Convergence of flux vector splitting schemes with single entropy inequality for hyperbolic systems of conservation laws

F. Berthelin

Laboratoire J. A. Dieudonné
UMR CNRS 6621
Université de Nice Sophia-Antipolis
Parc Valrose
06108 Nice cedex 2, France
e-mail: Florent.Berthelin@unice.fr

Abstract

We prove the convergence of flux vector splitting schemes associated to hyperbolic systems of conservation laws with a single compatible entropy η_c. We prove estimate on the L^2 norm of the gradient of the numerical approximation in the inverse square root of the space increment Δx. This estimate is related to the notion of (strictly) η_c-dissipativity on F^+, $-F^-$ and $Id - \lambda(F^+ - F^-)$, where F^+, F^- is the flux-decomposition. The second tool of the proof is a kinetic formulation of the flux-splitting scheme with three velocities. Then we get a control for all entropies and apply the compensated compactness theory.

Key-words: convergence of numerical scheme – conservation laws – flux vector splitting scheme – entropic schemes – entropy dissipativity – kinetic formulation

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Contents

1 Introduction

1.1 Context

Flux vector splitting schemes are now well-known numerical schemes for hyperbolic systems of conservation laws [15], [10] since they are easy to implement and effective. This last property is nevertheless related to there links with entropy functions, thus it leads to an important study. Furthermore, the convergence of these schemes is also related to entropies since it can be obtained via the Tartar's compensated compactness theory [20]. Di Perna [12] establishes the convergence of approximated solutions to one-dimensional nonlinear hyperbolic systems of conservation laws by a now classical argument, getting estimate on the gradient of the approximate solution from compatible entropy and applying this to control all entropies in the compensated compactness context. Following this idea, the convergence from a kinetic equation to weak solutions of isentropic gas dynamics satisfying a single entropy inequality is proved in [4]. In the present article, we study the numerical counterpart of this work. The importance of such a study comes in particular from physics relevant models in which only one entropy is compatible with the system and one have to remove hypothesis with full family of compatible entropies to be able to treat realistic schemes. Previous works in this numerical point of view can be found in [10] and [18], where relaxation of such schemes are performed in the case where all the entropies are compatible. The article [16] considers the case of only one compatible entropy for a relaxation scheme associated to the system of elastodynamics. In the continuous case, we refer to [19], [21], [2] and [3] for the case with all entropies and to [12], [17] and [4] for the case with a single entropy.

The framework in which we are placing us is the following. We consider a system of N conservation laws in one space variable

$$\partial_t U + \partial_x F(U) = 0, \qquad t \in \mathbb{R}^+,\ x \in \mathbb{R},\ U(t,x) \in \mathbb{R}^N, \tag{1.1}$$

where the flux $F : \mathbb{R}^N \to \mathbb{R}^N$ is assume to have a flux-decomposition

$$F = F^+ + F^-, \tag{1.2}$$

where

$$F^+ \text{ and } F^- \text{ are Lipschitz continuous.} \tag{1.3}$$

Furthermore F^+ and $-F^-$ have to satisfy a property similar to

$$\pm\nabla F^\pm \text{ have real and non-negative eigenvalues.} \tag{1.4}$$

The compactness argument we use to prove that the approximate numerical solutions converge to a solution of (1.1) forces the system to have a structure such that the compensated compactness theory is able to be applied. We recall that this theory gives the compactness on a bounded sequence of approximated solutions of the system (U_ε) which satisfies the relations

$$\partial_t \eta(U_\varepsilon) + \partial_x G(U_\varepsilon) \in \text{ compact of } H^{-1}_{loc}, \tag{1.5}$$

on a sufficiently large family of entropies. As a consequence, the system have to be endowed with a family of entropies to be able to apply this theory. We resume this assumption as:

$$\begin{gathered}\text{there exists a family } \mathcal{F} \text{ of } C^2 \text{ couples entropy-entropy flux } (\eta, G)\\ \text{such that (1.5) for any } (\eta, G) \text{ of the family } \mathcal{F},\\ \text{implies, on a bounded sequence } (U_\varepsilon)_{\varepsilon>0},\\ \text{that, up to an extraction, } U_\varepsilon \to U \text{ a.e..}\end{gathered} \tag{Hc}$$

We prove here that bounds as (1.5) on **all** entropies can be obtained for systems in which only **one** entropy is compatible in a sense we are going to detail in section 2.1. To explain briefly, the entropy η_c is compatible with the flux decomposition if there exists $\vartheta^\pm$ such that $(\vartheta^\pm)' = \eta_c'(F^\pm)'$. It gives discrete entropy inequalities and the dissipativity of the scheme. In our work, the single compatible entropy gives an estimate on the gradient of the approximate numerical solutions (U_Δ):

$$\|\partial_t U_\Delta\|_{L^2_{tx}} \le \frac{C}{\sqrt{\Delta x}}, \qquad \|\partial_x U_\Delta\|_{L^2_{tx}} \le \frac{C}{\sqrt{\Delta x}}, \tag{1.6}$$

where Δx is the space increment of the scheme. This is a similar estimate as that of [12] and [4] but in a numerical point of view. The estimates (1.6) are enough as in Di Perna approximation technique to control all entropies as (1.5). This is the key point of the method. Thus, we need for our system a notion of entropy dissipativity with respect to the compatible entropy η_c for F^+, F^- and $Id - \lambda(F^+ - F^-)$, that is hypotheses $(\mathcal{H}^1_{\eta_c})$ and $(\mathcal{H}^2_{\eta_c})$ in section 2.1. These hypotheses are related to (1.4) and a CFL condition. In other terms, this link between the flux decomposition and the main entropy η_c is the starting point to have a sufficient entropy compatibility to control all the information. We refer to weak BV estimates for finite volume method for scalar conservation laws ([9], [13], [14]) which are also related to gradient estimate from entropy dissipation.

At the limit, we obtain a solution of (1.1) which satisfies only one entropy inequality, this related to the compatible one:

$$\partial_t \eta_c(U) + \partial_x G_c(U) \leq 0. \tag{1.7}$$

An other tool we are using to prove the convergence of the flux splitting scheme is the possibility to have a kinetic formulation with three velocities of it. It allows to follow precisely where the information is located. We refer to [10], [1], [6] and the references therein for this aspect. This kinetic formulation is based on a transport-projection method, also called transport-collapse method introduced in [7], [8]. We also refer to [11], [22] and [23] for convergence results with this idea.
We present now more precisely the numerical scheme we are dealing with, then we expose the convergence result of this paper.

1.2 The flux-splitting scheme and the convergence result

Having given a time-step Δt and an uniform grid with space increment Δx, we set $\lambda = \Delta t/\Delta x$ constant and $x_j = j\Delta x$, $t_n = n\Delta t$. Let $U^0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ be an initial data. We consider the discretization of the initial data as

$$U_j^0 = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} U^0(y)\,dy, \qquad \text{for } x \in]x_j, x_{j+1}[. \tag{1.8}$$

The scheme writes

$$U_j^{n+1} - U_j^n + \lambda(F_{j+1/2}^n - F_{j-1/2}^n) = 0, \tag{1.9}$$

with the flux

$$F_{j+1/2}^n = F^+(U_j^n) + F^-(U_{j+1}^n). \tag{1.10}$$

We define the approximate solution, at fixed Δx, by

$$\begin{aligned} U_\Delta(t,x) \quad = \quad & \frac{1}{\Delta t}\left[\frac{U_{j+1}^{n+1} - U_j^{n+1} - U_{j+1}^n + U_j^n}{\Delta x}(x - x_j) + U_j^{n+1} - U_j^n\right](t - t_n), \\ & + \frac{U_{j+1}^n - U_j^n}{\Delta x}(x - x_j) + U_j^n \\ & \qquad\qquad \text{for } x_j < x < x_{j+1},\ t_n \leq t < t_{n+1}. \end{aligned}$$

We need the following hypothesis on the sequence (U_j^n) that is a limit property at infinity:

$$\text{there exists } (U_\infty^n)_n \text{ such that } U_j^n \underset{j\to\pm\infty}{\rightarrow} U_\infty^n, \forall n \in \mathbb{N}. \tag{1.11}$$

This hypothesis can be removed if we assume periodicity in x instead of beeing with $x \in \mathbb{R}$, see Remark 2.2. We state now the main result of this article, which is the proof of the convergence of the numerical scheme.

Theorem 1.1 *Let the system (1.1) be such that (Hc) is valid. Let η_c be one entropy of the system, and $U^0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ such that $\eta_c(U^0) \in L^1(\mathbb{R})$. We assume that the system have a flux decomposition such that (1.3) is valid, and such that the hypothesis $(\mathcal{H}^1_{\eta_c})$ and $(\mathcal{H}^2_{\eta_c})$ (see section 2.1) are satisfied and that it leads to a bounded sequence $(U^n_j)_{n,j}$ verifying (1.11). Then, up to a subsequence, $U_\Delta \to U$ as $\Delta \to 0$ where U is a solution of (1.1) satisfying the entropy inequality (1.7) with initial data U^0.*

Remark 1.1 By $\Delta \to 0$, we mean the convergence of a sequence $\Delta_k x \to 0$ with $\lambda = \Delta_k t/\Delta_k x$ kept constant. Thus $U_\Delta \to U$ corresponds to the convergence of $U_{\Delta_k x} \to U$ as $k \to +\infty$.

Remark 1.2 In multi-dimensional situations ($x \in \mathbb{R}^d$), there is generally only one entropy thus a fortiori at most one compatible entropy. The consequence of this fact is the necessity to perform similar methods where the information comes from a single entropy. Nevertheless, since there is not a family of entropies (even non compatible) the compactness of the sequence needs an other tool than compensated compactness.

We have to precise hypotheses $(\mathcal{H}^1_{\eta_c})$ and $(\mathcal{H}^2_{\eta_c})$ which are related to the notion of (strict) η_c-dissipativity. It will be done in the next section and it gives the wanted estimate on the gradient. In section 3, we present the kinetic formulation with three velocities of the flux-splitting scheme and then we prove the convergence result (Theorem 1.1). Finally in section 4, we detail our notion of entropy dissipativity for the p-system.

2 Estimate on the gradient of approximate solution

In this section, we first define the notion of entropy dissipativity, then we expose the consequence on discrete entropy inequalities and the non degeneracy hypothesis. It allows then to obtain the estimates on $\partial_t U_\Delta$ and $\partial_x U_\Delta$.

2.1 Notion of entropy dissipativity

We recall the following definition and the associated result for the discrete entropy inequalities ([6]).

Definition 2.1 (Bouchut) *Let $\eta(u)$ be a convex function. We say that W is η-dissipative in $\mathcal{U} \subset \mathbb{R}^N$ if*

$$W'(u)^t \eta''(u) \text{ is symmetric,} \tag{2.1}$$

and if, defining $G_\eta[W]$ by

$$G_\eta[W]'(u) = \eta'(u)W'(u), \tag{2.2}$$

the elementary entropy dissipation

$$D_\eta[W](u,v) = G_\eta[W](u) - G_\eta[W](v) + \eta'(u)(W(v) - W(u)) \tag{2.3}$$

is non-positive for any $u, v \in \mathcal{U}$.

This notion of η-dissipativity is related to $W \in \mathcal{M}_+^\eta$ (see [5], [6]), where

$$\mathcal{M}_+^\eta = \{W : \mathcal{U} \to \mathbb{R}^N \,;\, (W')^t \eta'' \text{ is symmetric}, \geq 0 \text{ in } \mathcal{U}\}. \tag{2.4}$$

We notice that this entropy dissipation property is locally equivalent to the entropy condition of [10].
We set now the first assumption for our hyperbolic system which is the dissipativity property of the flux-splitting with respect to the compatible entropy η_c:

$$F^+, -F^- \text{ and } Id - \lambda(F^+ - F^-) \text{ are } \eta_c - \text{dissipative.} \tag{$\mathcal{H}^1_{\eta_c}$}$$

With this, we get the discrete entropy inequalities:

Theorem 2.2 (Bouchut) *Let $\eta_c(u)$ be a convex function. We assume that $(\mathcal{H}^1_{\eta_c})$ is satisfied. Then the flux vector splitting scheme (1.9)-(1.10) satisfies the discrete entropy inequalities*

$$\eta_c(U_j^{n+1}) - \eta_c(U_j^n) + \lambda(\vartheta^n_{\eta_c,j+1/2} - \vartheta^n_{\eta_c,j-1/2}) = S^{n+1}_{\eta_c,j} \leq 0, \tag{2.5}$$

where

$$\vartheta^n_{\eta_c,j+1/2} = \vartheta^+_{\eta_c}(U_j^n) + \vartheta^-_{\eta_c}(U^n_{j+1}), \tag{2.6}$$

$$(\vartheta^\pm_{\eta_c})' = \eta_c'(F^\pm)'. \tag{2.7}$$

Remark 2.1 In this sense η_c is said to be a compatible entropy for the flux vector splitting scheme.

We define here a new notion we call strict η-dissipativity.

Definition 2.3 *Let $\eta(u)$ be a convex function. We say that W is strictly η-dissipative in $\mathcal{U}$ if it is η-dissipative and if there exists $\alpha_W > 0$ such that*

$$D_\eta[W](u,v) \leq -\alpha_W \|u-v\|^2, \qquad \text{for any } u, v \in \mathcal{U}, \tag{2.8}$$

where $\|.\|$ is the euclidian norm.

This notion can be seen as a non degeneracy property. More precisely, we have:

Lemma 2.4 *Let $\eta(u)$ be a convex function. If W is strictly η-dissipative in $\mathcal{U}$, then $W \in \mathcal{M}^\eta_{\alpha_W}$, where*

$$\mathcal{M}^\eta_\alpha = \{W : \mathcal{U} \to \mathbb{R}^N \,;\, W \in \mathcal{M}^\eta_+ \text{ and } (W')^t\eta'' \geq 2\alpha \text{ in } \mathcal{U}\}.$$

Proof. From (2.4), we have just to prove that $(W')^t\eta'' \geq 2\alpha$. The property (2.8) implies that $D_\eta[W](u,.) + \alpha_W(u-.)^2$ admits a maximum at u. Its second derivative is $-W'(u)^t\eta''(u) + 2\alpha_W$ and must be nonpositive. This gives the result. □

We set now the second hypothesis for our hyperbolic system, that is:

$$Id - \lambda(F^+ - F^-) \text{ and } (F^+ \text{ or } \ -F^-) \text{ are strictly } \eta_c - \text{dissipative.} \quad (\mathcal{H}^2_{\eta_c})$$

These notions are what we need to get the estimates on the derivates of the approximation U_Δ. We notice that in the continuous case [4], the estimates were obtained by a rather different way, that is from an averaging lemma adapted to a kinetic equation with a BGK relaxation right-hand side term.

Scalar case. In the scalar case ($N = 1$), if η_c is strictly convex, $(\mathcal{H}^1_{\eta_c})$ is related to the classical conditions $(F^+)' \geq 0$, $(-F^-)' \geq 0$ and $\lambda[(F^+)' - (F^-)'] \leq 1$. The last inequality is a refined way to state that the eigenvalues of $(F^+)' - (F^-)'$ are less than $\Delta x/\Delta t$. Condition $(\mathcal{H}^1_{\eta_c})$ is related to non degeneracy in the sense that we avoid the eigenvalue 0 and the possibility for the CFL to be egal to 1. This is classical since the stability of a scheme requires λ to satisfy a CFL condition as $2\lambda \max(|\lambda_i^+(u)|, |\lambda_i^-(u)|) < 1$, where $\lambda_i^\pm(u)$ are the eigenvalues of $F^\pm$.

2.2 Proof of the estimate on the gradient

We have the following estimate on the approximate solution.

Proposition 2.5 *Let $\eta_c(u)$ be a convex function and $U^0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ be an initial data such that $\eta_c(U^0) \in L^1(\mathbb{R})$. We assume that $(\mathcal{H}^1_{\eta_c})$, $(\mathcal{H}^2_{\eta_c})$ and (1.11) are satisfied. Then there exists a constant C_1 such that*

$$\left(\int_0^T \int_{\mathbb{R}} |\partial_x U_\Delta(t,x)|^2 \, dx \, dt\right)^{1/2} \leq \frac{C_1}{\sqrt{\Delta x}}, \tag{2.9}$$

$$\left(\int_0^T \int_{\mathbb{R}} |\partial_t U_\Delta(t,x)|^2 \, dx \, dt\right)^{1/2} \leq \frac{C_1}{\sqrt{\Delta x}}. \tag{2.10}$$

Proof. We use the following decomposition form [6],

$$S^{n+1}_{\eta_c,j} = S^{n+1}_{\eta_c,j}(-1) + S^{n+1}_{\eta_c,j}(0) + S^{n+1}_{\eta_c,j}(1), \tag{2.11}$$

where

$$S^{n+1}_{\eta_c,j}(-1) = \lambda\left[-\vartheta^-_{\eta_c}(U^{n+1}_j) + \vartheta^-_{\eta_c}(U^n_{j+1}) - \eta_c'(U^{n+1}_j)(F^-(U^n_{j+1}) - F^-(U^{n+1}_j)\right],$$

$$S^{n+1}_{\eta_c,j}(1) = \lambda\left[\vartheta^+_{\eta_c}(U^{n+1}_j) - \vartheta^+_{\eta_c}(U^n_{j-1}) + \eta_c'(U^{n+1}_j)(F^+(U^n_{j-1}) - F^+(U^{n+1}_j)\right],$$

and

$$\begin{aligned} S^{n+1}_{\eta_c,j}(0) \;=\;& \eta_c(U^{n+1}_j) - \lambda(\vartheta^+_{\eta_c}(U^{n+1}_j) - \vartheta^-_{\eta_c}(U^{n+1}_j)) \\ & - \left[\eta_c(U^n_j) - \lambda(\vartheta^+_{\eta_c}(U^n_j) - \vartheta^-_{\eta_c}(U^n_j))\right] \\ & + \eta_c'(U^{n+1}_j)\left[U^n_j - \lambda(F^+(U^n_j) - F^-(U^n_j))\right. \\ & \qquad \left. - U^{n+1}_j + \lambda(F^+(U^{n+1}_j) - F^-(U^{n+1}_j))\right]. \end{aligned}$$

From the hypothesis $(\mathcal{H}^1_{\eta_c})$, we notice that $S^{n+1}_{\eta_c,j}(k) \le 0$ for any $k = -1, 0, 1$ since each term has the form of an entropy dissipation term (2.3) associated to the M of (3.7).
Summing with respect to j the equalities (2.5) gives

$$E_n - E_{n+1} = -\sum_j S^{n+1}_{\eta_c,j}(-1)\Delta x - \sum_j S^{n+1}_{\eta_c,j}(1)\Delta x - \sum_j S^{n+1}_{\eta_c,j}(0)\Delta x, \quad (2.12)$$

where

$$E_n = \sum_j \eta_c(U_j^n)\Delta x.$$

We notice that the flux terms vanished thanks to the hypothesis on the limit in $\pm\infty$. We obtain

$$E_n \le E_0,$$

and

$$0 \le -\sum_n \sum_j S^{n+1}_{\eta_c,j}(k)\Delta x \le E_0, \qquad \text{for } k = -1, 0, 1, \quad (2.13)$$

by summing (2.12) with respect to n.
Since $G_{\eta_c}[F^+] = \vartheta^+_{\eta_c}$, this property with $k = 1$ can be rewritten as

$$0 \le -\sum_n \sum_j D_{\eta_c}[F^+](U_j^{n+1}, U_{j-1}^n)\Delta x \le \frac{E_0}{\lambda}.$$

If F^+ is strictly η_c-dissipative, it gives

$$\sum_n \sum_j (U_j^{n+1} - U_{j-1}^n)^2 \Delta x \le \frac{E_0}{\lambda \alpha_{F^+}}, \quad (2.14)$$

in the sense that it is true for all the components, and if it is F^-, we have

$$\sum_n \sum_j (U_j^{n+1} - U_{j+1}^n)^2 \Delta x \le \frac{E_0}{\lambda \alpha_{F^-}}. \quad (2.15)$$

The property, this time for $W = Id - \lambda(F^+ - F^-)$, leads to

$$\sum_n \sum_j (U_j^{n+1} - U_j^n)^2 \Delta x \le \frac{E_0}{\lambda \alpha_W}. \quad (2.16)$$

Now, E_0 is bounded since, by the convexity of η_c,

$$\eta_c(U_j^0) \le \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \eta_c(U^0(y))\,dy,$$

thus

$$E_0 \le \int_{\mathbb{R}} \eta_c(U^0(y))\,dy.$$

We get

$$\sum_n \sum_j (U_{j+1}^n - U_j^n)^2 \leq \frac{C}{\Delta x}, \qquad \sum_n \sum_j (U_j^{n+1} - U_j^n)^2 \leq \frac{C}{\Delta x}, \tag{2.17}$$

for a constant C. Now

$$\partial_x U_\Delta = \frac{t - t_n}{\Delta t} \frac{U_{j+1}^{n+1} - U_j^{n+1} - U_{j+1}^n + U_j^n}{\Delta x} + \frac{U_{j+1}^n - U_j^n}{\Delta x},$$

on $]t_n, t_{n+1}[\times]x_j, x_{j+1}[$, thus

$$\begin{aligned} & \int_{t_n}^{t_{n+1}} \int_{x_j}^{x_{j+1}} |\partial_x U_\Delta(t,x)|^2 \, dx \, dt \\ \leq \;& 3\lambda \left[(U_{j+1}^n - U_j^n)^2 + (U_{j+1}^{n+1} - U_{j+1}^n)^2 + (U_j^n - U_j^{n+1})^2 \right], \end{aligned}$$

and using (2.17), we get the first result by summing this over j and n. Similarly, we have

$$\partial_t U_\Delta = \frac{1}{\Delta t} \left[\frac{U_{j+1}^{n+1} - U_j^{n+1} - U_{j+1}^n + U_j^n}{\Delta x} (x - x_{j-1/2}) + U_j^{n+1} - U_j^n \right],$$

on $]t_n, t_{n+1}[\times]x_j, x_{j+1}[$, thus

$$\begin{aligned} & \int_{t_n}^{t_{n+1}} \int_{x_j}^{x_{j+1}} |\partial_t U_\Delta(t,x)|^2 \, dt \, dx \\ \leq \;& 3\lambda \left[(U_{j+1}^{n+1} - U_j^{n+1})^2 + (U_{j+1}^n - U_j^n)^2 + (U_j^{n+1} - U_j^n)^2 \right], \end{aligned}$$

and using (2.17), we get the second result. □

Remark 2.2 The hypothesis (1.11) is related to the limit when $M, N \to +\infty$ of

$$\theta_{\eta_c}^{\pm}(U_M^n) - \theta_{\eta_c}^{\pm}(U_{-N-1}^n),$$

thus we can remove this hypothesis in placing us in a periodic context with respect to x.

We turn now to the convergence result. In order to do this, we introduce the kinetic formulation of the flux vector splitting scheme.

3 Kinetic formulation of the flux-splitting scheme and convergence result

3.1 The kinetic formulation

The transport-projection method is a time-discrete version of a kinetic BGK equation. Furthermore, it is possible to represent a flux-splitting scheme with such a kinetic model with three velocities, see [6]. In fact, in this reference, it is proved that any kinetic scheme is first order equivalent to a three velocity relaxation model. We also refer to [1] and the references therein for discrete kinetic schemes. In our present work, the kinetic model can be written as:

$$\partial_t f + a(\xi)\partial_x f = \sum_{n=1}^{+\infty} \delta(t-t_n)(Mf^n - f^n), \tag{3.1}$$

where

$$Mf^n(x,\xi) = M(U_j^n,\xi) \qquad \text{for } x \in]x_j, x_{j+1}[, \tag{3.2}$$

with

$$U_j^n = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \tilde{U}_j^n(x)\,dx, \qquad \text{for } x \in]x_j, x_{j+1}[, \tag{3.3}$$

and

$$\tilde{U}_j^n(x) = \int f^n(x,v)\,dv, \qquad \text{for } x \in]x_j, x_{j+1}[, \tag{3.4}$$

where

$$f^n(x,\xi) = \lim_{t\to t_n^-} f(t,x,\xi), \text{ for } n \geq 1, \tag{3.5}$$

and the initial condition

$$f^0(x,\xi) = M(U_j^0,\xi), \qquad x \in]x_j, x_{j+1}[, \qquad U_j^0 \text{ defined by (1.8)}, \tag{3.6}$$

and with $\xi \in \{-1,0,1\}$, $a(-1) = -c$, $a(0) = 0$, $a(1) = c$, $c = 1/\lambda$ and the Maxwellian defined by

$$\begin{array}{rcl} M(u,-1) & = & -\lambda F^-(u), \\ M(u,0) & = & u - \lambda(F^+(u) - F^-(u)), \\ M(u,1) & = & \lambda F^+(u). \end{array} \tag{3.7}$$

We notice that the integral with respect to the ξ variable means here the sum over $\xi \in \{-1,0,1\}$. The maxwellian satisfies the consistency relations

$$\int M(U,\xi)\,d\xi = U, \qquad \int a(\xi)M(U,\xi)\,d\xi = F(U), \tag{3.8}$$

$$\int_{\pm a(\xi)>0} a(\xi)M(U,\xi)\,d\xi = F^{\pm}(U). \tag{3.9}$$

On $[t_n, t_{n+1}[$, we have

$$\begin{array}{rcl} f(t,x,-1) & = & f(t_n, x + c(t-t_n), -1), \\ f(t,x,0) & = & f(t_n,x,0), \\ f(t,x,1) & = & f(t_n, x - c(t-t_n), 1), \end{array}$$

thus, for $x \in]x_j, x_{j+1}[$,

$$\begin{array}{rcl} f^{n+1}(x,-1) = & f(t_n, x+\Delta x, -1) & = M(U_{j+1}^n, -1), \\ f^{n+1}(x,0) = & f(t_n,x,0) & = M(U_j^n, 0), \\ f^{n+1}(x,1) = & f(t_n, x-\Delta x, 1) & = M(U_{j-1}^n, 1). \end{array}$$

With this velocities, $f^n(x,v)$ is constant in x on $]x_j, x_{j+1}[$, thus $U_j^n = \tilde{U}_j^n(x)$ on $]x_j, x_{j+1}[$. Now we compute on $]x_j, x_{j+1}[$, for $n \geq 0$,

$$\begin{aligned}
U_j^{n+1} &= \int f^{n+1}(x,v)\,dv \\
&= f^{n+1}(x,-1) + f^{n+1}(x,0) + f^{n+1}(x,1) \\
&= M(U_{j+1}^n,-1) + M(U_j^n,0) + M(U_{j-1}^n,1) \\
&= -\lambda F^-(U_{j+1}^n) + U_j^n - \lambda(F^+(U_j^n) - F^-(U_j^n)) + \lambda F^+(U_{j-1}^n).
\end{aligned}$$

Thus we recover the flux vector splitting scheme and the kinetic model is a formulation of it.
We turn now to estimates between U_Δ and f.

3.2 Dissipation of the flux

As in the continuous case, we need to prove an estimate on $F(U_\Delta) - \int a(\xi) f\, d\xi$.

Proposition 3.1 *Let $\eta_c(u)$ be a convex function and $U^0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ be an initial data such that $\eta_c(U^0) \in L^1(\mathbb{R})$. We assume that (1.3), $(\mathcal{H}^1_{\eta_c})$ and $(\mathcal{H}^2_{\eta_c})$ are satisfied. Then there exists a constant C_2 such that*

$$\left(\int_0^T \int_{\mathbb{R}} \left| F(U_\Delta(t,x)) - \int a(\xi) f(t,x,\xi)\,d\xi \right|^2 dx\,dt\right)^{1/2} \leq C_2\sqrt{\Delta x}. \tag{3.10}$$

Proof. On $[t_n, t_{n+1}[\times]x_j, x_{j+1}[$, we have

$$\begin{aligned}
\int a(\xi) f(t,x,\xi)\,d\xi &= -cf(t,x,-1) + cf(t,x,1) \\
&= -cf(t_n, x + c(t-t_n), -1) + cf(t_n, x - c(t-t_n), 1).
\end{aligned}$$

For $t \in]t_n, t_n + \Delta t/2[$ and $x \in]x_j, x_{j+1}[$, we have

$$\int a(\xi) f(t,x,\xi)\,d\xi = \begin{cases}
-cM(U_j^n,-1) + cM(U_{j-1}^n,1) \\
\qquad\qquad \text{for } x \in]x_j, x_j + c(t-t_n)[, \\
-cM(U_j^n,-1) + cM(U_j^n,1) \\
\qquad\qquad \text{for } x \in]x_j + c(t-t_n), x_{j+1} - c(t-t_n)[, \\
-cM(U_{j+1}^n,-1) + cM(U_j^n,1) \\
\qquad\qquad \text{for } x \in]x_{j+1} - c(t-t_n), x_{j+1}[,
\end{cases}$$

thus

$$\int a(\xi) f(t,x,\xi)\,d\xi = \begin{cases}
F^+(U_{j-1}^n) + F^-(U_j^n) \\
\qquad\qquad \text{for } x \in]x_j, x_j + c(t-t_n)[, \\
F(U_j^n) \\
\qquad\qquad \text{for } x \in]x_j + c(t-t_n), x_{j+1} - c(t-t_n)[, \\
F^+(U_j^n) + F^-(U_{j+1}^n) \\
\qquad\qquad \text{for } x \in]x_{j+1} - c(t-t_n), x_{j+1}[,
\end{cases}$$

and for $t \in]t_n + \Delta t/2, t_{n+1}[$ and $x \in]x_j, x_{j+1}[$, we have

$$\int a(\xi) f(t,x,\xi)\, d\xi = \begin{cases} -cM(U_j^n, -1) + cM(U_{j-1}^n, 1) \\ \qquad\qquad \text{for } x \in]x_j, x_{j+1} - c(t-t_n)[, \\ -cM(U_{j+1}^n, -1) + cM(U_{j-1}^n, 1) \\ \qquad\qquad \text{for } x \in]x_{j+1} - c(t-t_n), x_j + c(t-t_n)[, \\ -cM(U_{j+1}^n, -1) + cM(U_j^n, 1) \\ \qquad\qquad \text{for } x \in]x_j + c(t-t_n), x_{j+1}[, \end{cases}$$

thus

$$\int a(\xi) f(t,x,\xi)\, d\xi = \begin{cases} F^+(U_{j-1}^n) + F^-(U_j^n) \\ \qquad\qquad \text{for } x \in]x_j, x_{j+1} - c(t-t_n)[, \\ F^+(U_{j-1}^n) + F^-(U_{j+1}^n) \\ \qquad\qquad \text{for } x \in]x_{j+1} - c(t-t_n), x_j + c(t-t_n)[, \\ F^+(U_j^n) + F^-(U_{j+1}^n) \\ \qquad\qquad \text{for } x \in]x_j + c(t-t_n), x_{j+1}[. \end{cases}$$

Now, from the Lipschitz property, we get the existence of K_1 such that

$$\begin{aligned} \left| F(U_\Delta) - \int a(\xi) f\, d\xi \right| &\leq K_1 \left[|U_\Delta - U_{j+1}^n| + |U_\Delta - U_{j-1}^n| + |U_\Delta - U_j^n| \right] \\ &\leq 2K_1 \left[|U_j^n - U_{j+1}^n| \left| 1 - \frac{x - x_j}{\Delta x} \right| \right. \\ &\qquad \left. + |U_{j-1}^n - U_{j+1}^n| + |U_{j+1}^n - U_j^n| \right] \\ &\leq 4K_1 \left[|U_j^n - U_{j+1}^n| + |U_{j-1}^n - U_{j+1}^n| \right], \end{aligned}$$

and thus on $[t_n, t_{n+1}[\times]x_j, x_{j+1}[$, we obtain

$$\left| F(U_\Delta) - \int a(\xi) f\, d\xi \right|^2 \leq 32 K_1^2 \left[|U_j^n - U_{j+1}^n|^2 + |U_{j-1}^n - U_{j+1}^n|^2 \right], \tag{3.11}$$

and

$$\begin{aligned} &\int_{t_n}^{t_{n+1}} \int_{x_j}^{x_{j+1}} \left| F(U_\Delta) - \int a(\xi) f\, d\xi \right|^2 dx\, dt \\ \leq\ & 32 \lambda K_1^2 (\Delta x)^2 \left[|U_j^n - U_{j+1}^n|^2 + |U_{j-1}^n - U_{j+1}^n|^2 \right]. \end{aligned}$$

It gives finally summing with respect to j and n and using the relation (2.17),

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \left| F(U_\Delta(t,x)) - \int a(\xi) f(t,x,\xi)\, d\xi \right|^2 dx\, dt \\ \leq\ & \sum_n \sum_j (\Delta x)^2 \lambda 32 K_1^2 \left[|U_j^n - U_{j+1}^n|^2 + |U_{j-1}^n - U_{j+1}^n|^2 \right] \\ \leq\ & K_2 \Delta x \end{aligned}$$

for a constant K_2 . $\Box$

Now we turn to the link between $\int f\,d\xi$ and U_Δ. On contrary to the continuous case where the approximation is just the integral of the kinetic function with respect to the kinetic variable, we do not have such relation here, and we have to estimate the difference between these two quantities, that is to say that $\int f\,d\xi$ is closed to U_Δ. It is the following proposition:

Proposition 3.2 *Let $\eta_c(u)$ be a convex function and $U^0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ be an initial data such that $\eta_c(U^0) \in L^1(\mathbb{R})$. We assume that (1.3), $(\mathcal{H}^1_{\eta_c})$ and $(\mathcal{H}^2_{\eta_c})$ are satisfied. Then there exists a constant C_3 such that*

$$\left(\int_0^T \int_{\mathbb{R}} \left| U_\Delta(t,x) - \int f(t,x,\xi)\,d\xi \right|^2 dx\,dt \right)^{1/2} \leq C_3 \sqrt{\Delta x}. \tag{3.12}$$

Proof. We have

$$\int f(t,x,\xi)\,d\xi = f(t_n, x + c(t-t_n), -1) + f(t_n, x, 0) + f(t_n, x - c(t-t_n), 1),$$

therefore, for $t \in]t_n, t_n + \Delta t/2[$ and $x \in]x_j, x_{j+1}[$, we obtain

$$\int f(t,x,\xi)\,d\xi = \begin{cases} M(U_j^n,-1) + M(U_j^n,0) + M(U_{j-1}^n,1) \\ \qquad\qquad \text{for } x \in]x_j, x_j + c(t-t_n)[, \\ M(U_j^n,-1) + M(U_j^n,0) + M(U_j^n,1) \\ \qquad\qquad \text{for } x \in]x_j + c(t-t_n), x_{j+1} - c(t-t_n)[, \\ M(U_{j+1}^n,-1) + M(U_j^n,0) + M(U_j^n,1) \\ \qquad\qquad \text{for } x \in]x_{j+1} - c(t-t_n), x_{j+1}[, \end{cases}$$

thus

$$\int f(t,x,\xi)\,d\xi = \begin{cases} U_j^n + \lambda[F^+(U_{j-1}^n) - F^+(U_j^n)] \\ \qquad\qquad \text{for } x \in]x_j, x_j + c(t-t_n)[, \\ U_j^n \\ \qquad\qquad \text{for } x \in]x_j + c(t-t_n), x_{j+1} - c(t-t_n)[, \\ U_j^n + \lambda[F^-(U_j^n) - F^-(U_{j+1}^n)] \\ \qquad\qquad \text{for } x \in]x_{j+1} - c(t-t_n), x_{j+1}[, \end{cases}$$

and for $t \in]t_n + \Delta t/2, t_{n+1}[$ and $x \in]x_j, x_{j+1}[$, we have

$$\int f(t,x,\xi)\,d\xi = \begin{cases} M(U_j^n,-1) + M(U_j^n,0) + M(U_{j-1}^n,1) \\ \qquad\qquad \text{for } x \in]x_j, x_{j+1} - c(t-t_n)[, \\ M(U_{j+1}^n,-1) + M(U_j^n,0) + M(U_{j-1}^n,1) \\ \qquad\qquad \text{for } x \in]x_{j+1} - c(t-t_n), x_j + c(t-t_n)[, \\ M(U_{j+1}^n,-1) + M(U_j^n,0) + M(U_j^n,1) \\ \qquad\qquad \text{for } x \in]x_j + c(t-t_n), x_{j+1}[, \end{cases}$$

thus

$$\int f(t,x,\xi)\,d\xi = \begin{cases} U_j^n + \lambda[F^+(U_{j-1}^n) - F^+(U_j^n)] \\ \qquad\qquad\qquad \text{for } x \in]x_j, x_{j+1} - c(t-t_n)[, \\ U_j^n + \lambda[F^+(U_{j-1}^n) - F^+(U_j^n)] + \lambda[F^-(U_j^n) - F^-(U_{j+1}^n)] \\ \qquad\qquad\qquad \text{for } x \in]x_{j+1} - c(t-t_n), x_j + c(t-t_n)[, \\ U_j^n + \lambda[F^-(U_j^n) - F^-(U_{j+1}^n)] \\ \qquad\qquad\qquad \text{for } x \in]x_j + c(t-t_n), x_{j+1}[, \end{cases}$$

which can also be rewritten as

$$\int f(t,x,\xi)\,d\xi = \begin{cases} U_j^{n+1} - \lambda[F^-(U_j^n) - F^-(U_{j+1}^n)] \\ \qquad\qquad\qquad \text{for } x \in]x_j, x_{j+1} - c(t-t_n)[, \\ U_j^{n+1} \\ \qquad\qquad\qquad \text{for } x \in]x_{j+1} - c(t-t_n), x_j + c(t-t_n)[, \\ U_j^{n+1} - \lambda[F^+(U_{j-1}^n) - F^+(U_j^n)] \\ \qquad\qquad\qquad \text{for } x \in]x_j + c(t-t_n), x_{j+1}[. \end{cases}$$

We set

$$U_\Delta^1(t,x) = \begin{cases} U_j^n & \text{for } (t,x) \in]t_n, t_n + \Delta t/2[\times]x_j, x_{j+1}[, \\ 0 & \text{for } (t,x) \in]t_n + \Delta t/2, t_{n+1}[\times]x_j, x_{j+1}[, \end{cases}$$

and

$$U_\Delta^2(t,x) = \begin{cases} 0 & \text{for } (t,x) \in]t_n, t_n + \Delta t/2[\times]x_j, x_{j+1}[, \\ U_j^{n+1} & \text{for } (t,x) \in]t_n + \Delta t/2, t_{n+1}[\times]x_j, x_{j+1}[. \end{cases}$$

On $t \in]t_n, t_n + \Delta t/2[$, $\int f\,d\xi$ is near U_j^n:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left| \int f(t,x,\xi)\,d\xi \, 1\!\!1_{]t_n, t_n+\Delta t/2[}(t) - U_\Delta^1(t,x) \right|^2 dx\,dt \\ = \; & \sum_n \sum_j \int_{t_n}^{t_n+\Delta t/2} \int_{x_j}^{x_{j+1}} \left| \int f(t,x,\xi)\,d\xi - U_j^n \right|^2 dx\,dt \\ = \; & \sum_n \sum_j \frac{\Delta t}{2} \Bigg[\int_{x_j}^{x_j + c(t-t_n)} \lambda^2 \left| F^+(U_{j-1}^n) - F^+(U_j^n) \right|^2 dx \\ & \qquad + \int_{x_{j+1}-c(t-t_n)}^{x_{j+1}} \lambda^2 \left| F^-(U_j^n) - F^-(U_{j+1}^n) \right|^2 dx \Bigg] \\ \leq \; & \sum_n \sum_j \lambda^2 \left| F^+(U_{j-1}^n) - F^+(U_j^n) \right|^2 \frac{(\Delta x)^2}{2} \\ & + \sum_n \sum_j \lambda^2 \left| F^-(U_j^n) - F^-(U_{j+1}^n) \right|^2 \frac{(\Delta x)^2}{2} \end{aligned}$$

$$\begin{aligned} &\leq \quad K\lambda^2 \Delta x \sum_n \sum_j \left| U_j^n - U_{j+1}^n \right|^2 \Delta x \\ &\leq \quad K' \Delta x, \end{aligned}$$

and on $t \in]t_n + \Delta t/2, t_{n+1}[$, $\int f \, d\xi$ is near U_j^{n+1} and we have similarly

$$\int_0^T \int_{\mathbb{R}} \left| \int f(t,x,\xi)\, d\xi \, \mathbb{1}_{]t_n+\Delta t/2, t_{n+1}[}(t) - U_\Delta^2(t,x) \right|^2 dx\, dt \leq K' \Delta x.$$

It gives

$$\int_0^T \int_{\mathbb{R}} \left| \int f(t,x,\xi)\, d\xi - U_\Delta^1(t,x) - U_\Delta^2(t,x) \right|^2 dx\, dt \leq 2K' \Delta x.$$

Now

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \left| U_\Delta(t,x) - U_\Delta^1(t,x) - U_\Delta^2(t,x) \right|^2 dx\, dt \\ \leq \quad & \sum_n \sum_j \int_{t_n}^{t_n+\Delta t/2} \int_{x_j}^{x_{j+1}} \left| \frac{U_{j+1}^n - U_j^n}{\Delta x} (x - x_j) \right|^2 dx\, dt \\ & + \sum_n \sum_j \int_{t_n+\Delta t/2}^{t_{n+1}} \int_{x_{j+1}}^{x_{j+2}} \left| \frac{U_{j+2}^n - U_{j+1}^n}{\Delta x} (x - x_{j+1}) \right|^2 dx\, dt \\ \leq \quad & \frac{\lambda}{3} \sum_n \sum_j |U_{j+1}^n - U_j^n|^2 (\Delta x)^2 \\ \leq \quad & K'' \Delta x, \end{aligned}$$

therefore

$$\int_0^T \int_{\mathbb{R}} \left| \int f(t,x,\xi)\, d\xi - U_\Delta(t,x) \right|^2 dx\, dt \leq C \Delta x. \ \square$$

We are now able to prove the convergence result.

3.3 Proof of Theorem 1.1

An integration with respect to ξ of (3.1) gives

$$\partial_t U_\Delta + \partial_x F(U_\Delta) = \partial_t (U_\Delta - \int f \, d\xi) + \partial_x (F(U_\Delta) - \int a(\xi) f \, d\xi), \qquad (3.13)$$

since

$$\int M f^n \, d\xi - \int f^n \, d\xi = 0.$$

We want now to control all entropies of the family $\mathcal{F}$ in order to apply the compensated compactness. Thus we multiplty (3.13) by $\eta'(U_\Delta)$ and we get, for any entropy-entropy flux (η, G), the following decomposition

$$\begin{aligned} \partial_t \eta(U_\Delta) + \partial_x G(U_\Delta) \quad &= \quad \eta'(U_\Delta) \cdot \partial_t \left(U_\Delta - \int f \, d\xi \right) \\ &\quad\ + \eta'(U_\Delta) \cdot \partial_x \left(F(U_\Delta) - \int a(\xi) f \, d\xi \right) \\ &= \quad R_1 + M_1 + R_2 + M_2, \end{aligned}$$

where

$$
\begin{aligned}
R_1 &= \partial_t \left[\eta'(U_\Delta) \cdot \left(U_\Delta - \int f \, d\xi \right) \right], \\
M_1 &= -\eta''(U_\Delta) \cdot \partial_t U_\Delta \cdot \left(U_\Delta - \int f \, d\xi \right), \\
R_2 &= \partial_x \left[\eta'(U_\Delta) \cdot \left(F(U_\Delta) - \int a(\xi) f \, d\xi \right) \right], \\
M_2 &= -\eta''(U_\Delta) \cdot \partial_x U_\Delta \cdot \left(F(U_\Delta) - \int a(\xi) f \, d\xi \right).
\end{aligned}
$$

First we have using Proposition 3.1,

$$
\begin{aligned}
& \int_0^T \int_{-R}^R \left| \eta'(U_\Delta) \cdot \left(F(U_\Delta) - \int a(\xi) f \, d\xi \right) \right|^2 dx \, dt \\
\leq \; & \|\eta'(U_\Delta)\|^2_{L^\infty(]0,T[\times]-R,R[)} \int_0^T \int_{\mathbb{R}} \left| F(U_\Delta) - \int a(\xi) f \, d\xi \right|^2 dx \, dt \\
\leq \; & C_R \sqrt{\Delta x},
\end{aligned}
$$

thus R_2 goes to zero in H^{-1}_{loc} as $\Delta x \to 0$. Similarly, using Proposition 3.2, R_1 goes to zero in H^{-1}_{loc} as $\Delta x \to 0$.
Furthermore, using propositions 2.5 and 3.2, we have

$$
\begin{aligned}
& \int_0^T \int_{-R}^R |M_1| \, dx \, dt \\
\leq \; & \|\eta''(U_\Delta)\|_{L^\infty(]0,T[\times]-R,R[)} \left(\iint |\partial_t U_\Delta|^2 \, dx \, dt \right)^{1/2} \left(\iint \left| U_\Delta - \int f \, d\xi \right|^2 dx \, dt \right)^{1/2} \\
\leq \; & \|\eta''(U_\Delta)\|_{L^\infty(]0,T[\times]-R,R[)} \frac{C_1}{\sqrt{\Delta x}} C_3 \sqrt{\Delta x} \\
\leq \; & C_R,
\end{aligned}
$$

thus M_1 is bounded in $\mathcal{M}_{loc}([0,T]\times\mathbb{R})$. Similarly, M_2 is bounded in $\mathcal{M}_{loc}([0,T]\times\mathbb{R})$. As a consequence, $\partial_t \eta(U_\Delta) + \partial_x G(U_\Delta)$ is compact in H^{-1}_{loc}. Furthermore, $(U_\Delta)_{\Delta>0}$ is bounded since we assume that $(U^n_j)_{n,j}$ is a bounded sequence. We are now able to apply compensated compactness and we get that up to a subsequence $U_\Delta \to U$ a.e. and in L^1_{loc}. We pass to the limit in (3.13) using propositions 3.1 and 3.2 and we get that U is a solution to the system. By classical arguments, since we have the discrete entropy inequality (2.5), we get (1.7). $\square$

4 Application to the p-system

In Section 2.1, we apply the result to the scalar case. Following [6], we study now the structure of the p-system in order to explicite in this example the properties $(\mathcal{H}^1_{\eta_c})$ and $(\mathcal{H}^2_{\eta_c})$. We recall that the p-system is

$$
\begin{cases} \partial_t \rho - \partial_x u = 0, \\ \partial_t u + \partial_x p(\rho) = 0, \end{cases} \tag{4.1}
$$

where $\rho, u \in \mathbb{R}$ and the nonlinearity $p(\rho)$ satisfies $p'(\rho) < 0$. The entropy is given by

$$\eta_c(\rho, u) = \frac{u^2}{2} + E(\rho), \qquad E'(\rho) = p(\rho), \tag{4.2}$$

and the flux-splitting we use is

$$F^+(\rho, u) = \left(\frac{1}{2c}p(\rho) + \frac{u}{2}\right)(-1, c), \qquad F^-(\rho, u) = \left(\frac{1}{2c}p(\rho) - \frac{u}{2}\right)(1, c), \tag{4.3}$$

for a constant c. We have

$$D_{\eta_c}[F^+]((\rho_1, u_1), (\rho_2, u_2)) = -\frac{1}{4c}(p(\rho_1) + cu_1 - p(\rho_2) - cu_2)^2, \tag{4.4}$$

$$D_{\eta_c}[-F^-]((\rho_1, u_1), (\rho_2, u_2)) = -\frac{1}{4c}(p(\rho_1) - cu_1 - p(\rho_2) + cu_2)^2, \tag{4.5}$$

and

$$D_{\eta_c}[Id - \lambda(F^+ - F^-)]((\rho_1, u_1), (\rho_2, u_2)) = Z(\rho_1, \rho_2) - \frac{1}{2}(1 - \lambda c)(u_1 - u_2)^2, \tag{4.6}$$

where

$$Z(\rho_1, \rho_2) = E(\rho_1) - E(\rho_2) - p(\rho_1)(\rho_2 - \rho_1) + \frac{\lambda}{2c}(p(\rho_2) - p(\rho_1))^2. \tag{4.7}$$

We notice first that F^+ and $-F^-$ are η_c-dissipative. We compute that

$$\partial_{\rho_2} Z(\rho_1, \rho_2) = (p(\rho_2) - p(\rho_1))\left(1 + \frac{\lambda}{c}p'(\rho_2)\right),$$

therefore assuming that

$$0 < \lambda\, c < 1, \tag{4.8}$$

and the existence of $\gamma \in]0, 1[$ and $\beta < 0$ such that

$$(\gamma - 1)\frac{c}{\lambda} \leq p'(\rho) \leq \beta < 0, \tag{4.9}$$

we get

$$D_{\eta_c}[Id - \lambda(F^+ - F^-)]((\rho_1, u_1), (\rho_2, u_2)) \leq -\frac{1}{2}(1 - \lambda c)(u_1 - u_2)^2 + \frac{1}{2}\gamma\beta(\rho_2 - \rho_1)^2, \tag{4.10}$$

thus $Id - \lambda(F^+ - F^-)$ is strictly η_c-dissipative. Now, for the strictly η_c-dissipativity of F^+ or $-F^-$, we notice that

$$\begin{aligned}
& D_{\eta_c}[\pm F^\pm]((\rho_1, u_1), (\rho_2, u_2)) \\
= \; & -\frac{1}{4c}\Big((p(\rho_1) - p(\rho_2))^2 + c^2(u_1 - u_2)^2 \pm 2c(p(\rho_1) - p(\rho_2))(u_1 - u_2)\Big), \\
\leq \; & -\frac{1}{4c}\Big(\beta^2(\rho_1 - \rho_2)^2 + c^2(u_1 - u_2)^2 \pm 2c(p(\rho_1) - p(\rho_2))(u_1 - u_2)\Big),
\end{aligned}$$

thus if the approximations stay in a zone such that

$$(p(\rho_1) - p(\rho_2))(u_1 - u_2) \text{ keeps a constant sign,} \tag{4.11}$$

$(\mathcal{H}^2_{\eta_c})$ is also satisfied. The property (4.11) can be relaxed by the existence of $\theta, \tilde{\theta} \in]0,1[$ such that

$$\theta(p(\rho_1) - p(\rho_2))^2 + \tilde{\theta}c^2(u_1 - u_2)^2 \pm 2c(p(\rho_1) - p(\rho_2))(u_1 - u_2) \geq 0, \tag{4.12}$$

which can be explicited and gives a less restrictive condition.

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