

HYPERPLANE SECTIONS OF CUBIC THREEFOLDS

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ABSTRACT. Let $X \subset \mathbb{P}^4$ be a smooth cubic hypersurface. Using the weak Lefschetz property for the Jacobian ring of X , we prove that a general cubic surface is isomorphic to a hyperplane section of X .

1. INTRODUCTION

A classical result of Sylvester states that a general cubic surface $S \subset \mathbb{P}^3$ admits an equation of the form $L_0^3 + \dots + L_4^3 = 0$, where the L_i are linear forms (see e.g. [D, §9.4]). An equivalent formulation is that S is isomorphic to a hyperplane section of the Fermat cubic threefold $X_0^3 + \dots + X_4^3 = 0$. We will show in this note that this actually holds for *any* smooth cubic threefold. Surprisingly, the proof reduces to a particular case of the *weak Lefschetz property* for the Jacobian ring of X .

Recall that a graded finite-dimensional algebra R satisfies the weak Lefschetz property if the multiplication map $\times \ell : R_p \rightarrow R_{p+1}$ by a general element ℓ of R_1 has maximal rank for all p . It is conjectured that all complete intersection algebras $k[x_1, \dots, x_n]/(P_1, \dots, P_n)$ have this property. The conjecture holds for $n \leq 3$ [HMNW], but little is known for $n \geq 4$. The case we need is $n = 5$, $\deg(P_i) = 2$, which is the main result of [A-R] (see also [B-F-P]).

2. PROOFS

Let $X \subset \mathbb{P}^4 = \mathbb{P}$ be a smooth cubic hypersurface, defined by an equation $F = 0$. Let $X^* \subset \mathbb{P}^*$ be the dual hypersurface. For any $H \in \mathbb{P}^* \setminus X^*$, the hyperplane section $X \cap H$ is a smooth cubic surface; we get in this way a morphism s_X from $\mathbb{P}^* \setminus X^*$ into the moduli space \mathcal{M}_3 of smooth cubic surfaces. Our aim is to determine when this map is dominant.

Let R denote the graded ring $\mathbb{C}[X_0, \dots, X_4]$. Let J be the ideal (F'_0, \dots, F'_4) in R , where we put $F'_i := \frac{\partial F}{\partial X_i}$, and let $\mathfrak{J} := R/J$ be the Jacobian ring of F .

Proposition. *Let $H \in \mathbb{P}^* \setminus X^*$, given by a linear form $L \in R_1$. The map $s_X : \mathbb{P}^* \setminus X^* \rightarrow \mathcal{M}_3$ is étale at H if and only if the multiplication map $\times L : \mathfrak{J}_2 \rightarrow \mathfrak{J}_3$ is injective.*

Proof. We put $S := X \cap H$. The tangent space to \mathbb{P}^* at H is naturally isomorphic to $H^0(H, \mathcal{O}_H(1)) = H^0(S, \mathcal{O}_S(1))$, and the tangent map to s_X at H is the coboundary map

$$\partial : H^0(S, \mathcal{O}_S(1)) \rightarrow H^1(S, T_S)$$

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deduced from the exact sequence

$$0 \rightarrow T_S \rightarrow T_{X|S} \rightarrow \mathcal{O}_S(1) \rightarrow 0.$$

Since $\dim H^0(S, \mathcal{O}_S(1)) = \dim H^1(S, T_S) = 4$ and $H^0(S, T_S) = 0$, s_X is étale at H if and only if $H^0(S, T_{X|S}) = 0$.

By restricting to S the exact sequence

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}|X} \rightarrow \mathcal{O}_X(3) \rightarrow 0,$$

we see that $H^0(S, T_{X|S})$ is the kernel of the map $\varphi : H^0(S, T_{\mathbb{P}|S}) \rightarrow H^0(S, \mathcal{O}_S(3))$. We choose the coordinates in \mathbb{P} so that $L = X_0$. The space $H^0(S, T_{X|S})$ is generated by the vector fields (restricted to S) $X_i \frac{\partial}{\partial X_j}$ ($i > 0, j \geq 0$), with the relation

$$\sum_{i \geq 1} X_i \frac{\partial}{\partial X_i} = 0; \text{ we have } \varphi(X_i \frac{\partial}{\partial X_j}) = X_i F'_j.$$

Let $V = \sum_{i=0}^4 L_i \frac{\partial}{\partial X_i}$ be a nonzero element of $H^0(S, T_{X|S})$, where the L_i are linear forms in X_1, \dots, X_4 . Assume that $\varphi(V) = 0$. This means that there exists $Q \in R_2$ and $a \in \mathbb{C}$ such that

$$(1) \quad \sum_{i=0}^4 L_i F'_i = X_0 Q + 3aF;$$

this implies that the class of Q in \mathfrak{J}_2 is mapped to zero in \mathfrak{J}_3 under multiplication by X_0 .

Let us show that this class is nonzero. Suppose $Q \in J_2$, so that $Q = \sum a_i F'_i$. Using the Euler relation $\sum X_i F'_i = 3F$, (1) becomes

$$\sum_{i=0}^4 (L_i - a_i X_0 - a X_i) F'_i = 0.$$

Now the only nontrivial relation $\sum M_i F'_i = 0$ with the M_i in R_1 is the Euler relation (this follows from the cohomology exact sequence associated to the exact sequence $0 \rightarrow T_X \rightarrow T_{\mathbb{P}|X} \rightarrow \mathcal{O}_X(3) \rightarrow 0$, plus the fact that $H^0(X, T_X) = 0$).

Therefore there exists $b \in \mathbb{C}$ such that $L_i - a_i X_0 - a X_i = b X_i$ for all i ; this implies $a_i = 0$ for $i \geq 1$, hence $L_i = (a+b)X_i$, and $L_0 = 0$. Thus $V = (a+b) \sum_{i \geq 1} X_i \frac{\partial}{\partial X_i} = 0$, a contradiction.

Conversely, if $\times X_0 : \mathfrak{J}_2 \rightarrow \mathfrak{J}_3$ is not injective, there exist forms $Q \in R_2 \setminus J_2$ and L_0, \dots, L_4 in R_1 such that

$$(2) \quad X_0 Q = \sum L_i F'_i.$$

Replacing Q by $Q - \sum a_i F'_i$, where a_i is the coefficient of X_0 in L_i , we can assume that the L_i are linear forms in X_1, \dots, X_4 . Then the vector field $V := \sum L_i \frac{\partial}{\partial X_i}$ satisfies $\varphi(V) = 0$.

We just have to check that V is nonzero. If it is, there exists $b \in \mathbb{C}$ such that $L_i = bX_i$ for $i \geq 1$ and $L_0 = 0$. Then (2) becomes

$$X_0(Q + bF'_0) = \sum_{i \geq 0} bX_i F'_i = 3bF.$$

Since F is irreducible, this implies $b = 0$ and $Q = 0$, contradiction. □

Remark. It is easy to give examples of pairs (X, H) such that s_X is not étale at H . For instance if X is the Fermat cubic and H is the hyperplane $X_0 = 0$, the elements $X_i \frac{\partial}{\partial X_0}$ ($i \geq 1$) of $H^0(S, T_{\mathbb{P}^3|S})$ belong to $\text{Ker } \varphi$, so the tangent map to s_X at H is zero. In fact, s_X contracts the lines in \mathbb{P}^* consisting of the hyperplanes $X_0 = tX_i$, $t \in \mathbb{C}$, $i \geq 1$.

Theorem. *The map $s_X : \mathbb{P}^* \setminus X^* \rightarrow \mathcal{M}_3$ is dominant.*

Proof. By [A-R, Theorem 1], the multiplication map $\times L : \mathfrak{J}_2 \rightarrow \mathfrak{J}_3$ is injective for L general in R_1 (weak Lefschetz property in degree 2). Thus by the Proposition, s_X is étale at a general point of \mathbb{P}^* , hence dominant. □

Remarks.

(1) The proof of the Proposition applies identically to a smooth hypersurface X of degree d in \mathbb{P}^n , with $d, n \geq 3$. With the previous notation, we get that s_X is unramified at H if and only if the multiplication map $\times L : \mathfrak{J}_{d-1} \rightarrow \mathfrak{J}_d$ is injective. If \mathfrak{J} satisfies the weak Lefschetz property in degree $d - 1$, we conclude that s_X is generically finite onto its image. This has been proved for $d \geq n + 3$ in [P-R-T], and more recently in [B-M] for $d \geq n + 2$. The cases $3 < d < n + 2$ and $d = 3$, $n \geq 5$ remain open.

(2) The proof of the Proposition works over any algebraically closed field k . However, the weak Lefschetz property used in the Theorem does not always hold. For instance if X is the Fermat cubic threefold in characteristic 2, the Jacobian ideal is generated by the squares of all linear forms; if ℓ and m are linearly independent linear forms, we have $\ell \cdot m \neq 0$ in \mathfrak{J}_2 , but $\ell \cdot (\ell \cdot m) = 0$ in \mathfrak{J}_3 , so the map $\times \ell : \mathfrak{J}_2 \rightarrow \mathfrak{J}_3$ is not injective.

In fact in this case all smooth hyperplane sections of X are isomorphic, see [B, Théorème 2].

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