High-field limit from a kinetic equation to multidimensional scalar conservation laws

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Abstract

We study the relaxation of kinetic BGK models involving a high-field term in the transport operator. This leads to multidimensional scalar conservation laws with a flux which is perturbed with respect to classical relaxation. The proof of the relaxation limit makes appear modified Maxwellians. We consider "pseudo distribution functions" which can take also negative values and introduce appropriate admissible states. This is a first step towards adapting this kind of analysis to quantum kinetic BGK models.

Key-words: high field limit – scalar conservation laws – admissible states – hydrodynamics limit – entropy solutions – modified Maxwellian

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Contents

1 Introduction

"Fluid dynamics limits" are motivated by the derivation of fluid mechanics as the zero mean free path limit of kinetic equations. A lot of significant progress has been made in the last years (for example, we can cite [2], [3], [23], [17], [32], [33], ...). The passage to the limit in the case of a BGK model with an operator of transport of the form $\partial_t + a(v) \cdot \partial_x$ towards a scalar conservation law was obtained in [30], the case of systems is solved only in particular cases (see for example [37], [35], [25], [4], [32], [17], [5], [6],...). See also [26] for kinetic formulation in the multidimensional scalar case. Recently, different scalings of the Vlasov equation make appear new interesting models [1], in particular, the case of high-field equations. For example this yields important new semiconductor models. This approach first appears in [31] and we refer to [29], [13] and the references therein for such problems. Similar problems are in the domain of strong magnetic field ([15], [18], [34]), quasi-neutral limit for plasmas ([19], [9], [10]) and strong friction ([21]).

In this paper we study the high-field limit of a kinetic model towards a scalar conservation law, that is the relaxation limit when $\varepsilon \to 0$ of the BGK model

$$\partial_t f + \mathrm{div}_x(a(x,v)\, f) + \frac{F(x)}{\varepsilon}\,\partial_v f = \frac{\chi_\rho - f}{\varepsilon}, \quad (t,x,v) \in \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}, \tag{1.1}$$

where

$$\chi_\rho(v) = \begin{cases} \mathrm{sgn}(\rho) & \text{if } (\rho - v)v \geq 0, \\ 0 & \text{for all other } v, \end{cases}$$

with

$$\rho(t,x) = \int_{\mathbb{R}} f(t,x,v)\, dv.$$

The function sgn is defined as usual by $\mathrm{sgn}(u) = 1$ for $u > 0$, $\mathrm{sgn}(u) = -1$ for $u < 0$ and $\mathrm{sgn}(0) = 0$. The velocity field $a : \mathbb{R}^D \times \mathbb{R} \to \mathbb{R}^D$ and the scalar force field $F : \mathbb{R}^D \to \mathbb{R}$ are given. Note that the "kinetic variable" $v \in \mathbb{R}$ is a scalar which means that (x, v) has the character of a true "phase space" only in the case $D = 1$.

The important new point is that the term F/ε modifies the obtained hydrodynamic limit. Indeed, in the traditional case, that is to say the case with $F = 0$, the obtained limit is

$$\partial_t \rho + \operatorname{div}_x A(x, \rho) = 0, \qquad A(x, \rho) = \int_0^\rho a(x, u)\, du, \tag{1.2}$$

where

$$\operatorname{div}_x A(x, \rho) = \sum_{i=1}^{D} \partial_{x_i} A_i(x, \rho),$$

whereas in the case $F \neq 0$, we obtain here

$$\partial_t \rho + \operatorname{div}_x B(x, \rho) = 0, \tag{1.3}$$

with

$$B(x, \rho) = \int_0^\rho \int_0^{+\infty} a(x, v + F(x)u)\, e^{-u}\, du\, dv. \tag{1.4}$$

Moreover, we give a first approach to make it possible to take into account quantum effect in such a situation: we are able to deal with the the possibility for the microscopic kinetic distribution f to have oscillations (i.e. negative values $f < 0$, like it is typical for "Wigner functions" ([24], [16]), although observable macroscopic quantities, like the density $\rho(t, x) = \int f(t, x, v)\, dv$, remain positive.
An other new aspect is that since the obtained scalar equation is modified, the thermodynamic equilibrium is also modified. Indeed, when $\varepsilon \to 0$, the distribution f does not converge to the Maxwellian χ_ρ but to a modified Maxwellian M_ρ.
To our knowledge this work is also the first time that the convergence of kinetic equations towards conservation laws with x-dependent flux is proved.

We note $C_b^1(\Omega)$ the set of continously differentiable real functions on Ω which are bounded with bounded derivatives. We define $C_b^{2,1}(\mathbb{R}^D \times \mathbb{R})$ as the space of bounded continuous functions which are two times continuously differentiable with respect to x and one times with respect to v with bounded derivatives. The space of bounded measures on Ω is denoted by $\mathcal{M}_1(\Omega)$. The set $BV(\Omega)$ is the set of integrables functions whose distribution derivatives belong to $\mathcal{M}_1(\Omega)$. The sets L^p, $W^{1,p}$, ... are the usual Sobolev spaces.

We obtain the two following results. The first one deals with the case $F(x) = F$ constant for $x \in \mathbb{R}^D$:

Theorem 1.1 *Let $f_I \in L^1(\mathbb{R}^D \times \mathbb{R})$, $F \in \mathbb{R}$ a constant and $a \in C_b^{2,1}(\mathbb{R}^D \times \mathbb{R})$. Then $\rho_\varepsilon = \int_{\mathbb{R}} f_\varepsilon\, dv$, where f_ε is the solution of the BGK model (1.1) with the Cauchy data $f_\varepsilon(0) = f_I$, satisfies $\rho_\varepsilon \to \rho$ in $L^1([0, T] \times K))$ for any $T > 0$ and any compact subset $K \in \mathbb{R}^D$. The limit ρ is a weak solution of*

$$\partial_t \rho + \operatorname{div}_x B(x, \rho) = 0, \quad \rho(0) = \rho_I = \int_{\mathbb{R}} f_I\, dv, \tag{1.5}$$

with B given by (1.4). The distribution function $f_\varepsilon \to M_\rho$ in $L^1([0,T]\times K\times\mathbb{R}))$. In particular we have the Kružkov entropy relations:

$$\forall k \in \mathbb{R}, \quad \partial_t|\rho - k| + \mathrm{div}_x\Big((B(x,\rho) - B(x,k))\,\mathrm{sgn}(\rho - k)\Big) \\ +(\mathrm{div}_x B)(x,k)\,\mathrm{sgn}(\rho - k) \leq 0. \tag{1.6}$$

The second result deals with the case of a $x-$dependent force field $F(x)$ but in one space dimension ($x \in \mathbb{R}$). Furthermore, we have to assume that the sequence

$$\left(\int_{\mathbb{R}} |f_\varepsilon(t,x,v)|\,dv\right)_{\varepsilon>0} \text{ is bounded in } L^\infty([0,+\infty[\times\mathbb{R}). \tag{1.7}$$

Theorem 1.2 *Let $f_I \in L^1(\mathbb{R}\times\mathbb{R})$, $F \in C_b^1(\mathbb{R})$, $F' \in L^1(\mathbb{R})$, and $a \in C_b^{2,1}(\mathbb{R}\times\mathbb{R})$. Let $\rho_\varepsilon = \int_{\mathbb{R}} f_\varepsilon\,dv$, where f_ε is the solution of the BGK model (1.1) with Cauchy data f_I. We moreover assume that (1.7) holds. Then $\rho_\varepsilon \to \rho$ in $L^\infty(\mathbb{R}^+\times\mathbb{R})$ weak * (up to a subsequence). The limit ρ is a weak solution of (1.5) with B given by (1.4).*
If we assume that $k \mapsto B(x,k)$ are nonlinear (in the sense that there exists no interval on which the function is linear), $\rho_\varepsilon \to \rho$ holds for the whole sequence in $L^1([0,T]\times K))$ for any $T > 0$ and any compact subset $K \in \mathbb{R}^D$. The distribution function $f_\varepsilon \to M_\rho$ in $L^1([0,T]\times K\times\mathbb{R}))$. In particular we have the Kružkov entropy inequalities (1.6).

The organisation of the paper is: In Section 2, we present the kinetic model we use. We prove the wellposedness of this model and we introduce the concept of "quantum admissible states". In Section 3, we give the formal asymptotics which leads from the kinetic model to the conservation law (1.3). In particular, we introduce the modified Maxwellians. The Section 4 is devoted to some Remarks about transport equations and Kružkov entropies. Finally, in Section 5, we give the rigorous proofs of the convergence of kinetic equations towards conservation laws in two cases. In the multidimensional case we assume $F =$ constant. We use BV techniques and Kružkov entropies. In the one-dimensional case, we can consider general $F(x)$ by using compensated compactness techniques. However we have to assume the uniform L^∞ bounds (1.7).

2 The kinetic model, for pseudo distribution functions

In this section, we study the kinetic equation (1.1) which is the basis of our high-field model. In particular we prove the existence of solutions and we introduce the concept of admissible states.

2.1 BGK model

We consider the following BGK model

$$\partial_t f + \mathrm{div}_x(a(x,v)\cdot f) + \frac{F(x)}{\varepsilon}\cdot\partial_v f = \frac{\chi_\rho - f}{\varepsilon}, \tag{2.1}$$

with $f = f(t,x,v) \in \mathbb{R}$ with $t \geq 0$, $x \in \mathbb{R}^D$, $v \in \mathbb{R}$, $a : \mathbb{R}^D \times \mathbb{R} \to \mathbb{R}^D$, $F : \mathbb{R}^D \to \mathbb{R}$, $\varepsilon \in]0, +\infty[$ and where the density is

$$\rho(t,x) = \int_{\mathbb{R}} f(t,x,v)\, dv, \tag{2.2}$$

and χ is the classical Maxwellian for scalar conservation laws

$$\chi_\rho(v) = \begin{cases} \mathrm{sgn}(\rho) & \text{if } (\rho - v)v \geq 0, \\ 0 & \text{if not.} \end{cases} \tag{2.3}$$

This Maxwellian satisfies

$$\int_{\mathbb{R}} \chi_k(v)\, dv = k, \tag{2.4}$$

$$\int_{\mathbb{R}} |\chi_k(v) - \chi_{k'}(v)|\, dv = |k - k'|, \tag{2.5}$$

$$\forall B \in C^1(\mathbb{R}), \quad \int_{\mathbb{R}} \chi_k(v) B'(v)\, dv = B(k) - B(0). \tag{2.6}$$

For kinetic models that generate L^1-contractions, see for example [38]. We also refer to [7] for the construction of BGK models.

2.2 Well-posedness of the kinetic model

We prove here the existence of solutions for the kinetic equation (2.1) for any $a(x,v)$ and any $F(x)$.

Theorem 2.1 *For $a \in C_b^1(\mathbb{R}^D \times \mathbb{R})$ and $F \in C_b^1(\mathbb{R}^D)$, the kinetic model (2.1)-(2.3) completed by the Cauchy data $f(0) = f_I \in L^1(\mathbb{R}^D \times \mathbb{R})$ is well-posed in $C(\mathbb{R}, L^1(\mathbb{R}^D \times \mathbb{R}))$.*

Proof. It is well known that the transport operator $\mathcal{T}^\varepsilon = \mathrm{div}_x(a\cdot) + \partial_v(\dfrac{F}{\varepsilon}\cdot)$ generates a continuous group $(\mathcal{G}_t^\varepsilon)_{t\in\mathbb{R}}$ in $L^1(\mathbb{R}^D \times \mathbb{R})$. Moreover this group satisfies

$$\forall f \in L^1(\mathbb{R}^D \times \mathbb{R}), \quad \forall t \in \mathbb{R},$$

$$\int_{\mathbb{R}^D\times\mathbb{R}} \mathcal{G}_t^\varepsilon(f)\, dx\, dv = \int_{\mathbb{R}^D\times\mathbb{R}} f\, dx\, dv, \qquad \|\mathcal{G}_t^\varepsilon(f)\|_{L^1_{xv}} = \|f\|_{L^1_{xv}}, \tag{2.7}$$

$$f \geq 0 \ \text{ implies } \mathcal{G}_t^\varepsilon(f) \geq 0. \tag{2.8}$$

Then the existence and uniqueness result of the above Theorem holds because the (non linear) operator, $f \mapsto \frac{1}{\varepsilon}(\chi_\rho - f)$ with $\rho = \int_{\mathbb{R}} f\, dv$, is Lipschitz in $L^1(\mathbb{R}^D \times \mathbb{R})$. Indeed, thanks to (2.5) we get

$$\|\frac{1}{\varepsilon}(\chi_{\rho_1} - f_1) - \frac{1}{\varepsilon}(\chi_{\rho_2} - f_2)\|_{L^1_{xv}} \leq \frac{2}{\varepsilon}\, \|f_1 - f_2\|_{L^1_{xv}}.$$

We also remark that the solution $f(t)$ of (2.1)-(2.3) solves

$$\partial_t(e^{t/\varepsilon}f(t)) + \mathcal{T}^\varepsilon(e^{t/\varepsilon}f(t)) = \frac{e^{t/\varepsilon}}{\varepsilon}\chi_{\rho(t)},$$

then the Duhamel formula yields

$$f(t) \;=\; e^{-t/\varepsilon}\mathcal{G}_t^\varepsilon(f(0)) + \int_0^t \mathcal{G}_{t-s}^\varepsilon(\chi_{\rho(s)})\frac{e^{-(t-s)/\varepsilon}}{\varepsilon}\,ds. \tag{2.9}$$

It follows that $f(t)$ can be obtained as the limit in $L^1(\mathbb{R}^D\times\mathbb{R})$ of the following sequence

$$\begin{aligned} f^0(t) &= f(0),\\ \rho_n(t) &= \int_{\mathbb{R}} f_n(t)\,dv,\\ f_{n+1}(t) &= e^{-t/\varepsilon}\mathcal{G}_t^\varepsilon(f(0)) + \int_0^t \mathcal{G}_{t-s}^\varepsilon(\chi_{\rho_n(s)})\,\frac{e^{-(t-s)/\varepsilon}}{\varepsilon}\,ds, \end{aligned} \tag{2.10}$$

and the convergence is uniform with respect to time on every bounded intervals. $\square$

2.3 Admissible states

We now define the admissible states which are compatible with this model. We first introduce the characteristics related to the transport operator $\mathcal{T}^\varepsilon$:

$$\begin{cases} \partial_t X_\varepsilon(t,x,v) = a(X_\varepsilon(t,x,v), V_\varepsilon(t,x,v)),\\ \partial_t V_\varepsilon(t,x,v) = \dfrac{1}{\varepsilon}F(X_\varepsilon(t,x,v)),\\ X_\varepsilon(0,x,v) = x, \qquad V_\varepsilon(0,x,v) = v. \end{cases} \tag{2.11}$$

The classical theory of transport equations leads to the following facts.

- The jacobian of the change of variables $(x,v)\mapsto\Big(X_\varepsilon(t,x,v), V_\varepsilon(t,x,v)\Big)$ is given by $J_\varepsilon(t,x,v)>0$ which solves

$$\begin{cases} \partial_t J_\varepsilon(t,x,v) = J_\varepsilon(t,x,v)\;\mathrm{div}_x\, a\Big(X_\varepsilon(t,x,v), V_\varepsilon(t,x,v)\Big),\\ J_\varepsilon(0,x,v) = 1. \end{cases} \tag{2.12}$$

- The group $(\mathcal{G}_t^\varepsilon)_{t\in\mathbb{R}}$ is explicitly given by

$$\mathcal{G}_t^\varepsilon(f)(x,v) = f\Big(X_\varepsilon(-t,x,v), V_\varepsilon(-t,x,v)\Big)\;J_\varepsilon(-t,x,v). \tag{2.13}$$

We are now ready to give the following definition

Definition 2.2 *"Admissible states" :*

a) We define as "admissible initial state" for the kinetic model (2.1) a function f_I such that

$$\int_{\mathbb{R}} f_I\big(X_\varepsilon(-t,x,v), V_\varepsilon(-t,x,v)\big)\, dv \geq 0, \qquad \forall x \in \mathbb{R}^D, t \in \mathbb{R}^+. \tag{2.14}$$

We note $\mathcal{I}_F^\varepsilon$ the set of all admissible initial state.

b) The admissible solutions $f(t)$ have to satisfy the following property : We define as "admissible state" a function $f(t,x,v)$ such that

$$\int_{\mathbb{R}} f(t,x,v)\, dv \geq 0, \quad \forall t \geq 0, \text{ and for a.e. } x \in \mathbb{R}^D. \tag{2.15}$$

We note $\mathcal{A}$ the set of all admissible states.

These definitions can be understood by looking at the free transport equation $\partial_t f(t) + \mathcal{T}^\varepsilon(f(t)) = 0$ whose solution is $f(t) = \mathcal{G}_t^\varepsilon(f(0))$. Indeed, due to (2.13), the admissibility condition (2.14) is equivalent to

$$\int_{\mathbb{R}} \mathcal{G}_t^\varepsilon(f(0))\, dv \geq 0, \quad \forall t \geq 0.$$

We prove now that this property is kept even in the case of a BGK right-hand side. We use the iteration (2.10). We have $f^0 \in \mathcal{A}$ and assuming that $f_n \in \mathcal{A}$ we get $\rho_n(s) \geq 0$ for every $s \geq 0$. Since $\mathcal{G}_s^\varepsilon$ and $k \mapsto \chi_k$ preserve the non negativity, we have

$$\int_0^t \mathcal{G}_s^\varepsilon(\chi_{\rho_n(s)})\, ds \geq 0,$$
$$f_{n+1} \geq e^{-t/\varepsilon}\mathcal{G}_t^\varepsilon(f(0)), \quad \rho_{n+1} \geq e^{-t/\varepsilon} \int_{\mathbb{R}} \mathcal{G}_t^\varepsilon(f(0))\, dv \geq 0.$$

Therefore $f_{n+1} \in \mathcal{A}$. Then f is the limit of a sequence of $\mathcal{A}$ which is closed in $L^1(\mathbb{R}^D \times \mathbb{R})$. It ends the proof.

Remark 2.1 The admissible property means that we allow for oscillations with negative values for f, but that the macroscopic observable ρ have to remain positive. These conditions are not enough to get classical physical quantum admissible states but it is a first step towards such model.

Remark 2.2 There exists admissible initial states such that $f^0 < 0$ occurs. For example, with $F = 1$ and $a(x,v) = v$, we can take

$$f^0(x,v) = (b|x|^2 + cv)\mathbb{1}_{|v| \leq R}, \;\; \text{with } b, R > 0, \; c < 0.$$

Remark 2.3 A natural extension of our model to a quantum kinetic model would be the Wigner equation

$$\partial_t f + v \cdot \partial_x f + \frac{1}{\varepsilon}\Theta^\hbar[U] \cdot f = \frac{1}{\varepsilon}(\chi_\rho - f), \tag{2.16}$$

where $(x,v) \in \mathbb{R}^D \times \mathbb{R}^D$ and f the Wignerfunction that can take negative values. In the "relaxation term" χ_ρ is defined as previously and the classical field term becomes now the pseudo-differential operator

$$(\Theta^\hbar[U]\cdot f)(t,x,v) = \frac{i}{2\pi}\int_{\mathbb{R}} \frac{U(x+\hbar\xi/2)-U(x-\hbar\xi/2)}{\hbar}\; e^{iv\xi}\; \hat{f}(t,x,\xi)\,d\xi, \tag{2.17}$$

with U smooth and $\hat{.}$ the Fourier transform in the variable v.
This equation is related to the Schrödinger equation with a potential U by a Wigner transform ([16]).
Formally, we have

$$\Theta^\hbar[U]\cdot f \underset{\hbar\to 0}{\longrightarrow} -\partial_x U\cdot\partial_v f,$$

(see [24], [27] for the mathematical study of this last fact).
The problem with the model (2.16) is that for a mathematical study the term χ needs an L^1 theory and the term $\Theta^\hbar[U]$ needs an approach in L^2...

3 Formal relaxation and modified Maxwellian

This section is devoted to the formal hydrodynamic limit $\varepsilon \to 0$. We notice first that the case $F=0$ or $|F| \ll 1$ can be treated as in [30], but we give the improvement that a also depends on x. However in the case $F \neq 0$, the high-field force modifies the kinetic equilibrium, that is to say the Maxwellians. As a consequence it modifies the limit conservation law.

In the first Subsection we present a formal derivation of this conservation law. This derivation will be made rigorous in the next Section for two cases: in any $D \geq 1$ if the force field is constant, for any force field if $D=1$.

In the second Subsection we introduce the modified Maxwellian which will be useful in the next Section.

3.1 Formal limit

We present here the formal limit which is obtained from the BGK model of the previous section. Formally, passing to the limit as $\varepsilon \to 0$ in (2.1), gives

$$F(x)\,\partial_v f = \chi_\rho - f. \tag{3.1}$$

On the other hand, an integration with respect to v of (2.1) yields

$$\partial_t \int_{\mathbb{R}} f\,dv + \mathrm{div}_x \int_{\mathbb{R}} a(x,v) f\,dv = 0. \tag{3.2}$$

The task is now to compute the flux $\int_{\mathbb{R}} a(x,v) f\,dv$ in term of the density $\rho = \int_{\mathbb{R}} f\,dv$. For any function $b(x,v)$, by using (3.1), we get

$$\begin{aligned} \int_{\mathbb{R}} b(x,v)F(x)\partial_v f\,dv &= \int_{\mathbb{R}} (\chi_\rho - f) b(x,v)\,dv \\ &= B(x,\rho) - \int_{\mathbb{R}} f b(x,v)\,dv, \end{aligned} \tag{3.3}$$

where we set

$$B(x,v) = \int_0^v b(x,u)\,du. \tag{3.4}$$

We notice that the last equality comes from (2.6). Then we have

$$\int_{\mathbb{R}} \Big(b(x,v) - F(x)\partial_v b(x,v) \Big) f\,dv = B(x,\rho).$$

It remains to compute the solution of the equation

$$b(x,v) - F(x)\,\partial_v b(x,v) = a(x,v). \tag{3.5}$$

If we impose that b is bounded, we obtain

$$b(x,v) = \begin{cases} \displaystyle\int_v^{+\infty} \frac{a(x,u)}{F(x)} e^{(v-u)/F(x)}\,du & \text{for } \ F(x) > 0, \\ a(x,v) & \text{for } \ F(x) = 0, \\ \displaystyle -\int_{-\infty}^{v} \frac{a(x,u)}{F(x)} e^{(v-u)/F(x)}\,du & \text{for } \ F(x) < 0. \end{cases}$$

By using the change of variables $u \to uF + v$ we get

$$b(x,v) = \int_0^{+\infty} a(x, v + uF(x))\,e^{-u}\,du. \tag{3.6}$$

Since B is defined by (3.4), we obtain that B is given by (1.4). Taking now the function b and reporting in (3.5), gives

$$B(x,\rho(t,x)) = \int_{\mathbb{R}} f(t,x,v) a(x,v)\,dv.$$

Finally (3.2) becomes

$$\partial_t \rho + \operatorname{div}_x B(x,\rho) = 0. \tag{3.7}$$

We notice that the high-field has an impact on the obtained scalar equation because classically we obtain the flux A defined by

$$A(x,\rho(t,x)) = \int_0^{\rho(t,x)} a(x,u)\,du$$

instead of B given by (1.4).

3.2 Modified Maxwellian

The equation (3.1) shows that f does not converge to χ_ρ but to a modified Maxwellian. In this section, we compute this Maxwellian M_ρ. Two cases appear.

First, in the case of a constant force F, the modified Maxwellians $M_k(v)$ does not depend on x. It results that if a is divergence free ($\operatorname{div}_x a = 0$), M_k is a particular solution of the kinetic equation. Then we can use the classical approximated entropy inequalities.

Secondly, for general $F(x)$, the Maxwellian $M_k(x,v)$ has an unusual dependence on x. Therefore M_k is no more a solution of the kinetic equation. It is also the case if a is non divergence free. Then the theory of classical entropies does not apply.

We remark that x is only a parameter in (3.1). Then the Maxwellian $M_k(x,v)$ have to satisfy

$$F(x)\,\partial_v M_k(x,v) + M_k(x,v) = \chi_k(v), \tag{3.8}$$

with

$$\int_{\mathbb{R}} M_k(x,v)\,dv = k. \tag{3.9}$$

By the same computation as in the previous Subsection we get

$$M_k(x,v) = \int_0^{+\infty} \chi_k(v + F(x)u)\,e^{-u}\,du. \tag{3.10}$$

From the above formula, it is easy to check that (3.9) holds by using (2.4) and Fubini's Theorem. By the same way using (2.5), we get, for any $k, k' \in \mathbb{R}$,

$$\begin{aligned}
\operatorname{sgn}(M_k(x,v) - M_{k'}(x,v)) &= \operatorname{sgn}(k-k'), && (3.11)\\
\int_{\mathbb{R}} |M_k(x,v) - M_{k'}(x,v)|\,dv &= |k-k'|, && (3.12)\\
\int_{\mathbb{R}} a(x,v)M_k(x,v)\,dv = \int_{\mathbb{R}} b(x,v)\chi_k(v)\,dv &= B(x,k). && (3.13)
\end{aligned}$$

4 Kružkov entropies and transport equations

We begin this Section with a general result on transport equations.

Lemma 4.1 *Let $c \in \left(W^{1,p}_{loc}(\mathbb{R}^N)\right)^N$ a vector field, $u \in L^1([0,T];L^q_{loc}(\mathbb{R}^N))$ a real function and $S \in L^1([0,T];L^1_{loc}(\mathbb{R}^N))$, $R \in \mathcal{M}([0,T]\times\mathbb{R}^N)$ source terms which satisfy*

$$\partial_t u(t,y) + \operatorname{div}_y(cu)(t,y) = S(t,y) + R(t,y), \qquad \text{on } [0,T]\times\mathbb{R}^N$$

in the distributions sense.
If $R = 0$, then for any real function $G \in C^1(\mathbb{R})$ satisfying $G' \in L^\infty(\mathbb{R})$, we have

$$\partial_t G(u) + \operatorname{div}_y(c\,G(u)) - \operatorname{div}_y(c)(G(u) - G'(u)u) - S\,G'(u) = 0, \qquad \text{on } [0,T]\times\mathbb{R}^N$$

in the distributions sense. The set $\{(t,x)\in[0,T]\times\mathbb{R}^N /\ u(t,x)=0,\ S(t,x)\neq 0\}$ is negligeable for the Lebesgue measure and

$$\partial_t|u| + \operatorname{div}_y(c\,|u|) = S\operatorname{sgn}(u), \qquad \text{on } [0,T]\times\mathbb{R}^N$$

in the distributions sense.
If $R \neq 0$, then for any real function $G \in C^1(\mathbb{R})$ satisfying $G' \in L^\infty(\mathbb{R})$, we still have

$$\partial_t G(u) + \operatorname{div}_y(c\,G(u)) - \operatorname{div}_y(c)(G(u) - G'(u)u) - S\,G'(u) \le \|G'\|_\infty\,|R|,$$

on $[0,T]\times\mathbb{R}^N$, *and*

$$\partial_t|u| + \mathrm{div}_y(c\,|u|) - S\,\mathrm{sgn}(u) \le |R|, \qquad \text{on } [0,T]\times\mathbb{R}^N$$

in the distributions sense.

Proof. The first assertion is now a classical result due to Di Perna and Lions, cf [14]. We recall the main ingredients of the proof. It is based on Friedrichs regularization and more precisely on the fact that, in the above conditions, we have for any compact subset $K\subset\mathbb{R}^N$

$$r^\delta = \mathrm{div}_y(cu) *_y \eta^\delta - \mathrm{div}_y(c(u *_y \eta^\delta)) \underset{\delta\to 0}{\longrightarrow} 0 \text{ in } L^1([0,T]\times K),$$

where η^δ is an approximation of unity

$$\eta^\delta(y) = \eta(y/\delta)/\delta^N, \quad \eta\in C_c^\infty(\mathbb{R}^N;\mathbb{R}^+), \quad \int_{\mathbb{R}^N}\eta(y)\,dy = 1, \quad \eta \text{ even}.$$

We can use the chain rule for $u^\delta = u *_y \eta^\delta$ which solves

$$\partial_t u^\delta + \mathrm{div}_y(cu^\delta) = S *_y \eta^\delta + R *_y \eta^\delta - r^\delta.$$

If $R=0$, the first equation of the Lemma is obtained by multiplying the above equation by $G'(u^\delta)$ and letting $\delta\to 0$.

We use now the first equation of the Lemma with $G' = \mathrm{sgn}^\alpha$ and $G(\lambda) = |\lambda|_\alpha = \int_0^\lambda \mathrm{sgn}^\alpha(s)\,ds$ where $\mathrm{sgn}^\alpha\in C^0(\mathbb{R};[-1,1])$ is a smooth regularization of the sign function. We impose that $\mathrm{sgn}^\alpha(0) = d\in[-1,1]$ in such a way that $\mathrm{sgn}^\alpha \underset{\alpha\to 0}{\longrightarrow} \mathrm{sgn}_d$ everywhere. The function sgn_d is defined as the function sgn except that $\mathrm{sgn}_d(0) = d$. (We have $\mathrm{sgn} = \mathrm{sgn}_0$.) We also remark that

$$G(\lambda) - \lambda\,G'(\lambda) = |\lambda|_\alpha - \lambda\,\mathrm{sgn}^\alpha(\lambda)$$

tends to 0 uniformly. Letting $\alpha\to 0$ we obtain

$$\partial_t|u| + \mathrm{div}_y(c\,|u|) = S\,\mathrm{sgn}_d(u).$$

Since $S\,\mathrm{sgn}_d(u)$ defines the same distribution for any d, the set $\{u=0,\ S\neq 0\}$ is negligeable.

If $R\neq 0$ the sequence $R*\eta^\delta G'(u*\eta^\delta)$ is possibly not converging but it remains bounded. Indeed let φ a non negative test function, we have

$$\int_0^T\int_K R*\eta^\delta\ G'(u*\eta^\delta)\varphi\,dt\,dx \le \|G'\|_\infty \int_0^T\int_K \varphi*\eta^\delta\ |R|\,dt\,dx.$$

It leads to the second part of the Lemma. □

The consequences for the kinetic entropy inequalities are detailed below.

We assume that $a \in [C_b^1(\mathbb{R}^D \times \mathbb{R})]^D$ and $F \in C_b^1(\mathbb{R}^D)$. Let f_1 and f_2 be two solutions of (2.1) belonging to $L^1([0,T] \times \mathbb{R}^D \times \mathbb{R})$ with source terms $S_1 + R_1$ and $S_2 + R_2$. We assume that S_1 and S_2 belong to $L^1([0,T] \times \mathbb{R}^D \times \mathbb{R})$ and that $R_1,\ R_2 \in \mathcal{M}_1([0,T] \times \mathbb{R}^D \times \mathbb{R})$. We have

$$\begin{aligned}\partial_t(f_1 - f_2) \quad &+ \quad \mathrm{div}_x(a(x,v)(f_1 - f_2)) + \frac{F(x)}{\varepsilon}\,\partial_v(f_1 - f_2)\\ &= \quad \frac{\chi_{\rho_1} - f_1 - \chi_{\rho_2} + f_2}{\varepsilon} + S_1 - S_2 + R_1 - R_2,\end{aligned}$$

with

$$\rho_1 = \int_{\mathbb{R}} f_1\,dv, \qquad \rho_2 = \int_{\mathbb{R}} f_2\,dv.$$

Using Lemma 4.1, and integrating in v, we get

$$\begin{aligned}\partial_t \int_{\mathbb{R}} |f_1 - f_2|\,dv &+ \mathrm{div}_x \int_{\mathbb{R}} a(x,v)|f_1 - f_2|\,dv - \int_{\mathbb{R}} (S_1 - S_2)\,\mathrm{sgn}(f_1 - f_2)\,dv\\ &\le \int_{\mathbb{R}} |R_1 - R_2|\,dv \;+\; \frac{1}{\varepsilon}\int_{\mathbb{R}} |\chi_{\rho_1} - \chi_{\rho_2}|\,dv - \frac{1}{\varepsilon}\int_{\mathbb{R}} |f_1 - f_2|\,dv.\end{aligned}$$

Now by (2.5),

$$\int_{\mathbb{R}} |\chi_{\rho_1} - \chi_{\rho_2}|\,dv = |\rho_1 - \rho_2| = |\int_{\mathbb{R}} (f_1 - f_2)\,dv| \le \int_{\mathbb{R}} |f_1 - f_2|\,dv.$$

We have obtained the following lemma.

Lemma 4.2 *Let $a \in [C_b^1(\mathbb{R}^D \times \mathbb{R})]^D$ and $F \in C_b^1(\mathbb{R}^D)$. Let f_1 and f_2 be two solutions of (2.1) with source terms $S_1 + R_1$ and $S_2 + R_2$. We assume that all the functions f_1, f_2, S_1, S_2 belong to $L^1([0,T] \times \mathbb{R}^D \times \mathbb{R})$ and that $R_1,\ R_2 \in \mathcal{M}_1([0,T] \times \mathbb{R}^D \times \mathbb{R})$. Then we get*

$$\begin{aligned}\partial_t \int_{\mathbb{R}} |f_1 - f_2|\,dv \quad &+ \quad \mathrm{div}_x \int_{\mathbb{R}} a\,|f_1 - f_2|\,dv - \int_{\mathbb{R}} (S_1 - S_2)\,\mathrm{sgn}(f_1 - f_2)\,dv\\ &\le \quad \int_{\mathbb{R}} |R_1 - R_2|\,dv \;+\; \frac{1}{\varepsilon}\Big(|\rho_1 - \rho_2| - \int_{\mathbb{R}} |f_1 - f_2|\,dv\Big)\\ &\le \quad \int_{\mathbb{R}} |R_1 - R_2|\,dv. \end{aligned} \tag{4.1}$$

It follows that, if $S_2 = \mathrm{div}_x(a(x,v)M_k(x,v)) \in L^1([0,T] \times \mathbb{R}^D \times \mathbb{R})$, we have the following entropy inequality for all solution f of (2.1) and all $k \in \mathbb{R}$,

$$\begin{aligned}\partial_t \int_{\mathbb{R}} |f - M_k|\,dv \quad &+ \quad \mathrm{div}_x \int_{\mathbb{R}} a\,|f - M_k|\,dv\\ &+ \quad \int_{\mathbb{R}} \mathrm{div}_x\Big(a\;M_k\Big)\,\mathrm{sgn}(f - M_k)\,dv \le 0. \end{aligned} \tag{4.2}$$

In the case where F is a constant and a is divergence free, M_k is a solution of (2.1). Then the last term in (4.2) vanishes. This simplifies considerably the proofs of the next Section. But in the general case, M_k is not a solution to the kinetic equation. An attempt to overcome this problem is to introduce the stationary solutions of (2.1) which satisfies

$$\varepsilon \operatorname{div}_x \Big(a(x,v) M_k^\varepsilon(x,v)\Big) + F(x)\,\partial_v M_k^\varepsilon(x,v) + M_k^\varepsilon(x,v) = \chi_k(v), \tag{4.3}$$

$$\int_{\mathbb{R}} M_k^\varepsilon(x,v)\,dv = k - \varepsilon \partial_x \int_{\mathbb{R}} a(x,v) M_k^\varepsilon(x,v)\,dv.$$

For example, the case $a(x,v) = a$ (constant) gives

$$\int_{\mathbb{R}} M_k^\varepsilon(x,v)\,dv = k + Ce^{-x/(\varepsilon a)},$$

and this approximation can have too large values to be a good way to pass to the limit in (2.1). Therefore we use the Maxwellian M_k instead of M_k^ε in the following.

5 Proofs of Theorems 1.1, 1.2

We are interested, in this Section, in the rigorous derivation of the relaxation limit. First, we consider the case $F =$ constant in any dimension $D \geq 1$. In this case it is possible to justify everything using BV technique and approximated entropy inequalities. Next, we consider the case of more general $F(x)$ in one dimension ($D = 1$) by using compensated compactness techniques. However we have to assume L^∞ bounds for $\int f_\varepsilon\,dv$.

We prove the following estimates in the general case, that is to say for $a(x,v) \in C_b^1(\mathbb{R}^D \times \mathbb{R}, \mathbb{R}^D)$ and $F(x) \in C_b^1(\mathbb{R}^D)$.

Proposition 5.1 *For $f_I \in L^1(\mathbb{R}_v \times \mathbb{R}_x^D)$, the sequence $(\int_{\mathbb{R}} |f_\varepsilon(t,x,v)|\,dv)_{\varepsilon>0}$ is bounded in $L^\infty(\mathbb{R}_t^+, L^1(\mathbb{R}_x^D))$. In particular, the sequence $(\rho_\varepsilon(t,x))_{\varepsilon>0}$ is bounded in $L^\infty(\mathbb{R}_t^+, L^1(\mathbb{R}_x^D))$.*

We assume moreover that $|\nabla_x F| \in L^1(\mathbb{R}^D)$. Let $k = k(t,x)$ be a $BV_{loc}(\mathbb{R} \times \mathbb{R}^D)$-function, we have for any $T > 0$ for any compact subset $K \subset \mathbb{R}^D$

$$\int_{\mathbb{R}} |f_\varepsilon - M_k|\,dv - |\rho_\varepsilon - k| \to 0, \qquad in\ L^1([0,T] \times K). \tag{5.1}$$

Moreover, if for a subsequence $\rho_\varepsilon \to \rho$ in $L^1([0,T] \times K)$ then

$$|f_\varepsilon - M_\rho| \to 0, \qquad in\ L^1([0,T] \times K \times \mathbb{R}). \tag{5.2}$$

Proof. We use (4.1) with $f_1 = f_\varepsilon$ and $f_2 = 0$. An integration with respect to x yields that

$$\int_{\mathbb{R}^D \times \mathbb{R}} |f_\varepsilon(t,x,v)|\,dx\,dv$$

is a non increasing function of time. This yields the L^1-estimates.

For the second part of the Proposition, let us first consider $k \in BV(\mathbb{R} \times \mathbb{R}^D)$. We use (4.1) with $f_1 = f_\varepsilon$ and $f_2 = M_k$. We have $S_1 = S_2 = R_1 = 0$ and $R_2 = \partial_t M_k + \operatorname{div}_x(a M_k)$. From (2.3), it is easy to check that

$$\partial_k \chi_k = \delta(v - k), \quad \partial_v \chi_k = \delta(v) - \delta(v - k).$$

Therefore, using (3.10), we get

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^D}\int_{\mathbb{R}}|\partial_t M_k|\,dv\,dx\,dt &\le \int_0^T\int_{\mathbb{R}^D}\int_{\mathbb{R}}\int_0^{+\infty}|\partial_t k(t,x)|\delta(v-F(x)u-k)\\ &\qquad e^{-u}\,du\,dv\,dx\,dt\\ &\le \int_0^T\int_{\mathbb{R}^D}|\partial_t k(t,x)|\,dx\,dt\end{aligned}$$

and

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^D}\int_{\mathbb{R}}|\nabla_x M_k|\,dv\,dx\,dt &\le \int_0^T\int_{\mathbb{R}^D}\int_{\mathbb{R}}\int_0^{+\infty}(|\nabla_x k(t,x)|\delta(v-F(x)u-k)\\ &\qquad +|\nabla_x F(x)|(\delta(v-F(x)u-k)+\delta(v-F(x)u)))\\ &\qquad e^{-u}du\,dv\,dx\,dt\\ &\le \int_0^T\int_{\mathbb{R}^D}|\nabla_x k(t,x)|+2|\nabla_x F(x)|\,dx\,dt. \qquad (5.3)\end{aligned}$$

Then R_2 is uniformly bounded in $\mathcal{M}_1([0,T]\times\mathbb{R}^D\times\mathbb{R})$. Then by integration with respect to t and x of (4.1) we get

$$\begin{aligned}0 &\le \int_0^T\int_{\mathbb{R}^D}\left|\int_{\mathbb{R}}|f_\varepsilon-M_k|\,dv-|\rho_\varepsilon-k|\right|\,dx\,dt\\ &\le \int_0^T\int_{\mathbb{R}^D}\Big(\int_{\mathbb{R}}|f_\varepsilon-M_k|\,dv-|\rho_\varepsilon-k|\Big)\,dx\,dt \qquad (5.4)\\ &\le \varepsilon\int_{\mathbb{R}^D}\int_{\mathbb{R}}|f_I-M_{k(0,.)}|\,dv\,dx+\varepsilon\int_0^T\int_{\mathbb{R}^D}\int_{\mathbb{R}}|R_2|\,dv\,dx\,dt.\end{aligned}$$

It leads to (5.1). (Multiply k by a cutoff function equal to 1 on K in the general case.)

We now prove (5.2) for a subsequence satisfying $\rho_\varepsilon\to\rho$ in $L^1([0,T]\times K)$. Let $(\rho^\delta)_{\delta>0}$ be a BV regularization of ρ:

$$\rho^\delta\in BV(\mathbb{R}^+\times\mathbb{R}^D),\qquad \rho^\delta\underset{\delta\to 0}{\to}\rho \text{ in } L^1(\mathbb{R}^+\times\mathbb{R}^D).$$

We have

$$\begin{aligned}\int_0^T\int_K\int_{\mathbb{R}}|f_\varepsilon-M_\rho|\,dv\,dx\,dt &\le \int_0^T\int_K\left[\int_{\mathbb{R}}|f_\varepsilon-M_{\rho^\delta}|\,dv-|\rho_\varepsilon-\rho^\delta|\right]\,dx\,dt\\ &\quad +\int_0^T\int_K 2|\rho^\delta-\rho|+|\rho-\rho_\varepsilon|\,dx\,dt.\end{aligned}$$

Using (5.1) and the convergence $\rho_\varepsilon\to\rho$, we obtain

$$\overline{\lim_{\varepsilon\to 0}}\|f_\varepsilon-M_\rho\|_{L^1([0,T]\times K\times\mathbb{R})}\le 2\|\rho^\delta-\rho\|_{L^1([0,T]\times K)}.$$

The limit $\delta\to 0$ concludes the proof of the Proposition. □

Let B and b defined by (3.6) and (3.4). A consequence of the above Proposition is the following result.

Lemma 5.2 *Let us define* $j_\varepsilon = \int_{\mathbb{R}} a(x,v)\, f_\varepsilon(t,x,v)\, dv$. *Under the same assumptions as in Proposition 5.1 we have*

$$j_\varepsilon - B(x,\rho_\varepsilon) \to 0, \qquad \textit{in the distribution sense}$$
$$\int_{\mathbb{R}} a(x,v)\, |f_\varepsilon - M_k|(t,x,v)\, dv - \operatorname{sgn}(\rho_\varepsilon - k)\Big(j_\varepsilon - B(x,k)\Big) \ \to \ 0$$

in $L^1([0,T]\times K)$ *for any* $T>0$ *and any* K *compact subset of* $\mathbb{R}^D$.

Proof. We use the computation made in Section 3.1 without neglecting the terms of order ε. We obtain

$$\begin{aligned} \int_{\mathbb{R}} a(x,v) M_k(x,v)\, dv &= B(x,k), \\ \int_{\mathbb{R}} a(x,v) f_\varepsilon(t,x,v)\, dv &= B(x,\rho_\varepsilon) - r_\varepsilon, \end{aligned} \tag{5.5}$$

$$\begin{aligned} r_\varepsilon \ = \ \varepsilon \Bigg(&\partial_t \int_{\mathbb{R}} b(x,v) f_\varepsilon(t,x,v)\, dv + \sum_i \partial_{x_i} \Big(\int_{\mathbb{R}} b(x,v) a_i(x,v)\, f_\varepsilon(t,x,v)\, dv \,\Big) \\ &- \sum_i \int_{\mathbb{R}} \partial_{x_i} b(x,v) a_i(x,v) f_\varepsilon(t,x,v)\, dv \Bigg). \end{aligned}$$

We remark that a, b and $\nabla_x b$ are bounded functions. Then thanks to the uniform estimate of Proposition 5.1, we get that r_ε tends to 0 in the distributions sense.

Now, we use (5.1). Setting $S(v) = \operatorname{sgn}(f_\varepsilon - M_k) - \operatorname{sgn}(\rho_\varepsilon - k)$, we have

$$\begin{aligned} &\left| \int_{\mathbb{R}} a(x,v) |f_\varepsilon - M_k|\, dv - \operatorname{sgn}(\rho_\varepsilon - k) \int_{\mathbb{R}} a(x,v)(f_\varepsilon - M_k)\, dv \right| \\ \leq\ & 2\|a\|_\infty \int_{\operatorname{supp} S} |f_\varepsilon - M_k|\, dv \\ \leq\ & \|a\|_\infty \left(\int_{\mathbb{R}} |f_\varepsilon - M_k|\, dv - |\rho_\varepsilon - k| \right) \to 0. \end{aligned}$$

It remains to use (5.5) to finish the proof.

5.1 The case of a constant force F (x in D dimensions)

In this section, we prove the validity of the previous formal relaxation limit in the case of a constant force. We take $F(x) = F \in \mathbb{R}$ and $a(x,v) \in C_b^1(\mathbb{R}^D \times \mathbb{R})$. Using (2.11) and (2.12), we obtain

$$\forall t \in \mathbb{R},\ \forall (x,v) \in \mathbb{R}^D \times \mathbb{R}, \quad V(t,x,v) = v + \frac{t}{\varepsilon} F. \tag{5.6}$$

Since $\operatorname{div}_x a$ is bounded, we also have that for any $T \geq 0$, there is a constant $C_T > 0$ independent on ε such that

$$\forall t \in [-T,T],\ \forall (x,v) \in \mathbb{R}^D \times \mathbb{R}, \quad \frac{1}{C_T} \leq J_\varepsilon(t,x,v) \leq C_T. \tag{5.7}$$

We use now the outline of the proof of [30] to justify the asymptotics of Section 3.1.

First, we prove the compactness of the sequence $(\rho_\varepsilon)_{\varepsilon>0}$.

Proposition 5.3 *We assume that F is constant and that $a \in C_b^{2,1}(\mathbb{R}^D \times \mathbb{R})$. For $f_I \in L^1(\mathbb{R}; BV(\mathbb{R}^D))$, the family $\{\rho_\varepsilon(t),\ \varepsilon > 0,\ 0 \le t \le T\}$ is bounded in $BV(\mathbb{R}^D)$. Moreover for all $T > 0$ and for all compact subset $K \subset \mathbb{R}^D$, the sequence $(\rho_\varepsilon)_{\varepsilon>0}$ is precompact in $C^0([0,T]; L^1(K))$. As a consequence, there is a subsequence still denoted by $(\rho_\varepsilon)_{\varepsilon>0}$ and $\rho \in C^0([0,T]; L^1(\mathbb{R}^D))$ such that*

$$\rho_\varepsilon(t,x) \underset{\varepsilon\to 0}{\to} \rho(t,x), \quad \forall t \ge 0,\ for\ a.e.\ x \in \mathbb{R}^D, \tag{5.8}$$

$$\int_K |\rho_\varepsilon - \rho|(t,x)\,dx \underset{\varepsilon\to 0}{\to} 0, \quad \text{for any compact subset } K \subset \mathbb{R}^D, \tag{5.9}$$

uniformly with respect to time on any intervals $[0,T], T > 0$.

Moreover ρ is a weak solution of the conservation law

$$\partial_t \rho(t,x) + \operatorname{div}_x B(x, \rho(t,x)) = 0, \quad \forall t > 0,\ \forall x \in \mathbb{R}^D, \tag{5.10}$$

$$\rho(0,x) = \rho_I(x) = \int_{\mathbb{R}} f_I(x,v)\,dv, \qquad \forall x \in \mathbb{R}^D. \tag{5.11}$$

Proof. In view of (5.6), we have

$$\partial_t D_x X_\varepsilon(t,x,v) = D_x a(X_\varepsilon(t,x,v), v + Ft/\varepsilon) \cdot D_x X_\varepsilon(t,x,v),$$

with $D_x X_\varepsilon(0,x,v) = I_D$. Therefore $D_x X_\varepsilon$ is uniformly bounded with respect to ε on $[-T,T] \times \mathbb{R}^D \times \mathbb{R}$. In the same way, differentiating (2.12), we get that $\nabla_x J$ is also uniformly bounded with respect to ε on $[-T,T] \times \mathbb{R}^D \times \mathbb{R}$.

Let us introduce the norm

$$\|f\|_{L^1(BV)} = \int_{\mathbb{R}^D \times \mathbb{R}} (|f| + |\nabla_x f|)(x,v)\,dx\,dv.$$

We now use (2.13) with (5.6) and (5.7) to obtain

$$\begin{aligned} |\nabla_x \mathcal{G}_t^\varepsilon(f)(x,v)| \;\le\; & C_T\, |\nabla_x f(X_\varepsilon(-t,x,v), v + Ft/\varepsilon)|\; |D_x X_\varepsilon(-t,x,v)| \\ & + |f(X_\varepsilon(-t,x,v), v + Ft/\varepsilon)|\; |\nabla_x J_\varepsilon(-t,x,v)|, \end{aligned} \tag{5.12}$$

and thus

$$\|\mathcal{G}_t^\varepsilon(f)\|_{L^1(BV)} \le C(T) \|f\|_{L^1(BV)}. \tag{5.13}$$

We also have $\nabla_x \chi_\rho = \nabla_x \rho\, \delta(v - \rho)$, therefore

$$\|\chi_\rho\|_{L^1(BV)} \le \|\rho\|_{L^1(BV)}.$$

Thanks to the above estimate and (5.13), the sequence defined in (2.10) belongs to $L^1(\mathbb{R}; BV(\mathbb{R}^D))$ and satisfies

$$\|f_{n+1}(t)\|_{L^1(BV)} \le C(T)\Big(e^{-t/\varepsilon}\|f_I\|_{L^1(BV)} + \int_0^t \|f_n(s)\|_{L^1(BV)} \frac{e^{-(t-s)/\varepsilon}}{\varepsilon}\,ds\Big).$$

It gives a bound in $L^1(\mathbb{R}; BV(\mathbb{R}^D))$ which unfortunately depends on ε. However, we obtain that $f_\varepsilon(t) \in L^1(\mathbb{R}; BV(\mathbb{R}^D))$. We have to work more to get a uniform estimate. For $h \in \mathbb{R}^D$, we use (4.1) with $f_1 = \tau_h f_\varepsilon$ and $f_2 = f_\varepsilon$ where $\tau_h f(t,x,v) = f(t, x+h, v)$. We have $S_1 = S_2 = R_2 = 0$ and $R_1 = \mathrm{div}_x\Big((a - \tau_h a)\tau_h f_\varepsilon\Big)$. Then by integrating with respect to x we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^D}\int_{\mathbb{R}} |\tau_h f_\varepsilon - f_\varepsilon|\, dv\, dx &\leq \int_{\mathbb{R}^D}\int_{\mathbb{R}} \Big|\,\mathrm{div}_x\Big((a - \tau_h a)\tau_h f_\varepsilon\Big)\Big|\, dv\, dx \\ &\leq C(a)\, \|f_\varepsilon(t)\|_{L^1(BV)}\, |h|. \end{aligned}$$

We pass to the limit $\tau \to 0$ with $h = \tau e$ for any fixed unit vector $e \in \mathbb{R}^D$ and we get

$$\int_{\mathbb{R}^D}\int_{\mathbb{R}} |\nabla_x f_\varepsilon|\, dv\, dx \leq \int_{\mathbb{R}^D}\int_{\mathbb{R}} |\nabla_x f_I|\, dv\, dx + C(a)\int_0^t \|f_\varepsilon(s)\|_{L^1(BV)}\, ds,$$

and thus

$$\|f_\varepsilon(t)\|_{L^1(BV)} \;\leq\; \|f_\varepsilon(t)\|_{L^1} + \|f_I\|_{L^1(BV)} + C(a)\int_0^t \|f_\varepsilon(s)\|_{L^1(BV)}\, ds\,.$$

Using Proposition 5.1 and a Gronwall Lemma show that for all $T > 0$ the sequences $(f_\varepsilon(t))_{\varepsilon>0},\ t \in [0,T]$ belong to a bounded set of $L^1(\mathbb{R}; BV(\mathbb{R}^D))$. In the same way the family $(\rho_\varepsilon(t))_{\varepsilon>0},\ t \in [0,T]$ belongs to a bounded set of $BV(\mathbb{R}^D)$ which is compact in $L^1(K)$ for any bounded subset $K \subset \mathbb{R}^D$.

We will see now that the family $t \in [0,T] \mapsto \rho_\varepsilon(t) \in L^1(K),\ \varepsilon > 0$ is equicontinuous. Indeed we deduce from the conservation law (3.2) that

$$|\partial_t \rho_\varepsilon(t)| \leq C\int_{\mathbb{R}} \left(|f_\varepsilon(t)| + |\nabla_x f_\varepsilon(t)|\right)\, dv. \tag{5.14}$$

It follows that

$$\|\rho_\varepsilon(t,.) - \rho_\varepsilon(s,.)\|_{L^1_x} \leq C(T)\|f_I\|_{L^1(BV)}\, |t-s|, \quad \forall t,s \in [0,T].$$

Thanks to Ascoli Theorem, it allows to conclude that $(\rho_\varepsilon)_{\varepsilon>0}$ is precompact in $C^0([0,T]; L^1(K))$. The Cantor diagonal process allows to end the proof of the compactness of $(\rho_\varepsilon)_{\varepsilon>0}$.

We now want to pass to the limit in (3.2). Moreover, since we have $|B(x,\rho_\varepsilon) - B(x,\rho)| \leq \|b\|_\infty |\rho - \rho_\varepsilon|$, (5.8) and (5.9) imply that for any $T > 0$, for any compact subset $K \subset \mathbb{R}^D$,

$$\begin{aligned} B(x, \rho_\varepsilon(t,x)) \underset{\varepsilon\to 0}{\to} B(x, \rho(t,x)), \quad \forall t \geq 0,\ \text{for a.e. } x \in \mathbb{R}^D \\ \text{and in } L^1(K) \text{ uniformly with respect to time on } [0,T]. \end{aligned} \tag{5.15}$$

Therefore using Lemma 5.2 we get

$$\int_{\mathbb{R}} a(x,v) f_\varepsilon(t,x,v)\, dv \underset{\varepsilon\to 0}{\to} B(x, \rho(t,x)),$$

in the distributions sense. Then we can pass to the limit in (3.2) and conclude the proof of the proposition. □

We are now able to prove the first main result of this paper.

Proof of Theorem 1.1 Since the BGK equation (2.1) generates a contraction group in $L^1(\mathbb{R}^D \times \mathbb{R})$ we can assume without loss of generality that the initial data is smooth, see [11]. Then the results of Proposition 5.3 hold. It remains to prove that the entropy inequalities are satisfied. In view of (4.2) we get for any $k \in \mathbb{R}$,

$$\partial_t \int_{\mathbb{R}} |f_\varepsilon - M_k|\, dv + \mathrm{div}_x \int_{\mathbb{R}} a(x,v)|f_\varepsilon - M_k|\, dv + \int_{\mathbb{R}} (\mathrm{div}_x\, a) M_k \mathrm{sgn}(f_\varepsilon - M_k)\, dv \le 0.$$

In view of the convergences stated in (5.9) and in (5.2) (cf Proposition 5.1), there is no problem to pass to the limit in the two first terms. The third one is much more delicate.

First, at the price of extracting an other subsequence, we may assume that $f_\varepsilon \to M_\rho$ almost every where on $\mathbb{R}^+ \times \mathbb{R}^D \times \mathbb{R}$. On the set $\{\rho \neq k\}$, we have $\mathrm{sgn}(M_\rho - M_k) = \mathrm{sgn}(\rho - k) \neq 0$. Thus we have $\mathrm{sgn}(f_\varepsilon - M_k) \to \mathrm{sgn}(\rho - k)$ a.e. on $\{\rho \neq k\}$. The problem is on the set $\{\rho = k\}$ where the sequence $\mathrm{sgn}(f_\varepsilon - M_k)$ may oscillate between -1, 0 and 1. We have

Lemma 5.4 *Let u be a measurable real function on Ω a measurable set of $\mathbb{R}^N$. Then there is a countable set $\mathcal{C} \subset \mathbb{R}$ such that $\forall k \in \mathbb{R} \setminus \mathcal{C}$, the set $\{u = k\}$ is negligeable.*

Proof : Let $\Omega_n = \Omega \cap B_n$ where B_n is the ball of radius n. Since the measurable sets $\{u = k\} \cap B_n$ are disjoint, we have $\sum_{i=1}^{I} \mathrm{meas}(\{u = k_i\} \cap B_n) \le \mathrm{meas}(B_n)$ for all distinct k_i, $i = 1, ..., I$. Therefore the familly $(\mathrm{meas}(\{u = k\} \cap B_n))_{k \in \mathbb{R}}$ is summable. Then there is a countable set $\mathcal{C}_n$ such that for any $k \in \mathbb{R} \setminus \mathcal{C}_n$, $\mathrm{meas}(\{u = k\} \cap B_n) = 0$. We thus get the result with $\mathcal{C} = \cup_{n \ge 1} \mathcal{C}_n$. □

We use this Lemma with $\Omega = \mathbb{R}^+ \times \mathbb{R}^D$ and $u = \rho$. We obtain by this way that, except for $k \in \mathcal{C}$, $\mathrm{sgn}(f_\varepsilon - M_k) \to \mathrm{sgn}(\rho - k)$ a.e. on $\mathbb{R}^+ \times \mathbb{R}^D$ and

$$\partial_t \int_{\mathbb{R}} |M_\rho - M_k|\, dv + \mathrm{div}_x \int_{\mathbb{R}} a(x,v)|M_\rho - M_k|\, dv + \int_{\mathbb{R}} (\mathrm{div}_x\, a) M_k \mathrm{sgn}(\rho - k)\, dv \le 0.$$

Using (3.11)- (3.13), the above inequality becomes

$$\forall k \in \mathbb{R} \setminus \mathcal{C}, \quad \partial_t |\rho - k| + \mathrm{div}_x \Big((B(x,\rho) - B(x,k))\, \mathrm{sgn}(\rho - k) \Big) + (\mathrm{div}_x\, B)(x,k)\, \mathrm{sgn}(\rho - k) \le 0. \tag{5.16}$$

Thanks to the Lemma below, we finally obtain that the entropy inequalities (5.16) hold for **all** $k \in \mathbb{R}$. This result with the continuity of $\rho(t)$ at time $t = 0$ in $L^1(\mathbb{R}^D)$ allows to use the uniqueness result of Kružkov, [22]. Therefore the whole sequences $(f_\varepsilon)_{\varepsilon > 0}$ and $(\rho_\varepsilon)_{\varepsilon > 0}$ converge. It ends the proof of the Theorem. □

Lemma 5.5 *Let $\rho \in C^0(\mathbb{R}^+; L^1(K))$ for all compact subset $K \subset \mathbb{R}^D$, a weak solution of (1.5) which satisfies the entropy inequalities (5.16) for a.e. $k \in \mathbb{R}$. Then it satisfies (5.16) for all $k \in \mathbb{R}$.*

Proof. For a.e. $k \in \mathbb{R}$, we have

$$\begin{aligned}\partial_t|\rho - k| + \operatorname{div}_x \Big((B(x,\rho) - B(x,k)) \operatorname{sgn}(\rho - k)\Big)&\\ +\Big((\operatorname{div}_x B)(x,k) - (\operatorname{div}_x B)(x,\rho)\Big) \operatorname{sgn}(\rho - k)&\\ +(\operatorname{div}_x B)(x,\rho) \operatorname{sgn}(\rho - k) \leq 0.&\end{aligned}$$

We take the convolution of these inequalities with respect to k with

$$\eta^\alpha(k) = \frac{1}{\alpha}\eta\Big(\frac{k}{\alpha}\Big),\ \eta \in C_c^\infty(\mathbb{R}; \mathbb{R}^+),\ \int_{\mathbb{R}} \eta(k)\, dk = 1,\ \eta \text{ is even.}$$

By this way we obtain for all k

$$\begin{aligned}&\partial_t|\rho - k| *_k \eta^\alpha(k) + \operatorname{div}_x \Big(((B(x,\rho) - B(x,k)) \operatorname{sgn}(\rho - k)) *_k \eta^\alpha(k)\Big)\\ &+\Big(((\operatorname{div}_x B)(x,k) - (\operatorname{div}_x B)(x,\rho)) \operatorname{sgn}(\rho - k)\Big) *_k \eta^\alpha(k)\\ &+(\operatorname{div}_x B)(x,\rho) \operatorname{sgn}^\alpha(\rho - k) \leq 0,\end{aligned}$$

where $\operatorname{sgn}^\alpha = \operatorname{sgn} *_k \eta^\alpha$. We remark that the first three terms are convolutions of η^α with functions which are continuous with respect to k. Therefore they converge as $\alpha \to 0$ for every $k \in \mathbb{R}$. For the last term we use that $\operatorname{sgn}^\alpha(0) = 0$ therefore $\operatorname{sgn}^\alpha$ converges everywhere which implies that $\operatorname{sgn}^\alpha(\rho - k) \underset{\alpha\to 0}{\to} 0$ for every k. Then the limit $\alpha \to 0$ gives the desired result. □

Admissible states
We assume here that a does not depend on x. For admissible states, we want that the solution of the relaxation satisfies $\rho \geq 0$. In this sense, the solution is said to be admissible. If the initial condition f_I satisfies

$$\int_{\mathbb{R}} f_I \left(x - \int_0^t a(v + \alpha u)\, du, v\right) dv \geq 0,\ \ \forall x \in \mathbb{R}^D, \forall t \in \mathbb{R}^+, \forall \alpha \text{ s.t. } F\alpha > 0, \tag{5.17}$$

then $f_I \in \mathcal{I}_F^\varepsilon$, $\forall \varepsilon > 0$, thus $f_\varepsilon \in \mathcal{A}$, $\forall \varepsilon > 0$, that is to say $\rho_\varepsilon = \int f_\varepsilon\, dv \geq 0$, and the limit satisfies $\rho \geq 0$. Hence we obtain :

Corollary 5.6 *If f_I is an admissible initial data in the sense of (5.17), then $\rho \geq 0$.*

Remark 5.1 There exists admissible initial data such that $f_I < 0$ is possible. The function given in Remark 2.2 is such an example.

5.2 Relaxation for $F(x)$ (x in one dimension)

In this section, we prove Theorem 1.2, that is to say we consider the rigorous relaxation proof for more general $F \in C^1(\mathbb{R})$, but only in one-dimension. Indeed, since F depends on x, we can't get BV estimates in this case and we turn to compensated compactness argument.
We restrict our proof to the following case: we will assume that (1.7) holds.

We denote by f_ε the solution of Theorem 2.1 associated to an initial data f_I and

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}} f_\varepsilon(t,x,v)\, dv.$$

Using (4.1), we have for any $k \in C^1(\mathbb{R} \times \mathbb{R}^D)$

$$\partial_t \int_{\mathbb{R}} |f_\varepsilon - M_k|\, dv + \partial_x \int_{\mathbb{R}} a(x,v)|f_\varepsilon - M_k|\, dv \le \mathcal{M}_\varepsilon^k, \tag{5.18}$$

where

$$\mathcal{M}_\varepsilon^k = -\int_{\mathbb{R}} \partial_x \left[a(x,v)\, M_k(x,v)\right] \operatorname{sgn}(f_\varepsilon - M_k)\, dv.$$

If we assume that $F' \in L^1(\mathbb{R})$, then we get by (5.3), that $\mathcal{M}_\varepsilon^k$ is a bounded sequence of measures. Furthermore the first term of (5.18) is bounded in $W^{-1,\infty}$ and the second one is bounded as a measure, thus we can apply the compensated compactness argument of [28], [36], as it was done in [30]. The inequality (5.18) combined with the conservation law (3.2) gives that, up to a subsequence,

$$\begin{aligned} &< \rho_\varepsilon \int_{\mathbb{R}} a(x,v)|f_\varepsilon - M_k|\, dv > - < \int_{\mathbb{R}} a(x,v) f_\varepsilon\, dv \int_{\mathbb{R}} |f_\varepsilon - M_k|\, dv >= \\ &\rho < \int_{\mathbb{R}} a(x,v)|f_\varepsilon - M_k|\, dv > - < \int_{\mathbb{R}} a(x,v) f_\varepsilon\, dv >< \int_{\mathbb{R}} |f_\varepsilon - M_k|\, dv > \end{aligned}$$

where $< u_\varepsilon >$ denotes the weak $*$ L^∞ limits of the sequence u_ε and where $\rho =< \rho_\varepsilon >$. Thus using (5.1) and Lemma 5.2 we get

$$< (\rho_\varepsilon - \rho) \operatorname{sgn}(\rho_\varepsilon - k)\, (j_\varepsilon - B(x,k)) > = \Big\langle |\rho_\varepsilon - k|(j_\varepsilon - < B(x,\rho_\varepsilon) >)\Big\rangle$$

that is

$$< (k - \rho) \operatorname{sgn}(\rho_\varepsilon - k)\, (j_\varepsilon - B(x,k)) > = \Big\langle |\rho_\varepsilon - k|(B(x,k) - < B(x,\rho_\varepsilon) >)\Big\rangle.$$

Then letting $k \to \rho$ we have

$$\Big\langle |\rho_\varepsilon - \rho|(B(x,\rho) - < B(x,\rho_\varepsilon) >)\Big\rangle = 0.$$

Arguing as in [36], we obtain

$$< B(x,\rho_\varepsilon) >= B(x,\rho). \tag{5.19}$$

Thus using Lemma 5.2 we pass to the limit in (3.2). By this way, we obtain that ρ is a solution of (1.3).

We turn now to the strong convergence and the entropy inequalities. We assume that $k \mapsto B(x,k)$ is nonlinear. We denote by μ the Young measure associated to the sequence (ρ_ε). We have

$$\rho_\varepsilon \rightharpoonup \rho = \int \lambda \, d\mu(\lambda) =< \mu, \lambda >, \qquad B(x,\rho_\varepsilon) \rightharpoonup \xi =< \mu, B(x,\lambda) >,$$

where $<,>$ is the distribution duality bracket. The relation (5.19) gives that $\xi = B(x,\rho)$. Now from (5.18), we have

$$\partial_t |\rho_\varepsilon - k| + \partial_x \left(\text{sgn}(\rho_\varepsilon - k)(B(x,\rho_\varepsilon) - B(x,k))\right) \leq \mathcal{M}_\varepsilon^k + \partial_t E_1^k + \partial_x E_2^k, \quad (5.20)$$

with

$$E_1^k = |\rho_\varepsilon - k| - \int_{\mathbb{R}} |f_\varepsilon - M_k| \, dv,$$

and

$$E_2^k = \text{sgn}(\rho_\varepsilon - k)(B(x,\rho_\varepsilon) - B(x,k)) - \int_{\mathbb{R}} a(x,v)|f_\varepsilon - M_k| \, dv.$$

These two last terms tend to 0 in H^{-1}_{loc} from Proposition 5.1, Lemma 5.2 and the L^∞ bounds. Therefore the div-curl lemma applied between (5.20) and (3.2) gives

$$\begin{aligned} & < \mu, \lambda \, \text{sgn}(\lambda - k)(B(x,\lambda) - B(x,k)) - |\lambda - k| \, B(x,\lambda) > \\ = \; & < \mu, \lambda >< \mu, \text{sgn}(\lambda - k)(B(x,\lambda) - B(x,k)) > - < \mu, |\lambda - k| >< \mu, B(x,\lambda) >, \end{aligned}$$

We are now able to perform a rather similar proof as in [36] to get that $\text{Conv}(\text{supp}\mu) = [\alpha, \beta]$ is reduced to a single point. By a translation, we assume that $\rho = B(x,\rho) = 0$. The difficult case is for $\alpha < 0 < \beta$. We define M and N by 0 outside $[\alpha,\beta]$ and such that $M' = \lambda\mu$, $N' = B(x,\lambda)\mu$. The relation

$$< \mu, \lambda \, \text{sgn}(\lambda - k)(B(x,\lambda) - B(x,k)) - |\lambda - k| \, B(x,\lambda) >= 0$$

can be rewritten as

$$< M', \text{sgn}(\lambda - k)(B(x,\lambda) - B(x,k)) >=< N', |\lambda - k| >,$$

or also

$$< M \partial_\lambda B(x,\lambda) - N, \phi(\lambda) >= 0,$$

with $\phi(\lambda) = \partial_\lambda |\lambda - k|$. This relation can be extended to any continuous ϕ with a compact support, and thus we get

$$M \partial_\lambda B(x,\lambda) - N = 0.$$

By the definition of M and N, we also have $N'\lambda = M'B(x,\lambda)$, and thus

$$\partial_\lambda (B(x,\lambda)M - \lambda N) = 0.$$

With the compact support property, it gives

$$B(x,\lambda)M - \lambda N = 0,$$

and thus

$$B(x,\lambda) - \lambda\partial_\lambda B(x,\lambda) = 0,$$

which leads to

$$B(x,\lambda) = c(x)\lambda, \text{ on } [\alpha,\beta],$$

and finally the Young measure μ is a Dirac measure since there is no interval on which $\lambda \mapsto B(x,\lambda)$ is linear. We obtain that ρ_ε converges strongly in L^1_{loc}. Then we proceed as in the case of F constant and we get (1.6).

About uniqueness of the solution
The obtained solution is in $L^1([0,+\infty[\times\mathbb{R})$ (not necessary continuous in time at 0) and satisfies the entropy inequalities, thus the question to know if it leads to the unique solution of Kružkov is important. This problem has been solved for one dimension in [12], then for the multi-dimensional case in [39] with a non-degeneracy assumption on $A(\rho)$.

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Final remark. This work was essentially finished when the third author passed away, so his inclusion as co-author "post mortem" is appropriate.

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