Existence and weak stability for a pressureless model with unilateral constraint

F. Berthelin

Université d'Orléans et CNRS, UMR 6628
Département de Mathématiques
BP 6759
45067 Orléans cedex 2, France
e-mail: florent.berthelin@labomath.univ-orleans.fr

Abstract

We consider a two-phase model described by a pressureless gas system with unilateral constraint. We prove weak stability and existence of weak solutions by passing to the limit in the sticky-blocks dynamics. We obtain the maximum principle on the velocity, the Oleinik entropy condition and local entropy inequalities. Initial data are taken in a very weak sense since the solution can jump initially in time.

Key-words: two-phase flow – conservation laws with constraint – pressureless gas – sticky blocks – jump in time

Mathematics Subject Classification: 35L85, 76T10, 35L65, 76N15, 35A35

Contents

1 Introduction and main results

1.1 The model

We consider a two-phase flow of gas and liquid where the liquid is assumed to be incompressible. As proposed in [3], we denote by α the volume fraction of liquid and u its velocity ($1-\alpha$ represents the volume fraction of gas), and we study the model

$$\begin{cases} \partial_t \alpha + \partial_x(\alpha u) = 0, \\ \partial_t(\alpha u) + \partial_x(\alpha u^2 + \pi) = 0, \end{cases} \tag{1.1}$$

with the constraints

$$0 \le \alpha \le 1, \tag{1.2}$$

$$\pi \ge 0, \tag{1.3}$$

and the exclusion relation

$$(1-\alpha)\pi = 0. \tag{1.4}$$

The system (1.1)-(1.4) is a system of pressureless gas with constraint. An important difficulty in such a system is to express the constraint $\alpha\pi = \pi$ which roughly speaking sets π working when $\alpha = 1$, and makes the liquid fraction stay between 0 and 1. Unfortunately, the product of α and π is not well-defined with π a measure and α in L^∞. Besides, if we could apply the classical compensated-compactness theory, we would get with standard notations $\overline{\alpha_n(\alpha_n u_n^2 + \pi_n) - (\alpha_n u_n)^2} = \overline{\alpha_n} \cdot \overline{\alpha_n u_n^2 + \pi_n} - \left(\overline{\alpha_n u_n}\right)^2$, which gives $\overline{\alpha_n \pi_n} = \overline{\alpha_n} \cdot \overline{\pi_n}$ if u_n is known to be compact. The fact that π is only a measure is an obstacle to this computation and to overcome this problem, we decompose $\alpha(\alpha u^2+\pi)-(\alpha u)^2$ in two different ways whether we privilege α or αu. Thus we consider the constraint

$$R_1 = R_2 = \pi, \tag{1.5}$$

where R_1 and R_2 are defined by

$$R_1 = -\partial_t(\alpha Q) - \partial_x(\alpha u Q), \tag{1.6}$$

$$\partial_x Q = \alpha u,\ \partial_t Q = -(\alpha u^2+\pi), \tag{1.7}$$

and

$$R_2 = \partial_t(\alpha u M) + \partial_x((\alpha u^2+\pi)M), \tag{1.8}$$

$$\partial_x M = \alpha,\ \partial_t M = -\alpha u. \tag{1.9}$$

We have to notice that in the case of smooth solutions to (1.1), we have $R_1 = R_2 = \alpha\pi$. We shall come back to this point in Section 6. The reference [12] considered the same problem but with viscosity. We refer to [10] for more details about modelling and two phase-flows. Other hyperbolic problems with constraints have been studied in [1], [9] and [11]. Existence and properties of the system of pressureless gas have been studied in [4], [8] and [6].
We give initial conditions for our problem,

$$\alpha(0,x) = \alpha^0(x),\ \alpha(0,x)u(0,x) = \alpha^0(x)u^0(x). \tag{1.10}$$

Let us define precisely the weak solutions we shall consider. We look for solutions satisfying

$$\alpha \in L^\infty_t(0,\infty; L^\infty_x(\mathbb{R}) \cap L^1_x(\mathbb{R})) \cap C_t([0,\infty[; L^\infty_{w*}(\mathbb{R})), \tag{1.11}$$

$$u \in L^\infty_t(0,\infty; L^\infty_x(\mathbb{R})), \tag{1.12}$$

$$\pi \in \mathcal{M}_{loc}([0,\infty[\times\mathbb{R}) \cap L^\infty(\mathbb{R}_x, \mathcal{M}([0,T])). \tag{1.13}$$

Hence, (1.1), (1.10) must be satisfied in the sense of distributions:
for all $\varphi \in \mathcal{D}([0,+\infty[\times\mathbb{R})$,

$$\int_{[0,+\infty[}\int_{\mathbb{R}}[\alpha\partial_t\varphi + \alpha u\partial_x\varphi]\,dx\,dt + \int_{\mathbb{R}}\alpha^0(x)\varphi(0,x)\,dx = 0, \tag{1.14}$$

$$\int_{[0,+\infty[}\int_{\mathbb{R}}[\alpha u\partial_t\varphi + \alpha u^2\partial_x\varphi]\,dx\,dt + \int_{[0,+\infty[}\int_{\mathbb{R}}\partial_x\varphi\,\pi + \int_{\mathbb{R}}(\alpha^0u^0)(x)\varphi(0,x)\,dx = 0. \tag{1.15}$$

The fact that α lies in $C_t([0,\infty[; L^\infty_{w*}(\mathbb{R}))$ implies with (1.14) that $\alpha(0,x) = \alpha^0(x)$. It is not the case for u and in general αu can have jumps in time, even initially, so that $\alpha(0,x)u(0,x) \neq \alpha^0(x)u^0(x)$. Indeed, $\alpha(t,x) = \mathbb{1}_{[-1,1]}(x)$, $u(t,x) = 0$, $\pi(t,x) = (1-|x|)\mathbb{1}_{-1<x<1}\delta(t)$ is a solution for initial conditions $\alpha^0 = \mathbb{1}_{[-1,1]}$, $\alpha^0u^0 = \mathbb{1}_{[-1,0]} - \mathbb{1}_{[0,1]}$. This represents a collision between two sticky blocks at time 0 and it gives a Dirac distribution at this time. As a result, (1.10) will be considered only in this weak sense (1.15). Similarly, the constraint (1.5) has to be understood as

$$\int_{[0,+\infty[}\int_{\mathbb{R}}[\alpha Q\partial_t\varphi + \alpha uQ\partial_x\varphi]\,dx\,dt + \int_{\mathbb{R}}(\alpha^0Q^0)(x)\varphi(0,x)\,dx = \int_{[0,+\infty[}\int_{\mathbb{R}}\varphi\,\pi \tag{1.16}$$

and

$$-\int_{[0,+\infty[}\int_{\mathbb{R}}[\alpha u M\partial_t\varphi+\alpha u^2 M\partial_x\varphi]\,dx\,dt-\int_{[0,+\infty[}\int_{\mathbb{R}} M\partial_x\varphi\;\pi$$
$$-\int_{\mathbb{R}}(\alpha^0 u^0 M^0)(x)\varphi(0,x)\,dx=\int_{[0,+\infty[}\int_{\mathbb{R}}\varphi\;\pi \tag{1.17}$$

for all $\varphi \in \mathcal{D}([0,+\infty[\times\mathbb{R})$.

Remark 1.1 Distributions Q and M are defined up to an additive constant. In (1.6) and (1.8), it is clearly not a trouble thanks to (1.1), and the same remark holds for the weak formulations (1.16) and (1.17) thanks to (1.14) and (1.15).

We also notice that the relation (1.9) with α, αu in L^∞ gives that M is Lipschitz continuous, so that $M^0 = M(0,.)$ is well-defined and the above writing makes sense. However, the meaning of Q^0 will be precised in Proposition 2.5.

Remark 1.2 There is no uniqueness without further conditions. Indeed, if we consider the initial datum $\alpha^0 = \mathbb{1}_{[-1,1]}$, $\alpha^0 u^0 = 0$, we can define

$$\alpha(t,x) = \mathbb{1}_{-1-t<x<-t} + \mathbb{1}_{t<x<1+t},$$
$$\alpha u(t,x) = -\mathbb{1}_{-1-t<x<-t} + \mathbb{1}_{t<x<1+t}$$

and $\pi = \delta(t)(1-|x|)\mathbb{1}_{-1<x<1}$. Then (α,u,π) solves (1.14)-(1.17). This represents a block which is divided in two at time 0. We also have the trivial solution $\alpha = \mathbb{1}_{[-1,1]}$, $\alpha u = 0$.

1.2 Main results

The main results we prove in this paper are the two following theorems.

Theorem 1.1 *If $\alpha^0 \in L^1(\mathbb{R})$ and $u^0 \in L^\infty(\mathbb{R})$, $0 \le \alpha^0 \le 1$, then there exists (α,u,π) with regularities (1.11) -(1.13), satisfying (1.14)-(1.15) with the constraints (1.2),(1.3),(1.16), (1.17).*

Theorem 1.2 *In addition, the solution (α,u,π) of Theorem 1.1 satisfies*

$$\partial_x u(t,x) \le \tfrac{1}{t}, \tag{1.18}$$
$$\text{essinf}\; u^0(x) \le u(t,x) \le \text{esssup}\; u^0(x), \tag{1.19}$$
$$TV_{[a,b]}(u(t,.)) \le \tfrac{2(b-a)}{t} + 2\|u^0\|_{L^\infty} \quad \forall a<b, \tag{1.20}$$
$$\partial_t(\alpha S(u)) + \partial_x(\alpha u S(u) + \pi_S) \le 0 \; in \;]0,\infty[\times\mathbb{R} \tag{1.21}$$

for every $S \in C^1(\mathbb{R})$ convex and Lipschitz continuous, where π_S lies in $L^\infty(\mathbb{R}_x, \mathcal{M}([0,T]))$ and satisfies

$$|\pi_S| \le Lip(S)\pi. \tag{1.22}$$

The inequality (1.20) is classically a consequence of (1.18) and (1.19). The last inequality (1.21) is a rigorous analogue of the formal computation

$$\partial_t(\alpha S(u)) + \partial_x(\alpha u S(u) + S'(u)\pi) = S''(u)\pi\partial_x u \le 0.$$

In Section 2, we establish the compactness (or stability) of sequences of solutions. In Section 3, we study a class of special solutions, called the sticky blocks. Using the convergence of those functions, we prove in Section 4 our existence theorem. Finally in Section 5, we prove the entropy inequalities of Theorem 1.2 and discuss the meaning of the relation $\alpha\pi = \pi$ in Section 6.

2 Compactness

The result we prove in this section has two specific purposes. First, it will prove the existence of solutions by passage to the limit in some particular solutions which initial data can approximate every initial data in $L^1(\mathbb{R})$. Secondly, it will prove stability of the obtained solutions.

Theorem 2.1 *Let us consider a sequence of solutions (α_n, u_n, π_n) with regularities (1.11) -(1.13), satisfying (1.14)-(1.15) with the constraints (1.2), (1.3), (1.16), (1.17). Initial data α_n^0, u_n^0 are supposed to satisfy*

$$0 \le \alpha_n^0 \le 1, \qquad (\alpha_n^0)_{n\ge 0} \text{ is bounded in } L^1(\mathbb{R}), \tag{2.1}$$

$$(u_n^0)_{n\ge 0} \text{ is bounded in } L^\infty(\mathbb{R}). \tag{2.2}$$

We also suppose that (α_n, u_n, π_n) satisfies (1.18), (1.19), and that

$$(\pi_n)_{n\ge 0} \text{ is bounded in } \mathcal{M}_{loc}([0,\infty[\times\mathbb{R}) \cap L^\infty(\mathbb{R}, \mathcal{M}([0,T])). \tag{2.3}$$

Then, extracting a subsequence if necessary, as $n \to \infty$, $(\alpha_n, u_n, \pi_n) \rightharpoonup (\alpha, u, \pi)$ in the sense of (2.4)-(2.5) and Proposition 2.4, where (α, u, π) has regularities (1.11) -(1.13) and satisfies (1.14)-(1.15) with the constraints (1.2),(1.3),(1.16), (1.17), and where α^0 and u^0 are defined by (2.6).

What follows in the section is the proof of this result. To achieve it, we suppose that the theorem's hypothesis are satisfied. Therefore, Properties (2.2) and (1.19) imply that $(u_n)_{n\ge 0}$ is bounded in $L^\infty_{t,x}(]0,\infty[\times\mathbb{R})$. Then, since the first equation in (1.1) gives time compactness, extracting a subsequence if necessary, we can assume that as $n \to \infty$,

$$\alpha_n \to \alpha \text{ in } C_t([0,T]; L^\infty_{w*}(\mathbb{R})), \qquad u_n \rightharpoonup u \text{ in } L^\infty_{w*}(]0,\infty[\times\mathbb{R}), \tag{2.4}$$

$$\pi_n \rightharpoonup \pi \text{ in } \mathcal{M}_{loc}([0,\infty[\times\mathbb{R})w*, \tag{2.5}$$

$$\alpha_n^0 \rightharpoonup \alpha^0 \text{ in } L^\infty_{w*}(\mathbb{R}), \quad \text{and} \quad \alpha_n^0 u_n^0 \rightharpoonup m^0 \equiv \alpha^0 u^0 \text{ in } L^\infty_{w*}(\mathbb{R}). \tag{2.6}$$

Passing to the limit in $\mathcal{D}'$, the limit u also satisfies the Oleinik entropy condition

$$\partial_x u(t,x) \le 1/t.$$

The only non-trivial facts that need to be proved are (1.14), (1.15), (1.16) and (1.17). Let ζ_ε be a sequence of mollifiers in $\mathbb{R}$. We shall use in all this section the notation $z^\varepsilon = z \underset{x}{*} \zeta_\varepsilon$. We have the following technical result that allows to pass to the limit in most of the products we are going to deal with.

Lemma 2.2 *Let us suppose that $(\gamma_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbb{R})$ that tends to γ in L^∞_{w*}, and satisfies for any $\Gamma \in \mathcal{D}(\mathbb{R})$*

$$\int_{\mathbb{R}} (\gamma_n - \gamma)(t,x)\Gamma(x)\,dx \to 0, \quad n \to \infty, \tag{2.7}$$

either i) a.e. $t \in]0,T[$ or ii) in $L^1_t(]0,T[)$. Let us also suppose that $(\omega_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbb{R})$ that tends to ω in L^∞_{w*} and such that for all compact interval $K = [a,b]$, there exists $C > 0$ such that for $\beta = \omega$ or $\beta = \omega_n$,

$$TV_K(\beta(t,.)) \le f(t) \quad a.e. \ t, \tag{2.8}$$

where $f(t) = C(\frac{1}{t} + 1)$. Then $\gamma_n\omega_n \rightharpoonup \gamma\omega$ in L^∞_{w*}, as $n \to \infty$.

We first prove a partial result.

Lemma 2.3 *Let $\varepsilon > 0$. Let us suppose that $(\gamma_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbb{R})$ that tends to γ in L^∞_{w*} and satisfies (2.7), either i) a.e. $t \in]0,T[$ or ii) in $L^1_t(]0,T[)$. Let us also suppose that $(\omega_n)_{n\in\mathbb{N}}$ is a bounded family in $L^\infty(]0,T[\times\mathbb{R})$. Then we have $(\gamma_n - \gamma)\omega_n^\varepsilon \rightharpoonup 0$ in L^∞_{w*}, as $n \to \infty$.*

Proof of Lemma 2.3. We treat the case *i*), the proof is very similar in the other case. We only need to consider a test function $\varphi \in C_c^\infty(]0,T[\times\mathbb{R})$, $\varphi(t,x) = \varphi_t(t)\varphi_x(x)$, $\varphi_t \in C_c^\infty(]0,T[)$, $\varphi_x \in C_c^\infty(\mathbb{R})$. In order to prove that

$$\int_0^T \int_{\mathbb{R}} (\gamma_n - \gamma)(t,x)\omega_n^\varepsilon(t,x)\varphi(t,x)\,dx\,dt \to 0, \quad n \to \infty,$$

we write this integral as $\int_{[0,T]}\int_{\mathbb{R}} I_n(t,y)\,dy\,dt$ where

$$I_n(t,y) = \omega_n(t,y)\left(\int_{\mathbb{R}}(\gamma_n - \gamma)(t,x)\zeta_\varepsilon(x-y)\varphi(t,x)\,dx\right).$$

We are going to prove the convergence of this integral using Lebesgue's theorem. Since, at y beeing fixed, $x \mapsto \zeta_\varepsilon(x-y)\varphi_x(x) \in C_c^\infty(\mathbb{R})$ and together with the fact that $(\omega_n)_{n\geq 0}$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$, we deduce from the property *i*) of γ, that for *a.e.* t, y, $I_n(t,y) \to 0$ as $n \to \infty$. We also have the following estimate,

$$|I_n(t,y)| \leq \sup_n \|\omega_n\|_{L^\infty_{t,x}}(\sup_n \|\gamma_n\|_{L^\infty_{t,x}} + \|\gamma\|_{L^\infty_{t,x}})\int_{\mathbb{R}}\zeta_\varepsilon(x-y)|\varphi(t,x)|\,dx,$$

where $(t,y) \mapsto \int_{\mathbb{R}}\zeta_\varepsilon(x-y)|\varphi(t,x)|\,dx \in L^1(]0,T[\times\mathbb{R})$. Therefore, by dominated convergence, we have that $I_n(t,y) \to 0$ in $L^1_{t,y}(]0,T[\times\mathbb{R})$, which gives the desired convergence. $\square$

Proof of Lemma 2.2. Let us write the decomposition

$$\gamma_n\omega_n - \gamma\omega = \gamma_n(\omega_n - \omega_n^\varepsilon) + (\gamma_n - \gamma)\omega_n^\varepsilon + \gamma(\omega_n^\varepsilon - \omega^\varepsilon) + \gamma(\omega^\varepsilon - \omega). \tag{2.9}$$

We are first going to control the first and fourth terms of this decomposition for ε small enough and uniformly in n. Then, fixing ε, we shall pass to the limit, when n tends towards infinity, in the second and third terms. Let $\varphi \in C_c^\infty(]0,T[\times\mathbb{R})$.

– Let $\eta > 0$. The term $\int_0^T\int_{\mathbb{R}}\gamma(\omega^\varepsilon - \omega)\varphi\,dx\,dt$ is controlled in the following way. Since $\omega \in BV_x$, we have

$$\int_{[a,b]}|\omega(t,x-y) - \omega(t,x)|\,dx \leq |y|TV_{[a-2|y|,b+2|y|]}(\omega(t,.)),$$

hence for $\varepsilon < 1$,

$$\begin{aligned}\int_a^b\left|\int_{\mathbb{R}}(\omega(t,x-y) - \omega(t,x))\zeta_\varepsilon(y)\,dy\right|dx \quad &\leq \quad \int_{-\varepsilon}^\varepsilon |y|C(\frac{1}{t}+1)\zeta_\varepsilon(y)\,dy \\ &\leq \quad \varepsilon C(1/t+1),\end{aligned}$$

where C is a constant independent of ε and t, obtained from the hypothesis with the compact $[a-2, b+2]$. Then, $K = [a, b]$ beeing a fixed compact containing the support in x of φ, we get

$$\|\omega^\varepsilon - \omega\|_{L^1_x(K)} \leq C(1/t+1)\varepsilon, \quad \text{a.e. } 0 < t \leq T.$$

There exists $\delta > 0$ such that $supp_t \varphi \subset [\delta, T]$, thus

$$\begin{aligned}
|\int_0^T \int_{\mathbb{R}} \gamma(\omega^\varepsilon - \omega)\varphi \, dx \, dt| &\leq \int_\delta^T \int_{\mathbb{R}} |\gamma(\omega^\varepsilon - \omega)\varphi| \, dx \, dt \\
&\leq \int_\delta^T C\varepsilon(1/t+1) \, dt \|\varphi\|_\infty \|\gamma\|_\infty \\
&\leq \varepsilon C(1/\delta^2 + T)\|\varphi\|_\infty \|\gamma\|_\infty.
\end{aligned}$$

This is less than η if ε is small enough. We have the same bound uniformly in n for $(\gamma_n)_{n\geq 0}$ and $(\omega_n)_{n\geq 0}$, thus for such ε,

$$|\int_0^T \int_{\mathbb{R}} \gamma(\omega^\varepsilon - \omega)\varphi \, dx \, dt| \leq \eta \quad \text{and} \quad |\int_0^T \int_{\mathbb{R}} \gamma_n(\omega_n^\varepsilon - \omega_n)\varphi \, dx \, dt| \leq \eta, \quad \forall n \in \mathbb{N}.$$

– Let now ε be fixed as above. For the third term of the decomposition (2.9), obviously $\omega_n^\varepsilon - \omega^\varepsilon \rightharpoonup 0$ in $L^\infty_{w*}(]0,T[\times\mathbb{R})$, thus $\gamma(\omega_n^\varepsilon - \omega^\varepsilon) \rightharpoonup 0$. Lemma 2.3 establishes the convergence $(\gamma_n - \gamma)\omega_n^\varepsilon \rightharpoonup 0$ in $L^\infty_{w*}(]0,T[\times\mathbb{R})$, thus we have proved that $\gamma_n\omega_n - \gamma\omega \rightharpoonup 0$ in $L^\infty_{w*}(]0,T[\times\mathbb{R})$. □

We prove now that the limit (α, u, π) is a solution to (1.14)-(1.15).

Proposition 2.4 *We have the following weak convergences:*

$$\alpha_n u_n \rightharpoonup \alpha u \quad \textit{and} \quad \alpha_n u_n^2 \rightharpoonup \alpha u^2 \quad \textit{in } L^\infty_{w*}(]0,T[\times\mathbb{R}), \quad n \to \infty.$$

Proof. From the convergence $\alpha_n \to \alpha$ in $C_t([0,T]; L^\infty_{w*}(\mathbb{R}))$, we have

$$\sup_{t\in[0,T]} \left| \int_{\mathbb{R}} (\alpha_n - \alpha)(t,x)\Gamma(x) \, dx \right| \to 0, \quad \forall \Gamma \in L^1(\mathbb{R}).$$

From Oleinik's inequality on u, we get

$$\int_{[a,b]} |\partial_x u| \leq 2\frac{b-a}{t} + 2\|u_0\|_\infty.$$

A similar result holds for u_n, namely $\int_{[a,b]} |\partial_x u_n| \leq 2\frac{b-a}{t} + 2\sup_n \|u_n^0\|_\infty$.
We apply Proposition 2.2 with $\gamma_n = \alpha_n$ and $\omega_n = u_n$ and we get that $\alpha_n u_n - \alpha u \rightharpoonup 0$ in $L^\infty_{w*}(]0,T[\times\mathbb{R})$.
We turn now to the proof of the convergence $\alpha_n u_n^2 - \alpha u^2 \rightharpoonup 0$.
Let $\varphi \in \mathcal{D}(\mathbb{R})$. We have

$$\frac{d}{dt}\int_{\mathbb{R}} (\alpha_n u_n)(t,x)\varphi(x) \, dx - \int_{\mathbb{R}} \alpha_n u_n^2 \varphi'(x) \, dx - \int_{\mathbb{R}} \varphi'(x)\pi_n = 0,$$

thus the sequence $(\int_{\mathbb{R}}(\alpha_n u_n)(t,x)\varphi(x)\,dx)_{n\in\mathbb{N}}$ is bounded in BV_t. Therefore we can extract a subsequence such that

$$\int_{\mathbb{R}}(\alpha_n u_n)(t,x)\varphi(x)\,dx \to \int_{\mathbb{R}}(\alpha u)(t,x)\varphi(x)\,dx \text{ in } L^1(]0,T[),\ \forall\varphi\in\mathcal{D}(\mathbb{R}).$$

We can now apply Proposition 2.2 in the case *ii)* with $\gamma_n = \alpha_n u_n$, $\omega_n = u_n$, and get that for a subsequence $(\alpha_n u_n)u_n \rightharpoonup (\alpha u)u$ in $L^\infty_{w*}(]0,T[\times\mathbb{R})$. □
Combining the previous Proposition and convergences (2.4)-(2.6), we get that (α,u,π) is a solution to (1.14)-(1.15). It remains now to prove the weak form of (1.16)-(1.17).

Proposition 2.5 *There exists $Q_n\in\mathcal{D}'_{t,x}$ such that $\alpha_n u_n = \partial_x Q_n$, $\partial_t Q_n = -\alpha_n u_n^2 - \pi_n$ and such that $(Q_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(]0,T[\times\mathbb{R})\cap BV_{loc}([0,\infty[\times\mathbb{R})$.*

Proof. We take $Q_n\in\mathcal{D}'_{t,x}$ such that $\alpha_n u_n = \partial_x Q_n$ and $\alpha_n u_n^2 + \pi_n = -\partial_t Q_n$. The distribution Q_n is defined up to an additive constant that will be fixed later on. We have $\partial_x Q_n = \alpha_n u_n$ and $(\alpha_n u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty_{t,x}$. Then $\partial_t Q_n = -\alpha_n u_n^2 - \pi_n$ gives that $(Q_n)_{n\in\mathbb{N}}$ is bounded in BV_{loc}. We set

$$\widetilde{Q_n} = Q_n - \int_0^x \alpha_n u_n\,dy.$$

Then $\partial_x\widetilde{Q_n} = 0$, thus $\widetilde{Q_n}$ is a function of t only, and

$$\partial_t\widetilde{Q_n} = \partial_t Q_n - \partial_t\int_0^x \alpha_n u_n\,dy = -\alpha_n u_n^2 - \pi_n - \partial_t\int_0^x \alpha_n u_n\,dy.$$

Let $\varphi\in\mathcal{D}(\mathbb{R})$ such that $\int_{\mathbb{R}}\varphi(x)\,dx = 1$. Then $\partial_t\widetilde{Q_n} = \int_{\mathbb{R}}\partial_t\widetilde{Q_n}\varphi(x)\,dx$, and

$$\partial_t\widetilde{Q_n} = -\int_{\mathbb{R}}\alpha_n u_n^2\varphi(x)\,dx - \int_{\mathbb{R}}\varphi(x)\pi_n - \partial_t(\int_{\mathbb{R}}\int_0^x \alpha_n u_n\,dy\varphi(x)\,dx).$$

We obtain, fixing now the unknow constant, that

$$\widetilde{Q_n} = -\int_0^t\int_{\mathbb{R}}\alpha_n u_n^2\varphi\,dx\,ds - \int_0^t\int_{\mathbb{R}}\varphi\pi_n - \int_{\mathbb{R}}(\int_0^x \alpha_n u_n\,dy)\varphi\,dx.$$

We have

$$|\int_0^t\int_{\mathbb{R}}\alpha_n u_n^2\varphi(x)\,dx\,ds| \le T\ meas(\mathrm{supp}\varphi)\ (\sup_n\|u_n^0\|_\infty)^2\|\varphi\|_\infty < +\infty,$$

$$|\int_0^t\int_{\mathbb{R}}\varphi(x)\pi_n| \le \|\varphi\|_\infty\sup_n\int_0^T\int_{\mathrm{supp}\varphi}|\pi_n| < +\infty,$$

hence we get that $(\widetilde{Q_n})_{n\in\mathbb{N}}$ is bounded in $L^\infty_t(]0,T[)$. Now, according to the conservation relation $\int_{\mathbb{R}}\alpha_n(t,y)\,dy = \int_{\mathbb{R}}\alpha_n^0(y)\,dy$, we get

$$|Q_n(t,x)| \le \sup_n\|\widetilde{Q_n}\|_\infty + \sup_n\|u_n^0\|_\infty\int_{\mathbb{R}}\alpha_n^0(y)\,dy,$$

and therefore $(Q_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$ because of (2.1) and (2.2). $\square$
We can now precise the meaning of Q_n^0, which is defined by the relation

$$\int_{[0,+\infty[}\int_{\mathbb{R}} Q_n\partial_t\varphi\,dx\,dt + \int_{\mathbb{R}} Q_n^0(x)\varphi(0,x)\,dx = \int_{[0,+\infty[}\int_{\mathbb{R}}\varphi\,\left(\alpha_n u_n^2+\pi_n\right) \quad (2.10)$$

for any $\varphi\in\mathcal{D}([0,\infty[\times\mathbb{R})$. Since π_n is bounded in $L^\infty(\mathbb{R},\mathcal{M}([0,T]))$, we get that Q_n^0 is bounded in $L^\infty(\mathbb{R})$.
As in Proposition 2.5 and with the fact that $\partial_x M_n=\alpha_n$ and $\partial_t M_n=-\alpha_n u_n$ are bounded uniformly in n, we have the following result.

Proposition 2.6 *There exists $M_n\in\mathcal{D}'_{t,x}$ such that $\partial_x M_n=\alpha_n$, $\partial_t M_n=-\alpha_n u_n$ and such that $(M_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(]0,T[\times\mathbb{R})\cap Lip(]0,\infty[\times\mathbb{R})$.*

Remark 2.1 From the relations $\partial_x Q_n=\alpha_n u_n$ and $\partial_x M_n=\alpha_n$, we get that

$$\int_{[a,b]}|\partial_x Q_n|\le\sup_n\|u_n^0\|_\infty(b-a)\quad\text{and}\quad\int_{[a,b]}|\partial_x M_n|\le(b-a).$$

This gives property (2.8) with $f(t)=C$.

The following result is devoted to the proof of some results that deals with initial regularities.

Proposition 2.7 *i) Let us consider a sequence of solutions (α_n,u_n,π_n) with regularities (1.11)-(1.13) satisfying hypothesis of Theorem 2.1, in particular $(\alpha_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ are bounded sequences in L^∞ and $(\pi_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\mathbb{R}_x,\mathcal{M}([0,T]))$. We consider the distribution Q_n obtained in Proposition 2.5 and Q_n^0 defined by relation (2.10). Then $(Q_n^0)_{n\in\mathbb{N}}$ is bounded in $Lip(\mathbb{R})$.*
ii) Let us assume we have some functions $\alpha^0\in L^1(\mathbb{R})$, $u^0\in L^\infty(\mathbb{R})$, $0\le\alpha^0\le 1$ and (α,u,π) with regularities (1.11) -(1.13), satisfying (1.14)-(1.15) with the constraints (1.2),(1.3),(1.16), (1.17). Then we have the relation $\alpha^0(x)\pi(\{0\},x)=\pi(\{0\},x)$ a.e. x and the equality $\frac{dQ^0}{dx}=\alpha^0u^0$ holds in $\mathcal{D}'$.

Proof. 0) We suppose that (α,u,π) is a solution to (1.15) with regularities (1.11)-(1.13). Let us consider a test function $\varphi(t,x)=\chi(t)\psi(x)$, $\chi\in C_c^\infty([0,\infty[)$, $\psi\in C_c^\infty(\mathbb{R})$. The relation (1.15) gives

$$\int_{\mathbb{R}}\left(\int_0^\infty\alpha u\chi'(t)\,dt+\alpha^0u^0\chi(0)\right)\psi(x)\,dx$$
$$+\int_{\mathbb{R}}\left(\int_0^\infty\alpha u^2\chi(t)\,dt+\int_{t\in[0,\infty[}\chi(t)\pi(\,dt,x)\right)\psi'(x)\,dx=0$$

for any ψ, therefore in $\mathcal{D}'(\mathbb{R})$ we have

$$\frac{d}{dx}f_\chi(x)=\int_0^\infty\alpha u\chi'(t)\,dt+\alpha^0(x)u^0(x)\chi(0), \quad (2.11)$$

where

$$f_\chi(x)=\int_0^\infty\alpha u^2\chi(t)\,dt+\int_{t\in[0,\infty[}\chi(t)\pi(\,dt,x).$$

We notice that $f_\chi \in Lip(\mathbb{R})$ with $Lip(f_\chi) \le C(|\chi(0)| + \int_0^\infty |\chi'(t)|\,dt)$.
i) The relation (2.10) means

$$Q_n^0(x) = Q_n(0^+, x) + \pi_n(\{0\}, x). \tag{2.12}$$

We have the regularity $\pi_n \in L^\infty(\mathbb{R}_x, \mathcal{M}([0,T]))$, thus for a.e. x, $\pi_n(.,x)$ is a measure and $\pi_n(\{0\},x) = \int_{t\in\{0\}} \pi_n(\,dt, x) \in L_x^\infty$.
First, according to $\partial_x Q_n(0^+,x) = (\alpha_n u_n)(0^+,x)$, we have $(Q_n(0^+,x))_{n\in\mathbb{N}}$ bounded in $Lip(\mathbb{R})$. Thus we need to control $\pi_n(\{0\},x)$ in $Lip(\mathbb{R})$.
We use 0) with χ_ε nonincreasing on $[\varepsilon/2, \varepsilon]$ and such that $\chi_\varepsilon(t) = 1$ on $[0, \varepsilon/2]$, $\chi_\varepsilon(t) = 0$ for $t \ge \varepsilon$. We have $\int_0^\infty |\chi_\varepsilon'(t)|\,dt = 1$, thus $(f_{\chi_\varepsilon})_{\varepsilon>0}$ is bounded in $Lip(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Now, $f_{\chi_\varepsilon} \to \int_{t\in\{0\}} \pi(\,dt, x)$ a.e. as $\varepsilon \to 0$, therefore $\int_{t\in\{0\}} \pi(\,dt,x) \in Lip(\mathbb{R})$ with $Lip(\int_{t\in\{0\}} \pi(\,dt,x)) \le 2C$. Using (2.12), we get that $(Q_n^0)_{n\in\mathbb{N}}$ is bounded in $Lip(\mathbb{R})$.
ii) The relation (1.16) yields

$$\frac{d}{dx}\left(\int_0^\infty \alpha u Q \chi_\varepsilon(t)\,dt\right) = \alpha^0(x)Q^0(x) + \int_0^\infty \alpha Q \chi_\varepsilon'(t)\,dt - \int_{[0,\infty[} \chi_\varepsilon(t)\pi(\,dt, x),$$

thus, when $\varepsilon \to 0$, we get

$$\alpha^0(x)Q^0(x) - \alpha^0(x)Q(0^+,x) = \int_{t\in\{0\}} \pi(\,dt,x),$$

using (2.12), it means

$$\alpha^0(x)\pi(\{0\},x) = \pi(\{0\},x).$$

The relation (2.11) gives

$$\frac{df_{\chi_\varepsilon}}{dx} \to -(\alpha u)(0^+,x) + \alpha^0(x)u^0(x), \qquad \text{as } \varepsilon \to 0,$$

thus

$$\frac{d}{dx}\int_{t\in\{0\}} \pi(\,dt,x) = \alpha^0(x)u^0(x) - (\alpha u)(0^+,x),$$

and

$$\frac{dQ^0}{dx} = \alpha^0 u^0. \ \square$$

Using these various regularities, we now pass to the limit in the constraints (1.16)-(1.17). Since $(Q_n)_{n\in\mathbb{N}}$ is bounded in $BV_{loc} \cap L^\infty_{t,x}$ and $(\alpha_n^0 Q_n^0)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R})$, there exists $Q \in BV_{loc} \cap L^\infty_{t,x}$, $P^0 \in L^\infty(\mathbb{R})$ and a subsequence still noted $(Q_n)_{n\in\mathbb{N}}$ such that, as $n \to \infty$,

$$Q_n \to Q \text{ in } L^1_{loc},\ Q_n \rightharpoonup Q \text{ in } L^\infty_{w*} \text{ and } \alpha_n^0 Q_n^0 \rightharpoonup P^0 \text{ in } L^\infty_{w*}.$$

We notice that $\partial_x Q = \alpha u$ and $\partial_t Q = -\alpha u^2 - \pi$. We can also pass to the limit in (2.10) and get $Q_n^0 \rightharpoonup Q^0$ in L^∞_{w*}. Then since $Q_n \to Q$ strongly, we can pass to the limit in (1.16), but we need to prove that $P^0 = \alpha^0 Q^0$. This last point comes from the fact that $(Q_n^0)_{n\in\mathbb{N}}$ is bounded in $Lip(\mathbb{R})$ which is proved in Proposition 2.7. Now, concerning R_2, the proof is straightforward because $M_n \to M$ locally uniformly. This ends the proof of Theorem 2.1.

3 Sticky blocks

In this section, we are going to provide explicit solutions for some particular initial data. Nevertheless, we shall have enough solutions in the weak sense to pass to the limit by using the stability result of the previous section. With these solutions, we shall be able to prove the existence theorem for arbitrary initial data. The class of initial data which gives simple solutions are the sticky blocks. These sticky blocks generalize the sticky particles involved in the pressureless gas system of [7] and were introduced in [3].
Let us consider a volume fraction $\alpha(t,x)$ and a momentum density $\alpha(t,x)u(t,x)$ given by

$$\alpha(t,x) = \sum_{i=1}^{n} \mathbb{1}_{a_i(t)<x<b_i(t)}, \quad \alpha(t,x)u(t,x) = \sum_{i=1}^{n} u_i(t)\mathbb{1}_{a_i(t)<x<b_i(t)}, \tag{3.1}$$

with $a_1(t) < b_1(t) < a_2(t) < b_2(t) < \cdots < b_n(t)$. The time evolution is defined as follows. The number of blocks n indeed depends on t, but is piecewise constant. As long as the blocks do not meet, they move at constant velocity $u_i(t)$. When two or more blocks collide, they get stuck, building a new block, with volume the sum of the volumes, and with momentum the sum of momenta. Obviously at most $n^0 - 1$ collisions can occur, where n^0 is the initial number of blocks. We have to underline that we only consider in this section initial data such that u^0 is constant in each block, so that π contains no Dirac at $t = 0$. Generalized sticky blocks with non constant u^0 in a block, such as in the example of Section 1.1, can nevertheless appear in the limit of Theorem 2.1. In [3], the following Theorem was obtained.

Theorem 3.1 *There exists a nonnegative measure $\pi(t,x)$ such that with $\alpha(t,x)$ and $u(t,x)$ defined by (3.1), and with the above defined dynamics, we get a solution to (1.1)-(1.3).*

The expression of π on a shock at time t^* is given by the formula

$$\pi(t,x) = \begin{cases} \delta(t-t^*)(u_l - u^*)(x - a^*) & if\ a^* \le x \le x^*, \\ \delta(t-t^*)(u^* - u_r)(b^* - x) & if\ x^* \le x \le b^*, \\ 0 & otherwise, \end{cases} \tag{3.2}$$

with the notations of Figure 1.

Figure 1 :

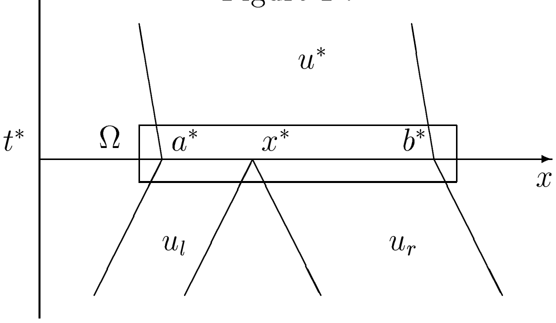
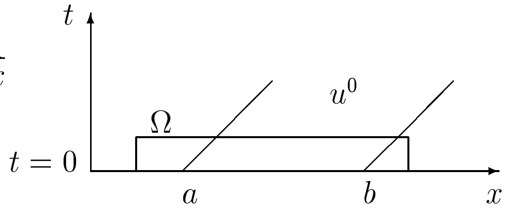


Figure 2 :



We also set $a(t) = a^* + u^*(t - t^*)$, $b(t) = b^* + u^*(t - t^*)$, $a_l(t) = a^* + u_l(t - t^*)$, $b_l(t) = x^* + u_l(t - t^*)$, $a_r(t) = x^* + u_r(t - t^*)$ and $b_r(t) = b^* + u_r(t - t^*)$.
We define the value of $u(t, x)$ for all x by asking that u is Lipschitz continuous, takes the value $u_i(t)$ in each block i and is extended linearly between each block, and by a constant at infinity. As was proved in [3], the sticky blocks dynamics satisfies Oleinik's condition.

Proposition 3.2 *For any $t > 0$, we have $\partial_x u(t, x) \leq 1/t$.*

Notice that we also have the properties (1.2), (1.3), (1.19), (1.11), (1.12), (1.13). We prove now that the sticky blocks satisfy the weak formulation (1.14)-(1.15) and the constraint $R_1 = R_2 = \pi$ in the sense of (1.16)-(1.17).

Proposition 3.3 *The solution (α, u, π) satisfies (1.14)-(1.17).*

Proof. First, according to Theorem 3.1, we only have to prove the result (1.14)-(1.15) for $\varphi \in \mathcal{D}(\Omega)$ where $\Omega = [0; \varepsilon[\times]a - \eta; b + \eta[$ where $]a; b[$ is a block at $t = 0$ and ε such that there is no collisions between $t = 0$ and $t = \varepsilon$. The situation is displayed on Figure 2. We have

$$
\begin{aligned}
& \int_{[0,+\infty[} \int_{\mathbb{R}} [-\alpha u \partial_t \varphi - (\alpha u^2 + \pi)\partial_x \varphi] \, dx \, dt \\
= \; & -\int_0^\varepsilon \int_{a(t)}^{b(t)} (u^0 \partial_t \varphi + (u^0)^2 \partial_x \varphi) \, dx \, dt \\
= \; & -\int_0^\varepsilon \frac{d}{dt}(u^0 \int_{a(t)}^{b(t)} \varphi \, dx) \, dt + \int_0^\varepsilon (\varphi(t, b(t)) - \varphi(t, a(t)))(u^0)^2 \, dt \\
& -\int_0^\varepsilon (\varphi(t, b(t)) - \varphi(t, a(t)))(u^0)^2 \, dt \\
= \; & u^0 \int_{a(0)}^{b(0)} \varphi(0, x) \, dx - u^0 \int_{a(\varepsilon)}^{b(\varepsilon)} \varphi(\varepsilon, x) \, dx \\
= \; & \int_a^b u^0 \varphi(0, x) \, dx = \int_{\mathbb{R}} (\alpha^0 u^0)(x) \varphi(0, x) \, dx,
\end{aligned}
$$

that is (1.15). We have similarly (1.14). Concerning (1.16) and (1.17), we have to work with Figure 1 and Figure 2. We shall detail the case R_1 with Figure 2, the case R_2 with Figure 1 and the two similar cases are left to the reader.
– About R_1. A useful expression of $Q(t, x)$ is

$$
Q(t, x) = \sum_{i=1}^n u_i(t) T_{a_i(t), b_i(t)}(x), \quad \text{where} \quad T_{a,b}(x) = \begin{cases} 0 & if \ x \leq a \\ x - a & if \ a \leq x \leq b \\ b - a & if \ x \geq b. \end{cases}
$$

In the neighbourhood of the block, up to a constant, we can take $Q(t, x) = u^0 T_{a(t), b(t)}(x)$. We compute R_1,

$$
\begin{aligned}
< R_1, \varphi > \; = \; & \int_0^\varepsilon \int_{\mathbb{R}} \mathbb{I}_{a(t) < x < b(t)} Q(t, x) \partial_t \varphi \, dx \, dt + \\
& \int_0^\varepsilon \int_{\mathbb{R}} \mathbb{I}_{a(t) < x < b(t)} u(t, x) Q(t, x) \partial_x \varphi \, dx \, dt + \int_{\mathbb{R}} (\alpha Q)^0 \varphi(0, x) \, dx.
\end{aligned}
$$

Then the expression of R_1 is $< R_1, \varphi >= A + A^0$, where

$$A = \int_0^\varepsilon \int_{a(t)}^{b(t)} u^0(x - a(t))\partial_t\varphi \, dx \, dt + \int_0^\varepsilon \int_{a(t)}^{b(t)} (u^0)^2(x - a(t))\partial_x\varphi \, dx \, dt,$$
$$A^0 = \int_a^b u_i^0(x - a)\varphi^0 \, dx.$$

We have

$$\begin{aligned} A &= \int_0^\varepsilon u^0 \left[\frac{d}{dt}\int_{a(t)}^{b(t)} x\varphi(t,x)\,dx - [b(t)\varphi(t,b(t)) - a(t)\varphi(t,a(t))]u^0 \right. \\ &\quad \left. -a(t)\frac{d}{dt}\int_{a(t)}^{b(t)} \varphi(t,x)\,dx + [b(t)\varphi(t,b(t)) - a(t)\varphi(t,a(t))]u^0 \right] dt \\ &\quad + \int_0^\varepsilon (u^0)^2[b(t)\varphi(t,b(t)) - a(t)\varphi(t,a(t)) - \int_{a(t)}^{b(t)} \varphi(t,x)\,dx \\ &\quad -a(t)[\varphi(t,b(t)) - \varphi(t,a(t))]]\,dt \\ &= -u^0\int_{a(0)}^{b(0)} x\varphi(0,x)\,dx - (u^0)^2\int_0^\varepsilon (b(t) - a(t))\varphi(t,b(t))\,dt \\ &\quad -u^0\int_0^\varepsilon (a^0 + tu^0)\left(\frac{d}{dt}\int_{a(t)}^{b(t)} \varphi(t,x)\,dx\right) dt \\ &\quad +(u^0)^2\int_0^\varepsilon (b(t) - a(t))\varphi(t,b(t))\,dt - (u_0)^2\int_0^\varepsilon \int_{a(t)}^{b(t)} \varphi(t,x)\,dx\,dt, \end{aligned}$$

now using

$$\begin{aligned} &\int_0^\varepsilon (a^0 + tu^0)\left(\frac{d}{dt}\int_{a(t)}^{b(t)} \varphi(t,x)\,dx\right) dt \\ &= -a^0\int_{a(0)}^{b(0)} \varphi(0,x)\,dx + u^0\left[t\int_{a(t)}^{b(t)} \varphi(t,x)\,dx\right]_0^\varepsilon - u^0\int_0^\varepsilon \int_{a(t)}^{b(t)} \varphi(t,x)\,dx\,dt \\ &= -a^0\int_a^b \varphi(0,x)\,dx - u^0\int_0^\varepsilon \int_{a(t)}^{b(t)} \varphi(t,x)\,dx\,dt, \end{aligned}$$

the expression A finally simplifies in $\int_a^b u^0(a - x)\varphi(0,x)\,dx$. Thus we have $< R_1, \varphi >= 0 =< \pi, \varphi > \forall\varphi \in \mathcal{D}(\Omega)$ with Ω as in Figure 2.

– About R_2. We use now $M(t,x) = \sum_{i=1}^n T_{a_i(t),b_i(t)}(x)$. Let $\varphi \in \mathcal{D}(\Omega)$ with Ω as in Figure 1. In this domain, we can use

$$M(t,x) = \begin{cases} T_{a_l(t),b_l(t)}(x) + T_{a_r(t),b_r(t)}(x) & if\ t < t^*, \\ T_{a(t),b(t)}(x) & if\ t > t^*. \end{cases}$$

With similar technics, we obtain successively that

$$\iint_\Omega \alpha u^2 M\partial_x\varphi\,dx\,dt + \iint_\Omega \alpha u M\partial_t\varphi\,dx\,dt$$

$$
\begin{aligned}
&= \int^{t^*}\left(\int_{a_l(t)}^{b_l(t)}(x-a_l(t))\left[u_l^2\partial_x\varphi+u_l\partial_t\varphi\right]\,dx\right)\,dt \\
&\quad+\int^{t^*}\left(\int_{a_r(t)}^{b_r(t)}(b_l(t)-a_l(t)+x-a_r(t))\left[u_r^2\partial_x\varphi+u_r\partial_t\varphi\right]\,dx\right)\,dt \\
&\quad+\int_{t^*}\left(\int_{a(t)}^{b(t)}(x-a(t))\left[(u^*)^2\partial_x\varphi+u^*\partial_t\varphi\right]\,dx\right)\,dt \\
&= -u_l^2\int^{t^*}\int_{a_l(t)}^{b_l(t)}\varphi(t,x)\,dx\,dt-\int^{t^*}a_l(t)u_l\left[\frac{d}{dt}\int_{a_l(t)}^{b_l(t)}\varphi(t,x)\,dx\right]\,dt \\
&\quad+\int^{t^*}u_l\left[\frac{d}{dt}\int_{a_l(t)}^{b_l(t)}x\varphi(t,x)\,dx\right]\,dt-u_r^2\int^{t^*}\int_{a_r(t)}^{b_r(t)}\varphi(t,x)\,dx\,dt \\
&\quad+\int^{t^*}(b_l(t)-a_l(t)-a_r(t))u_r\left[\frac{d}{dt}\int_{a_r(t)}^{b_r(t)}\varphi(t,x)\,dx\right]\,dt \\
&\quad+\int^{t^*}u_r\left[\frac{d}{dt}\int_{a_r(t)}^{b_r(t)}x\varphi(t,x)\,dx\right]\,dt-(u^*)^2\int_{t^*}\int_{a(t)}^{b(t)}\varphi(t,x)\,dx\,dt \\
&\quad-\int_{t^*}a(t)u^*\left[\frac{d}{dt}\int_{a(t)}^{b(t)}\varphi(t,x)\,dx\right]\,dt+\int_{t^*}u^*\left[\frac{d}{dt}\int_{a(t)}^{b(t)}x\varphi(t,x)\,dx\right]\,dt \\
&= -a^*\left[u_l\int_{a^*}^{x^*}\varphi(t^*,x)\,dx+u_r\int_{x^*}^{b^*}\varphi(t^*,x)\,dx-u^*\int_{a^*}^{b^*}\varphi(t^*,x)\,dx\right] \\
&\quad+u_l\int_{a^*}^{x^*}x\varphi(t^*,x)\,dx+u_r\int_{x^*}^{b^*}x\varphi(t^*,x)\,dx-u^*\int_{a^*}^{b^*}x\varphi(t^*,x)\,dx.
\end{aligned}
$$

On an other hand, a computation gives that the expression of $\int_\Omega M\pi\partial_x\varphi$ is

$$
-\int_{\mathbb{R}}[2(x-a^*)(u_l-u^*)\mathbb{1}_{[a^*,x^*]}(x)+(u^*-u_r)(b^*+a^*-2x)\mathbb{1}_{[x^*,b^*]}(x)]\varphi(t^*,x)\,dx,
$$

and finally after simplifications we get

$$
\iint_\Omega M\alpha u^2\partial_x\varphi\,dx\,dt+\iint_\Omega \alpha uM\partial_t\varphi\,dx\,dt+\iint_\Omega M\pi\partial_x\varphi=-\iint_\Omega\varphi\pi.\ \square
$$

4 Existence of a solution

In this section, we prove the existence of a solution. To reach this aim, we prove that every initial data can be approched by blocks in $\mathcal{D}'(\mathbb{R})$ and then we pass to the limit in the sticky blocks solution of Section 3 using the compactness result of Section 2. We first prove a lemma about weak limit of sticky blocks.

Lemma 4.1 *Let $\alpha^0\in L^1(\mathbb{R})$ such that $0\le\alpha^0\le 1$ and $u^0\in L^\infty(\mathbb{R})$, then there exist block initial data $(\alpha_n^0)_{n\ge0}$ and $(\alpha_n^0u_n^0)_{n\ge0}$ such that $\int_{\mathbb{R}}\alpha_n^0(x)\,dx\le\int_{\mathbb{R}}\alpha^0(x)\,dx$ and essinf $u^0\le u_n^0\le$ esssup u^0 for which the convergences $\alpha_n^0\rightharpoonup\alpha^0$ and $\alpha_n^0u_n^0\rightharpoonup\alpha^0u^0$ hold in $\mathcal{D}'_x$.*

Proof. We set

$$\alpha_n^0(x) = \sum_{\substack{i=-n^2 \\ m_{in}>0}}^{n^2} \mathbb{I}_{]\frac{i}{n},\frac{i}{n}+m_{in}[} \qquad \text{and} \qquad \alpha_n^0 u_n^0(x) = \sum_{\substack{i=-n^2 \\ m_{in}>0}}^{n^2} u_{in}\mathbb{I}_{]\frac{i}{n},\frac{i}{n}+m_{in}[},$$

where

$$m_{in} = \int_{i/n}^{(i+1)/n-1/n^2} \alpha^0(x)\,dx \qquad \text{and} \qquad u_{in} = \frac{1}{m_{in}}\int_{i/n}^{(i+1)/n-1/n^2} \alpha^0(x)u^0(x)\,dx.$$

We notice that $0 \le m_{in} \le 1/n - 1/n^2 < 1/n$ and that u_{in} is defined when $m_{in} > 0$. Let $\varphi \in \mathcal{D}(\mathbb{R})$, let n_0 be an integer such that $supp\varphi \subset [-n_0, n_0]$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \alpha_n^0(x)\varphi(x)\,dx &= \sum_{i=-n^2}^{n^2} \int_{i/n}^{i/n+m_{in}} \varphi(x)\,dx = \sum_{i=-n^2}^{n^2} \Big[\varphi(i/n)m_{in} + \varphi'(x_n^i)m_{in}^2/2\Big] \\
&= \sum_{i=-n^2}^{n^2} \int_{i/n}^{(i+1)/n-1/n^2} \alpha^0(x)\varphi(i/n)\,dx + \sum_{i=-n^2}^{n^2} \varphi'(x_n^i)m_{in}^2/2,
\end{aligned}$$

where $x_n^i \in [i/n; i/n + m_{in}]$. Thus we get for $n > n_0$,

$$\begin{aligned}
\left|\int_{\mathbb{R}} \alpha_n^0(x)\varphi(x)\,dx - \int_{\mathbb{R}} \alpha^0(x)\varphi(x)\,dx\right| \le\ & \sum_{i=-nn_0}^{nn_0-1} \int_{i/n}^{(i+1)/n-1/n^2} |\varphi(i/n) - \varphi(x)|\,dx \\
&+ \sum_{i=-nn_0}^{nn_0-1} \int_{(i+1)/n-1/n^2}^{(i+1)/n} |\varphi(x)|\,dx \\
&+ \sum_{i=-nn_0}^{nn_0-1} \frac{1}{2}|\varphi'(x_n^i)|m_{in}^2,
\end{aligned}$$

according to $[i/n, (i+1)/n] \subset [-n_0, n_0] \Leftrightarrow -nn_0 \le i \le nn_0 - 1$. We finally get that, as $n \to \infty$,

$$< \alpha_n^0, \varphi > \to < \alpha^0, \varphi > \qquad \text{and} \qquad < \alpha_n^0 u_n^0, \varphi > \to < \alpha^0 u^0, \varphi > . \ \Box$$

Proof of Theorem 1.1. Let $\alpha^0 \in L^1(\mathbb{R})$, $0 \le \alpha^0 \le 1$, $u^0 \in L^\infty(\mathbb{R})$. Let α_n^0 and $\alpha_n^0 u_n^0$ be the block initial data of Lemma 4.1. Then using Section 3, we get (α_n, u_n, π_n) with regularity (1.11)-(1.13) solution to (1.14)-(1.17) such that these solutions satisfy (1.18), (1.19), (1.2), (1.3). From the inequality $essinf\ u^0 \le u_n^0 \le esssup\ u^0$, we get that $essinf\ u^0 \le u_n(t,x) \le esssup\ u^0$. Thus, taking the limit (2.4), we obtain the maximum principle (1.19). We have (2.1), (2.2) and (1.19), we get that the nonnegative measures $(\pi_n)_{n\ge 0}$ are bounded in $\mathcal{D}'_{t,x}$ thus bounded in $\mathcal{M}_{loc}([0,\infty[\times\mathbb{R})$. Using now (1.15), we get (2.3). Thus we can apply Theorem 2.1. $\Box$

5 Entropy inequalities

We first prove results on sticky blocks, then we shall pass to the limit.

5.1 The case of sticky blocks

Proposition 5.1 *The sticky blocks solution of Section 3 satisfies*

$$TV(u(t,.)) \leq TV(u^0) \quad \forall t \geq 0, \tag{5.1}$$

$$\partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)) + \partial_x(\pi_S) \leq 0 \tag{5.2}$$

for every $S \in C^1(\mathbb{R})$ *convex Lipschitz,*

where $\pi_S(t,x) = \begin{cases} S'(u^*)\pi(t,x) & \text{on a shock (see (3.2)),} \\ 0 & \text{otherwise,} \end{cases}$ *and* $|\pi_S| \leq Lip(S)\pi$.

Proof. *i)* We have $TV(u^0) = \sum_{i=2}^{n^0} |u_i^0 - u_{i-1}^0|$ initially. When a collision happens, for exemple at time t^* between blocks k and $k+1$ to simplify the presentation, we have after the collision $u^* = u_k^0 \Delta_a/\Delta + u_{k+1}^0 \Delta_b/\Delta$, where $\Delta_a = x^* - a^*$, $\Delta_b = b^* - x^*$ and $\Delta = b^* - a^* = \Delta_a + \Delta_b$ using the notation of Figure 1. As

$$TV(u(t^*)) = \sum_{i=2}^{k-1} |u_i^0 - u_{i-1}^0| + |u^* - u_{k-1}^0| + |u_{k+2}^0 - u^*| + \sum_{i=k+3}^{n^0} |u_i^0 - u_{i-1}^0|,$$

in order to obtain $TV(u(t^*)) \leq TV(u^0)$, we have to prove that

$$|u^* - u_{k-1}^0| + |u_{k+2}^0 - u^*| \leq |u_k^0 - u_{k-1}^0| + |u_{k+1}^0 - u_k^0| + |u_{k+2}^0 - u_{k+1}^0|.$$

First, we have

$$\begin{aligned} |u_{k+2}^0 - u^*| &\leq |u_{k+2}^0 - u_{k+1}^0| + |u_{k+1}^0 - u^*| \\ &\leq |u_{k+2}^0 - u_{k+1}^0| + |u_{k+1}^0 \frac{\Delta_a}{\Delta} + u_{k+1}^0 \frac{\Delta_b}{\Delta} - u_k^0 \frac{\Delta_a}{\Delta} - u_{k+1}^0 \frac{\Delta_b}{\Delta}| \\ &\leq |u_{k+2}^0 - u_{k+1}^0| + \frac{\Delta_a}{\Delta} |u_{k+1}^0 - u_k^0|. \end{aligned}$$

Similarly, we get $|u^* - u_{k-1}^0| \leq |u_k^0 - u_{k-1}^0| + \frac{\Delta_b}{\Delta} |u_{k+1}^0 - u_k^0|$, and by adding these two inequalities, we get the desired inequality. Finally, collision after collision, the total variation is nonincreasing.

ii) We have to prove the entropy inequality on a shock. We use again the notation of Figure 1. Let $\varphi(t,x)$ be a smooth function with support in Ω. We compute as in [3] that

$$\begin{aligned} &< \partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)), \varphi > \\ = &\; - \int_{a^*}^{x^*} S(u_l)\varphi(t^*,x)\,dx - \int_{x^*}^{b^*} S(u_r)\varphi(t^*,x)\,dx + \int_{a^*}^{b^*} S(u^*)\varphi(t^*,x)\,dx. \end{aligned}$$

Besides we also have

$$< \partial_x \pi_S, \varphi >= S'(u^*) \left[u_l \int_{a^*}^{x^*} \varphi(t^*,x)\,dx + u_r \int_{x^*}^{b^*} \varphi(t^*,x)\,dx - u^* \int_{a^*}^{b^*} \varphi(t^*,x)\,dx \right]$$

thanks to (3.2). Thus we get

$$\begin{aligned}
& < \partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)) + \partial_x \pi_S, \varphi > \\
= & \int_{a^*}^{x^*} [S(u^*) - S(u_l) - S'(u^*) \cdot (u^* - u_l)] \, \varphi(t^*, x) \, dx \\
& + \int_{x^*}^{b^*} [S(u^*) - S(u_r) - S'(u^*) \cdot (u^* - u_r)] \, \varphi(t^*, x) \, dx.
\end{aligned}$$

Now by convexity of S, for every $\varphi \geq 0$, we get

$$< \partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)) + \partial_x \pi_S, \varphi > \leq 0,$$

that is $\partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)) + \partial_x \pi_S \leq 0$ in $\mathcal{D}'$. □

5.2 Entropy inequalities for a general solution

We want to pass to the limit in (5.2). Let (α, u, π) be the solution of Theorem 1.1 and (α_n, u_n, π_n) the sticky blocks that tends to (α, u, π) as in Section 4.

Lemma 5.2 *For any $S : \mathbb{R} \to \mathbb{R}$ continuous, $\alpha_n S(u_n) \rightharpoonup \alpha S(u)$.*

Proof. Obviously, it is enough to get the result for $S_k(u) = u^k$, $k \in \mathbb{N}$. We have, for $S \in C^1(\mathbb{R})$ convex,

$$\partial_t(\alpha_n S(u_n)) + \partial_x(\alpha_n u_n S(u_n)) + \partial_x \pi_S^n \leq 0$$

and $(\pi_S^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{M}_{t,x}$, thus we get that $(\alpha_n S(u_n))_{n\in\mathbb{N}}$ is bounded in $BV_t(\mathcal{D}'_x)$.
For $S \in C^1(\mathbb{R})$ concave, we have $\partial_t(\alpha_n S(u_n)) + \partial_x(\alpha_n u_n S(u_n)) + \partial_x \pi_S^n \geq 0$. Hence, similarly, $(\alpha_n S(u_n))_{n\in\mathbb{N}}$ is bounded in $BV_t(\mathcal{D}'_x)$.
Now, $x \mapsto x^{2k}$ is convex, of class C^1 on $\mathbb{R}$ for every $k \geq 0$, therefore

$$(\alpha_n u_n^{2k})_{n\in\mathbb{N}} \text{ is bounded in } BV_t(\mathcal{D}'_x).$$

For $k \geq 0$, $x^{2k+1} = S_1(x) + S_2(x)$ where $S_1(x) = x^{2k+1}\mathbb{1}_{x\geq 0}$ and $S_2(x) = x^{2k+1}\mathbb{1}_{x<0}$. The function S_1 is convex of class C^1 on $\mathbb{R}$ and S_2 is concave of class C^1 on $\mathbb{R}$, hence $(\alpha_n S_1(u_n))_{n\in\mathbb{N}}$ is bounded in $BV_t(\mathcal{D}'_x)$ and $(\alpha_n S_2(u_n))_{n\in\mathbb{N}}$ is bounded in $BV_t(\mathcal{D}'_x)$, thus

$$(\alpha_n u_n^{2k+1})_{n\in\mathbb{N}} \text{ is bounded in } BV_t(\mathcal{D}'_x).$$

Finally,

$$(\alpha_n u_n^k)_{n\in\mathbb{N}} \text{ is bounded in } BV_t(\mathcal{D}'_x), \quad \forall k \geq 0.$$

We may now proceed by induction on k and apply Lemma 2.2. Thus we obtain

$$\alpha_n u_n^k \rightharpoonup \alpha u^k, \text{ as } n \to \infty, \text{ for every } k \geq 0. \ \square$$

Proof of Theorem 1.2. It only remains to prove the entropy inequality (1.21). For $S \in C^1(\mathbb{R})$ convex, we can write the inequality

$$\partial_t(\alpha_n S(u_n)) + \partial_x(\alpha_n u_n S(u_n)) + \partial_x \pi_S^n \leq 0.$$

By Lemma 5.2, $\alpha_n S(u_n) \rightharpoonup \alpha S(u)$ and $(\alpha_n S(u_n))u_n \rightharpoonup (\alpha S(u))u$. Now $(\pi_S^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{M}_{t,x}$ hence for a subsequence, $\pi_S^n \rightharpoonup \pi_S$ and we get, passing to the limit, the relation

$$\partial_t(\alpha S(u)) + \partial_x(\alpha u S(u)) + \partial_x \pi_S \leq 0.$$

Finally, from the estimate $|\pi_S^n| \leq C\pi^n$ that is to say $- < C\pi^n, \varphi > \leq < \pi_S^n, \varphi > \leq < C\pi^n, \varphi >$ for every $\varphi \in C_c^\infty$, $\varphi \geq 0$, by passing to the limit, we get the localisation $|\pi_S| \leq C\pi$. $\square$

6 The exclusion relation $\alpha\pi = \pi$

The aim of this section is to give a more classical sense than that of (1.16)-(1.17) to the relation $\alpha\pi = \pi$ in a special class of solutions.

6.1 Definition of $\alpha\pi$ for π in the class $\mathcal{ML}$

We say that π is in $\mathcal{ML}$ if π lies in $\mathcal{M}_{loc}([0,\infty[\times\mathbb{R})$ and if there exists C such that

$$\left| \int_{[0,\infty[} \int_{\mathbb{R}} \phi(t,x)\pi \, dt \, dx \right| \leq C \|\phi\|_{L_t^\infty(]0,\infty[, L_x^1(\mathbb{R}))}, \quad \forall \phi \in C_c([0,\infty[\times\mathbb{R}). \qquad (6.1)$$

If π is in $\mathcal{ML}$, then thanks to the density of $C_c([0,\infty[\times\mathbb{R})$ in $C_c([0,\infty[, L^1(\mathbb{R}))$ for $\|.\|_{L_t^\infty(]0,\infty[, L_x^1(\mathbb{R}))}$, we can define $< \pi, \phi >$ for $\phi \in C_c([0,\infty[, L^1(\mathbb{R}))$ by classical argument. We obtain besides that

$$| < \pi, \phi > | \leq C \|\phi\|_{L_t^\infty(]0,\infty[, L_x^1(\mathbb{R}))}, \quad \forall \phi \in C_c([0,\infty[, L^1(\mathbb{R})).$$

Definition 6.1 *Let $\alpha \in C_t([0,\infty[, L^1_{loc}(\mathbb{R}))$ and $\pi \in \mathcal{ML}$. Then the product $\alpha\pi$ is defined as a measure by $< \alpha\pi, \phi > = < \pi, \alpha\phi >$ for $\phi \in C_c([0,\infty[\times\mathbb{R})$.*

We notice that if $\pi \in \mathcal{M}_{loc}([0,\infty[\times\mathbb{R})$ satisfies

$$\int_{[0,\infty[} \int_{\mathbb{R}} |\phi(t,x)\pi| \, dt \, dx \leq C \|\phi\|_{L_t^\infty(]0,\infty[, L_x^1(\mathbb{R}))}, \quad \forall \phi \in C_c([0,\infty[, L^1(\mathbb{R})), \quad (6.2)$$

then $\pi \in \mathcal{ML}$ and

$$< \pi, \phi > = \int_{[0,\infty[} \int_{\mathbb{R}} \phi(t,x)\pi \, dt \, dx, \quad \forall \phi \in C_c([0,\infty[, L^1(\mathbb{R})).$$

We also remark that $\mathcal{ML}$ looks like $\mathcal{M}_t(L_x^\infty)$ but we shall not discuss here the relations furthermore. The two following Propositions prove that the sticky blocks of Section 3 have this regularity.

Proposition 6.2 *For the sticky blocks of Section 3, the pressure π satisfies (6.2).*

Proof. This is obvious with (3.2) and we obtain (6.2) with a constant C which depends on the number of blocks and on the size of the support of the solution. $\square$

Proposition 6.3 *For the sticky blocks, we have* $\alpha \in C([0,\infty[, L^1(\mathbb{R}))$ *and* $\alpha\pi = \pi$ *in* $\mathcal{M}_{loc}([0,\infty[\times\mathbb{R})$.

Proof. With the notations of Figure 1, for any $\varphi \in C_c(\Omega)$, we have

$$\begin{aligned} < \alpha\pi, \varphi > &= < \pi, \alpha\varphi >= \iint_\Omega \pi\alpha\varphi = \int_{a^*}^{b^*} \alpha(t^*, x)\varphi(t^*, x)\pi(t^*, x)\, dx \\ &= \int_{a^*}^{b^*} \varphi(t^*, x)\pi(t^*, x)\, dx =< \pi, \varphi >, \end{aligned}$$

and it is enough to get our result. $\square$

6.2 Approximation in $\mathcal{ML}$

We prove here different technical lemma about approximation of measures with the following sequence of mollifiers ζ_ε : $\zeta_\varepsilon(t,x) = \zeta_\varepsilon^1(t)\zeta_\varepsilon^2(x)$ where $supp(\zeta_\varepsilon^1) \subset]0,\varepsilon[$ and $supp(\zeta_\varepsilon^2) \subset]-\varepsilon,\varepsilon[$. We shall use this sequence in all Section 6.

Lemma 6.4 *Let* $\mu \in \mathcal{ML}$ *and set* $\mu_\varepsilon(t,x) = \int_{[0,\infty[}\int_\mathbb{R} \zeta_\varepsilon(t-s, x-y)\, d\mu(s,y)$. *Then* $\mu_\varepsilon \in \mathcal{ML}$ *with a constant independant of* ε.

Proof. Let $\phi \in C_c([0,\infty[\times\mathbb{R})$. We set $\phi_\varepsilon(s,y) = \int_0^\infty \int_\mathbb{R} \zeta_\varepsilon(t-s, x-y)\phi(t,x)\, dt\, dx$. We notice that

$$\begin{aligned} \int_\mathbb{R} |\phi_\varepsilon(s,y)|\, dy &= \int_\mathbb{R} |\int_0^\infty \int_\mathbb{R} \zeta_\varepsilon(t-s, x-y)\phi(t,x)\, dt\, dx|\, dy \\ &\le \int_0^\infty \int_\mathbb{R} |\phi(t,x)|\zeta_\varepsilon^1(t-s)\, dx\, dt. \end{aligned}$$

Now from the estimate $\int_0^\infty F(t)\zeta_\varepsilon^1(t-s)\, dt \le \sup\limits_{t\ge 0} F(t)$ with $F(t) = \int_\mathbb{R} |\phi(t,x)|\, dx$, and taking the supremum over $s \ge 0$, we obtain that

$$\sup_{s\ge 0} \int_\mathbb{R} |\phi_\varepsilon(s,y)|\, dy \le \sup_{t\ge 0} \int_\mathbb{R} |\phi(t,x)|\, dx,$$

that is to say $\|\phi_\varepsilon\|_{L_t^\infty(L_x^1)} \le \|\phi\|_{L_t^\infty(L_x^1)}$. Since $\phi_\varepsilon \in C_c([0,\infty[\times\mathbb{R})$, we have

$$\begin{aligned} |\int_{[0,\infty[}\int_\mathbb{R} \phi(t,x)\mu_\varepsilon(t,x)\, dt\, dx| &= |\int_{[0,\infty[}\int_\mathbb{R} \phi_\varepsilon(s,y)\, d\mu(s,y)| \\ &\le C\|\phi_\varepsilon\|_{L_t^\infty(L_x^1)} \le C\|\phi\|_{L_t^\infty(L_x^1)}, \end{aligned}$$

thus $\mu_\varepsilon \in \mathcal{ML}$. $\square$

Lemma 6.5 *Let* $\mu \in \mathcal{ML}$ *and set* $\mu_\varepsilon(t,x) = \int_{[0,\infty[}\int_\mathbb{R} \zeta_\varepsilon(t-s, x-y)\, d\mu(s,y)$. *Then* $< \mu_\varepsilon, \varphi > \to < \mu, \varphi >$ *as* $\varepsilon \to 0$, *for any* $\varphi \in C_c([0,\infty[, L^1(\mathbb{R}))$.

Proof. Let $\varphi \in C_c([0,\infty[, L^1_x(\mathbb{R}))$. Thanks to the density of $C_c([0,\infty[\times\mathbb{R})$ in $C_c([0,\infty[, L^1(\mathbb{R}))$ for the norm $\|.\|_{L^\infty_t(L^1_x)}$, there exists $\phi(t,x)$ in $C_c([0,\infty[\times\mathbb{R})$ such that $\|\varphi-\phi\|_{L^\infty_t(L^1_x)} \leq \eta$. We write

$$< \mu_\varepsilon, \phi >= \int_{[0,\infty[}\int_{\mathbb{R}} \phi_\varepsilon(s,y)\, d\mu(s,y),$$

where $\phi_\varepsilon(s,y) = \int_0^\infty \int_{\mathbb{R}} \zeta_\varepsilon(t-s,x-y)\phi(t,x)\, dt\, dx$. For $s \geq 0$, $y \in \mathbb{R}$, we have

$$\phi_\varepsilon(s,y) - \phi(s,y) = \int_0^\varepsilon \int_{-\varepsilon}^\varepsilon (\phi(u+s,y+z) - \phi(s,y))\zeta_\varepsilon(u,z))\, du\, dz. \qquad (6.3)$$

Thus $\phi_\varepsilon(s,y) \to \phi(s,y)$, as $\varepsilon \to 0$, uniformly for $(s,y) \in [0,\infty[\times\mathbb{R}$, and we get

$$\int_{[0,\infty[}\int_{\mathbb{R}} \phi_\varepsilon(s,y)\, d\mu(s,y) \to \int_{[0,\infty[}\int_{\mathbb{R}} \phi(s,y)\, d\mu(s,y), \quad \varepsilon \to 0.$$

We use now the decomposition

$$< \mu_\varepsilon, \varphi > - < \mu, \varphi >=< \mu_\varepsilon, \varphi - \phi > + < \mu_\varepsilon, \phi > - < \mu, \phi > + < \mu, \phi - \varphi > .$$

We have just proved that $< \mu_\varepsilon, \phi > - < \mu, \phi > \underset{\varepsilon\to 0}{\to} 0$. Now $\phi - \varphi \in C_c([0,\infty[, L^1(\mathbb{R}))$, therefore

$$| < \mu, \phi - \varphi > | \leq C\|\phi-\varphi\|_{L^\infty_t(L^1_x)} \leq C\eta.$$

Using Lemma 6.4, we have similarly $| < \mu_\varepsilon, \varphi - \phi > | \leq C\eta$.
Finally $| < \mu_\varepsilon, \varphi > - < \mu, \varphi > | \leq 3C\eta$, for ε small enough, that is our result. □

6.3 Relation $\alpha\pi = \pi$ for solutions with π in the class $\mathcal{ML}$

Proposition 6.6 *If (α, u, π) is a solution to (1.1) -(1.4), (1.10) in the sense of Theorem 1.1 and if we suppose furthermore that $\alpha \in C_t([0,\infty[, L^1_{loc}(\mathbb{R}))$ and $\pi \in \mathcal{ML}$, then the exclusion relation $\alpha\pi = \pi$ holds in the sense of definition 6.1.*

Proof. We set $\alpha_\varepsilon = \alpha * \zeta_\varepsilon$, $(\alpha u)_\varepsilon = (\alpha u) * \zeta_\varepsilon$, $(\alpha u^2)_\varepsilon = (\alpha u^2) * \zeta_\varepsilon$ where $g * \zeta_\varepsilon(t,x) = \int_0^\infty \int_{\mathbb{R}} \zeta_\varepsilon(t-s,x-y)g(s,y)\, ds\, dy$, and $\pi_\varepsilon(t,x) = \int_{[0,\infty[}\int_{\mathbb{R}} \zeta_\varepsilon(t-s,x-y)\, d\pi(s,y)$. We have from (1.14)-(1.15)

$$\begin{cases} \partial_t\alpha_\varepsilon + \partial_x(\alpha u)_\varepsilon = (\delta(t)\alpha^0(x)) * \zeta_\varepsilon, \\ \partial_t(\alpha u)_\varepsilon + \partial_x((\alpha u^2)_\varepsilon + \pi_\varepsilon) = (\delta(t)\alpha^0(x)u^0(x)) * \zeta_\varepsilon. \end{cases} \qquad (6.4)$$

We define Q_ε in $]\varepsilon,\infty[\times\mathbb{R}$ by $\partial_x Q_\varepsilon = (\alpha u)_\varepsilon$, $\partial_t Q_\varepsilon = -(\alpha u^2)_\varepsilon - \pi_\varepsilon$. Since all functions are smooth, we have

$$\begin{aligned} \partial_t(\alpha_\varepsilon Q_\varepsilon) + \partial_x((\alpha u)_\varepsilon Q_\varepsilon) &= \alpha_\varepsilon\partial_t Q_\varepsilon + Q_\varepsilon\partial_t\alpha_\varepsilon + (\partial_x(\alpha u)_\varepsilon)Q_\varepsilon + (\partial_x Q_\varepsilon)(\alpha u)_\varepsilon \\ &= -\alpha_\varepsilon(\alpha u^2)_\varepsilon - \alpha_\varepsilon\pi_\varepsilon + (\alpha u)^2_\varepsilon + Q_\varepsilon(\partial_t\alpha_\varepsilon + \partial_x(\alpha u)_\varepsilon), \end{aligned}$$

and thanks to (6.4), we obtain

$$\partial_t(\alpha_\varepsilon Q_\varepsilon) + \partial_x((\alpha u)_\varepsilon Q_\varepsilon) = -\alpha_\varepsilon(\alpha u^2)_\varepsilon - \alpha_\varepsilon \pi_\varepsilon + (\alpha u)_\varepsilon^2. \tag{6.5}$$

From the definition of Q_ε, we have that $(Q_\varepsilon)_{\varepsilon>0}$ is bounded in $BV_{t,x}$. Therefore, passing if necessary to a subsequence and adding eventually a constant, Q_ε converges in L^1_{loc} to some Q satisfying $\partial_x Q = \alpha u$, $\partial_t Q = -\alpha u^2 - \pi$. Then, in (6.5) all the terms pass to the limit trivially except $\alpha_\varepsilon \pi_\varepsilon$. For $\varphi \in \mathcal{D}(]0,+\infty[\times\mathbb{R})$, we get, passing to the limit in (1.16) and using (6.5), that

$$\int_{[0,\infty[}\int_{\mathbb{R}} \varphi\pi = \lim_{\varepsilon\to 0}\int_{[0,\infty[}\int_{\mathbb{R}} \alpha_\varepsilon \pi_\varepsilon \varphi = \lim_{\varepsilon\to 0} < \pi_\varepsilon, \alpha_\varepsilon\varphi > .$$

We can write, since $\pi \in \mathcal{ML}$ and $\alpha \in C([0,\infty[, L^1_{loc}(\mathbb{R}))$,

$$< \pi_\varepsilon, \alpha_\varepsilon\varphi > - < \pi, \alpha\varphi > = < \pi_\varepsilon, (\alpha_\varepsilon - \alpha)\varphi > + < \pi_\varepsilon - \pi, \alpha\varphi > .$$

On one hand, $\alpha\varphi \in C_c([0,\infty[, L^1_x(\mathbb{R}))$ hence, by Lemma 6.5, $< \pi_\varepsilon - \pi, \alpha\varphi > \underset{\varepsilon\to 0}{\to} 0$. On the other hand,

$$\begin{aligned} | < \pi_\varepsilon, (\alpha_\varepsilon - \alpha)\varphi > | &\leq C \sup_t \int_{\mathbb{R}} |(\alpha_\varepsilon - \alpha)\varphi|\, dx \\ &\leq C\|\varphi\|_{L^\infty_{t,x}} \|\alpha_\varepsilon - \alpha\|_{L^\infty_t(L^1_{loc,x})} \to 0 \end{aligned}$$

because $\alpha \in C_t([0,\infty[, L^1_x(\mathbb{R}))$. Therefore we get $\lim_{\varepsilon\to 0} < \pi_\varepsilon, \alpha_\varepsilon\varphi > = < \alpha\pi, \varphi >$. Finally, we have proved that $\int_{[0,\infty[}\int_{\mathbb{R}} \varphi\pi = < \alpha\pi, \varphi >$, $\forall\varphi \in \mathcal{D}(]0,+\infty[\times\mathbb{R})$ and thus $\pi = \alpha\pi$ in $]0,+\infty[\times\mathbb{R}$. We can extend the result to 0 because for $t = 0$, the property is always satisfied according to Proposition 2.7. □

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