The LWR model on a network

Mathematical Models of Traffic Flow (October 28—November 1, 2007)

Mauro Garavello

DiSTA
Università del Piemonte Orientale
Via Bellini, 25/G
15100 Alessandria
mauro.garavello@mfn.unipmn.it

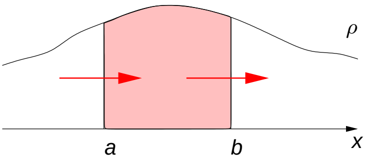
Benedetto Piccoli

I.A.C.
C.N.R.
Viale del Policlinico, 137
00161 Roma
b.piccoli@iac.rm.cnr.it

October 30, 2007

First order model (Lighthill, Whitham 1955 — Richards 1956)

$$\frac{d}{dt}\int_a^b \rho(t,x)dx = v(\rho(t,a))\cdot\rho(t,a) - v(\rho(t,b))\cdot\rho(t,b)$$
$$= \int_a^b [v(\rho(t,x))\cdot\rho(t,x)]_x dx.$$



First order model (Lighthill, Whitham 1955 — Richards 1956)

$$\rho_t + f(\rho)_x = 0$$

- ρ density of cars
- $f(\rho) = \rho \cdot v$ flux
- v average speed of cars

First order model (Lighthill, Whitham 1955 — Richards 1956)

$$\rho_t + f(\rho)_x = 0$$

- non linear conservation law
- discontinuities in finite time $\leftrightarrow$ queue formation
- in case of small diffusion: car may have negative speed

Second order models (H. J. Payne 1971 — G. B. Whitham 1974)

$$\rho_t + (\rho v)_x = 0$$

$$+$$

equation for the speed

Second order models (H. J. Payne 1971 — G. B. Whitham 1974)

$$\rho_t + (\rho v)_x = 0$$

$$+$$

$$v_t + v v_x + \frac{1}{\rho}(A_e(\rho))_x = \frac{1}{\tau}(v_e(\rho) - v)$$

Requiem and resurrection

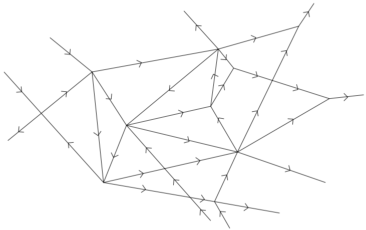
- C. Daganzo, *Requiem for second-order fluid approximation to traffic flow*, 1995

Requiem and resurrection

- C. Daganzo, *Requiem for second-order fluid approximation to traffic flow*, 1995
- A. Aw, M. Rascle, *Resurrection of "second order" models of traffic flow?*, 2000

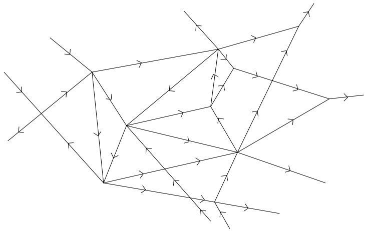
$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ (v + \rho^\gamma)_t + v(v + \rho^\gamma)_x = 0 \end{cases}$$

Road network with the LWR model



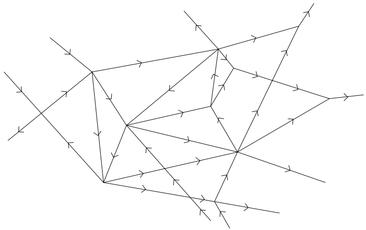
- finite collection of roads connected by junctions

Road network with the LWR model



- finite collection of roads connected by junctions
- each road is modeled by an interval $[a_i, b_i]$

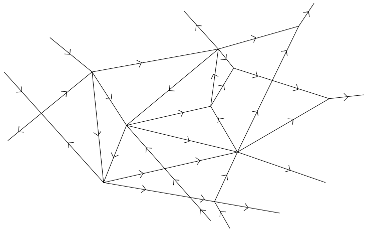
Road network with the LWR model



- finite collection of roads connected by junctions
- each road is modeled by an interval $[a_i, b_i]$
- in each road we consider the LWR model

$$\rho_t + f(\rho)_x = 0$$

Road network with the LWR model



- finite collection of roads connected by junctions
- each road is modeled by an interval $[a_i, b_i]$
- in each road we consider the LWR model

$$\rho_t + f(\rho)_x = 0$$

- solution at junctions?

Technical hypotheses

- the density ρ satisfies $0 \leq \rho \leq 1$

Technical hypotheses

- the density ρ satisfies $0 \leq \rho \leq 1$
- the speed v depends only by ρ

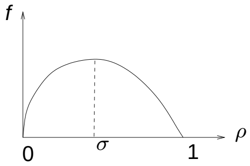
Technical hypotheses

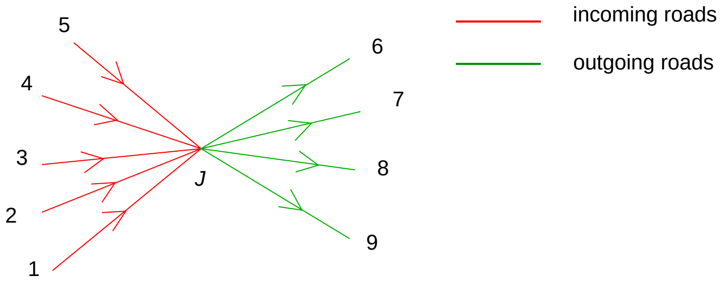
- the density ρ satisfies $0 \leq \rho \leq 1$
- the speed v depends only by ρ
- the speed v is a strictly decreasing function w.r.t. ρ

Technical hypotheses

- the density ρ satisfies $0 \leq \rho \leq 1$
- the speed v depends only by ρ
- the speed v is a strictly decreasing function w.r.t. ρ
- the flux f is concave and

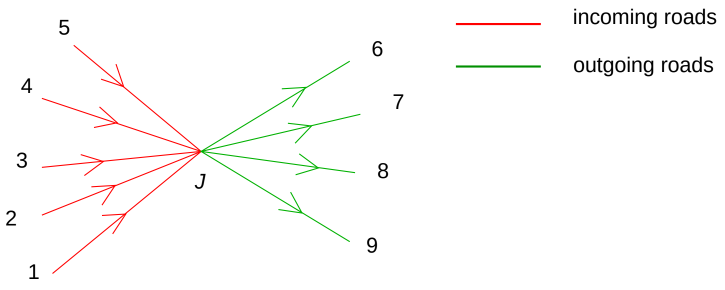
$$f(0) = f(1) = 0, \quad f'(\sigma) = 0 \ \exists ! \, \sigma \in [0, 1]$$



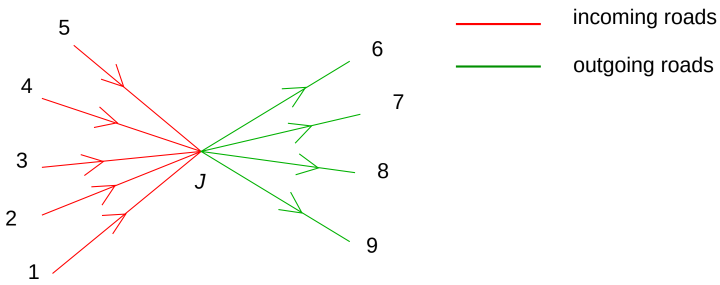


- incoming roads $]-\infty, 0]$

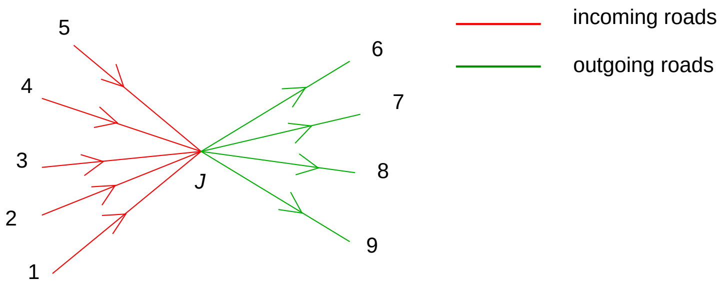
Riemann problem at junctions



- incoming roads $]-\infty, 0]$
- outgoing roads $[0, +\infty[$

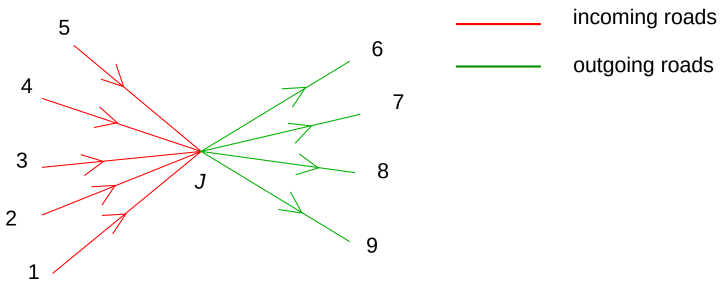


- incoming roads $]-\infty, 0]$
- outgoing roads $[0, +\infty[$
- in each road we consider a constant initial condition



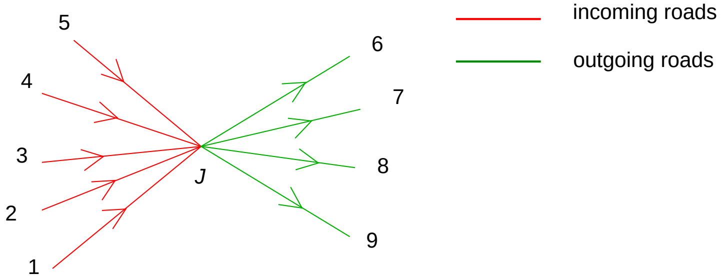
- negative speed of waves in incoming roads

Riemann problem at junctions



- negative speed of waves in incoming roads
- positive speed of waves in outgoing roads

Riemann problem at junctions



- negative speed of waves in incoming roads
- positive speed of waves in outgoing roads
- conservation of the number of cars

Riemann solver *RS*1

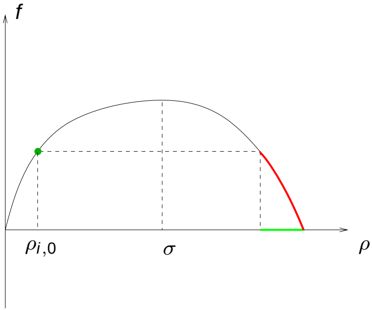
- Preferences of drivers
- Maximization of flux

Riemann solver *RS*2 D'Apice-Manzo-Piccoli

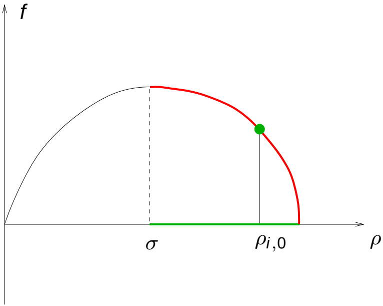
- Maximization of the flux
- Distribution over roads

negative speed in incoming roads

- $0 \leq \rho_{i,0} \leq \sigma$



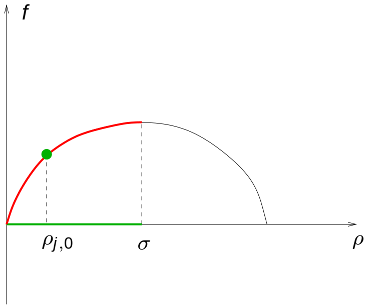
- $0 \leq \rho_{i,0} \leq \sigma$
- $\sigma \leq \rho_{i,0} \leq 1$



$$\Omega_i = \begin{cases} [0, f(\rho_{i,0})], & \text{se } 0 \leq \rho_{i,0} \leq \sigma, \\ [0, f(\sigma)], & \text{se } \sigma \leq \rho_{i,0} \leq 1. \end{cases}$$

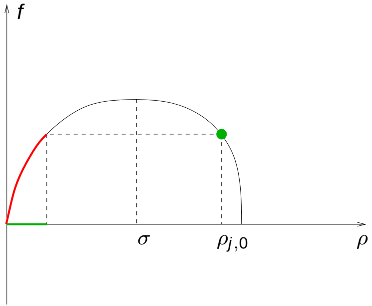
positive speed in outgoing roads

- $0 \leq \rho_{j,0} \leq \sigma$



positive speed in outgoing roads

- $0 \le \rho_{j,0} \le \sigma$
- $\sigma \le \rho_{j,0} \le 1$



$$\Omega_j = \begin{cases} [0, f(\sigma)], & \text{se } 0 \le \rho_{j,0} \le \sigma, \\ [0, f(\rho_{j,0})], & \text{se } \sigma \le \rho_{j,0} \le 1. \end{cases}$$

Conservation of the number of cars

incoming flux = $\sum_{i=1}^{n} f(\rho_i)$

outgoing flux = $\sum_{j=n+1}^{n+m} f(\rho_j)$

$$\sum_{i=1}^{n} f(\rho_i) = \sum_{j=n+1}^{n+m} f(\rho_j)$$

“Rankine-Hugoniot” condition at junctions.

Existence of solution to Riemann problems

Theorem

Each Riemann solver (RS1 or RS2) provides a unique solution to the Riemann problem at the junction.

junctions 2x2, 2x1, 1x2

- waves from incoming roads: the total variation of flux does not increase

junctions 2x2, 2x1, 1x2

- waves from incoming roads: the total variation of flux does not increase
- waves from outgoing roads: the total variation of flux increases

$$\text{Tot.Var.}_f^+ \leq C\,\text{Tot.Var.}_f^-$$

Fundamental estimates

junctions 2x2, 2x1, 1x2

- waves from incoming roads: the total variation of flux does not increase
- waves from outgoing roads: the total variation of flux increases

$$\text{Tot.Var.}_f^+ \leq C\,\text{Tot.Var.}_f^-$$

- the flux variation of each single wave is finite

$$\text{Tot.Var.}_f(0,T) \leq M$$

each type of junctions

- waves from incoming roads: the total variation of flux does not increase

each type of junctions

- waves from incoming roads: the total variation of flux does not increase
- waves from outgoing roads: the total variation of flux does not increase

each type of junctions

- For each junction, consider the functional

$$\Gamma(t) = \sum_{i=1}^{n} f(\rho_i(t,0))$$

each type of junctions

- For each junction, consider the functional

$$\Gamma(t) = \sum_{i=1}^{n} f(\rho_i(t, 0))$$

- The total variation of Γ is bounded

$$\text{Tot.Var.}\Gamma(\cdot) \leq nf(\sigma) + \text{Tot.Var.}_f(0+)$$

each type of junctions

- For each junction, consider the functional

$$\Gamma(t) = \sum_{i=1}^{n} f(\rho_i(t,0))$$

- The total variation of Γ is bounded

$$\text{Tot.Var.}\Gamma(\cdot) \leq nf(\sigma) + \text{Tot.Var.}_f(0+)$$

- The total variation of flux produced at the junction is bounded

each type of junctions

- For each junction, consider the functional

$$\Gamma(t) = \sum_{i=1}^{n} f(\rho_i(t,0))$$

- The total variation of Γ is bounded

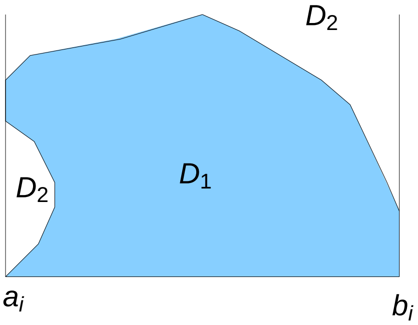
$$\text{Tot.Var.}\Gamma(\cdot) \leq nf(\sigma) + \text{Tot.Var.}_f(0+)$$

- The total variation of flux produced at the junction is bounded
- The total variation of flux is bounded

$$\text{Tot.Var.}_f(t) \leq C_1 nf(\sigma) + C_2 \text{Tot.Var.}_f(0+)$$

$$\text{su } [0, T] \text{ si ha } TV(f(\rho_\nu(t, \cdot))) \leq C$$

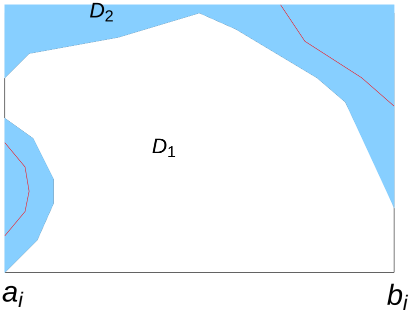
- classical existence theorems



Existence of solutions

$$\text{su } [0,T] \text{ si ha } TV(f(\rho_\nu(t,\cdot))) \leq C$$

- $f(\rho_\nu) \to \bar{f}$
- $\rho_\nu \rightharpoonup \rho$ in L^1
- invertibility of f
- strong convergence of ρ_ν



Continuous dependence RS1

- Continuous dependence? OPEN PROBLEM

- Continuous dependence? OPEN PROBLEM
- Lipschitz continuous dependence does not hold

$$\xi_j\left(\rho_j^+ - \rho_j^-\right) = \frac{\Delta\gamma_j}{\Delta\gamma_i}\xi_i\left(\rho_i^+ - \rho_i^-\right)$$

Continuous dependence

- Continuous dependence? OPEN PROBLEM
- Lipschitz continuous dependence does not hold

$$\xi_j \left(\rho_j^+ - \rho_j^-\right) = \frac{\Delta\gamma_j}{\Delta\gamma_i} \xi_i \left(\rho_i^+ - \rho_i^-\right)$$

- counterexample in the case of a junction with 2 incoming and 2 outgoing roads

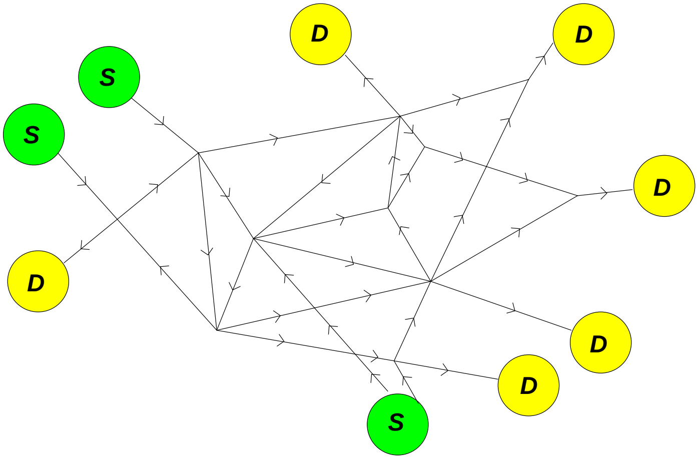
Continuous dependence RS2

- Lipschitz continuous dependence holds

Continuous dependence RS2

- Lipschitz continuous dependence holds
- the proof is based on the shifts of waves

Source-destination model



Source-destination model

- traffic-distribution functions

$$r_J : \mathrm{Inc}(J) \times \mathcal{S} \times \mathcal{D} \longrightarrow \mathrm{Out}(J)$$

Source-destination model

- traffic-distribution functions

$$r_J : \mathrm{Inc}(J) \times \mathcal{S} \times \mathcal{D} \longrightarrow \mathrm{Out}(J)$$

- single path for each car

Source-destination model

- traffic-distribution functions

$$r_J : \mathrm{Inc}(J) \times \mathcal{S} \times \mathcal{D} \to \mathrm{Out}(J)$$

- single path for each car
- traffic-composition functions

$$\pi^i : [0, \infty[\times [a_i, b_i] \times \mathcal{S} \times \mathcal{D} \to [0, 1]$$

Source-destination model

- traffic-distribution functions

$$r_J : \mathrm{Inc}(J) \times \mathcal{S} \times \mathcal{D} \to \mathrm{Out}(J)$$

- single path for each car
- traffic-composition functions

$$\pi^i : [0, \infty[\times [a_i, b_i] \times \mathcal{S} \times \mathcal{D} \to [0, 1]$$

- evolution of π^i

$$\partial_t \pi^i(t, x, s, d) + \partial_x \pi^i(t, x, s, d) \cdot v^i(\rho^i(t, x)) = 0$$

Existence of solutions

Theorem. Let $(\bar{\rho}, \bar{\Pi})$ be a constant solution on the network. For every $T > 0$ there exists $\varepsilon > 0$ with the following property. For every small perturbation $(\tilde{\rho}, \tilde{\Pi})$ of the equilibrium solution

$$\|\tilde{\rho}\|_{BV} \le \varepsilon, \quad \left\|\tilde{\Pi}\right\|_{BV} \le \varepsilon,$$

and

$$\|\tilde{\rho} - \bar{\rho}\|_{\infty} + \left\|\tilde{\Pi} - \bar{\Pi}\right\|_{\infty} \le \varepsilon,$$

there exists a solution (ρ, Π) defined on $t \in [0, T[$ such that at $t = 0$ coincides with $(\tilde{\rho}, \tilde{\Pi})$.