Lyapunov stability analysis of networks of conservation laws

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Outline

2x2 hyperbolic systems of conservation laws

Steady-state and characteristic form

Networks of conservation laws

Boundary conditions

Exponential stability

Lyapunov stability analysis

Real-life application

Final comments

Open channels, St Venant equations.

$$\partial_t h + \partial_x (hv) = 0$$

$$\partial_t v + \partial_x (gh + \frac{1}{2} v^2) = gS - Cv^2/h$$

h = water level, v = water velocity,
g= gravity, S = canal slope, C = friction coefficient

Road traffic, Aw-Rascle equations.

$$\partial_t \rho + \partial_x (\rho v) = 0$$

$$\partial_t (v + p(\rho)) + v\partial_x \big(v + p(\rho)\big) = (V(\rho) - v)/\tau$$

ρ = traffic density, v = traffic velocity,
$p(\rho)$ = "traffic pressure" , $V(\rho)$ = preferential velocity,
σ = constant

2x2 hyperbolic systems

Space $x \in [0, L]$

Time $t \in [0, +\infty)$

State

$$Y(t,x) \triangleq \begin{pmatrix} y_1(t,x) \\ y_2(t,x) \end{pmatrix}$$

$$\partial_t Y + \partial_x f(Y) = g(Y)$$

$$\boxed{\partial_t Y + A(Y)\partial_x Y = g(Y)}$$

$A(Y)$ has 2 distinct real eigenvalues

Steady state

$$\partial_t Y + A(Y)\partial_x Y = g(Y)$$

A steady-state is a constant solution $Y(t,x) \equiv \bar{Y}$

which satisfies the equation $g(\bar{Y}) = 0$

and (obviously) the state equation $\partial_t \bar{Y} + A(\bar{Y})\partial_x \bar{Y} = g(\bar{Y})$

Road traffic

$\rho =$ density

$v =$ velocity

Steady state

$$\bar{v} = V(\bar{\rho})$$

Open channels

$h =$ water depth

$v =$ velocity

Steady state

$$\bar{v} = \sqrt{\frac{gS}{C}\bar{h}}$$

Characteristic form

- Hyperbolic system : $\partial_t Y + A(Y)\partial_x Y = g(Y)$

↓

- Change of coordinates :

$$\xi(Y) = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \qquad \partial_t \xi + \begin{pmatrix} c_1(\xi) & 0 \\ 0 & c_2(\xi \end{pmatrix} \partial_x \xi = h(\xi)$$

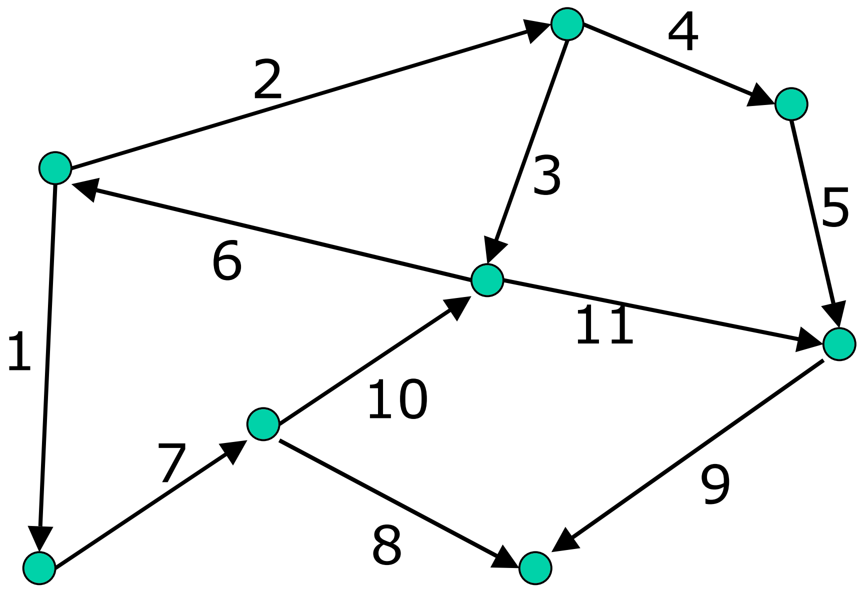
(Riemann)

with $c_1(\xi) \neq c_2(\xi)$
eigenvalues of $A(Y)$

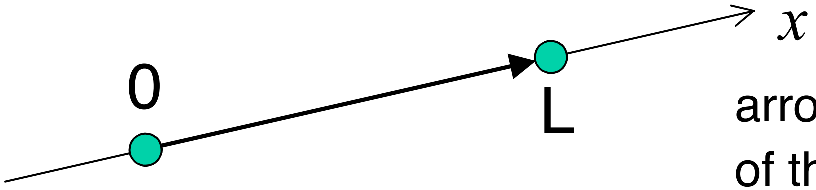
The change of coordinates $\xi(Y)$ is defined up to a constant.
It can therefore be selected such that $\xi(\bar{Y}) = 0 \quad \Rightarrow h(0) = 0.$

Generalisation to netwoks of 2x2 hyperbolic systems

(e.g. hydraulic networks (irrigation, waterways) or road traffic networks)



- directed graph
- n arcs
- one system of two conservation laws attached to each arc



arrow = orientation of the x axis

$$\boxed{\partial_t \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} + \begin{pmatrix} c_i(\xi) & 0 \\ 0 & c_{n+i}(\xi) \end{pmatrix} \partial_x \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} = h \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix}} \qquad (i = 1, \ldots, n)$$

System

$$\partial_t \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} + \begin{pmatrix} c_i(\xi) & 0 \\ 0 & c_{n+i}(\xi) \end{pmatrix} \partial_x \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} = h \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix}$$

$$\boxed{\partial_t \xi + C(\xi)\partial_x \xi = h(\xi)}$$

$$C(\xi) = \mathrm{diag}(c_i(\xi))$$
$$c_i(0) > 0$$

$$x \in [0, L]$$

Boundary conditions $\xi(t, 0) = G\big(\xi(t, L)\big)$

Problem G such that boundary conditions are dissipative ?

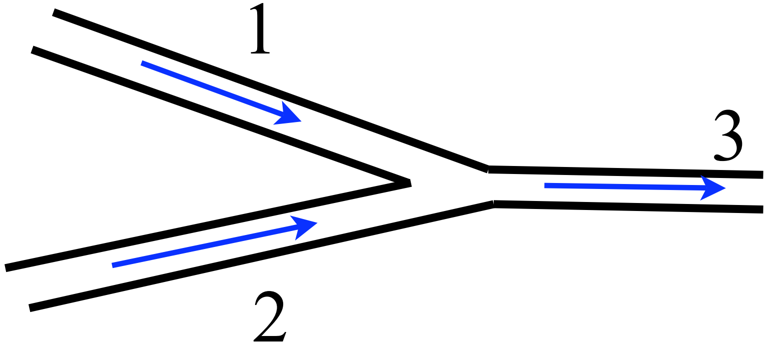
i.e. steady-state $\xi \equiv 0$ is exponentially stable

Boundary conditions = Physical constraints

$$\boxed{\xi(t,0) = G\big(\xi(t,L)\big)} \iff \boxed{F(Y(t,0), Y(t,L)) = 0}$$

Road traffic

ρ = density
v = velocity

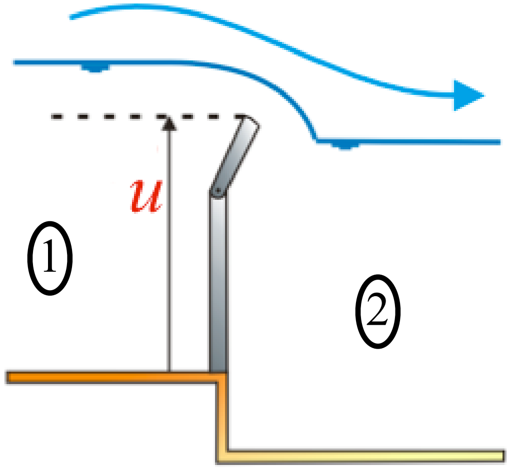


Flow conservation at a junction

$$\rho_3(t,0)v_3(t,0) = \\ \rho_1(t,L)v_1(t,L) + \rho_2(t,L)v_2(t,L)$$

Open channels

h = water depth
v = velocity



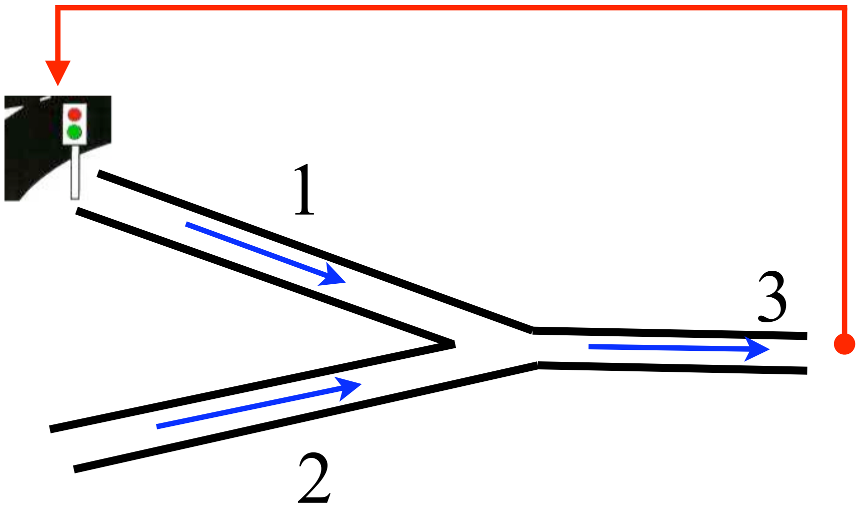
Modelling of hydraulic gates

$$h_2(t,0)v_2(t,0) = \alpha\big(h_1(t,L) - u\big)^{3/2}$$

Boundary conditions = boundary feedback control

Road traffic

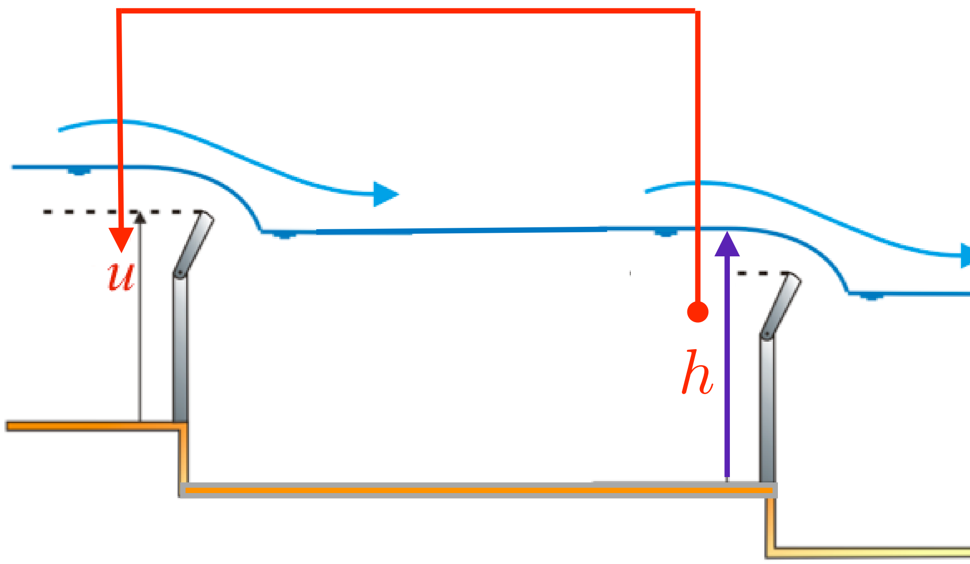
$\rho =$ density
$v =$ velocity
$q = \rho v =$ flux



Feedback implementation of ramp metering

Open channels

$h =$ water depth
$v =$ velocity



Feedback control of water depth in navigable waterways

G function of the control tuning parameters : How to design the control laws to make the boundary conditions dissipative ?

$$\partial_t \xi + C(\xi)\partial_x \xi = h(\xi)$$

$$\xi(t,0) = G\big(\xi(t,L)\big)$$

$$t \in [0,+\infty) \quad x \in [0,L]$$

G such that boundary conditions are dissipative, i.e. $\xi = 0$ is exp. stable ?

Definition of exponential stability

$\exists\, \varepsilon, \gamma, \nu > 0$ such that, classical solutions on $[0,L]$ satisfy

$$\|\xi(t,0)\| \leqslant \varepsilon \;\Rightarrow\; \|\xi(t,x)\|_{H^2} \leqslant \gamma e^{-\nu t}\|\xi(t,0)\|_{H^2}$$

Sufficient exponential stability conditions

$$\partial_t \xi + C(\xi)\partial_x \xi = \mathbf{0}$$

$$\xi(t,0) = G\big(\xi(t,L)\big)$$

$$t \in [0,+\infty) \qquad x \in [0,L]$$

Slemrod (1983)
Greenberg and Li (1984)
Qin, Yu and Li (1985)
etc...

$$\boxed{\rho(|G'(0)|) < 1}$$

A weaker condition (2007)

$$\boxed{\rho_1\big(G'(0)\big) \triangleq \underset{\Delta}{\mathrm{Inf}}\ \{\|\Delta G'(0)\Delta^{-1}\|\} < 1}$$

strictly positive
diagonal matrices

Lyapunov stability analysis (first trial)

$$\partial_t \xi + C(\xi)\partial_x \xi = \mathbf{0}$$

$$\xi(t,0) = G\big(\xi(t,L)\big)$$

$$t \in [0,+\infty) \qquad x \in [0,L]$$

notations:

$\Lambda \triangleq C(0)$

$K \triangleq G'(0)$

$\boxed{\text{Stability condition } \rho_1(K) < 1 \Rightarrow \exists\ D \text{ (diag) s.t. } \|DKD^{-1}\| < 1}$

Lyapunov function candidate

$$V = \int_0^L E(\xi)dx \qquad \text{with} \qquad E(\xi) \triangleq \xi^T D^2 \Lambda^{-1} \xi$$

Entropy

$$\partial_t E(\xi) + \partial_x F(\xi) = 0 \quad \Longrightarrow \quad \frac{dV}{dt} = -\Big[F(\xi)\Big]_0^L$$

Entropy Flux

Lyapunov stability analysis (first trial continued)

$$\frac{dV}{dt} = -\Big[F(\xi)\Big]_0^L$$

depends only on B.C. $\xi(t,0) = G\big(\xi(t,L)\big)$

$$\Big\Downarrow$$

$$\frac{dV}{dt} = |D\xi(t,L)|^2\Big(\underset{\text{stab. cond.} < 1}{\|DKD^{-1}\|^2} - 1\Big) + \text{H.O.T.}$$

But time derivative of Lyapunov function only
semi-negative definite ...

A strict Lyapunov function

$$V = \int_0^L (\xi^T D^2 \Lambda^{-1} \xi)\; e^{-\mu x} dx$$

$$\Downarrow$$

$$\frac{dV}{dt} \leqslant -\mu V - \left[F(\xi)\right]_0^L + \beta \int_0^L |\xi|^2 |\partial_x \xi| dx$$

O.K. for exponential stability

$\leqslant 0$

see above

Problem !

Extended model

notations $\zeta \triangleq \partial_x \xi \quad \eta \triangleq \partial_x \zeta$

$$\boxed{\begin{aligned}&\partial_t \xi + C(\xi)\partial_x \xi = 0\\&\partial_t \zeta + C(\xi)\partial_x \zeta + [C'(\xi)\zeta]\zeta = 0\\&\partial_t \eta + C(\xi)\partial_x \eta + [C'(\xi)\eta]\zeta + 2[C'(\xi)\zeta]\eta + [(C''(\xi)\eta)\eta]\zeta = 0\end{aligned}}$$

➡ Extended Lyapunov function

$$V = \int_0^L \Big[(\xi^T D^2 \Lambda^{-1} \xi) + (\zeta^T \Lambda D^2 \zeta) + (\eta^T \Lambda^2 D^2 \Lambda \eta)\Big] e^{-\mu x} dx$$

$\Rightarrow \quad \dfrac{dV}{dt} \leqslant -\lambda V \quad \Longrightarrow$ exponential stability in H_2 norm

$$\|\xi\|_{H_2} = \left(\int_0^L (|\xi|^2 + |\zeta|^2 + |\eta|^2) dx \right)^{1/2}$$

- **Theorem** If $\rho_1(G'(0)) < 1$, the equilibrium $\xi \equiv 0$ of the quasi-linear hyperbolic system

$$\partial_t \xi + C(\xi)\partial_x \xi = \mathbf{0} \qquad \xi(t,0) = G\big(\xi(t,L)\big)$$

is exponentially stable.

- The same Lyapunov function may be used for:

If $\rho_1(G'(0)) < 1$ and $h'(0) < \kappa$ sufficiently small, the equilibrium $\xi \equiv 0$ of the quasi-linear hyperbolic system

$$\partial_t \xi + C(\xi)\partial_x \xi = h(\xi) \qquad \xi(t,0) = G\big(\xi(t,L)\big)$$

is exponentially stable.

(References
Single 2x2 system : IEEE-TAC 2007
Network of scalar systems : accepted NHM
General result : submitted paper)

Positive and negative characteristic velocities

System

$$\partial_t \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} + \begin{pmatrix} c_i(\xi) & 0 \\ 0 & c_{n+i}(\xi) \end{pmatrix} \partial_x \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} = h \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix}$$

$$\boxed{c_{n+i}(0) < 0 < c_i(0)}$$

e.g. St Venant equations
(fluvial flow)

Notations

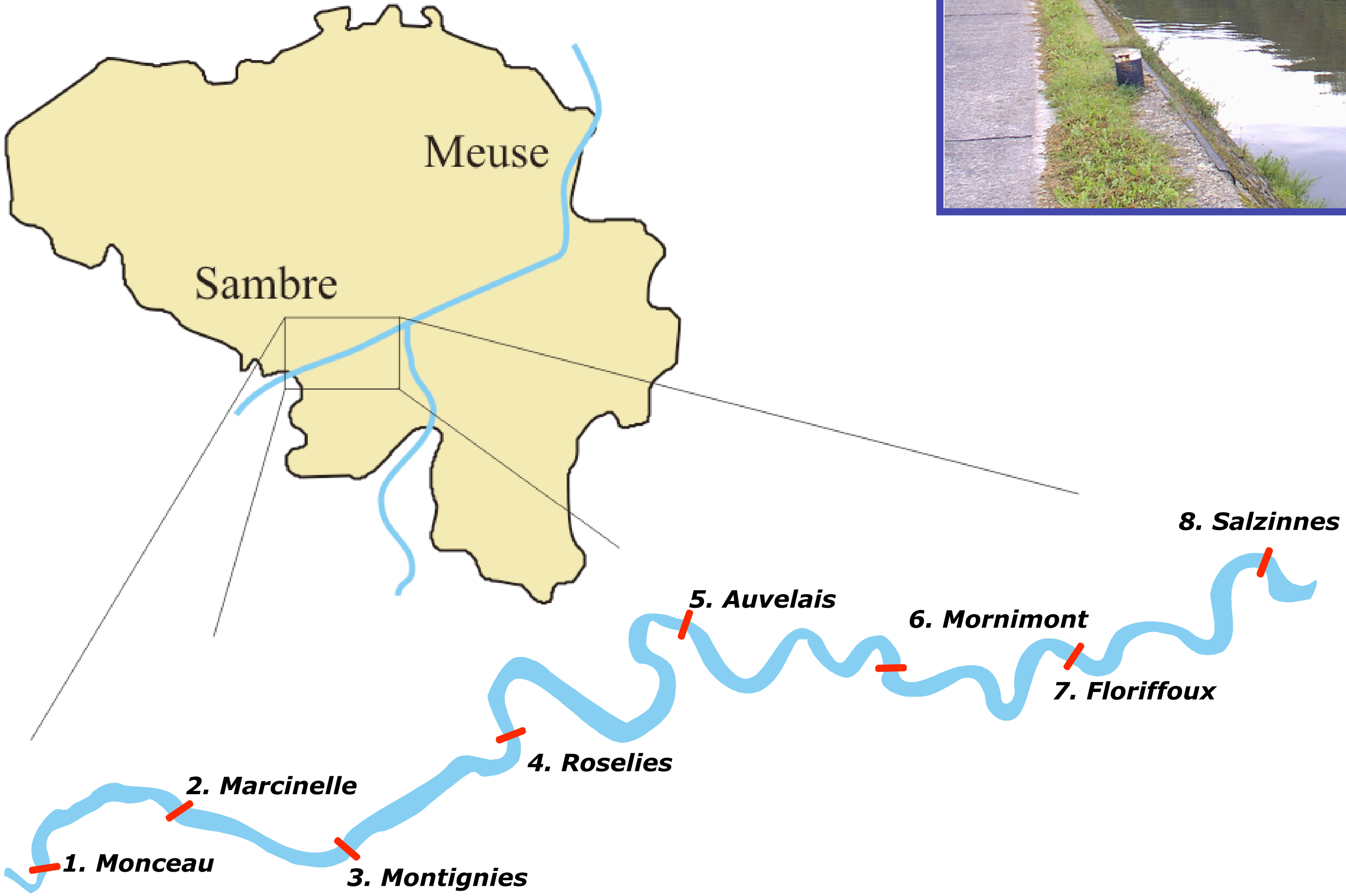
$$\xi = \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad \begin{aligned} \xi_+ &= (\xi_1, \ldots, \xi_n)^T \\ \xi_- &= (\xi_{n+1}, \ldots, \xi_{2n})^T \end{aligned}$$

Boundary conditions → **Same stability condition**

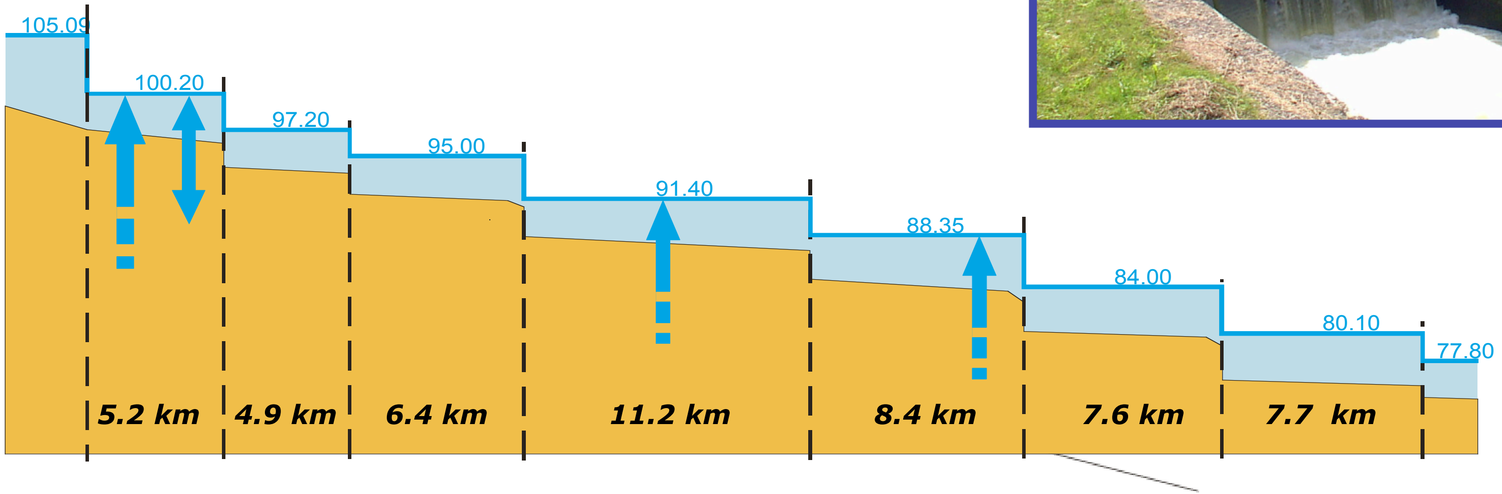
$$\begin{pmatrix} \xi_+(t,0) \\ \xi_-(t,L) \end{pmatrix} = G \begin{pmatrix} \xi_+(t,L) \\ \xi_-(t,0) \end{pmatrix}$$

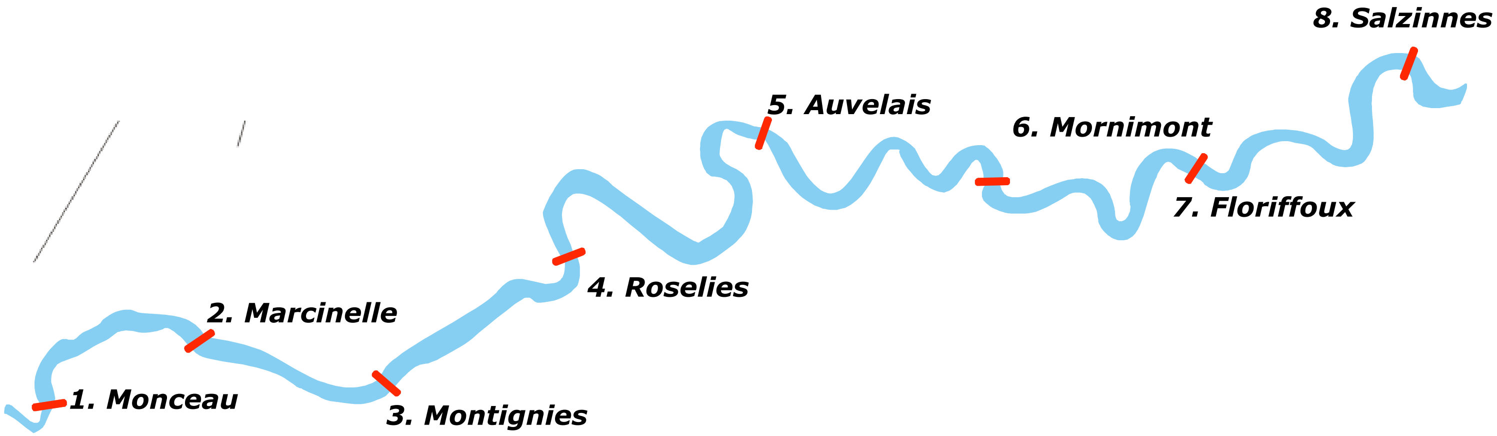
$$\boxed{\rho_1\big(G'(0)\big) < 1}$$

Real-life application : level and flow control in navigable waterways. Sambre river (Belgium).

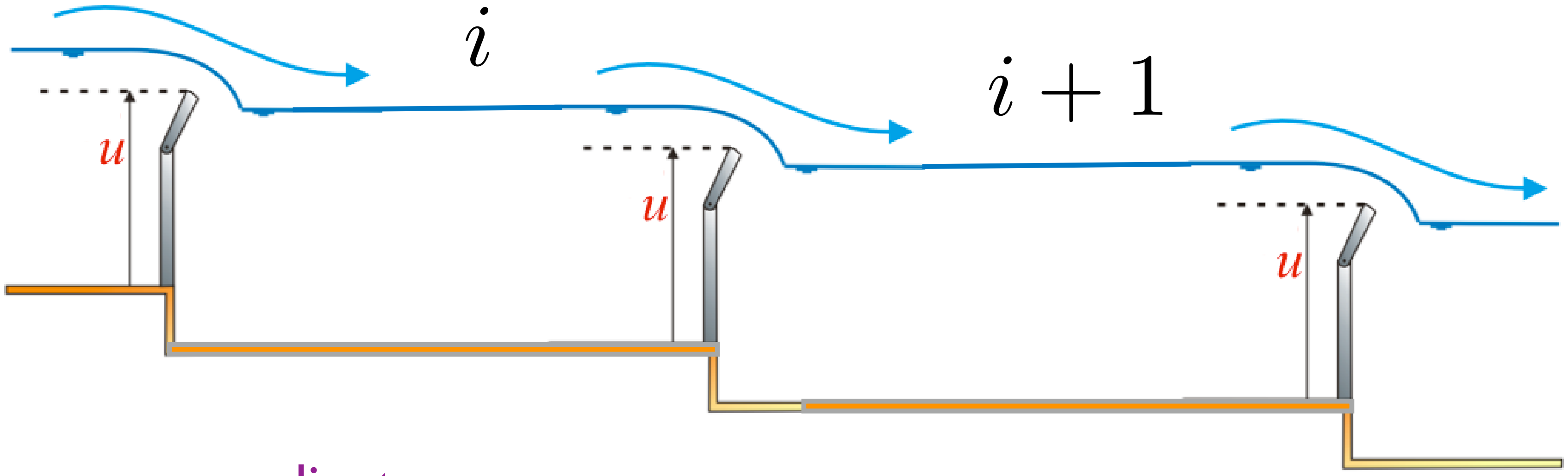


Navigable waterway
= cascade of pools





Control design and stability analysis



Riemann coordinates

$$\begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} = \begin{pmatrix} v_i + 2\sqrt{gh_i} - \bar{v} - 2\sqrt{g\bar{h}} \\ v_i - 2\sqrt{gh_i} - \bar{v} + 2\sqrt{g\bar{h}} \end{pmatrix}$$

Control design :

Nonlinear control laws such that

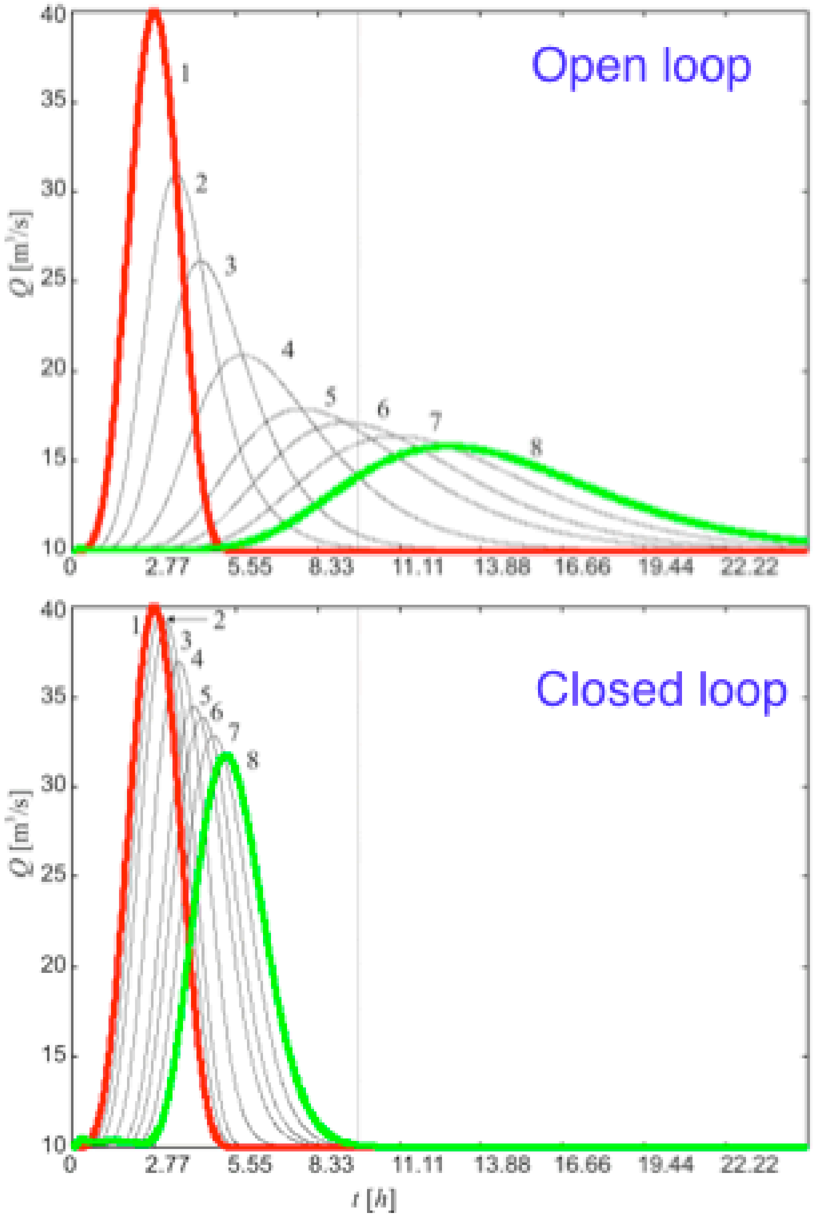
$$\xi_{n+i}(L) = {\color{red}k_{Li}}\xi_i(L) + {\color{red}k_{oi}}\xi_{n+i+1}(0)$$

$\Longrightarrow$

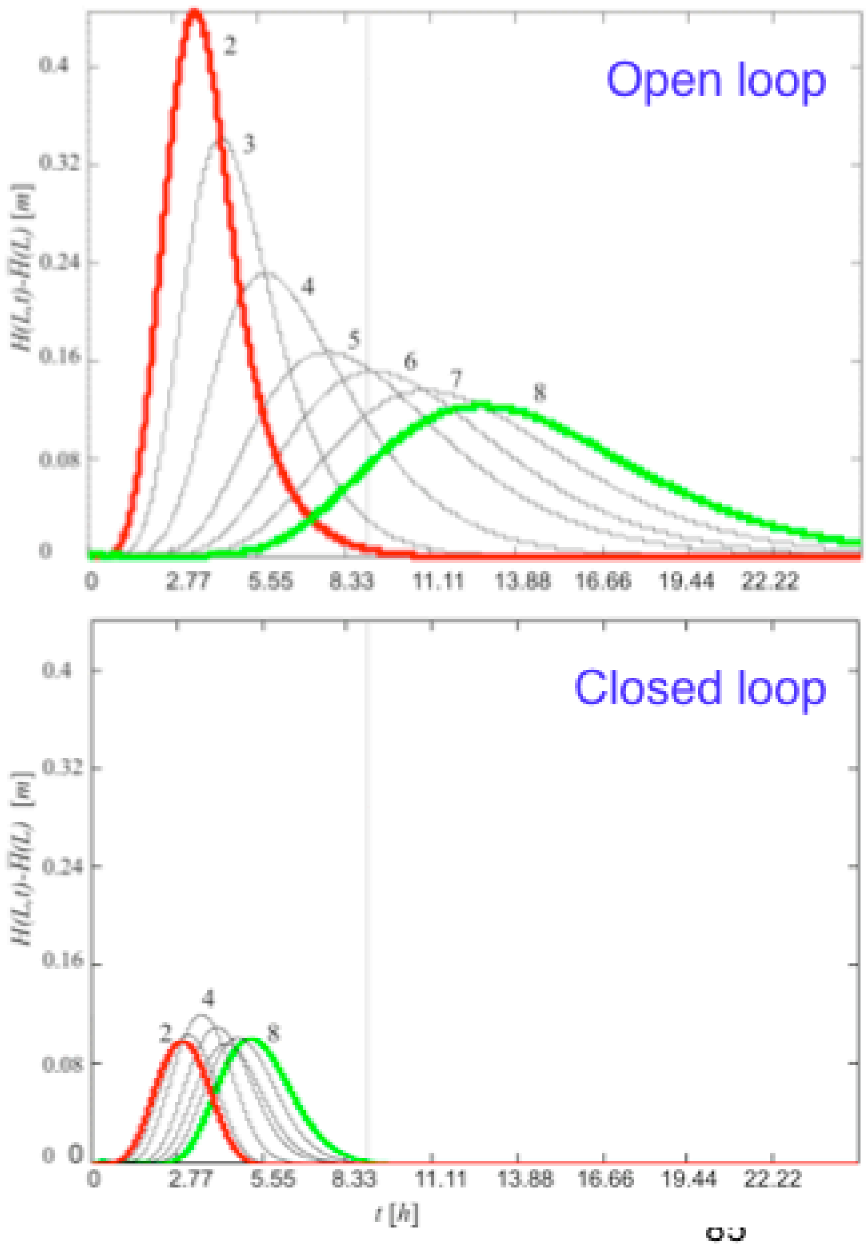
Stability condition

$$\rho_1(G'(0)) = \max_i(|k_{oi}k_{Li}|) < 1$$

Flow Rates Q



Water levels $H - \bar{H}$



Some final comments

- Lyapunov stability analysis with dynamic boundary control (PI-type) + experimental validation on a laboratory pilot plant. (paper accepted in Automatica).

- Riemann coordinates may also be useful for the design of feedforward controllers (cancellation of disturbances) (see IEEE CDC 2005) and for the design of exponentially converging observers (paper in preparation).

- Application to flow control in navigable water-ways. (Automatica 2003)