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Microscopic Models under a Macroscopic Perspective

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Outline:

- General
- Dynamics of the microscopic model (homogeneous case)
- Dynamics of the microscopic model (non-homogeneous case and roadworks)
- Macroscopic view
- Fundamental diagrams
- Future

Basic concept: Take a very simple microscopic model (Bando), study the full dynamics, take a macroscopic view on the results.

Microscopic Bando model on a circular road (scaled)

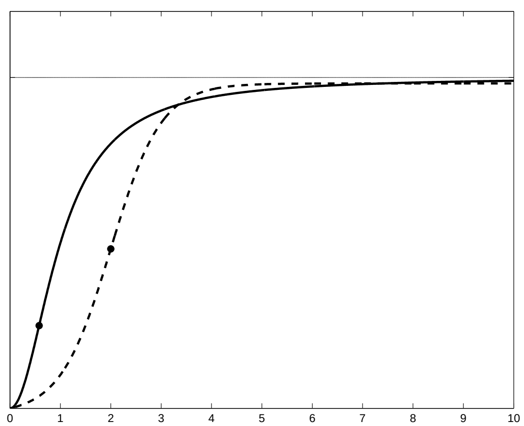
N **cars on a circular road of lenght** L:

Behaviour: x_j position of the j-th car

$$\ddot{x}_j(t) = -\left\{V\left(x_{j+1}(t) - x_j(t)\right) - \dot{x}_j(t)\right\}, \qquad j = 1, ..., N, \quad x_{N+1} = x_1 + L$$

$V = V(x)$ optimal velocity function:

$$V(0) = 0, \qquad V \text{ strictly monoton increasing }, \qquad \lim_{x\to\infty} V(x) = V^{max}$$



System for the headways: $y_j = x_{j+1} - x_j$

$$\begin{aligned} \dot{y}_j &= z_j \\ \dot{z}_j &= -\left\{V(y_{j+1}) - V(y_j) - \dot{z}_j\right\}, \qquad j = 1, \ldots, N, \quad y_{N+1} = y_1 \end{aligned}$$

Additional condition: $\sum_{j=1}^{N} y_j = L$

"quasistationary" solutions: $y_{s;j} = \frac{L}{N}, \quad z_{s;j} = 0, \quad j = 1, \ldots, N.$

Linear stability-analysis around this solution gives for the Eigen-values λ:

$$(\lambda^2 + \lambda + \beta)^N - \beta^N = 0, \qquad \beta = V'(\frac{L}{N})$$

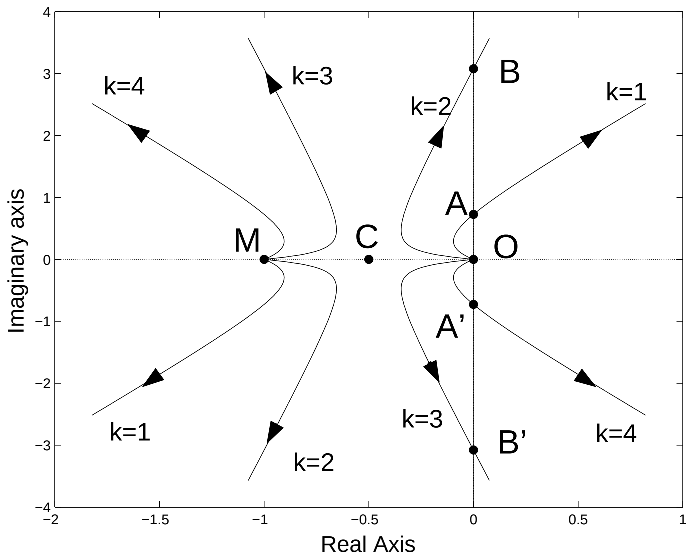
Result (Huijberts ('02)):

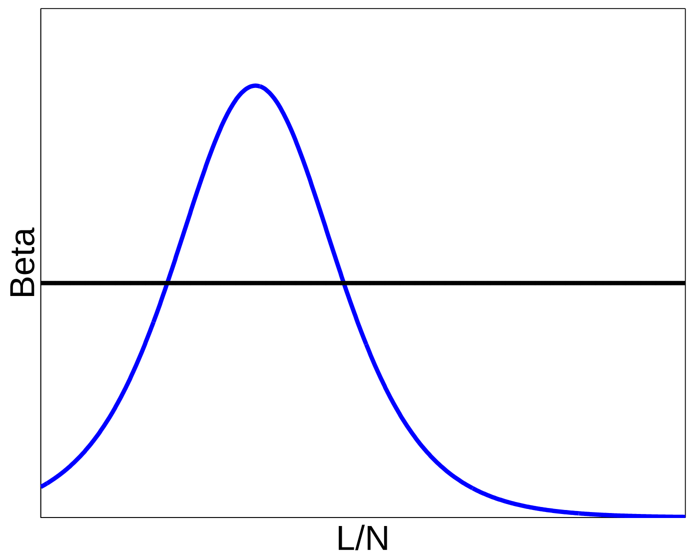
For $\frac{1}{1+\cos\frac{2\pi}{N}} > \beta^{max} = max_x V'(x)$ asymptotic stability

For $\frac{1}{1+\cos\frac{2\pi}{N}} = V'(\frac{L}{N})$ loss of stability

What kind of loss of stability? (I.G., G.Sirito, B. Werner '04):

Eigenvalues as functions of $\beta = V'(\frac{L}{N})$





Bifurcation analysis gives a Hopfbifurcation.

Therefore we have locally periodic solutions.

Are these solutions stable? (i.e. is the bifurcation sub- or super-critical?)

Criterion: Sign of the first Ljapunov-coefficient l

Theorem:

$$l = c^2 \left\{ V''' \left(\frac{L}{N} \right) - \frac{\left(V''(\frac{L}{N}) \right)^2}{V'(\frac{L}{N})} \right\}$$

Conclusion: For the mostly used (Bando et al (95))

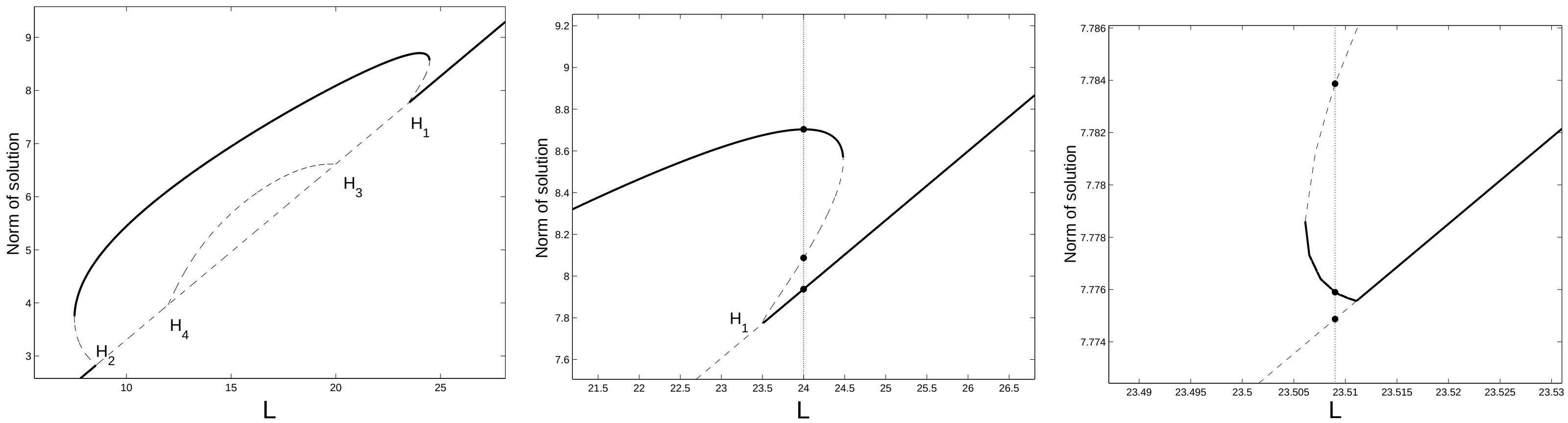
$$V(x) = V^{max} \frac{\tanh(a(x-1)) + \tanh a}{1 + \tanh a}$$

the bifurction is supercritical (i.e. stable periodic orbits).

But: “Similar” functions V give also subcritical bifurcations.

Problem: It seems to be very sensitive with respect to V

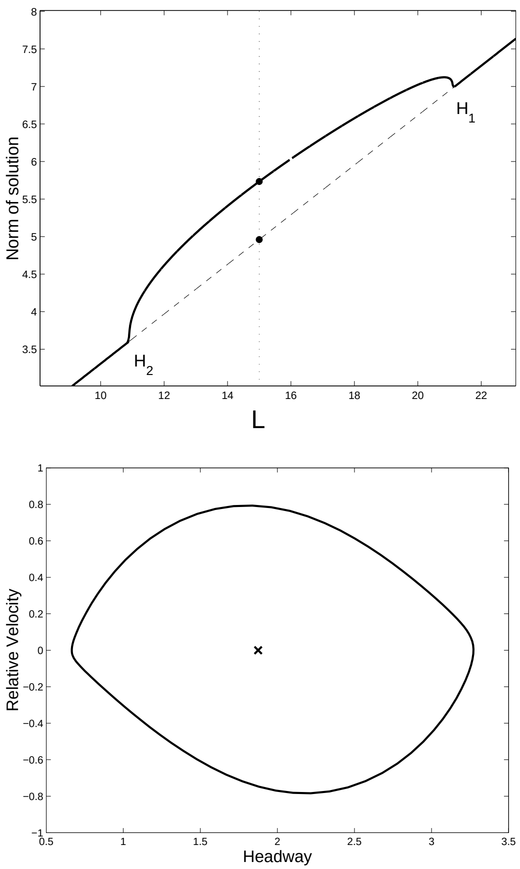
Global bifurcation analysis: numerical tool (AUTO2000)



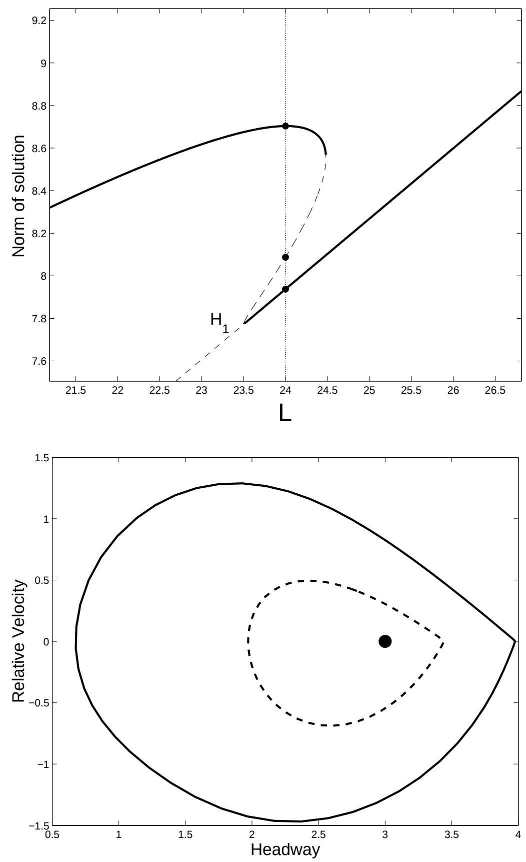
Conclusion: Globally "similar" functions V give similar behaviour. The bifurcation is "macroscopically" subcritical

Conclusion for the application: the critical parameters from the linear analysis are not relevant

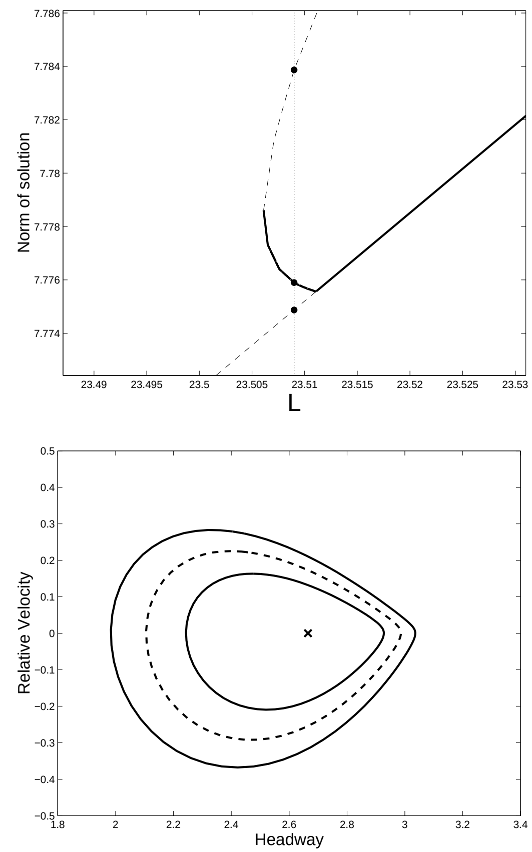
Example 1



Example 2

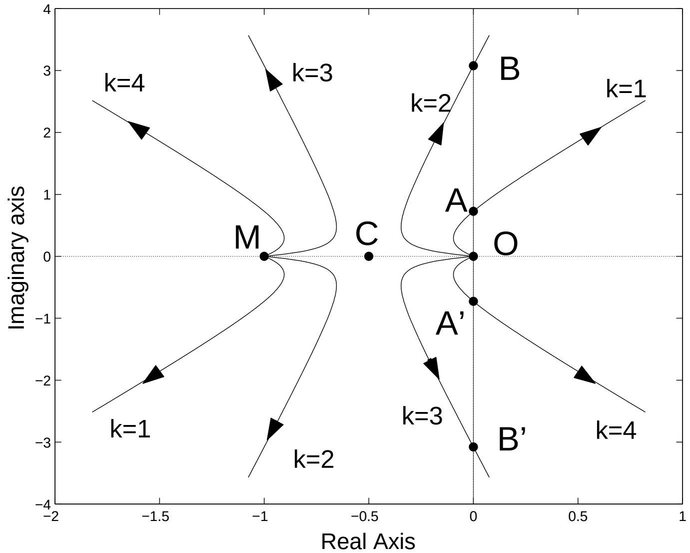


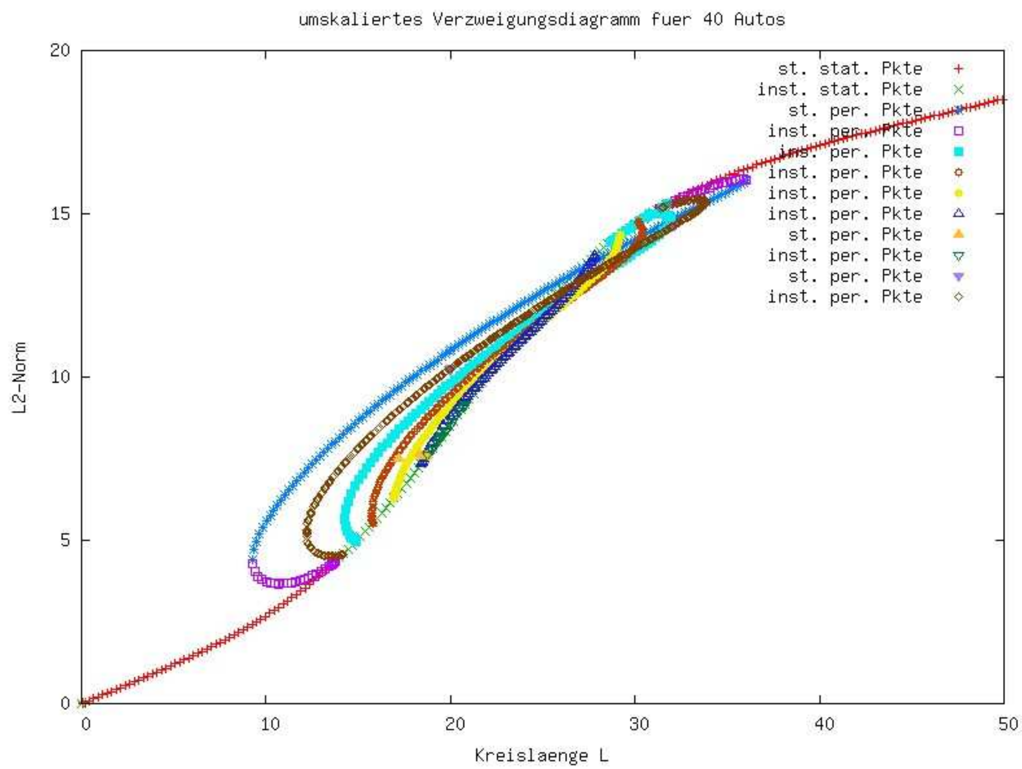
Example 3



More bifurcations:

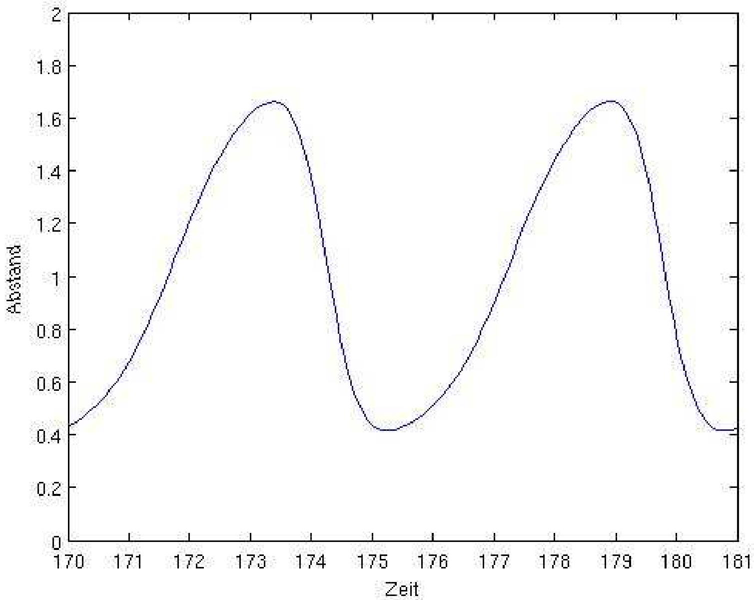
Eigenvalues as functions of $\beta = V'(\frac{L}{N})$

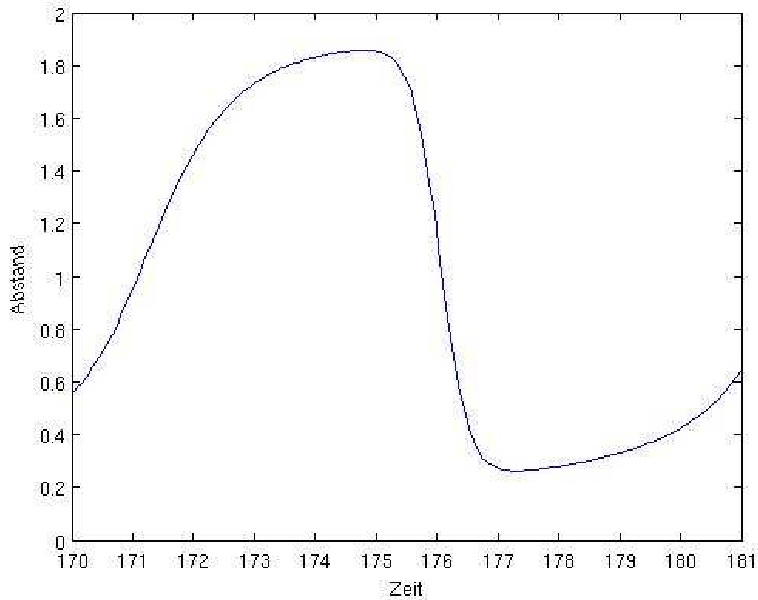




Conclusion: There are many other (weakly unstable) periodic solutions

(J.Greenberg '04,'07) Solutions with many oscillations finally tend to a solution with one oscillation





(G. Oroz, R.E. Wilson, B.Krauskopf '04, '05) Qualitatively the same global bifurcation diagram for the model with delay

Extension to “standart” microscopic model

Every driver is “aggressive” with weight α

$$\ddot{x}_j(t) = -\frac{1-\alpha}{\tau}\left\{V\left(x_{j+1}(t) - x_j(t)\right) - \dot{x}_j(t)\right\} + \alpha\left\{\dot{x}_{j+1}(t) - \dot{x}_j(t)\right\},$$

$j = 1, ..., N, \quad x_{N+1} = x_1 + L$ optimal velocity-function:

loss of stability similar

“Aggressive” drivers stabilize the fraffic flow!

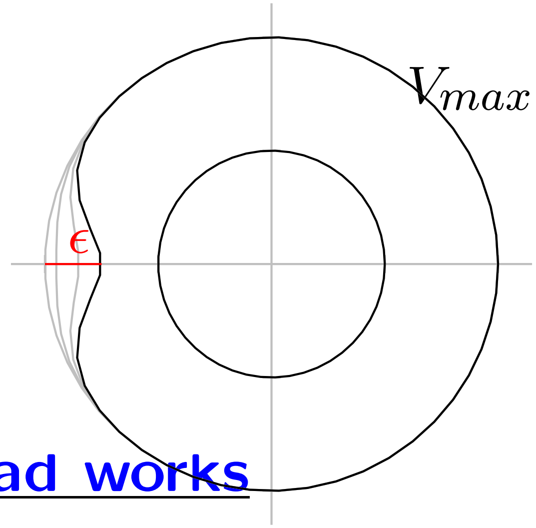
unfortunately also the number of accidents increases!

(Olmos & Munos, Condensed matter 2004)

"DIE ZEIT", 18.6.04

Raser fegen Straßen frei

Dass Bogotá, eine Stadt mit etwa sieben Millionen Einwohnern und mehr als einer Million Autos, nicht im Stau versinkt, soll ausgerechnet der rücksichtslosen Fahrweise der Kolumbianer zu verdanken sein. Wissenschaftler der städtischen Nationaluniversität haben das Fahrverhalten ihrer Landsleute aufgezeichnet und daraus ein Computerprogramm erstellt, das den Verkehrsfluss simuliert. Im Vergleich zu den Autofahrern in New York oder London verhalten sich jene in Bogotá wie Rallyefahrer. Die Folge der aggressiven Manöver: Die Zahl der Verkehrstoten steigt – doch die Gefahr von Staus sinkt.



Road works

Symmetry breaking, the above theory is not easily applicable

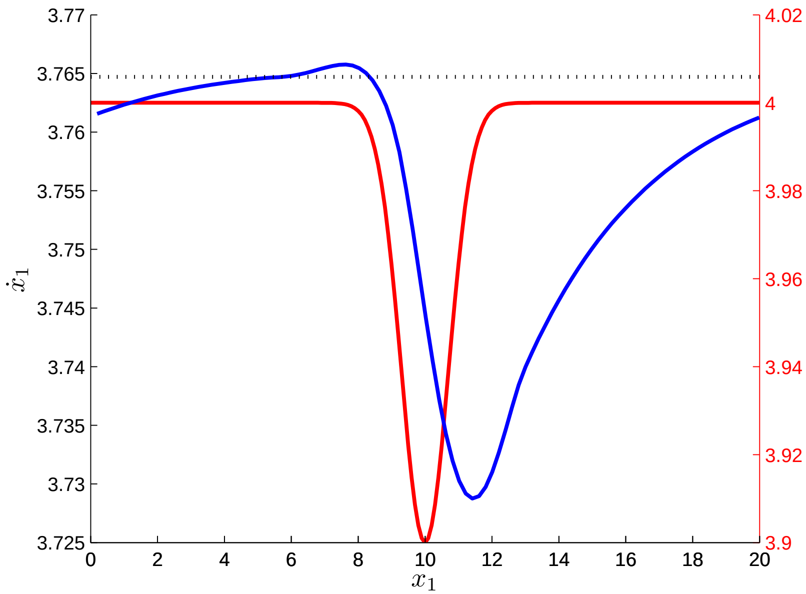
A solution is called ponies on a Merry-Go-Round solution (short POM), if there is a $T \in \mathbb{R}$, such that

(i) $x_i(t+T) = x_i(t) + L \quad (i = 1, \ldots, N)$

(ii) $x_i(t) = x_{i-1}\left(t + \frac{T}{N}\right) \quad (i = 1, \ldots, N)$

hold (Aronson, Golubitsky, Mallet-Paret '91). We call T rotation number and $\frac{T}{N}$ the phase (phase shift).

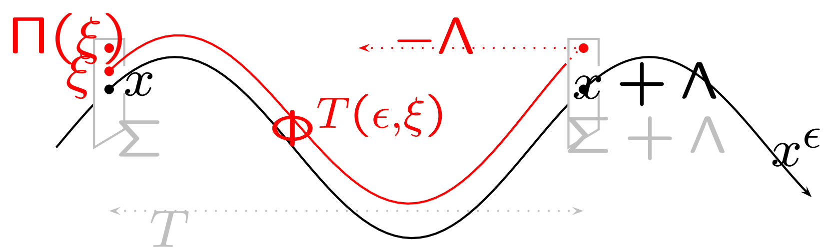
Theorem: The above model has POM solutions for small $\epsilon > 0$.



Velocity of the quasistationary solution (no roadwork) versus roadwork solution (The red line indicates maximum velocity).

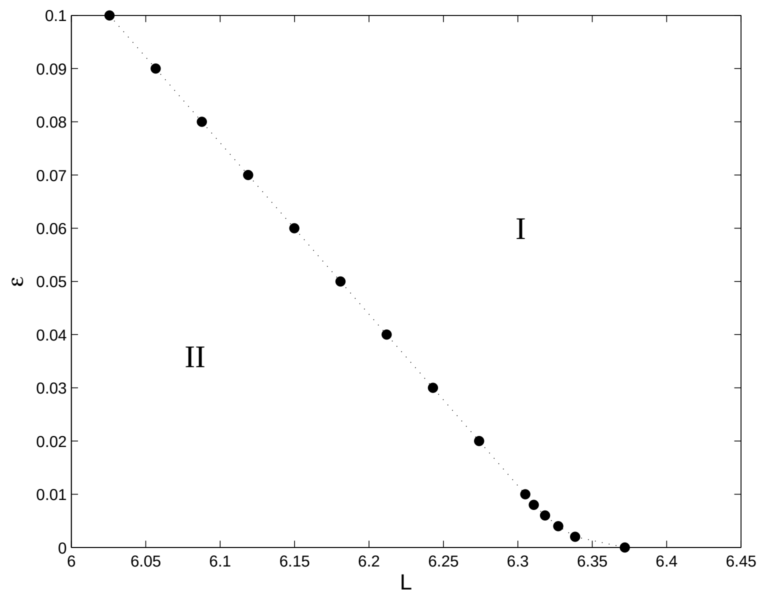
Technique: Poincare maps

$\Pi(\eta) = \Phi_{T(\eta)}(\eta) - \Lambda$, where Φ is the induced flow and Λ reduces the spacial components by L.



Study fixed points of the corresponding Poincare and reduced Poincare maps. Roadworks are (regular) perturbations.

Bifurcation diagram in (L, ϵ)-plane:



A curve of Neimark-Sacker bifurcations in the (L, ϵ)-plane for $N = 5$.

Four different attractors.:

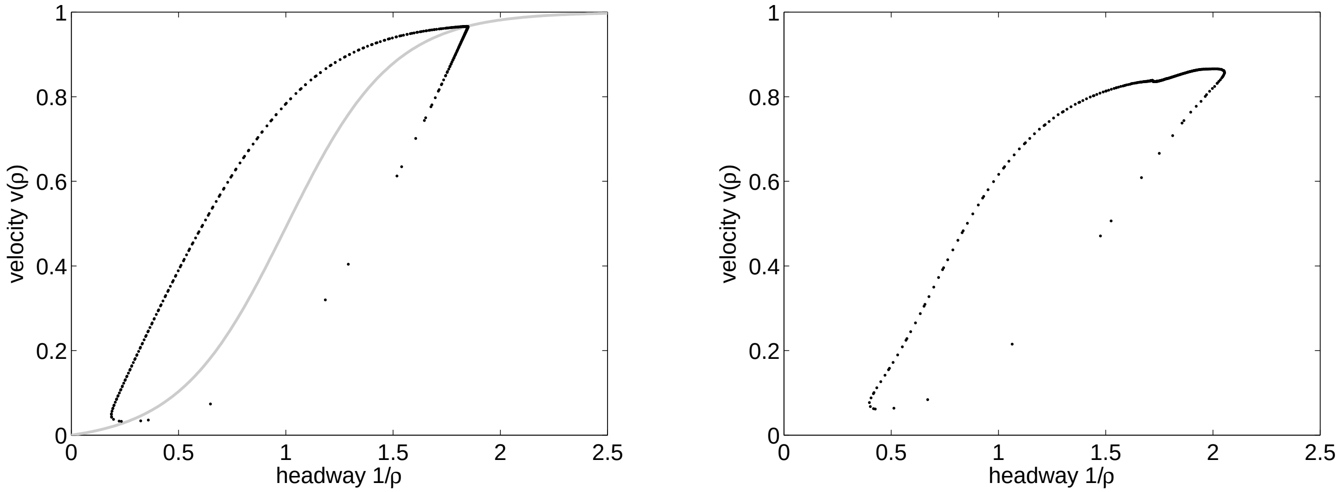
ϵ	I	II
$\epsilon = 0$	trivial POM x^0	Hopf periodic solution
$\epsilon > 0$	POM x^ϵ	quasi-POM

i.e. here

POM's are typically perturbed quasistationary solutions

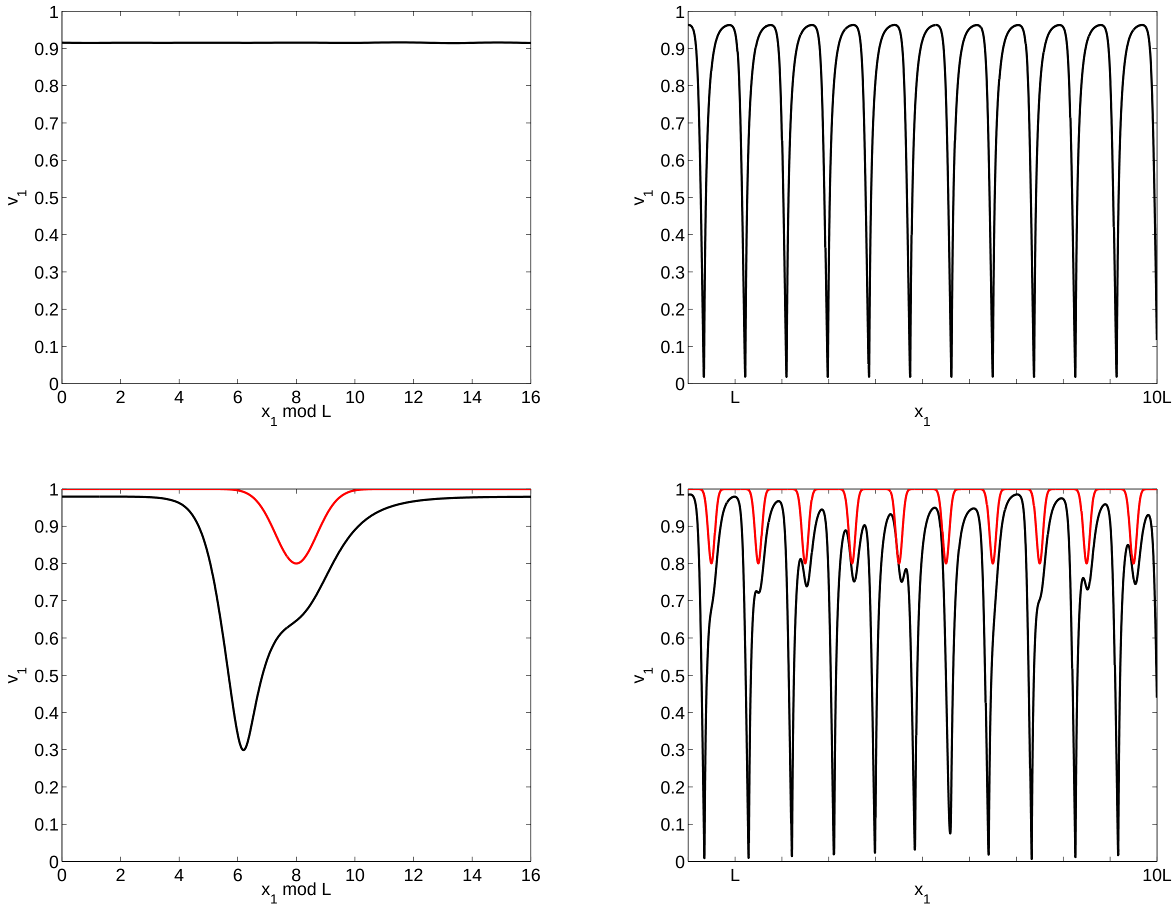
quasi-POM's are perturbed (Hopf) periodic solutions

Invariant curves:



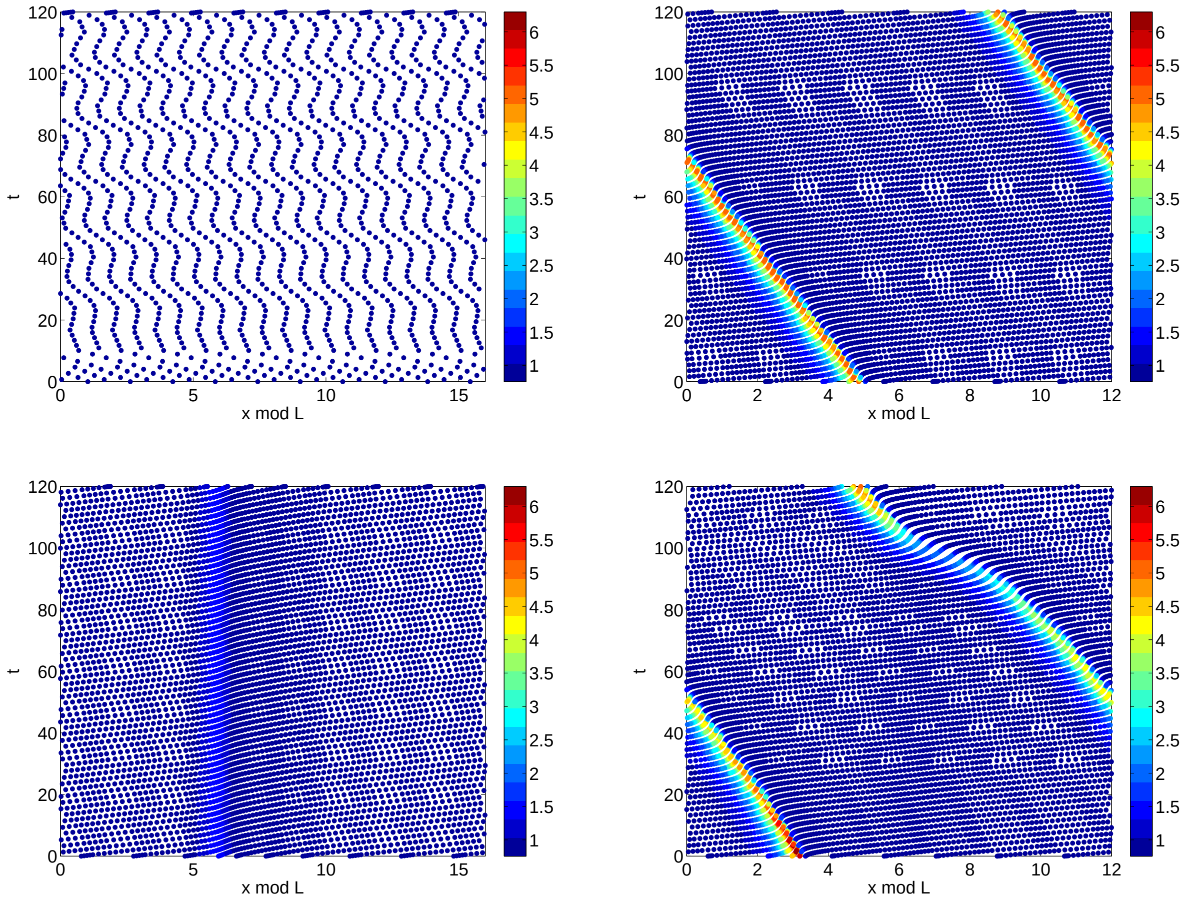
Two closed invariant curves ($\epsilon = 0$ and $\epsilon > 0$) of the reduced Poincaré map π. On the left also the optimal velocity function V_0 is given in gray.

The 4 different scenarios:



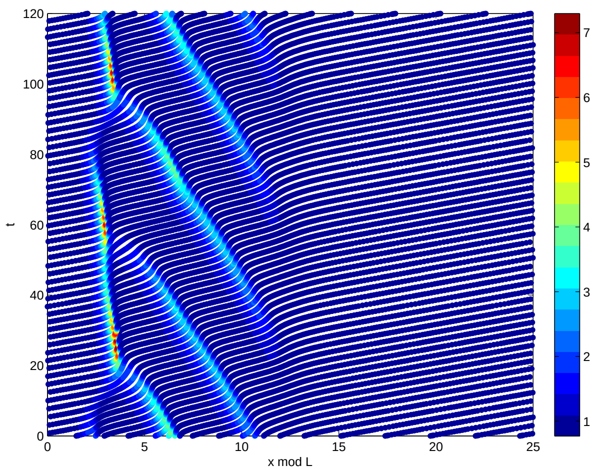
above: no roadworks, below: with roadworks

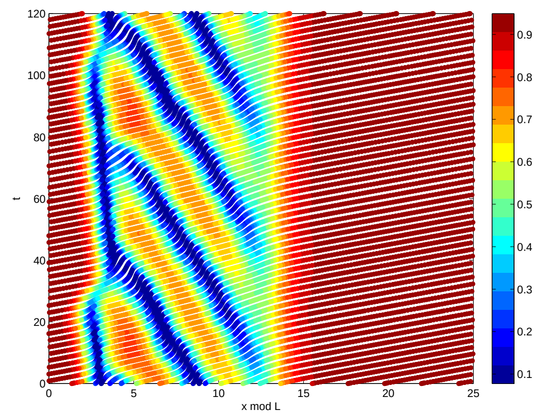
Macroscopic view of the 4 different scenarios:



above: no roadworks, below: with roadworks

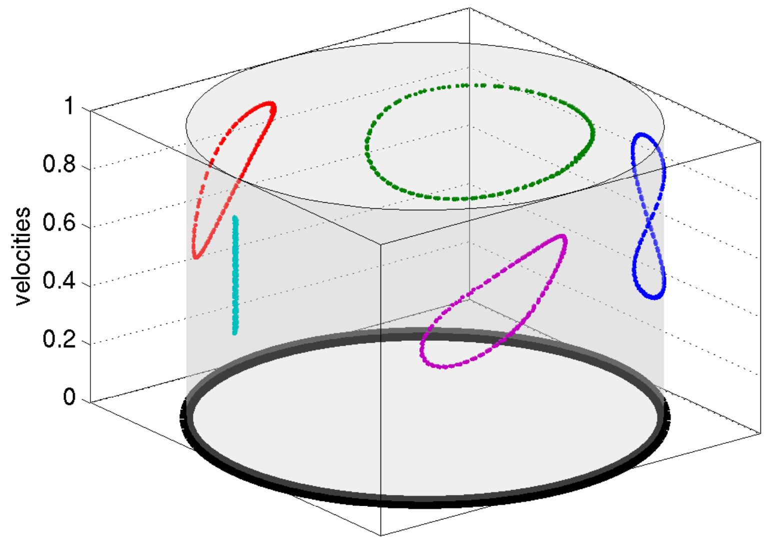
Macroscopic view of density and velocity:





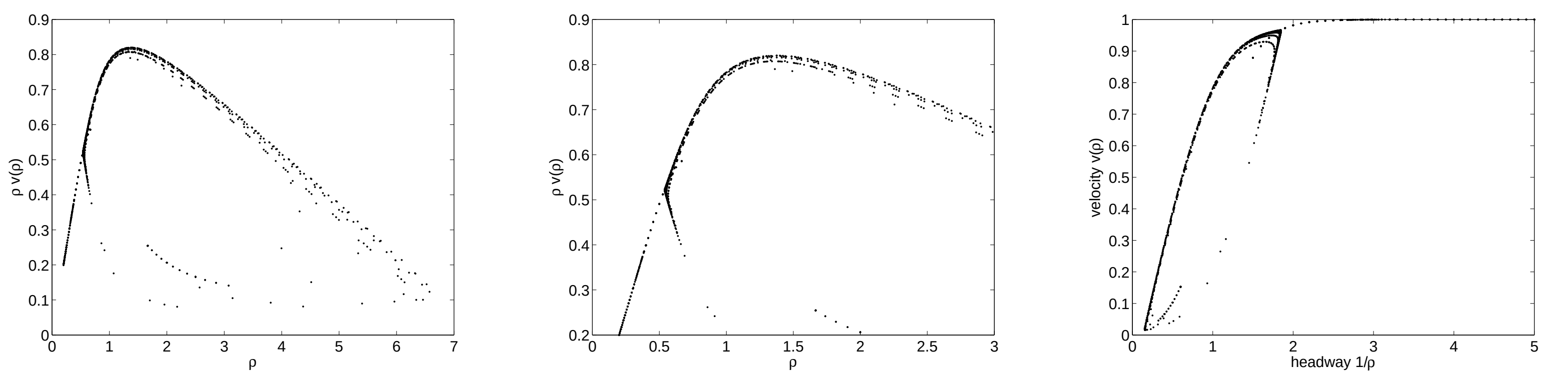
strong road work influence ($\epsilon = 0.32$)

Fundamental diagrams:



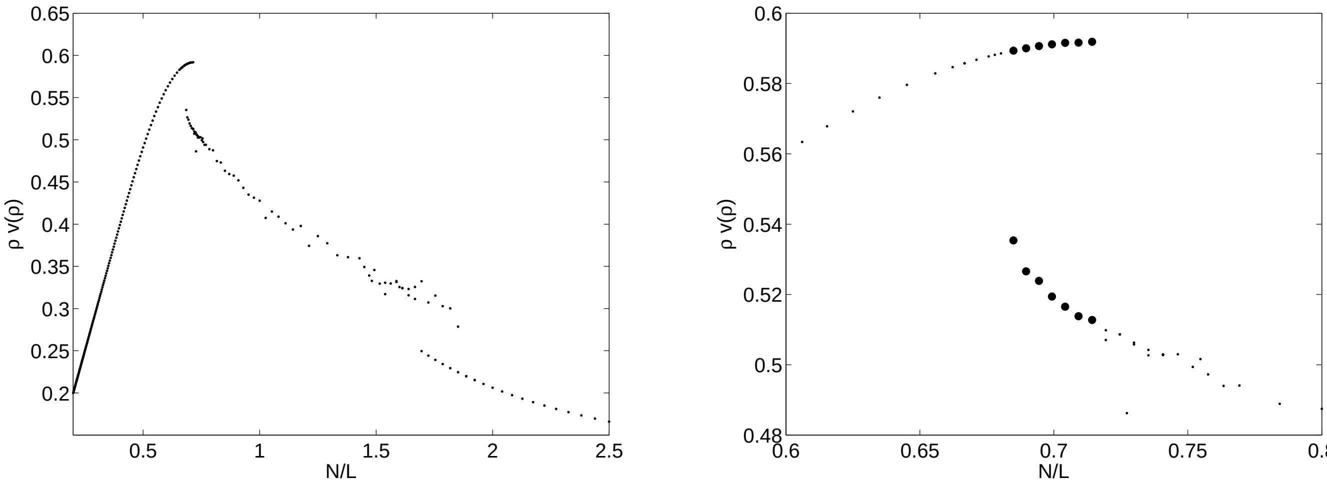
A real world point of view on the reduced Poincaré map π for $N = 10, \epsilon = 0$.

Fundamental diagrams I:



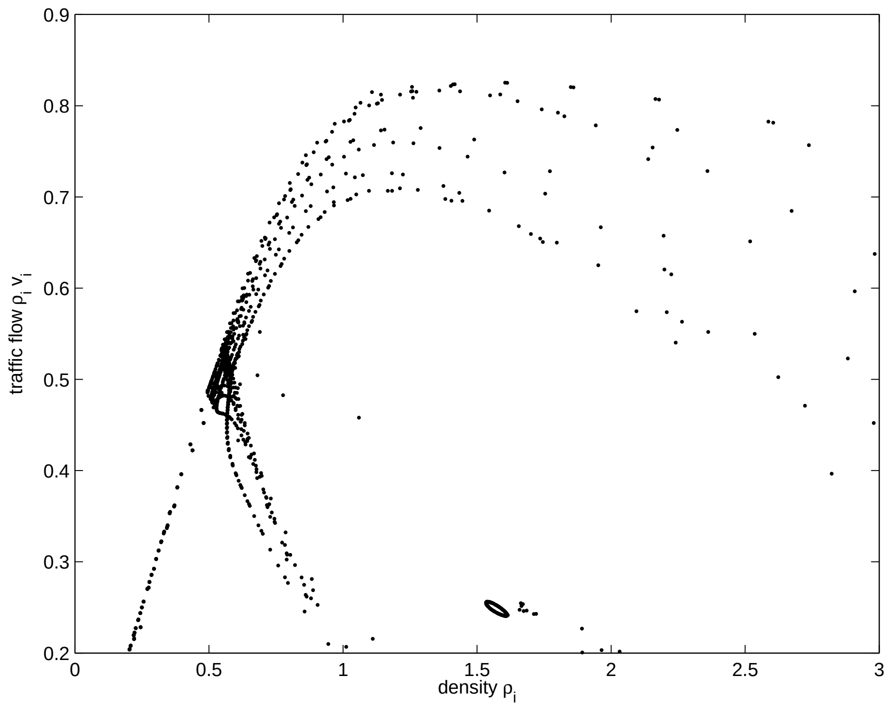
Overlapped fundamental diagrams for $N = 10, L = 50, ..., 4$ measuring at a fixed point.

Fundamental diagrams II:



Fundamental diagram of time-averaged flow versus average density for $N = 10, L = 50 \dots 4$.

Fundamental diagrams III (with roadworks):



Overlapped fundamental diagrams for
$N = 10, L = 50, ..., 4, \quad \epsilon = 0.1$ measuring at a fixed point.

Current and future work:

- Is this dynamics contained in macroscopic models?
- Which marcoscopic model has the same (rich) dynamics than the basic Bando model
- Micro-macro link (Aw, Klar, Materne, Rascle 2002)