Weak entropy boundary conditions for isentropic gas dynamics via kinetic relaxation

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Abstract

We consider a kinetic BGK model relaxing to isentropic gas dynamics previously introduced by the authors, but with Dirichlet boundary condition on the incoming velocities. We pass to the limit as the relaxation parameter tends to zero by compensated compactness inside the domain, and obtain that the limit satisfies entropy inequalities on the boundary involving weak traces of entropy fluxes. Our method is very general and could be applied to any entropy satisfying BGK model as soon as we have strong compactness of the macroscopic variables inside the domain.

Key-words: entropy boundary conditions – isentropic gas dynamics – kinetic BGK model – kinetic entropy – kinetic invariant domain – relaxation limit – weak trace

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Contents

1 Introduction and main results

We consider the one-dimensional system of isentropic gas dynamics

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \kappa \rho^\gamma) = 0, \end{cases} \qquad t > 0, x > 0, \tag{1.1}$$

with $\rho(t,x) \geq 0,\ u(t,x) \in \mathbb{R}$ and $\kappa > 0,\ 1 < \gamma < 3$, with initial data

$$\rho(0,x) = \rho^0(x), \quad \rho(0,x)u(0,x) = \rho^0(x)u^0(x), \qquad x > 0, \tag{1.2}$$

and Dirichlet boundary conditions

$$\rho(t,0) = \rho^b(t), \quad \rho(t,0)u(t,0) = \rho^b(t)u^b(t), \qquad t > 0. \tag{1.3}$$

Entropy solutions are defined as solutions to (1.1) with finite energy, which satisfy the entropy inequalities

$$\partial_t(\eta_S) + \partial_x(G_S) \leq 0, \qquad \text{in }]0,\infty[_t \times]0,\infty[_x, \tag{1.4}$$

for any η_S and G_S suitable entropy and entropy flux for (1.1) parametrized by a convex function S with subquadratic growth that will be defined precisely in the next section.

It is well-known that (1.3) cannot be satisfied everywhere, therefore one has to introduce a weaker formulation. In [12], Dubois and Le Floch proposed two formulations for the boundary condition; one based on a Riemann problem in a half-space, and another based on entropy boundary inequalities derived from a viscosity approximation. Here we consider boundary conditions of the second kind, which are defined as

$$\overline{G_S(\rho,u)} - G_S(\rho^b,u^b) - \eta_S'(\rho^b,u^b)\cdot(\overline{F(\rho,u)} - F(\rho^b,u^b)) \leq 0, \quad \text{in }]0,\infty[_t \times \{0\}_x, \tag{1.5}$$

where $F(\rho,u) = (\rho u, \rho u^2 + \kappa\rho^\gamma)$ is the flux of the system, overbars denote weak traces (and will be defined later on), and prime denotes differentiation with respect to conservative variables $(\rho, q \equiv \rho u)$. The weak formulation of (1.4)-(1.5) is

$$
\begin{aligned}
&-\iint_{]0,\infty[\times]0,\infty[} \eta_S(\rho,u)\partial_t\varphi\,dtdx - \iint_{]0,\infty[\times]0,\infty[} G_S(\rho,u)\partial_x\varphi\,dtdx\\
&-\int_{]0,\infty[}[G_S(\rho^b,u^b) + \eta_S'(\rho^b,u^b)\cdot(\overline{F(\rho,u)} - F(\rho^b,u^b))]\varphi(t,0)\,dt \le 0
\end{aligned}
\tag{1.6}
$$

for any $\varphi \in \mathcal{D}(]0,\infty[_t\times[0,\infty[_x)$, $\varphi \ge 0$. We notice that according to [1], (1.1) allows to define the weak trace of $\overline{F}(\rho,u)$, and the entropy inequalities (1.4) allow to define the weak traces of $G_S(\rho,u)$ for any S (see Section 6). Since we consider data that can contain vacuum $\rho^b = 0$, and since η_S is not differentiable at 0, this formula has to be generalized, see (1.22)-(1.23). We recall that in the scalar case, the weak entropy boundary conditions (1.5) give a unique solution, as proved by Otto [26].

The aim of this paper is to obtain the existence of a solution to the initial-boundary value problem, for arbitrary L^∞ initial-boundary value data. Indeed we are able to obtain it from the relaxation of an entropy compatible BGK model with Dirichlet condition on the incoming velocities, by a very general method that uses mainly the subdifferential inequality that characterizes the entropy compatibility of the kinetic model, as proved in [7]. We need only to know some a priori bound on the macroscopic variables, and their strong compactness. Here it is obtained by the existence of kinetic invariant domains and by compensated compactness. Our method could be used, for example, in the context of [25] to prove that the relaxation limit gives the right boundary conditions.

In the BV context, the gas dynamics in lagrangian coordinates was considered in [20]. The existence of a strong trace for BV functions is used for boundary conditions since [2], in which they got the first existence and uniqueness result in the scalar case. Other hyperbolic systems with boundary conditions in the BV context can be found in [14] for small data and together with a relaxation approximation in [32] for a uniqueness result, and in [23] for regular solutions. BGK approaches for scalar conservation laws with boundary conditions can be found in [21] and [25]. In the less restrictive context of L^1 or L^∞ functions, the existence of a strong trace is a difficult problem that has only be solved in the scalar case in [31], and in general weak trace formulations have to be taken into account, as introduced in [26].

Relations between the two sets of expected boundary values of [12] is an interesting problem. The fact that these sets are not equal was established in [4], in which they give a condition for the coincidence for 2×2 systems. General systems of conservations laws are considered in [16], where they give existence from Godunov schemes and viscous approximation and also prove in this context that the boundary values derived from the viscous approximation contains the one derived in terms of the boundary Riemann problems, and the falsity of the converse in general. The p-system in the interval $[0,1]$ is studied in [24] via a solution in $\mathbb{R}^+\times[0,1]$ and in [3] as the limit of an associated parabolic equation. A problem arising in chemical engineering using compensated compactness methods is studied in [15]. These problems use the entropy formulation. In [13], the formulation with Godunov schemes is used for hyperbolic systems in one space variable.

The kinetic model we use here is the one of [5], [6] with Dirichlet conditions on incoming velocities and can be written as

$$\partial_t f + \xi \partial_x f = \frac{M[f] - f}{\varepsilon} \quad \text{in }]0,\infty[_t \times]0,\infty[_x \times \mathbb{R}_\xi, \tag{1.7}$$

where $f = f(t,x,\xi) \in \mathbb{R}^2$, $t > 0$, $x > 0$, $\xi \in \mathbb{R}$, $M[f]$ is defined by (2.1)-(2.6),

$$f(t,x,\xi) \in D = \{(f_0, f_1) \in \mathbb{R}^2,\ f_0 > 0 \text{ or } f_1 = f_0 = 0\}, \tag{1.8}$$

with the initial data

$$f(0,x,\xi) = f^0(x,\xi), \qquad x > 0,\ \xi \in \mathbb{R}, \tag{1.9}$$

and the boundary condition on incoming velocities

$$f(t,0,\xi) = f^b(t,\xi), \qquad t > 0,\ \xi > 0. \tag{1.10}$$

It is well-known that the natural space associated to the traces of the kinetic equation (1.7) is the space L^1_μ with

$$d\mu = |\xi| d\xi dt. \tag{1.11}$$

We have the following existence theorem for the BGK model.

Theorem 1.1 *Assume that* $f^0 \in L^1(]0,\infty[_x \times \mathbb{R}_\xi)$, $f^b \in L^1_\mu(]0,T[_t \times]0,\infty[_\xi)$ *satisfy*

$$f^0(x,\xi) \in D,\ f^b(t,\xi) \in D \qquad a.e., \tag{1.12}$$

and with H *the kinetic entropy (2.14) associated with the physical energy,*

$$\iint_{]0,\infty[\times\mathbb{R}} H(f^0(x,\xi),\xi)\, dx d\xi < \infty, \tag{1.13}$$

$$\iint_{]0,T[\times]0,\infty[} H(f^b(t,\xi),\xi)\, d\mu(t,\xi) < \infty. \tag{1.14}$$

Then there exists a solution f *to (1.7)-(1.10) satisfying*

$$f \in C_t([0,T], L^1(]0,\infty[_x \times \mathbb{R}_\xi)) \cap C_x([0,\infty[, L^1_\mu(]0,T[_t \times \mathbb{R}_\xi)), \tag{1.15}$$

$$\forall t \geq 0,\ f(t,x,\xi) \in D \ a.e.\ in\ ]0,\infty[_x \times \mathbb{R}_\xi, \tag{1.16}$$

$$H(f(t,x,\xi),\xi) \in L^\infty_t(]0,T[, L^1(]0,\infty[_x \times \mathbb{R}_\xi)). \tag{1.17}$$

Furthermore, f *satisfies for any* $t \geq 0$ *the estimates*

$$\iint_{]0,\infty[\times\mathbb{R}} f_0(t,x,\xi)\, dx d\xi \leq \iint_{]0,\infty[\times\mathbb{R}} f_0^0(x,\xi)\, dx d\xi + \iint_{]0,t[\times]0,\infty[} f_0^b(s,\xi)\, d\mu(s,\xi), \tag{1.18}$$

$$\begin{aligned} \iint_{]0,\infty[\times\mathbb{R}} H(f(t,x,\xi),\xi)\, dx d\xi \leq &\iint_{]0,\infty[\times\mathbb{R}} H(f^0(x,\xi),\xi)\, dx d\xi \\ &+ \iint_{]0,t[\times]0,\infty[} H(f^b(s,\xi),\xi)\, d\mu(s,\xi). \end{aligned} \tag{1.19}$$

We obtain for these solutions a maximum principle and a compact support property. In order to pass to the limit, we impose the boundary condition to be maxwellian. We also assume that initial data and boundary conditions lie in an invariant domain $\tilde{D}$ of the system,

$$\tilde{D} = \{(\rho, u) \in \mathbb{R}^+ \times \mathbb{R};\ \rho = 0 \ or\ \omega_{min} \leq \omega_1 \leq \omega_2 \leq \omega_{max}\}, \tag{1.20}$$

where ω_1 and ω_2 are the Riemann invariants of the system, defined by

$$\omega_1 = u - \frac{2\sqrt{\gamma\kappa}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}, \qquad \omega_2 = u + \frac{2\sqrt{\gamma\kappa}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}. \tag{1.21}$$

We refer to the next section for more details and for the expression of $\tilde{D}_\xi$ the kinetic invariant domains associated to $\tilde{D}$. We have the following convergence result.

Theorem 1.2 *Let ρ^b, $u^b \in L^\infty(]0,\infty[_t)$, $\rho^b \geq 0$, and let us denote by f_ε the solution of Theorem 1.1 with the same initial data $f^0(x,\xi) \in L^1(]0,\infty[\times\mathbb{R})$ and the same boundary condition $f^b(t,\xi) = M(\rho^b(t), u^b(t), \xi)$ that satisfy $f^0(x,\xi) \in \tilde{D}_\xi$ a.e. x, ξ, $\rho^b, u^b \in \tilde{D}$ a.e. t for some $\omega_{min} < \omega_{max}$, and the energy bound (1.13). Then $(\rho_\varepsilon, u_\varepsilon)$ defined by (2.2) lie in $\tilde{D}$ and thus are uniformly bounded in L^∞, and passing if necessary to subsequences, $(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)$ converge a.e. in $]0,\infty[\times]0,\infty[$ when $\varepsilon \to 0$ to an entropy solution $(\rho, \rho u)$ to (1.1), (1.4) remaining in $\tilde{D}$ with initial data $(\rho^0, \rho^0 u^0) = \int f^0 d\xi$ and the boundary condition*

$$\overline{G_S(\rho,u)} - G_S(\rho^b,u^b) - T_S(\rho^b,u^b)\cdot(\overline{F(\rho,u)} - F(\rho^b,u^b)) \leq 0, \quad in\ ]0,\infty[_t\times\{0\}_x, \tag{1.22}$$

where

$$T_S(\rho^b,u^b) \text{ is defined by (2.17) and coincides with } \eta'_S(\rho^b,u^b) \text{ when } \rho^b > 0. \tag{1.23}$$

In particular, we can take for initial data in Theorem 1.2 any Maxwellian $f^0(x,\xi) = M(\rho^0(x), u^0(x), \xi)$ with $(\rho^0, u^0) \in \tilde{D}$ a.e. such that $\int_0^\infty \rho^0 dx < \infty$.

In Section 2, we define precisely the kinetic model. We prove in Section 3 the existence of a BGK solution with boundary condition. In Section 4, we give uniform bounds for the obtained sequence of solutions in order to pass to the limit in Section 5 and to get the boundary condition (1.22). Finally, in Section 6, we give basic features on weak traces that are used in this paper.

2 Kinetic model

In this section, we present in more detail the kinetic model we consider. This BGK model has been introduced in [7] and studied in [5], [6]. The Maxwellian is defined by

$$M[f](t,x,\xi) = M\left(\rho(t,x), u(t,x), \xi\right), \tag{2.1}$$

with

$$\rho(t,x) = \int_{\mathbb{R}} f_0(t,x,\xi)\, d\xi, \qquad \rho(t,x)u(t,x) = \int_{\mathbb{R}} f_1(t,x,\xi)\, d\xi, \tag{2.2}$$

and

$$M(\rho,u,\xi)=\Big(\chi(\rho,\xi-u),((1-\theta)u+\theta\xi)\chi(\rho,\xi-u)\Big), \tag{2.3}$$

$$\chi(\rho,\xi)=c_{\gamma,\kappa}\left(a_\gamma^2\rho^{\gamma-1}-\xi^2\right)_+^\lambda, \tag{2.4}$$

$$\theta=\frac{\gamma-1}{2},\quad \lambda=\frac{1}{\gamma-1}-\frac{1}{2},\quad c_{\gamma,\kappa}=a_\gamma^{-2/(\gamma-1)}/J_\lambda, \tag{2.5}$$

$$J_\lambda=\int_{-1}^1(1-z^2)^\lambda\,dz=\sqrt{\pi}\Gamma(\lambda+1)/\Gamma(\lambda+3/2),\quad a_\gamma=\frac{2\sqrt{\gamma\kappa}}{\gamma-1}. \tag{2.6}$$

The kinetic equilibrium χ has been introduced in [11] as a generating function for entropies and has been used in the stability analysis of [17]. This function is also involved in [19] for the so-called kinetic formulation. The Maxwellian has the moment properties

$$\int_{\mathbb{R}} M(\rho,u,\xi)\,d\xi=(\rho,\rho u),\quad \int_{\mathbb{R}}\xi M(\rho,u,\xi)\,d\xi=(\rho u,\rho u^2+\kappa\rho^\gamma)=F(\rho,u), \tag{2.7}$$

for every $\rho\geq 0$ and $u\in\mathbb{R}$. Kinetic entropies are parametrized by convex functions S and are defined from a kernel by

$$H_S(f,\xi)=\int_{\mathbb{R}}\Phi(\rho(f,\xi),u(f,\xi),\xi,v)S(v)\,dv,\ \text{for } f\neq 0,\quad H_S(0,\xi)=0, \tag{2.8}$$

where

$$u(f,\xi)=\frac{f_1/f_0-\theta\xi}{1-\theta},\quad \rho(f,\xi)=a_\gamma^{-\frac{2}{\gamma-1}}\left(\left(\frac{f_1/f_0-\xi}{1-\theta}\right)^2+(f_0/c_{\gamma,\kappa})^{1/\lambda}\right)^{\frac{1}{\gamma-1}} \tag{2.9}$$

is the inverse relation for $f=M(\rho,u,\xi)$. The kernel Φ is symmetric in ξ,v and satisfies in particular $\Phi\geq 0$ and $\int_{\mathbb{R}}(1,v)\cdot\Phi(\rho,u,\xi,v)\,dv=M(\rho,u,\xi)$. The entropy and entropy flux for (1.1) are defined by

$$\eta_S(\rho,u)=\int_{\mathbb{R}}\chi(\rho,v-u)S(v)\,dv=\int_{\mathbb{R}}H_S(M(\rho,u,\xi),\xi)\,d\xi, \tag{2.10}$$

$$G_S(\rho,u)=\int_{\mathbb{R}}[(1-\theta)u+\theta v)]\chi(\rho,v-u)S(v)\,dv=\int_{\mathbb{R}}\xi\,H_S(M(\rho,u,\xi),\xi)\,d\xi. \tag{2.11}$$

The kinetic Riemann invariants are defined for $f\neq 0$ by

$$\omega_1(f,\xi)=u(f,\xi)-a_\gamma\rho(f,\xi)^{\frac{\gamma-1}{2}},\qquad \omega_2(f,\xi)=u(f,\xi)+a_\gamma\rho(f,\xi)^{\frac{\gamma-1}{2}}, \tag{2.12}$$

and the kinetic invariant domains by

$$\tilde{D}_\xi=\{f\in D;\ f=0\ \mathit{or}\ \omega_{min}\leq\omega_1(f,\xi)\leq\omega_2(f,\xi)\leq\omega_{max}\}. \tag{2.13}$$

We notice that for $S(v)=v^2/2$, we get $H_S=H$ and $\eta_S=\eta$ with

$$H(f,\xi)=\frac{\theta}{1-\theta}\frac{\xi^2}{2}f_0+\frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda}+\frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0}-\frac{\theta}{1-\theta}\xi f_1, \tag{2.14}$$

and

$$\eta(\rho, u) = \rho u^2/2 + \frac{\kappa}{\gamma - 1}\rho^\gamma, \tag{2.15}$$

the physical energy for (1.1). Let us recall the following properties for this kinetic model, that are proved in [6].

Proposition 2.1 *If $S : \mathbb{R} \to \mathbb{R}$ is of class C^1 then we have $H'_S(M(\rho, u, \xi), \xi) = \eta'_S(\rho, u)$ whenever $(M(\rho, u, \xi))_0 > 0$.*

Proposition 2.2 *i) If $S : \mathbb{R} \to \mathbb{R}$ is convex and continuous, then $H_S(., \xi)$ is convex in D.*
ii) If $S : \mathbb{R} \to \mathbb{R}$ is bounded on compact sets, then $H_S(., \xi)$ is continuous at 0 in $\{f \in D\ ;\ |f_1| \leq A f_0\}$, for any $A > 0$.

Proposition 2.3 (Subdifferential inequality) *If $S : \mathbb{R} \to \mathbb{R}$ is convex, of class C^1, then for every $f \in D$, $\rho \geq 0$ and $u, \xi \in \mathbb{R}$, we have*

$$H_S(f, \xi) \geq H_S(M(\rho, u, \xi), \xi) + T_S(\rho, u) \cdot (f - M(\rho, u, \xi)), \tag{2.16}$$

with

$$T_S(\rho, u) = \frac{1}{J_\lambda}\int_{-1}^{1}(1-z^2)^\lambda \begin{pmatrix} S(u + a_\gamma\rho^\theta z) + (\theta a_\gamma\rho^\theta z - u)S'(u + a_\gamma\rho^\theta z)) \\ S'(u + a_\gamma\rho^\theta z)) \end{pmatrix} dz, \tag{2.17}$$

which coincides with $\eta'_S(\rho, u)$ when $\rho > 0$. We also have if $f \neq 0$,

$$(H'_S(f, \xi) - T_S(\rho, u)) \cdot (M(\rho, u, \xi) - f) \leq 0. \tag{2.18}$$

Corollary 2.4 (Entropy minimization principle) *Assume that $S : \mathbb{R} \to \mathbb{R}$ is convex, of class C^1 and such that $|S(v)| \leq B(1 + v^2)$ for some $B \geq 0$. Consider $f \in L^1(\mathbb{R}_\xi)$ such that $f \in D$ a.e. and $\int_\mathbb{R} H(f(\xi), \xi)\, d\xi < \infty$. Then $H_S(f(\xi), \xi)$ and $H_S(M[f](\xi), \xi)$ lie in $L^1(\mathbb{R}_\xi)$ with*

$$\int_\mathbb{R} H_S(M[f](\xi), \xi)\, d\xi \leq \int_\mathbb{R} H_S(f(\xi), \xi)\, d\xi. \tag{2.19}$$

Following [29] and [6], we consider the notion of kinetic invariant domain.

Proposition 2.5 1) *The convex sets $\tilde{D}_\xi$ are associated to the domains $\tilde{D}$ in the sense that*
i) For any $(\rho, u) \in \tilde{D}$, $M(\rho, u, \xi) \in \tilde{D}_\xi$ a.e. ξ,
ii) For any $f \in L^1(\mathbb{R}_\xi)$ such that $f(\xi) \in \tilde{D}_\xi$ a.e. ξ, the averages $(\rho, \rho u) = \int_\mathbb{R} f(\xi)\, d\xi$ verify $(\rho, u) \in \tilde{D}$.
2) *If $\xi \notin [\omega_{min}, \omega_{max}]$ then $\tilde{D}_\xi = \{0\}$.*

For more details about this model, we refer to [6]. For kinetic relaxation models and kinetic formulations, we refer in particular to [7], [29], [22], [27], [30], [18], [9], [19] and [28].

3 Solution to the BGK model

This section is devoted to the existence result for the BGK model with initial-boundary condition. We notice first the following characteristics formula for (1.7).

Lemma 3.1 *Let $h \in L^1(]0,T[, L^1(]0,\infty[\times\mathbb{R}))$, $f^0 \in L^1(]0,\infty[\times\mathbb{R})$ and $f^b \in L^1_\mu(]0,T[\times]0,\infty[)$. Then there exists a unique solution*

$$f \in C_t([0,T], L^1(]0,\infty[_x\times\mathbb{R}_\xi)) \cap C_x([0,\infty[, L^1_\mu(]0,T[_t\times\mathbb{R}_\xi)) \tag{3.1}$$

to the problem

$$\begin{cases} \partial_t f + \xi\partial_x f = \dfrac{h-f}{\varepsilon}, & x>0,\ t>0,\ \xi\in\mathbb{R}, \\ f(0,x,\xi) = f^0(x,\xi), & x>0,\ \xi\in\mathbb{R}, \\ f(t,0,\xi) = f^b(t,\xi), & t>0,\ \xi>0. \end{cases} \tag{3.2}$$

Furthermore, for any $t\geq 0$, a.e. $x>0$, a.e. $\xi\in\mathbb{R}$,

$$\begin{aligned} f(t,x,\xi) = &\left[f^0(x-t\xi,\xi)e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon}h(t-s,x-s\xi,\xi)\,ds\right]\mathbb{1}_{x>t\xi} \\ &+ \left[f^b(t-x/\xi,\xi)e^{-x/(\varepsilon\xi)} + \frac{1}{\varepsilon}\int_0^{x/\xi} e^{-s/\varepsilon}h(t-s,x-s\xi,\xi)\,ds\right]\mathbb{1}_{\xi>0,x<t\xi}, \end{aligned} \tag{3.3}$$

and

$$\|f\|_{C_x([0,\infty[,L^1_\mu(]0,T[_t\times\mathbb{R}_\xi))} \leq \|f^0\|_{L^1} + \|f^b\|_{L^1_\mu} + \frac{1}{\varepsilon}\|h\|_{L^1}. \tag{3.4}$$

Now, we explain briefly how we obtain Theorem 1.1. Fixing $T>0$, $f^0 \in L^1(]0,\infty[_x\times\mathbb{R}_\xi)$, $f^b \in L^1_\mu(]0,T[_t\times]0,\infty[_\xi)$ such that (1.12) and (1.13), (1.14) are satisfied, we set

$$C_H(t) = \iint\limits_{]0,\infty[\times\mathbb{R}} H(f^0(x,\xi),\xi)\,dxd\xi + \iint\limits_{]0,t[\times]0,\infty[} H(f^b(s,\xi),\xi)\,d\mu(s,\xi), \tag{3.5}$$

$$C_0(t) = \iint\limits_{]0,\infty[\times\mathbb{R}} f^0_0(x,\xi)\,dxd\xi + \iint\limits_{]0,t[\times]0,\infty[} f^b_0(s,\xi)\,d\mu(s,\xi). \tag{3.6}$$

Proceeding as in [5], we use the Tychonoff-Schauder fixed point theorem in the locally convex topological vector space $C([0,T], L^1(]0,\infty[_{loc,x}\times\mathbb{R}_\xi))$ and with the following convex $\tilde{C}$ and the following function F. Define C to be the set of all $g\in L^\infty(]0,T[,L^1(]0,\infty[\times\mathbb{R}))$ satisfying $(C1)-(C2)$ for a.e. $t\in[0,T]$, where

$(C1)$ $\quad g(t,x,\xi)\in D \quad$ *a.e.* in $]0,\infty[\times\mathbb{R}$,

$(C2)$ $\quad \displaystyle\iint\limits_{]0,\infty[\times\mathbb{R}} H(g(t,x,\xi),\xi)\,dxd\xi \leq C_H(t), \quad \iint\limits_{]0,\infty[\times\mathbb{R}} g_0(t,x,\xi)\,dxd\xi \leq C_0(t).$

Let us also introduce

$$\tilde{C} = \Big\{ g \in C([0,T], L^1(]0,\infty[\times\mathbb{R})) \text{ satisfying } \partial_t g + \xi \partial_x g + \frac{g}{\varepsilon} \in \frac{C}{\varepsilon}, \quad \text{and } (C1) - (C2) \text{ for all } t \in [0,T] \Big\}. \tag{3.7}$$

For any $g \in C$, we denote by $F(g)$ the solution to (3.1)-(3.2) with $h = M[g]$. Using Jensen's inequality with the convex function H, and with changes of variables, we get

$$\begin{aligned}
&\iint_{]0,\infty[\times\mathbb{R}} H(F(g)(t,x,\xi),\xi)\, dx d\xi \\
\leq\ & \iint_{\substack{x>0\\ x>t\xi}} H(f^0(x-t\xi,\xi),\xi) e^{-t/\varepsilon}\, dx d\xi + \iint_{\substack{\xi>0\\ 0<x<t\xi}} H(f^b(t-\frac{x}{\xi},\xi),\xi) e^{-\frac{x}{\varepsilon\xi}}\, dx d\xi \\
& + \frac{1}{\varepsilon} \iint_{]0,\infty[\times\mathbb{R}} \int_0^{\min(t,x/\xi_+)} e^{-s/\varepsilon} H(M[g](t-s,x-s\xi,\xi),\xi)\, ds dx d\xi \\
\leq\ & \left(\iint_{]0,\infty[\times\mathbb{R}} H(f^0(x,\xi),\xi)\, dx d\xi + \iint_{]0,t[\times]0,\infty[} H(f^b(s,\xi),\xi) e^{s/\varepsilon}\, d\mu(s,\xi) \right) e^{-t/\varepsilon} \\
& + \frac{1}{\varepsilon} \int_0^t \iint_{]0,\infty[\times\mathbb{R}} e^{-s/\varepsilon} H(M[g](t-s,x,\xi),\xi)\, dx d\xi ds.
\end{aligned}$$

With the entropy minimization principle, in the last integral we can replace $M[g]$ by g, and using $(C2)$ we finally get that $F(g)$ also satisfies $(C2)$. The estimation of the mass by $C_0(t)$ works the same. It gives the stability of the set $\tilde{C}$ by F. We have a similar compactness result than in [5], furthermore the definition of $\tilde{C}$ allows to apply compactness averaging lemma and to finally get the existence of a solution to (1.7)-(1.9) in $C([0,T], L^1(]0,\infty[_{loc}\times\mathbb{R}))$. Since $F(f) = f$, the function $F(f)$ is a solution with the desired regularity and we get Theorem 1.1.

4 Maximum principle

In this section, we obtain the uniform bounds for the solution of the BGK model. Let us first introduce the notion of kinetic invariant domains for an initial-boundary problem.

Definition 4.1 *A family $\tilde{D}_\xi$ of subsets of $\mathbb{R}^2$ is a family of kinetic invariant domains for an initial-boundary problem if, denoting by f the solution obtained in Theorem 1.1,*

$$f^0(x,\xi) \in \tilde{D}_\xi \ a.e.\ x,\xi,\ f^b(t,\xi) \in \tilde{D}_\xi \ a.e.\ t,\xi, \Rightarrow \forall t\ f(t,x,\xi) \in \tilde{D}_\xi \ a.e.\ x,\xi.$$

Proposition 4.2 *For any $\omega_{min} < \omega_{max}$, define $\tilde{D}_\xi$ by (2.13). Assume that the initial-boundary condition of Theorem 1.1 satisfies $f^0(x,\xi) \in \tilde{D}_\xi$, $f^b(t,\xi) \in \tilde{D}_\xi$ a.e. Then the system (1.7) has the property that $\tilde{D}_\xi$ is a family of convex kinetic invariant domains for the initial-boundary problem. Furthermore (ρ, u) defined by (2.2) verify $(\rho(t,x), u(t,x)) \in \tilde{D}$ $\forall t \geq 0$ and $M[f](t,x,\xi) \in \tilde{D}_\xi$ $\forall t \geq 0$. Consequently f has compact support with respect to ξ, $supp_\xi f \subset [\omega_{min}, \omega_{max}]$.*

Proof. We consider functions $S_M(v) = (v - \omega_{max})_+^2$ and $S_m(v) = (\omega_{min} - v)_+^2$ which are convex and C^1. These functions allow to give a correspondence between kinetic entropies and kinetic invariant domains, namely, for a fixed ξ,

$$f \in \tilde{D}_\xi \Leftrightarrow (H_{S_M}(f,\xi) \leq 0 \text{ and } H_{S_m}(f,\xi) \leq 0), \tag{4.1}$$

see [6] for this result. Then similarly as we did for H, we can obtain the inequality (1.19) with H_S instead of H, with $S = S_M$ or $S = S_m$. Now, supposing that f^0 and f^b are in $\tilde{D}_\xi$ a.e., it gives $H_S(f^0(x,\xi),\xi) = H_S(f^b(t,\xi),\xi) = 0$ a.e., thus $H_S(f(t,x,\xi),\xi) = 0$ a.e., therefore $f \in \tilde{D}_\xi$ $\forall t \geq 0$ and $\tilde{D}_\xi$ is kinetic invariant. Finally, Proposition 2.5 allows to conclude. □

It gives the following bounds with the formula (2.9).

Proposition 4.3 *If we suppose that $f^0(x,\xi) \in \tilde{D}_\xi$ a.e. x, ξ and $f^b(t,\xi) \in \tilde{D}_\xi$ a.e. t, ξ, then ρ_ε, u_ε, f_ε, $M[f_\varepsilon]$ are uniformly bounded in L^∞. Furthermore we have $supp_\xi f_\varepsilon \subset [\omega_{min}, \omega_{max}]$, $supp_\xi M[f_\varepsilon] \subset [\omega_{min}, \omega_{max}]$ and $|(f_\varepsilon)_1| \leq A(f_\varepsilon)_0$ for some A.*

Corollary 4.4 *i) The sequence of functions $(t,x,\xi) \mapsto H(f_\varepsilon(t,x,\xi),\xi)$ are bounded in $L_t^\infty(]0,T[, L^1(]0,\infty[_x \times \mathbb{R}_\xi))$.*
ii) Functions $H_S(f_\varepsilon(t,x,\xi),\xi)$ are bounded in $C_t([0,T], L^1(]0,\infty[_x \times \mathbb{R}_\xi))$ for any $S : \mathbb{R} \to \mathbb{R}$ convex, of class C^1 and such that $|S(v)| \leq B(1+v^2)$ for some $B \geq 0$.

Proof. The $i)$ comes from the definition of H and the previous proposition. Now, the estimate $|H_S(f,\xi)| \leq B(f_0 + 2H(f,\xi))$ and the bound on $(f_\varepsilon)_0$ give that $H_S(f_\varepsilon(t,x,\xi),\xi)$ is bounded in $L_t^\infty(]0,T[, L^1(]0,\infty[_x \times \mathbb{R}_\xi))$. Using Lebesgue's theorem and the continuity of H_S in D_A of Proposition 2.2, we conclude the $ii)$. □

5 Relaxation to the boundary condition

Let us now consider the limit $\varepsilon \to 0$. At first, we have that as in [6], $H'_S(f_\varepsilon,\xi) \in L^\infty(]0,T[\times]0,\infty[\times\mathbb{R})$, thus according to [8] and Corollary 4.4, $M[f_\varepsilon] - f_\varepsilon = 0$ a.e. where $f_\varepsilon = 0$, and

$$\partial_t(H_S(f_\varepsilon,\xi)) + \xi\partial_x(H_S(f_\varepsilon,\xi)) = H'_S(f_\varepsilon,\xi) \cdot \frac{M[f_\varepsilon] - f_\varepsilon}{\varepsilon}. \tag{5.1}$$

Let $\varphi \in \mathcal{D}([0,\infty[_t\times[0,\infty[_x)$. Using the continuity of $t \mapsto H_S(f_\varepsilon(t,x,\xi),\xi)$ stated in Corollary 4.4 and the similar fact that $t \mapsto H_S(f_\varepsilon(t,x,\xi),\xi) \in C_x([0,\infty[,L^1_\mu(]0,T[_t\times\mathbb{R}_\xi))$, we get

$$\begin{aligned}
&-\iiint\limits_{]0,\infty[^2\times\mathbb{R}} H_S(f_\varepsilon,\xi)\partial_t\varphi\,dtdxd\xi - \iint\limits_{]0,\infty[\times\mathbb{R}} H_S(f_\varepsilon,\xi)(t=0)\varphi(0,x)\,dxd\xi\\
&-\iiint\limits_{]0,\infty[^2\times\mathbb{R}} \xi H_S(f_\varepsilon,\xi)\partial_x\varphi\,dtdxd\xi - \iint\limits_{]0,\infty[\times\mathbb{R}} \xi H_S(f_\varepsilon,\xi)(x=0)\varphi(t,0)\,dtd\xi\\
=&\iiint\limits_{]0,\infty[^2\times\mathbb{R}} H'_S(f_\varepsilon,\xi)\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\varphi\,dtdxd\xi\\
=&\iiint\limits_{]0,\infty[^2\times\mathbb{R}} \left(H'_S(f_\varepsilon,\xi)-T_S(\rho_\varepsilon,u_\varepsilon)\right)\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\varphi\,dtdxd\xi.
\end{aligned}\tag{5.2}$$

Using the sign of the entropy dissipation for $S(v)=v^2/2$ (Proposition 2.3), we obtain

$$\begin{aligned}
-\iiint\limits_{]0,\infty[^2\times\mathbb{R}} H_S(f_\varepsilon,\xi)\partial_t\varphi\,dtdxd\xi - \iiint\limits_{]0,\infty[^2\times\mathbb{R}} \xi H_S(f_\varepsilon,\xi)\partial_x\varphi\,dtdxd\xi&\\
-\iint\limits_{]0,\infty[\times\mathbb{R}} \xi H_S(f_\varepsilon,\xi)(x=0)\varphi(t,0)\,dtd\xi\le 0&
\end{aligned}\tag{5.3}$$

for $\varphi \in \mathcal{D}(]0,\infty[_t\times[0,\infty[_x)$, $\varphi\ge 0$.

We notice that $H_S(f_\varepsilon,\xi)(x=0) = H_S(f_\varepsilon(x=0),\xi)$. We use now the hypothesis on the boundary condition to be maxwellian, this means that

$$f_\varepsilon(t,0,\xi) = f^b(t,\xi) = M^b(t,\xi),\qquad t>0,\ \xi>0,\tag{5.4}$$

where

$$M^b(t,\xi) = M(\rho^b(t),u^b(t),\xi),\qquad t>0,\ \xi\in\mathbb{R}.\tag{5.5}$$

Using the subdifferential inequality (Proposition 2.3), we have for a.e. $t>0$, $\xi\in\mathbb{R}$,

$$H_S(f_\varepsilon(x=0),\xi)\ge H_S(M^b,\xi)+T_S(\rho^b,u^b)\cdot(f_\varepsilon(x=0)-M^b),\tag{5.6}$$

with equality if $\xi>0$, thus

$$\int_{\mathbb{R}}\xi H_S(f_\varepsilon(x=0),\xi)d\xi \le \int_{\mathbb{R}}\xi H_S(M^b,\xi)d\xi + T_S(\rho^b,u^b)\cdot\int_{\mathbb{R}}\xi(f_\varepsilon(x=0)-M^b)d\xi,$$

and with (5.3) it yields

$$\begin{aligned}
&-\iiint\limits_{]0,\infty[^2\times\mathbb{R}} H_S(f_\varepsilon,\xi)\partial_t\varphi\,dtdxd\xi - \iiint\limits_{]0,\infty[^2\times\mathbb{R}} \xi H_S(f_\varepsilon,\xi)\partial_x\varphi\,dtdxd\xi\\
&-\iint\limits_{]0,\infty[\times\mathbb{R}} \xi H_S(M^b,\xi)\varphi(t,0)\,dtd\xi\\
&-\int_{]0,\infty[} T_S(\rho^b,u^b)\cdot\int_{\mathbb{R}}\xi(f_\varepsilon(x=0)-M^b)\,d\xi\varphi(t,0)\,dt\le 0,
\end{aligned}\tag{5.7}$$

that is to say by comparing f_ε to $M[f_\varepsilon]$

$$
\begin{aligned}
&-\iint\limits_{]0,\infty[^2} \eta_S(\rho_\varepsilon,u_\varepsilon)\partial_t\varphi\,dtdx - \iint\limits_{]0,\infty[^2} G_S(\rho_\varepsilon,u_\varepsilon)\partial_x\varphi\,dtdx\\
&-<R_{S,\varepsilon},\varphi> - \int_0^\infty G_S(\rho^b,u^b)\varphi(t,0)\,dt\\
&-\int_{]0,\infty[} T_S(\rho^b,u^b)\cdot\Big(\int_{\mathbb{R}}\xi f_\varepsilon(x=0)\,d\xi - F(\rho^b,u^b)\Big)\,\varphi(t,0)\,dt \le 0,
\end{aligned}
\tag{5.8}
$$

with

$$
\begin{aligned}
<R_{S,\varepsilon},\varphi>= &\iiint\limits_{]0,\infty[^2\times\mathbb{R}} (H_S(f_\varepsilon,\xi)-H_S(M[f_\varepsilon],\xi))\partial_t\varphi\,dtdxd\xi\\
&+\iiint\limits_{]0,\infty[^2\times\mathbb{R}} \xi(H_S(f_\varepsilon,\xi)-H_S(M[f_\varepsilon],\xi))\partial_x\varphi\,dtdxd\xi.
\end{aligned}
\tag{5.9}
$$

But (5.2) gives

$$
\begin{aligned}
&\iint\limits_{]0,\infty[\times\mathbb{R}} H_S(f_\varepsilon,\xi)(t=T)\,dxd\xi - \iint\limits_{]0,\infty[\times\mathbb{R}} H_S(f_\varepsilon,\xi)(t=0)\,dxd\xi\\
&\quad-\iint\limits_{]0,T[\times\mathbb{R}} \xi H_S(f_\varepsilon,\xi)(x=0)\,dtd\xi\\
&=\iiint\limits_{]0,T[\times]0,\infty[\times\mathbb{R}} (H_S'(f_\varepsilon,\xi)-T_S(\rho_\varepsilon,u_\varepsilon))\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\,dtdxd\xi,
\end{aligned}
\tag{5.10}
$$

therefore

$$
\iiint\limits_{]0,T[\times]0,\infty[\times\mathbb{R}} (H'(f_\varepsilon,\xi)-T_{\frac{v^2}{2}}(\rho_\varepsilon,u_\varepsilon))\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\,dtdxd\xi \quad \text{is bounded,} \tag{5.11}
$$

uniformly in ε. This result, together with the fact that f_ε, $M[f_\varepsilon]$ are bounded in $L^\infty_{t,x,\xi}$ and the property of uniform compact support allow to apply the dissipation result of [6] and to get that $f_\varepsilon - M[f_\varepsilon] \to 0$ a.e. t,x,ξ. Thus $<R_{S,\varepsilon},\varphi>\to 0$. Then, (5.8) gives in particular, for $\varphi\in\mathcal{D}(]0,\infty[^2_{t,x})$,

$$
\partial_t\eta_S(\rho_\varepsilon,u_\varepsilon)+\partial_x G_S(\rho_\varepsilon,u_\varepsilon)\le R_{S,\varepsilon}\to 0 \text{ in } W^{-1,p}_{loc} \tag{5.12}
$$

for any $1<p<\infty$, as $\varepsilon\to 0$, for $S:\mathbb{R}\to\mathbb{R}$ convex, of class C^1. Moreover, (5.12) becomes an equality if $S(v)=1$ or v. Since ρ_ε and u_ε are bounded in L^∞, we can then apply the stability result of [17] which gives the first part of Theorem 1.2: up to a subsequence, $(\rho_\varepsilon,\rho_\varepsilon u_\varepsilon)$ converge a.e. in $]0,\infty[\times\mathbb{R}$ when $\varepsilon\to 0$ to an entropy solution $(\rho,\rho u)$ to (1.1), (1.4) remaining in $\tilde{D}$, with initial data $(\rho^0,\rho^0u^0)=\int f^0 d\xi$.
It only remains to pass to the limit in the boundary condition (5.8). The only nontrivial term is the one involving

$$
\psi_\varepsilon(t)=\int_{\mathbb{R}}\xi f_\varepsilon(x=0)\,d\xi. \tag{5.13}
$$

The sequence $(\psi_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(]0,\infty[)$, thus there exists $\psi \in L^\infty$ such that for a subsequence

$$\psi_\varepsilon \rightharpoonup \psi \text{ in } L^\infty_{w*}(]0,\infty[), \qquad \text{as } \varepsilon \to 0. \tag{5.14}$$

We have from (1.7)

$$-\iiint\limits_{]0,\infty[^2\times\mathbb{R}} (f_\varepsilon \partial_t\varphi + \xi f_\varepsilon \partial_x\varphi)\,dtdxd\xi - \iint\limits_{]0,\infty[\times\mathbb{R}} \xi f_\varepsilon(x=0)\varphi(t,0)\,dtd\xi = 0 \tag{5.15}$$

for any $\varphi \in \mathcal{D}(]0,\infty[_t \times [0,\infty[_x)$, that is to say

$$\begin{aligned} -\iint\limits_{]0,\infty[^2} (\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)\partial_t\varphi\,dtdx - \iint\limits_{]0,\infty[^2} F(\rho_\varepsilon, u_\varepsilon)\partial_x\varphi\,dtdx - \int_0^\infty \psi_\varepsilon(t)\varphi(t,0)\,dt \\ + \iiint\limits_{]0,\infty[^2\times\mathbb{R}} \xi(M[f_\varepsilon] - f_\varepsilon)\partial_x\varphi\,dtdxd\xi = 0. \end{aligned} \tag{5.16}$$

Passing to the limit $\varepsilon \to 0$, we obtain

$$-\iint\limits_{]0,\infty[^2} (\rho, \rho u)\partial_t\varphi\,dtdx - \iint\limits_{]0,\infty[^2} F(\rho,u)\partial_x\varphi\,dtdx - \int_0^\infty \psi\varphi(t,0)\,dt = 0. \tag{5.17}$$

Now, according to Section 6, the relation $\partial_t(\rho,\rho u) + \partial_x F(\rho,u) = 0$ allows to consider the weak trace $\overline{F(\rho,u)}$, such that

$$-\iint\limits_{]0,\infty[^2} (\rho, \rho u)\partial_t\varphi\,dtdx - \iint\limits_{]0,\infty[^2} F(\rho,u)\partial_x\varphi\,dtdx - \int_{]0,\infty[} \overline{F(\rho,u)}\varphi(t,0)\,dt = 0. \tag{5.18}$$

By uniqueness of the weak trace, we obtain $\psi = \overline{F(\rho,u)}$, which yields

$$\psi_\varepsilon \rightharpoonup \overline{F(\rho,u)} \text{ in } L^\infty_{w*}(]0,\infty[), \qquad \text{as } \varepsilon \to 0. \tag{5.19}$$

Passing to the limit in (5.8), we finally obtain

$$\begin{aligned} -\iint\limits_{]0,\infty[\times]0,\infty[} \eta_S(\rho,u)\partial_t\varphi\,dtdx - \iint\limits_{]0,\infty[\times]0,\infty[} G_S(\rho,u)\partial_x\varphi\,dtdx \\ -\int_{]0,\infty[} [G_S(\rho^b,u^b) + T_S(\rho^b,u^b)\cdot(\overline{F(\rho,u)} - F(\rho^b,u^b))]\varphi(t,0)\,dt \le 0 \end{aligned} \tag{5.20}$$

for any $\varphi \in \mathcal{D}(]0,\infty[_t \times [0,\infty[_x)$, $\varphi \ge 0$.

In order to complete the proof of Theorem 1.2, it only remains to prove the equivalence of the weak formulation (5.20) with (1.4), (1.22), which is done in Proposition 6.2.

6 Weak trace

Let us recall here, with an elementary proof, the existence result for the weak trace of [1] and [10] in the case of a quarter space in $\mathbb{R}^2$.

Theorem 6.1 *Let $V = (V_0, V_1) \in L^\infty(]0,\infty[^2)$ be a vector field such that $\operatorname{div}_{t,x} V \in \mathcal{M}(]t_1,t_2[\times]0,R[)$ for any $0 < t_1 < t_2 < \infty$ and $R > 0$. Then there exists a unique solution $\overline{V_1} \in L^\infty_t(]0,\infty[)$ to*

$$-\iint\limits_{]0,\infty[^2} \varphi \operatorname{div} V - \iint\limits_{]0,\infty[^2} V_0 \partial_t \varphi \, dtdx - \iint\limits_{]0,\infty[^2} V_1 \partial_x \varphi \, dtdx - \int_{]0,\infty[} \overline{V_1} \varphi(t,0)\, dt = 0 \tag{6.1}$$

for any $\varphi \in C^\infty_c(]0,\infty[_t \times [0,\infty[_x)$. In fact $\overline{V_1}$ depends only of V_1 (see (6.2) below) and satisfies the bound $\|\overline{V_1}\|_\infty \le \|V_1\|_\infty$.

Proof. We take a smooth function φ_δ nonincreasing on $[0,\infty[$ and such that $\varphi_\delta(x) = 1$ for $x \le \delta/2$, $\varphi_\delta(x) = 0$ for $x \ge \delta$ and $|\varphi'_\delta(x)| \le C/\delta$. We have the decomposition

$$\begin{aligned}
& -\iint\limits_{]0,\infty[^2} \varphi \operatorname{div} V - \iint\limits_{]0,\infty[^2} V_0 \partial_t \varphi - \iint\limits_{]0,\infty[^2} V_1 \partial_x \varphi \\
= & -\iint\limits_{]0,\infty[^2} \varphi\varphi_\delta \operatorname{div} V - \iint\limits_{]0,\infty[^2} V_0 (\partial_t \varphi)\varphi_\delta - \iint\limits_{]0,\infty[^2} V_1 (\partial_x \varphi)\varphi_\delta - \iint\limits_{]0,\infty[^2} V_1 \varphi (\partial_x \varphi_\delta) \\
& -\iint\limits_{]0,\infty[^2} \varphi(1-\varphi_\delta) \operatorname{div} V - \iint\limits_{]0,\infty[^2} V_0 \partial_t(\varphi(1-\varphi_\delta)) - \iint\limits_{]0,\infty[^2} V_1 \partial_x(\varphi(1-\varphi_\delta)),
\end{aligned}$$

and the sum of the last three terms is 0 because $\varphi(1-\varphi_\delta) \in \mathcal{D}(]0,\infty[^2)$. Now

$$\iint\limits_{]0,\infty[^2} V_0 (\partial_t \varphi)\varphi_\delta \, dtdx \to 0, \qquad \iint\limits_{]0,\infty[^2} V_1 (\partial_x \varphi)\varphi_\delta \, dtdx \to 0,$$

as $\delta \to 0$, and using the dominated convergence theorem with the measure $\operatorname{div} V$, we have also

$$\iint\limits_{]0,\infty[^2} \varphi\varphi_\delta \operatorname{div} V \to 0, \qquad \text{as } \delta \to 0.$$

The sequence $(\int_0^\infty V_1(t,x)\varphi'_\delta(x)\, dx)_{\delta>0}$ is bounded in $L^\infty(]0,\infty[)$ by $\|V_1\|_{L^\infty}$, thus there exists $\overline{V_1} \in L^\infty_t(]0,\infty[)$ such that, up to a subsequence,

$$\int_0^\infty V_1(t,x)\varphi'_\delta(x)\, dx \rightharpoonup \overline{V_1}(t) \text{ in } L^\infty_{w*}(]0,\infty[). \tag{6.2}$$

Now

$$\begin{aligned}
\iint\limits_{]0,\infty[^2} V_1(t,x)\varphi(t,x)\varphi'_\delta(x)\, dtdx = & \iint\limits_{]0,\infty[^2} V_1(t,x)\varphi'_\delta(x)(\varphi(t,x) - \varphi(t,0))\, dtdx \\
& + \int_{]0,\infty[} (\int_0^\infty V_1(t,x)\varphi'_\delta(x)\, dx)\varphi(t,0)\, dt,
\end{aligned} \tag{6.3}$$

which yields

$$\iint_{]0,\infty[^2} V_1(t,x)\varphi(t,x)\varphi_\delta'(x)\,dtdx \to \int_{]0,\infty[} \overline{V_1}(t)\varphi(t,0)\,dt, \qquad \text{as } \delta\to 0, \tag{6.4}$$

and this ends the proof of existence. Uniqueness is obvious, and the bound follows from (6.2). □

Remark 6.1 In particular, it allows to define the weak trace of $G_S(\rho,u)$ for every S with subquadratic growth, by applying our result with $V=(\eta_S,G_S)$ thanks to (1.4).

Proposition 6.2 *The formulation (1.4), (1.22) is equivalent to (5.20).*

Proof. It is clear by (6.1) that (1.4), (1.22) implies (5.20). Let us assume that (5.20) holds. Then (1.4) comes from the use of a test function with support in $]0,\infty[^2$. Now, using the function φ_δ of the previous theorem, the weak formulation (5.20) with the test function $\varphi(t,x)=\psi(t)\varphi_\delta(x)$ gives for any $\psi\in C_c^\infty(]0,\infty[)$, $\psi\ge 0$,

$$\begin{aligned}&-\iint_{]0,\infty[\times]0,\infty[} \eta_S(\rho,u)\psi'\varphi_\delta\,dtdx - \iint_{]0,\infty[\times]0,\infty[} G_S(\rho,u)\psi\varphi_\delta'\,dtdx\\ &-\int_{]0,\infty[}[G_S(\rho^b,u^b)+T_S(\rho^b,u^b)\cdot(\overline{F(\rho,u)}-F(\rho^b,u^b))]\psi\,dt\le 0.\end{aligned} \tag{6.5}$$

By using (6.1), it gives

$$\begin{aligned}&\iint_{]0,\infty[\times]0,\infty[} \psi\varphi_\delta\,\mathrm{div}(\eta_S(\rho,u),G_S(\rho,u))\\ &+\int_{]0,\infty[}[\overline{G_S(\rho,u)}-G_S(\rho^b,u^b)-T_S(\rho^b,u^b)\cdot(\overline{F(\rho,u)}-F(\rho^b,u^b))]\psi\,dt\le 0.\end{aligned} \tag{6.6}$$

We finally get the boundary condition (1.22) as $\delta\to 0$ by using the dominated convergence theorem with the measure $\mathrm{div}(\eta_S(\rho,u),G_S(\rho,u))$. □

References

[1] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness, Ann. Mat. Pura Appl. (4) 135 (1983), 293–318 (1984).

[2] C. Bardos, A. Le Roux, J.-C. Nédélec, First order quasilinear equations with boundary conditions, Comm. Partial Differential Equations 4 (1979), no. 9, 1017–1034.

[3] A. Benabdallah, Le "p système" dans un intervalle, C. R. Acad. Sci. Paris Sér. I Math. 303 (1986), no. 4, 123–126.

[4] A. Benabdallah, D. Serre, Problèmes aux limites pour des systèmes hyperboliques non linéaires de deux équations à une dimension d'espace, C. R. Acad. Sci. Paris Sér. I Math. 305 (1987), no. 15, 677–680.

[5] F. Berthelin, F. Bouchut, Solution with finite energy to a BGK system relaxing to isentropic gas dynamics, to appear in Annales de la Faculté des Sciences de Toulouse (2001).

[6] F. Berthelin, F. Bouchut, Kinetic invariant domains and relaxation limit from a BGK model to isentropic gas dynamics, to appear in Asymptotic Analysis (2001).

[7] F. Bouchut, Construction of BGK models with a family of kinetic entropies for a given system of conservation laws, J. Statist. Phys. 95 (1999), 113–170.

[8] F. Bouchut, Renormalized solutions to the Vlasov equation with coefficients of bounded variation, Arch. for Ration. Mech. and Anal. 157 (2001), 75-90.

[9] Y. Brenier, L. Corrias, A kinetic formulation for multi-branch entropy solutions of scalar conservation laws, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998), 169-190.

[10] G.-Q. Chen, H. Frid, Divergence-measure fields and hyperbolic conservation laws, Arch. Ration. Mech. Anal. 147 (1999), no. 2, 89–118.

[11] R.J. DiPerna, Convergence of the viscosity method for isentropic gas dynamics, Comm. Math. Phys. 91 (1983), 1–30.

[12] F. Dubois, P. Le Floch, Boundary conditions for nonlinear hyperbolic systems of conservation laws, J. Differential Equations 71 (1988), no. 1, 93–122.

[13] M. Gisclon, D. Serre, Conditions aux limites pour un système strictement hyperbolique fournies par le schéma de Godunov, RAIRO Modèl. Math. Anal. Numér. 31 (1997), no. 3, 359–380.

[14] J.B. Goodman, Initial boundary value problems for hyperbolic systems of conservation laws, Thesis, Stanford University, 1981.

[15] F. James, Convergence results for some conservation laws with a reflux boundary condition and a relaxation term arising in chemical engineering, SIAM J. Math. Anal. 29 (1998), no. 5, 1200–1223.

[16] P.T. Kan, M. Santos, Z. Xin, Initial-boundary value problem for conservation laws, Comm. Math. Phys. 186 (1997), no. 3, 701–730.

[17] P.-L. Lions, B. Perthame, P.E. Souganidis, Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, Comm. Pure Appl. Math. 49 (1996), 599–638.

[18] P.-L. Lions, B. Perthame, E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations, J. Amer. Math. Soc. 7 (1994), 169-191.

[19] P.-L. Lions, B. Perthame, E. Tadmor, Kinetic formulation of the isentropic gas dynamics and p-systems, Comm. Math. Phys. 163 (1994), 415–431.

[20] T.P. Liu, Initial-boundary value problems for gas dynamics, Arch. Rational Mech. Anal. 64 (1977), no. 2, 137–168.

[21] V. Milišić, Stability and convergence of discrete kinetic approximations to an initial-boundary value problem for conservation laws, preprint.

[22] R. Natalini, A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws, J. Diff. Eq. 148 (1998), 292–317.

[23] S. Nishibata, The initial-boundary value problems for hyperbolic conservation laws with relaxation, J. Differential Equations 130 (1996), no. 1, 100–126.

[24] T. Nishida, J. Smoller, Mixed problems for nonlinear conservation laws, J. Differential Equations 23 (1977), no. 2, 244–269.

[25] A. Nouri, A. Omrane, J.P. Vila, Boundary conditions for scalar conservation laws from a kinetic point of view, J. Statist. Phys. 94 (1999), no. 5-6, 779–804.

[26] F. Otto, Initial-boundary value problem for a scalar conservation law, C. R. Acad. Sci. Paris Sér. I Math. 322 (1996), no. 8, 729–734.

[27] B. Perthame, E. Tadmor, A kinetic equation with kinetic entropy functions for scalar conservation laws, Comm. Math. Phys. 136 (1991), 501-517.

[28] B. Perthame, A.E. Tzavaras, Kinetic formulation for systems of two conservation laws and elastodynamics, Arch. Rational Mech. Anal. 155 (2000), 1-48.

[29] D. Serre, Relaxation semi-linéaire et cinétique des systèmes de lois de conservation, Ann. Inst. H. Poincaré Anal. non linéaire 17 (2000), 169-192.

[30] A.E. Tzavaras, Materials with internal variables and relaxation to conservation laws, Arch. Ration. Mech. Anal. 146 (1999), 129-155.

[31] A. Vasseur, Strong traces for solutions to multidimensional scalar conservation laws, preprint.

[32] W.-C. Wang, Z. Xin, Asymptotic limit of initial-boundary value problems for conservation laws with relaxational extensions, Comm. Pure Appl. Math. 51 (1998), no. 5, 505–535.