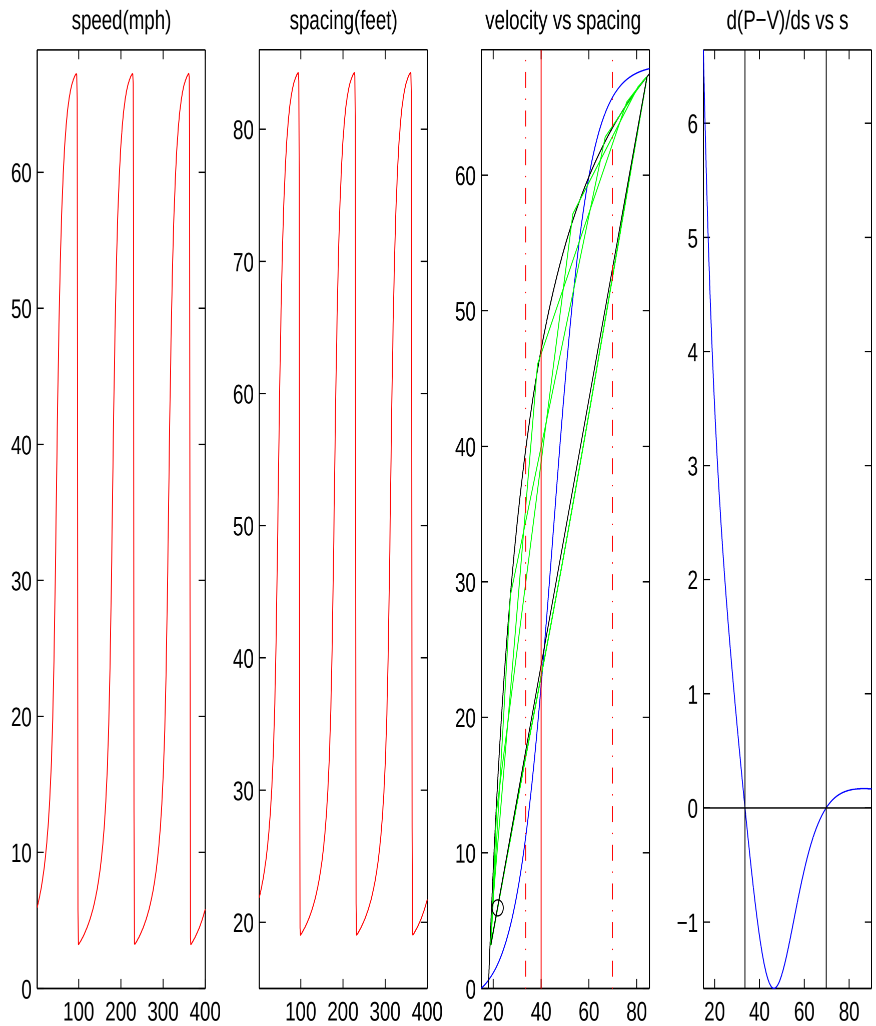
speed(mph)
spacing(feet)
velocity vs spacing
d(P−V)/ds vs s
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**Figure 3**