

Vector bundles on the cubic threefold

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1. Introduction

Let X be a smooth cubic hypersurface in \mathbf{P}^4 . In their seminal paper [C-G], Clemens and Griffiths showed that the intermediate Jacobian $J(X)$, an abelian variety defined analytically through Hodge theory, is a fundamental tool to understand the geometry of X . They studied the Fano surface F of lines contained in X , proving that the Abel-Jacobi map embeds F into $J(X)$ and induces an isomorphism $\text{Alb}(F) \xrightarrow{\sim} J(X)$. They were able to deduce from this the Torelli theorem and the non-rationality of X (a problem which had resisted the efforts of the Italian geometers)¹.

Mumford noticed that one can express $J(X)$ as a *Prym variety* and thus give an alternate proof for the non-rationality of X ([C-G], Appendix C); the other results of [C-G] can also be obtained via this approach [B2]. Later Clemens observed that one could use the twisted cubics as well, giving an elegant parametrization of the theta divisor (see (5.2) below). At this point the cubic threefold could be considered as well understood, and the emphasis shifted to other Fano threefolds.

On the other hand vector bundles on low-dimensional varieties attracted much attention in the last two decades, notably because of their relation with mathematical physics. Since rank 2 vector bundles are well-known to be connected to codimension 2 cycles, it seems obvious to try to relate them to the intermediate Jacobian. Surprisingly this was done only recently: to my knowledge the first attempt appears in [M-T], followed by [I-M] and [D]. I would like to explain these results in this paper and to give some applications.

We will look at the simplest possible case, namely stable rank 2 vector bundles on X with trivial determinant. Using the isomorphism $\text{deg} : H^4(X, \mathbf{Z}) \xrightarrow{\sim} \mathbf{Z}$ we consider the second Chern class c_2 as a number, which is easily seen to be ≥ 2 (see e.g. [B-M]). This leads us to consider the moduli space \mathcal{M} of stable rank 2 vector bundles on X with $c_1 = 0$, $c_2 = 2$. It is a quasi-projective variety; according to Maruyama [Ma1] it admits a natural compactification $\overline{\mathcal{M}}$ consisting of classes of semi-stable sheaves with $c_1 = 0$, $c_2 = 2$, $c_3 = 0$. The main result is:

Theorem 1.1. *The moduli space $\overline{\mathcal{M}}$ is isomorphic to the intermediate Jacobian of X blown up along the Fano surface.*

¹Except the last one, these results had been obtained independently by Tyurin [T1].

(See Thm. 7.3 below for a more intrinsic formulation, and Cor. 7.4 for the corresponding description of \mathcal{M} .)

The theorem as stated is due to Druel [D]; the proof relies heavily on the results of [M-T] and [I-M], which itself relies on [I]. Since these papers are rather technical I thought it worthwhile to write a simplified version, insisting on the geometric ideas. I have made this note independent of [I], [M-T] and [I-M], but not of [D] to which I will have to refer for some delicate technical points.

The last sections contain some applications, in particular the construction of a completely integrable Hamiltonian system related to the situation.

2. Vector bundles in \mathcal{M} and elliptic quintics

For the rest of the paper X denotes a smooth cubic threefold (over \mathbf{C}), and \mathcal{M} the moduli space of stable rank 2 vector bundles on X with $c_1 = 0$, $c_2 = 2$. In this situation the condition “ E stable” simply means $H^0(X, E) = 0$ (this follows from Riemann-Roch and the fact that $\text{Pic}(X) = \mathbf{Z}$).

Proposition 2.1. [D] *Let $[E] \in \mathcal{M}$. Then:*

- a) $H^1(X, E(n)) = H^2(X, E(n)) = 0$ for all $n \in \mathbf{Z}$;
- b) $E(1)$ is spanned by its global sections.

Sketch of proof: Let S be a general hyperplane section of X . By a result of Maruyama [Ma2], the restriction $E|_S$ satisfies $H^0(S, E(-1)|_S) = 0$. The crucial point is the vanishing of $H^0(S, E|_S)$. This is somewhat delicate: Druel shows that in any case the bundle $E(2)$ is spanned by its global sections. Thus the zero set of a general section of $E(2)$ is a smooth curve C . Using Chern classes and cohomology computation Druel proves that if $H^0(S, E|_S) \neq 0$, C is the projection in \mathbf{P}^4 (from an external point) of a curve of degree 14 and genus 5 in \mathbf{P}^5 . Such a curve is an extremal Castelnuovo curve, so that we know precisely how to describe it; from this one deduces that C cannot be contained in a smooth cubic threefold.

Thus we have $H^0(S, E|_S) = 0$, hence also $H^2(S, E|_S) = H^2(S, E(-1)|_S) = 0$ by Serre duality². Since $\chi(E|_S) = 0$ by Riemann-Roch, we have also $H^1(S, E|_S) = 0$. Thus the sheaf $F = E(1)|_S$ is 0-regular in the sense of Mumford, that is, satisfies $H^i(X, F(-i)) = 0$ for $i > 0$. By an easy induction argument using hyperplane sections, this implies that F is generated by its global sections ([M1], lect. 14).

Therefore a general section s of $E(1)|_S$ vanishes along a finite subscheme Z of S , consisting of $c_2(E(1)|_S) = 5$ distinct points. We have an extension

$$0 \rightarrow \mathcal{O}_S(-1) \xrightarrow{s} E|_S \rightarrow \mathcal{I}_Z(1) \rightarrow 0,$$

where \mathcal{I}_Z is the ideal sheaf of Z in S . Since $H^0(S, E|_S) = 0$ Z is not contained in a hyperplane; it follows easily that Z imposes independent conditions on degree d hypersurfaces for any $d \geq 2$, that is, $H^1(S, \mathcal{I}_Z(d)) = 0$. From the above exact sequence we deduce $H^1(S, E(n)|_S) = 0$ for $n \geq 1$; we have already seen that $H^1(S, E|_S) = 0$, so $H^1(S, E(n)|_S) = 0$ for all n by Serre duality. This together with the exact sequence

$$0 \rightarrow E(n-1) \rightarrow E(n) \rightarrow E(n)|_S \rightarrow 0$$

²Recall that a rank 2 vector bundle with trivial determinant is isomorphic to its dual.

shows that the natural map $H^i(X, E(n-1)) \rightarrow H^i(X, E(n))$ is surjective for $i = 1$ and injective for $i = 2$. Since $H^1(X, E(n))$ and $H^2(X, E(n))$ vanish for $|n| \gg 0$, assertion a) follows.

We have in particular $H^1(X, E) = H^2(X, E(-1)) = 0$; the space $H^3(X, E(-2))$ is dual to $H^0(X, E)$, which is 0 by stability. Thus the vector bundle $E(1)$ is again 0-regular, hence generated by its global sections. \square

Let me mention an amusing corollary; I refer to [B3] for the details. The assertion a) above means that E , viewed as an $\mathcal{O}_{\mathbf{P}^4}$ -module, is arithmetically Cohen-Macaulay, that is, the $\mathbf{C}[X_0, \dots, X_4]$ -module $\bigoplus_{n \in \mathbf{Z}} H^0(X, E(n))$ admits a length 1 resolution by graded free modules. Using the alternate form $\det : \Lambda^2 E \rightarrow \mathcal{O}_X$ one shows that the resolution can be chosen skew-symmetric; the outcome is:

Corollary 2.2. *The vector bundle E admits a resolution*

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^4}(-2)^6 \xrightarrow{M} \mathcal{O}_{\mathbf{P}^4}(-1)^6 \rightarrow E \rightarrow 0 ,$$

where M is a skew-symmetric matrix with linear entries, such that X is defined by the equation $\text{pf}(M) = 0$. \square

We will now relate our vector bundles to a certain type of curves on X through Serre's construction.

Proposition 2.3. *The zero locus of a non-zero section $s \in H^0(X, E(1))$ is a locally complete intersection quintic curve $\Gamma \subset \mathbf{P}^4$ spanning \mathbf{P}^4 , with trivial canonical bundle and $\dim H^0(\Gamma, \mathcal{O}_\Gamma) = 1$. Conversely, given a curve $\Gamma \subset X$ with the above properties, there exists a vector bundle E of \mathcal{M} and a section s of $E(1)$ whose zero locus is Γ ; the pair (E, s) is uniquely determined up to automorphism.*

If s is general enough, its zero locus is a smooth elliptic quintic curve.

We will refer to the curves with the properties stated in the Proposition as *elliptic quintics*. It is not difficult to list all possible configurations for such curves, but we will not need this.

PROOF. Since E is stable, any non-zero section s of $E(1)$ vanishes along a l.c.i. curve $\Gamma \subset X$, giving rise to an extension

$$0 \rightarrow \mathcal{O}_X \xrightarrow{s} E(1) \rightarrow \mathcal{I}_\Gamma(2) \rightarrow 0 ,$$

where \mathcal{I}_Γ is the ideal sheaf of Γ in X . From this exact sequence we get

(2.3.a) $H^0(X, \mathcal{I}_\Gamma(1)) \cong H^0(X, E) = 0$ by stability, hence Γ spans \mathbf{P}^4 ;

(2.3.b) $H^1(X, \mathcal{I}_\Gamma) \cong H^1(X, E(-1)) = 0$ by (2.1), hence $h^0(\mathcal{O}_\Gamma) = 1$;

(2.3.c) by restriction to Γ , an isomorphism $N_{\Gamma/X} \xrightarrow{\sim} E(1)|_\Gamma$, which gives by the adjunction formula $\omega_\Gamma \cong \mathcal{O}_\Gamma$.

The degree of Γ is $c_2(E(1)) = 5$. Since $E(1)$ is spanned by its global sections (2.1), Γ is smooth for s general.

Conversely, let $\Gamma \subset X$ be an elliptic quintic. Since ω_Γ is trivial, we can apply Serre's construction to Γ : there are canonical isomorphisms

$$\text{Ext}^1(\mathcal{I}_\Gamma, \omega_X) \xrightarrow{\sim} \text{Ext}^2(\mathcal{O}_\Gamma, \omega_X) \xrightarrow{\sim} H^0(\Gamma, \omega_\Gamma) ,$$

so up to isomorphism there is a unique non-trivial extension

$$0 \rightarrow \mathcal{O}_X(-1) \xrightarrow{s} E \rightarrow \mathcal{I}_\Gamma(1) \rightarrow 0 ;$$

since a generator of $H^0(\Gamma, \omega_\Gamma)$ is everywhere $\neq 0$, the sheaf E is a rank 2 vector bundle with $c_1 = 0$ and $c_2 = 2$. We have $H^0(X, E) \cong H^0(X, \mathcal{I}_\Gamma(1)) = 0$, so E is stable. \square

Remark 2.4. The cubic X contains smooth elliptic curves of degree 5 which are contained in a hyperplane, and are therefore *not* elliptic quintics in our sense: in fact, any smooth hyperplane section of X contains many 4-dimensional linear systems of such curves. One can still perform Serre's construction with these curves, but the resulting vector bundle E has a non-zero section, hence it is no longer stable.

3. The Abel-Jacobi map; infinitesimal study

3.1. Recall that $\text{Pic}(X) = H^2(X, \mathbf{Z}) = \mathbf{Z}h$, where h is the class of a hyperplane section; we identify $H^4(X, \mathbf{Z})$ with \mathbf{Z} via the isomorphism $\text{deg} : \xi \mapsto (\xi \cdot h)$. For each $d \in \mathbf{Z}$, we denote by $J^d(X)$ the translate of $J(X)$ which parametrizes 1-cycles of degree d on X . The variety $J^d(X)$ is non-canonically isomorphic to $J(X) = J^0(X)$; moreover the map $\gamma \mapsto \gamma + h^2$ provides canonical isomorphisms $J^d(X) \xrightarrow{\sim} J^{d+3}(X)$.

Let T be an algebraic variety and z an element of the Chow group $CH^2(X \times T)$, such that the restriction $z_t := z|_{X \times \{t\}}$ has degree d for each $t \in T$; the map $t \mapsto [z_t]$ of T into $J^d(X)$ is algebraic. In particular, for each algebraic family $(\Gamma_t)_{t \in T}$ of degree d curves on X , the *Abel-Jacobi map* $\alpha : T \rightarrow J^d(X)$ given by $\alpha(t) = [\Gamma_t]$ is algebraic.

Let \mathcal{F} be a coherent sheaf on X . Grothendieck has constructed Chern classes $\tilde{c}_i(\mathcal{F}) \in CH^i(X)$, which give back the usual Chern classes by applying the cycle map $CH^i(X) \rightarrow H^{2i}(X, \mathbf{Z})$. Let $d = c_2(\mathcal{F})$; the class $\tilde{c}_2(\mathcal{F})$ defines an element of $J^d(X)$, which we will denote by $\mathbf{c}_2(\mathcal{F})$. As above, any algebraic family $(\mathcal{F}_t)_{t \in T}$ of coherent sheaves such that $c_2(\mathcal{F}_t) = d$ for all $t \in T$ gives rise to a morphism $\mathbf{c}_2 : T \rightarrow J^d(X)$. We have in particular a morphism $\mathbf{c}_2 : \mathcal{M} \rightarrow J^2(X)$.

3.2. Let \mathcal{Q} be the Hilbert scheme of elliptic quintics contained in X . Serre's construction provides us with a morphism $p : \mathcal{Q} \rightarrow \mathcal{M}$, whose fibre at a point $[E]$ of \mathcal{M} is the projective space $\mathbf{P}(H^0(X, E(1)))$.

The Abel-Jacobi map $\alpha : \mathcal{Q} \rightarrow J^2(X)$ which associates to the curve Γ the class of the cycle $\Gamma - h^2$ factors as

$$\alpha : \mathcal{Q} \xrightarrow{p} \mathcal{M} \xrightarrow{\mathbf{c}_2} J^2(X).$$

Proposition 3.3. **[M-T]** *The moduli spaces \mathcal{M} and \mathcal{Q} are smooth; the morphism $\mathbf{c}_2 : \mathcal{M} \rightarrow J^2(X)$ is étale.*

PROOF. a) Let us first prove that \mathcal{Q} is smooth. Let $\Gamma \in \mathcal{Q}$, and let N be its normal bundle in X . Applying the functor $\text{Hom}_{\mathcal{O}_{\mathbf{P}^4}}(\cdot, \mathcal{O}_\Gamma(-1))$ to the resolution (2.2), and using the isomorphism $N^* \cong E^*(-1)|_\Gamma$ (2.3.c), we get an exact sequence

$$0 \rightarrow H^0(\Gamma, N^*) \rightarrow H^0(\Gamma, \mathcal{O}_\Gamma)^6 \xrightarrow{M} H^0(\Gamma, \mathcal{O}_\Gamma(1))^6$$

where the second arrow can be identified with

$$H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4})^6 \xrightarrow{M} H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1))^6.$$

Since it is injective, we get $H^0(\Gamma, N^*) = 0$, hence by duality $H^1(\Gamma, N) = 0$. This implies that \mathcal{Q} is smooth at $[\Gamma]$,

of dimension $h^0(N) = \deg N = \deg E(1)|_\Gamma = 10$. Since \mathcal{Q} is a \mathbf{P}^5 -bundle over \mathcal{M} , \mathcal{M} is smooth of dimension 5.

- b) Let us prove now that α is smooth; here I follow closely [M-T]. Let Γ be a smooth elliptic quintic in X . The tangent map $T_\Gamma(\alpha)$ maps $T_\Gamma(\mathcal{Q}) = H^0(\Gamma, N)$ into $T_{\alpha(\Gamma)}(J(X)) = H^1(X, \Omega_X^2)^*$; using Serre duality and the isomorphism $N^* \cong N(-2)$, we can view its transpose as ${}^tT(\alpha) : H^1(X, \Omega_X^2) \rightarrow H^1(\Gamma, N(-2))$. By Welters general results [We], this map fits into a commutative diagram:

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(1)) & \xrightarrow{R} & H^1(X, \Omega_X^2) \\ \downarrow r_\Gamma & & \downarrow {}^tT(\alpha) \\ H^0(\Gamma, \mathcal{O}_\Gamma(1)) & \xrightarrow{\partial} & H^1(\Gamma, N(-2)) \end{array}$$

where r_Γ is the restriction map, R the Griffiths residue isomorphism, and ∂ the coboundary homomorphism associated to the exact sequence of normal bundles

$$0 \rightarrow N \rightarrow N_{\Gamma/\mathbf{P}^4} \rightarrow \mathcal{O}_\Gamma(3) \rightarrow 0$$

twisted by $\mathcal{O}_\Gamma(-2)$.

Since r_Γ and R are bijective, we only have to prove that ∂ is injective; it is sufficient to prove $H^0(\Gamma, N_{\Gamma/\mathbf{P}^4}(-2)) = 0$ – that is, by duality, $H^1(\Gamma, N_{\Gamma/\mathbf{P}^4}^*(2)) = 0$. Using the exact sequence

$$0 \rightarrow N_{\Gamma/\mathbf{P}^4}^* \rightarrow \Omega_{\mathbf{P}^4}^1|_\Gamma \rightarrow \omega_\Gamma \rightarrow 0$$

and the vanishing of $H^1(\Gamma, \Omega_{\mathbf{P}^4}^1(2)|_\Gamma)$, this is equivalent to the surjectivity of the map $H^0(\Gamma, \Omega_{\mathbf{P}^4}^1(2)|_\Gamma) \rightarrow H^0(\Gamma, \omega_\Gamma(2))$; it is enough to prove that the restriction map $r : H^0(\mathbf{P}^4, \Omega_{\mathbf{P}^4}^1(2)) \rightarrow H^0(\Gamma, \omega_\Gamma(2))$ is onto. We have a commutative diagram

$$\begin{array}{ccc} \Lambda^2 H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1)) & \xrightarrow{w_{\mathbf{P}}} & H^0(\mathbf{P}^4, \Omega_{\mathbf{P}^4}^1(2)) \\ \downarrow \wr & & \downarrow r \\ \Lambda^2 H^0(\Gamma, \mathcal{O}_\Gamma(1)) & \xrightarrow{w_\Gamma} & H^0(\Gamma, \omega_\Gamma(2)) \end{array}$$

where the horizontal maps are the *Wahl maps* given by $L \wedge M \mapsto LdM - MdL$; since w_Γ is known to be surjective in this case ([W], Thm. 4.2), r is surjective.

- c) Since α and p are smooth, \mathfrak{c}_2 is smooth, hence étale because \mathcal{M} and $J(X)$ have the same dimension. □

4. Reminder: $J(X)$ as a Prym variety

4.1. We need to recall some standard facts about Prym varieties and the corresponding description of $J(X)$; the reader familiar with this is advised to go directly to the next section. A general reference is [M2].

Let C be a curve of genus g and $\pi : \tilde{C} \rightarrow C$ an étale (connected) double covering. We denote by σ the involution which exchanges the two sheets of the covering. The abelian variety $P = \text{Im}(1 - \sigma) \subset J(\tilde{C})$, of dimension $g - 1$, is called the *Prym variety* associated to π . It is equipped with a principal polarization θ such that the restriction to P of the principal polarization of $J(\tilde{C})$ equals 2θ .

To describe the corresponding theta divisor it is convenient to translate the situation in the Jacobian $J^{2g-2}(\tilde{C})$ parametrizing divisor classes of degree $2g - 2$. The divisor classes $[D]$ such that $\pi_*D \equiv K_C$ fit into two subvarieties P_0 and P_1 of $J^{2g-2}(\tilde{C})$, both isomorphic to P , characterized by

$$[D] \in P_\varepsilon \iff h^0(D) \equiv \varepsilon \pmod{2}.$$

In particular, the subvariety $W \subset \text{Sym}^{2g-2}(\tilde{C})$ of effective divisors D such that $\pi_*D \in |K_C|$ has two irreducible components W_0 and W_1 , according to the parity of $h^0(D)$. If $D = x_1 + \dots + x_{2g-2}$ belongs to the component W_ε , the divisor $\sigma x_1 + x_2 + \dots + x_{2g-2}$ belongs to $W_{1-\varepsilon}$. The map $\text{Sym}^{2g-2}(\tilde{C}) \rightarrow J^{2g-2}(\tilde{C})$ induces morphisms $w_\varepsilon : W_\varepsilon \rightarrow P_\varepsilon$, whose fibre at a point $[D]$ of P_ε is the linear system $|D|$. It follows that

- $w_1 : W_1 \rightarrow P_1$ is birational;
- w_0 maps W_0 onto a divisor $\Theta \subset P_0$, which is a theta divisor on $P_0 \cong P$; the map $W_0 \rightarrow \Theta$ is generically a \mathbf{P}^1 -bundle.

4.2. We now recall the description of the intermediate Jacobian $J(X)$ as a Prym variety. We pick a general line $\ell \subset X$, a plane $\mathbf{P}^2 \subset \mathbf{P}^4$ and project X from ℓ onto \mathbf{P}^2 ; we get a morphism $f : X_\ell \rightarrow \mathbf{P}^2$, where X_ℓ is obtained by blowing up X along ℓ . The fibre of f over a point $p \in \mathbf{P}^2$ is a conic, the residual intersection of X with the plane $\langle \ell, p \rangle$; this conic is smooth for general p , and becomes singular when p lies in the “discriminant curve” $\Delta \subset \mathbf{P}^2$, which is a plane quintic. When ℓ is general enough, Δ is smooth, and every singular fibre is a rank 2 conic, that is, the union of two distinct lines meeting at one point. These lines are parametrized by a curve $\tilde{\Delta}$ with a double étale covering $\pi : \tilde{\Delta} \rightarrow \Delta$; if $\pi^{-1}(p) = \{x, y\}$, we have $f^{-1}(p) = \ell_x \cup \ell_y$. As in (4.1) we denote by σ the involution which exchanges x and y .

The Abel-Jacobi map $\tilde{\Delta} \rightarrow J^1(X)$ gives rise to a morphism of abelian varieties $\psi : J(\tilde{\Delta}) \rightarrow J(X)$. For $x \in \tilde{\Delta}$, the 1-cycle $\ell_x + \ell_{\sigma x}$ is linearly equivalent to $h^2 - \ell$, hence is independent of p ; thus ψ annihilates $\pi^*JC = \text{Ker}(1 - \sigma)$, and factors as

$$\psi : J(\tilde{\Delta}) \xrightarrow{1-\sigma} P \xrightarrow{\varphi} J(X).$$

If $D = \sum n_i x_i$ is a divisor on $\tilde{\Delta}$ with $\pi_*D \equiv 0$, we have $\sigma D \equiv -D$, hence

$$\varphi(2D) = \sum n_i \ell_{x_i}.$$

The following result is stated in [C-G], App. C; a proof can be found for instance in [T2].

Theorem 4.3. (Mumford) φ is an isomorphism of principally polarized abelian varieties. \square

5. Low degree curves on the cubic

5.1. I noticed in [B1] that the Fano surface F of lines contained in X has a simple interpretation in terms of the Prym variety (it is isomorphic to one of the two components of the variety of divisors $D \in \text{Sym}^5(\tilde{\Delta})$ such that $\pi_* D \in |\mathcal{O}_\Delta(1)|$); this gives for instance an easy proof of the fact (already proved in [C-G]) that the Abel-Jacobi map embeds F into $J^1(X)$ as a surface with minimal cohomology class $\frac{\Theta^3}{3!}$. Iliev made the nice observation that the same method can be used to study higher degree curves as well [I]. His construction works for general conic bundles, but I will specialize to the cubic threefold for simplicity.

Conics are not interesting: any conic q determines a line ℓ_q (the residual intersection with X of the plane spanned by q) such that $q + \ell_q = h^2$ in $CH^2(X)$; thus the image $F^2 \subset J^2(X)$ of the variety of conics in X by the Abel-Jacobi map is isomorphic to the Fano surface $F \subset J^1(X)$ through the isomorphism $J^2(X) \xrightarrow{\sim} J^1(X)$ given by $\xi \mapsto h^2 - \xi$.

The next case is twisted cubics. Let \mathcal{T} be the variety of twisted cubics contained in X ; we denote by $\overline{\mathcal{T}}$ its closure in the Hilbert scheme of X . Let $\alpha : \overline{\mathcal{T}} \rightarrow J(X)$ be the Abel-Jacobi map $t \mapsto [t] - h^2$.

Proposition 5.2. *The image of $\alpha : \overline{\mathcal{T}} \rightarrow J(X)$ is a theta divisor $\Theta \subset J(X)$; its generic fibre is isomorphic to \mathbf{P}^2 .*

I learned this result from Herb Clemens some 25 years ago; it can be easily deduced from the parametrization $\Theta = F - F$ given in [C-G]. We will give a different proof, to illustrate Iliev's method and also to open the way for the case of quartic curves.

PROOF. (5.2.a) A twisted cubic $t \subset X$ is contained in a unique hyperplane section S_t of X ; we replace \mathcal{T} by the open and dense subset consisting of those twisted cubic for which S_t is smooth and does not contain ℓ . On S_t there are 72 nets (= 2-dimensional linear systems) of twisted cubics. Thus the Abel-Jacobi map factors as $\alpha : \mathcal{T} \rightarrow \mathcal{S} \rightarrow J(X)$, where \mathcal{S} parametrizes pairs (S, L) of a smooth hyperplane section S of X , not containing ℓ , and a net of twisted cubics $|L|$ on S . We may now add to \mathcal{T} the singular elements in each net of cubics, so that \mathcal{T} becomes a \mathbf{P}^2 -bundle over \mathcal{S} .

Let \mathcal{T}_ℓ be the subvariety of cubics in \mathcal{T} which intersect ℓ (that is, which pass through the unique point of $\ell \cap S$); then $\mathcal{T}_\ell \rightarrow \mathcal{S}$ is a sub- \mathbf{P}^1 -bundle of the \mathbf{P}^2 -bundle $\mathcal{T} \rightarrow \mathcal{S}$.

Let $t \in \mathcal{T}_\ell$; the projection p_ℓ maps t to a conic $c \subset \mathbf{P}^2$. The trace of c on Δ is a divisor $\sum p_i \in |K_\Delta|$. The two points (x_i, y_i) of $\tilde{\Delta}$ over a point p_i correspond to two lines (ℓ_{x_i}, ℓ_{y_i}) meeting ℓ ; since the projection $t \rightarrow c$ is one-to-one, there is exactly one of these two lines, say ℓ_{y_i} , which intersects t ; we put $D(t) := \sum x_i$. Using the notation of (4.1), this gives a rational map $D : \mathcal{T}_\ell \dashrightarrow W$.

(5.2.b) *Claim :* D maps \mathcal{T}_ℓ into one component W_ε of W ; the map $D : \mathcal{T}_\ell \dashrightarrow W_\varepsilon$ is *birational* (in particular, \mathcal{T}_ℓ is irreducible).

Let $D = \sum x_i$ be a general element of W ; its push-down $\sum \pi x_i$ on Δ is cut down by a smooth conic c transversal to Δ . Let S_c be the pull-back

to c of the conic bundle f :

$$\begin{array}{ccc} S_c & \hookrightarrow & X_\ell \\ \downarrow & & \downarrow f \\ c & \hookrightarrow & \mathbf{P}^2. \end{array}$$

It is a conic bundle over c with 10 singular fibres $f^{-1}(\pi x_i) = \ell_{x_i} \cup \ell_{\sigma x_i}$. Let $b: S_c \rightarrow S_D$ be the blowing down of $\ell_{x_1}, \dots, \ell_{x_{10}}$; S_D is a \mathbf{P}^1 -bundle over c , thus isomorphic to one of the ruled surfaces \mathbf{F}_n ($n \geq 0$). Since these surfaces have different topological type according to the parity of n , we see that this parity is constant when D varies in each component W_ε . On the other hand, let us observe that *the parity of n changes with ε* : consider the divisor $D' = \sigma x_1 + x_2 + \dots + x_{10}$, which belongs to $W_{1-\varepsilon}$ (4.1); the surface $S_{D'}$ is obtained from $S_D \cong \mathbf{F}_n$ by performing an elementary transformation on the fibre at $\pi(x_1)$, thus is isomorphic to $\mathbf{F}_{n\pm 1}$.

Let us now specialize to the case where D is the divisor $D(t)$ associated to a general twisted cubic $t \in \mathcal{T}_\ell$. Then t lies in S_c , and projects down isomorphically to a curve \bar{t} in S_D . Let E be the exceptional divisor of X_ℓ . Since the canonical bundle of X_ℓ is $f^* \mathcal{O}_{\mathbf{P}^2}(-2)(-E) \cong \mathcal{O}_{X_\ell}(-S_c - E)$, the adjunction formula gives $K_{S_c} \cong -E|_{S_c}$. Therefore $-K_{S_c} \cdot t = E \cdot t = 1$, hence by the adjunction formula $t^2 = -1$ in S_c , and therefore also $\bar{t}^2 = -1$ in S_D . But for $n > 0$ the surface \mathbf{F}_n contains a unique curve of negative square, namely the *exceptional section* of square $-n$. We conclude that S_D is isomorphic to \mathbf{F}_1 , and that \bar{t} is its exceptional section; moreover the points of S_D which are blown up by b do not lie on \bar{t} .

Thus all components of \mathcal{T}_ℓ are mapped into the same component W_ε of W . For a general D in W_ε , the surface S_D is isomorphic to \mathbf{F}_1 (a small deformation of \mathbf{F}_1 is again isomorphic to \mathbf{F}_1), and the exceptional section \bar{t} does not pass through the points which are blown up. Let t be its proper transform in S_c . We have $t^2 = -1$, hence $t \cdot E = 1$; since t projects isomorphically onto c , it is a twisted cubic intersecting ℓ at one point. This provides (birationally) the inverse mapping of D .

(5.2.c) *Claim* : We have a commutative diagram

$$\begin{array}{ccc} \mathcal{T}_\ell & \xrightarrow{\quad D \quad} & W_\varepsilon \\ \alpha \downarrow & & \downarrow w_\varepsilon \\ J(X) & \xleftarrow[\varphi]{\sim} & P \cong P_\varepsilon \end{array}$$

where the isomorphism $P \cong P_\varepsilon$ is the translation by an appropriate element $\xi \in P_\varepsilon$.

Let $t \in \mathcal{T}_\ell$, and $D(t) = \sum x_i$ the corresponding divisor on $\tilde{\Delta}$. The commutativity of the diagram means that the element $u(t) := \alpha(t) - \varphi(\sum x_i - \xi)$ of $J(X)$ is zero. Since ξ is arbitrary and \mathcal{T}_ℓ is irreducible, it suffices to prove that $2u(t)$ is constant when t varies. In view of the definition of φ (4.2), this means that *the class of $2t - \sum \ell_{x_i}$ in $J^{-4}(X)$ is independent of t* .

Let us denote by f the class of a fibre of the ruling in $\text{Pic}(S_D)$ and also in $\text{Pic}(S_c)$. We have $-K_{S_D} \equiv 2\bar{t} + kf$ for some integer $k \in \mathbf{Z}$, hence

$$2t - \sum \ell_{x_i} + kf \equiv -K_{S_c} \equiv E|_{S_c}.$$

In $CH^2(X_\ell)$ the classes $f = f^*[p]$ and $E \cdot S_c = E \cdot f^*[c]$ are independent of t , hence so is $2t - \sum \ell_{x_i}$; pushing down in $CH^2(X)$ gives our assertion.

(5.2.d) The commutativity of the diagram implies that the generic fibre of α is birationally isomorphic to that of w_ε . We know that α has positive-dimensional fibres ((5.2.a), hence so does w_ε ; therefore $\varepsilon = 0$. Now the image of w_0 is the canonical theta divisor in P_0 , and its general fibre is isomorphic to \mathbf{P}^1 (4.1); thus the image of α is a theta divisor $\Theta \subset J(X)$, and its general fibre is isomorphic to \mathbf{P}^1 . In the factorization $\alpha : \mathcal{T}_\ell \rightarrow \mathcal{S} \xrightarrow{\rho} J(X)$, we see that ρ is birational; therefore \mathcal{T} is birationally a \mathbf{P}^2 -bundle over Θ . □

Remark 5.3. The divisor Θ is the *canonical theta divisor* of $J(X)$: its unique singular point is 0 (see [B2]). The corresponding divisor $\varphi^*\Theta \subset P$ is the translate of the canonical theta divisor $w_0(W_0) \subset P_0$ by the element $\pi^*\mathcal{O}_\Delta(1)$ of P_0 .

We now pass to quartic curves. Let \mathcal{Q}_ℓ be the variety of rational normal quartic curves which are contained in X and intersect ℓ transversally in 2 points; let $\bar{\mathcal{Q}}_\ell$ be its closure in the Hilbert scheme of X .

Proposition 5.4. *The Abel-Jacobi map $\bar{\mathcal{Q}}_\ell \rightarrow J^4(X)$ is surjective; its general fibre is isomorphic to \mathbf{P}^1 .*

PROOF. The proof follows closely the proof of (5.2). Let q be a general element of \mathcal{Q}_ℓ . Again q projects from ℓ onto a smooth conic $c \subset \mathbf{P}^2$, and defines a section of the ruled surface S_c . Let $\sum p_i$ be the divisor cut down by c on Δ ; we write as above $f^{-1}(p_i) = \ell_{x_i} \cup \ell_{y_i}$, with $\ell_{x_i} \cap q = \emptyset$, and $D(q) = \sum x_i$.

As in (5.2.b) we consider the image \bar{q} of q by $b : S_c \rightarrow S_{D(q)}$; since $E \cdot q = 2$ we find $\bar{q}^2 = q^2 = 0$. This implies:

- 1) $S_{D(q)}$ is isomorphic to \mathbf{F}_0 , so by (5.2.b) $D(q)$ lives in W_1 ;
- 2) \bar{q} is one of the lines of the horizontal ruling.

Conversely, any general line of this ruling lifts to a quartic q of \mathcal{Q}_ℓ : thus D maps \mathcal{Q}_ℓ into W_1 , it is dominant, and its general fibre is a rational curve (parametrizing the horizontal ruling).

As in (5.2.c) we get a commutative diagram

$$\begin{array}{ccc} \mathcal{Q}_\ell & \xrightarrow{D} & W_1 \\ \alpha \downarrow & & \downarrow w_1 \\ J(X) & \xleftarrow[\varphi]{\sim} & P \cong P_1 \end{array}$$

Now w_1 is birational, so α is birationally equivalent to D ; the Proposition follows. □

6. c_2 is of degree one

We now turn back to our moduli space \mathcal{M} and the map $c_2 : \mathcal{M} \rightarrow J^2(X)$. We keep our general line $\ell \subset X$.

Lemma 6.1. *Let $[E] \in \mathcal{M}$. The subspace of sections in $H^0(X, E(1))$ which vanish along ℓ has dimension 2.*

PROOF. We use the exact sequence (2.2):

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^4}(-1)^6 \rightarrow \mathcal{O}_{\mathbf{P}^4}^6 \rightarrow E(1) \rightarrow 0.$$

Restricting to ℓ we get an exact sequence $\mathcal{O}_\ell(-1)^6 \rightarrow \mathcal{O}_\ell^6 \xrightarrow{p} E(1)|_\ell \rightarrow 0$. The kernel K of p is a rank 4 vector bundle on ℓ , of determinant $\mathcal{O}_\ell(-2)$, with a surjective map $\mathcal{O}_\ell(-1)^6 \rightarrow K$; this implies $H^1(\ell, K) = 0$, hence $\dim H^0(\ell, K) = 2$ by Riemann-Roch; but $H^0(\ell, K)$ can be identified with the kernel of the restriction map $H^0(X, E(1)) \rightarrow H^0(\ell, E(1)|_\ell)$. \square

Proposition 6.2. c_2 induces an isomorphism of \mathcal{M} onto an open subset of $J^2(X)$.

We will describe precisely this open subset later (Cor. 7.4).

PROOF. Recall that the variety \mathcal{Q} of elliptic quintics in X admits a fibration $p : \mathcal{Q} \rightarrow \mathcal{M}$, such that the fibre of p at $[E] \in \mathcal{M}$ is identified with $\mathbf{P}(H^0(X, E(1)))$. Let \mathcal{Q}'_ℓ be the subvariety of \mathcal{Q} parametrizing elliptic quintics containing ℓ ; by the lemma the restriction of p to \mathcal{Q}'_ℓ is a sub- \mathbf{P}^1 -bundle of $\mathcal{Q} \rightarrow \mathcal{M}$. A curve in \mathcal{Q}'_ℓ is the union of ℓ and a quartic curve, so we can view \mathcal{Q}'_ℓ as a variety parametrizing quartic curves in X , containing \mathcal{Q}_ℓ ; the Abel-Jacobi map $\mathcal{Q} \rightarrow J^2(X)$ induces on \mathcal{Q}'_ℓ , up to a constant, the Abel-Jacobi map for this family of quartics. Since \mathcal{Q}'_ℓ is smooth and 6-dimensional (3.3), an easy count of constants shows that \mathcal{Q}'_ℓ is contained in $\overline{\mathcal{Q}}_\ell$. In the diagram

$$\begin{array}{ccccc} \mathcal{Q}'_\ell & & & & \\ \downarrow & \searrow & \alpha & & \\ \mathcal{Q} & \xrightarrow{p} & \mathcal{M} & \xrightarrow{c_2} & J^2(X) \end{array}$$

we know that c_2 is étale (3.3) and that the fibres of α are connected (5.4), so c_2 has degree one, hence is an open embedding. \square

7. The boundary of $\overline{\mathcal{M}}$

7.1. To complete the description of the moduli space $\overline{\mathcal{M}}$, we must describe the sheaves in $\overline{\mathcal{M}}$ which are not locally free. Let us first describe examples of such sheaves (the systematic reference for this section is [D]):

- a) Let c be a smooth conic in X , L the positive generator of $\text{Pic}(c)$ (so that $\mathcal{O}_X(1)|_c \cong L^{\otimes 2}$). Let E be the kernel of the canonical map $H^0(c, L) \otimes_{\mathbf{C}} \mathcal{O}_X \rightarrow L$. Then E is a torsion free sheaf, with $c_1(E) = c_3(E) = 0$ and $c_2(E) = [c]$. It is not difficult to check that E is *stable*.

- b) Let ℓ, ℓ' be two lines in X (possibly equal), and let $\mathcal{I}_\ell, \mathcal{I}_{\ell'}$ be their ideal sheaves. The sheaf $\mathcal{I}_\ell \oplus \mathcal{I}_{\ell'}$ is a torsion free sheaf with $c_1(E) = c_3(E) = 0$ and $c_2(E) = [\ell] + [\ell']$. It is clearly semi-stable but not stable.

Sheaves of type a) are parametrized by smooth conics of X , hence they form a codimension 1 subvariety A of $\overline{\mathcal{M}}$. Sheaves of type b) are parametrized by the symmetric square of F , hence they form an irreducible divisor $B \subset \overline{\mathcal{M}}$.

Proposition 7.2. [D]

- a) $\overline{\mathcal{M}} - \mathcal{M} = A \cup B$.
 b) *The moduli space $\overline{\mathcal{M}}$ is smooth and connected.*

The proof of a) requires a thorough analysis of the relationship between a sheaf in $\overline{\mathcal{M}}$ and its bidual; we can only refer the reader to [D].

Let us give an idea of the proof of b). We already know that \mathcal{M} is smooth and connected; it remains to prove that $\overline{\mathcal{M}}$ is smooth of dimension 5 at each point of the boundary. This is easy for a point of A , which corresponds to a stable sheaf E of type a): the exact sequence

$$0 \rightarrow E \rightarrow \mathcal{O}_X^2 \rightarrow L \rightarrow 0$$

gives an inclusion of $\text{Ext}_X^2(E, E)$ into $\text{Ext}_X^3(L, E)$; by Serre duality, this last space is dual to $\text{Hom}_X(E, L(-2))$, which is easily seen to be zero. Thus $\text{Ext}_X^2(E, E) = 0$ and $\dim \text{Ext}_X^1(E, E) = 5$ by Riemann-Roch, hence our assertion.

Things become more complicated for a sheaf $E = \mathcal{I}_\ell \oplus \mathcal{I}_{\ell'}$, which is not stable. Recall that the moduli space $\overline{\mathcal{M}}$ is constructed as the GIT quotient of a Quot scheme by a reductive group; using Luna's slice theorem, one shows that $\overline{\mathcal{M}}$ is locally analytically isomorphic to the quotient of $\text{Ext}_X^1(E, E)$ by the automorphism group of E . This group is isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$ if $\ell \neq \ell'$ and to $GL(2, \mathbf{C})$ if $\ell = \ell'$; one can describe explicitly its action on $\text{Ext}_X^1(E, E)$. Then it is not difficult to check that the quotient is smooth. \square

Recall that we denote by $F^2 \subset J^2(X)$ the image of the variety of conics; it is isomorphic to the Fano surface $F \subset J^1(X)$ (5.1). The morphism c_2 maps B onto the divisor $F + F$ in $J^2(X)$ and thus contracts only one irreducible divisor, namely A which is mapped onto the smooth surface F^2 . By [Mo], Thm. 1, this implies:

Theorem 7.3. [D] *The map $c_2 : \overline{\mathcal{M}} \rightarrow J^2(X)$ is isomorphic to the blowing up of $J^2(X)$ along the Fano surface F^2 .* \square

Any conic in X is linearly equivalent to a sum of two (incident) lines, so F^2 is contained in $F + F$; from (7.2.a) and (7.3) we conclude:

Corollary 7.4. c_2 *induces an isomorphism of \mathcal{M} onto the complement of the divisor $F + F$ in $J^2(X)$.* \square

Remark 7.5. Let $s : \text{Sym}^2 F \rightarrow F + F$ be the sum map, and let $\theta \in H^2(J(X), \mathbf{Z})$ be the principal polarization. Since the class of F in $H^6(J(X), \mathbf{Z})$ is $\frac{\theta^3}{3!}$, a standard Pontrjagin product computation gives $(\deg s) \cdot [F + F] = 3\theta$. But s can be identified with the restriction of c_2 to B , hence it is generically one-to-one by the theorem; thus we find that *the divisor $F + F$ is algebraically equivalent to 3θ .*

In particular, $F + F$ is an ample divisor; thus:

Corollary 7.6. *The variety \mathcal{M} is affine.* \square

8. Applications

Our first application is a Torelli-type theorem:

Proposition 8.1. *Let X' be another cubic threefold, and $\mathcal{M}' \subset \overline{\mathcal{M}}'$ the corresponding moduli spaces. If \mathcal{M} and \mathcal{M}' , or $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}'$, are isomorphic, then X and X' are isomorphic.*

PROOF. Any isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ extends to an isomorphism $J^2(X) \xrightarrow{\sim} J^2(X')$ which maps the divisor $F + F$ onto $F' + F'$. By Remark 7.5, the principally polarized abelian varieties $J(X)$ and $J(X')$ are isomorphic, hence by the Torelli theorem X and X' are isomorphic.

Similarly, an isomorphism $\overline{\mathcal{M}} \xrightarrow{\sim} \overline{\mathcal{M}}'$ induces an isomorphism $J^2(X) \xrightarrow{\sim} J^2(X')$ mapping F^2 onto F'^2 , from which we deduce an isomorphism $J(X) \xrightarrow{\sim} J(X')$ mapping $F - F = \Theta$ onto $F' - F' = \Theta'$; again we conclude that X and X' are isomorphic. \square

Once we have a parametrization of the intermediate Jacobian, it is natural to ask if it can be used to express the theta divisor. Giving a theta divisor in $J^2(X)$ is equivalent to choosing a point of $J^2(X)$ (namely the singular point of the divisor); it is therefore natural to fix a smooth conic $c \subset X$. Let $\Theta_c \subset J^2(X)$ be the translate by c of the canonical theta divisor $\Theta \subset J(X)$ (5.3).

Proposition 8.2. *Let $\mathcal{M}_c \subset \mathcal{M}$ be the locus of vector bundles E such that $E(1)$ has a non-zero section vanishing along c . Then the closure $\overline{\mathcal{M}}_c$ of \mathcal{M}_c in $\overline{\mathcal{M}}$ is the proper transform of Θ_c .*

PROOF. We first observe that $h^0(c, E(1)|_c) = 6$ for all $[E] \in \mathcal{M}$: by an easy count of constants, a general section s of $E(1)$ does not vanish on c ; then the exact sequence (2.3) gives by restriction to c an exact sequence $0 \rightarrow \mathcal{O}_c \rightarrow E(1)|_c \rightarrow \mathcal{O}_c(2) \rightarrow 0$, hence our assertion.

Thus the elements $[E]$ of \mathcal{M}_c are those for which the restriction map $H^0(X, E(1)) \rightarrow h^0(c, E(1)|_c)$ is not bijective. By a well-known construction this condition defines a *divisor* in \mathcal{M} .

We now consider the factorization $\alpha : \mathcal{Q} \xrightarrow{p} \mathcal{M} \hookrightarrow J^2(X)$ (3.2). Let \mathcal{Q}_c be the subvariety of \mathcal{Q} consisting of elliptic quintics containing c ; we have by definition $p(\mathcal{Q}_c) = \mathcal{M}_c$. A curve in \mathcal{Q}_c is the union of c and a curve t of degree 3. Let \mathcal{T}_c be the variety of twisted cubics in X meeting c transversally at 2 points; we identify it with a subvariety of \mathcal{Q}_c by mapping t to $c \cup t$. It is 4-dimensional, and a simple count of constants shows that $\mathcal{Q}_c - \mathcal{T}_c$ has dimension ≤ 3 . It follows that $\mathcal{M}_c = \overline{p(\mathcal{T}_c)}$ in \mathcal{M} .

Now by (5.2) $\alpha(\mathcal{T}_c)$ is contained in the divisor $\Theta_c \subset J^2(X)$, and therefore the divisor \mathcal{M}_c is contained in the restriction of Θ_c to \mathcal{M} ; since the latter is irreducible, they coincide, and $\overline{\mathcal{M}}_c$ maps onto Θ_c in $J^2(X)$.

To conclude we just have to check that $\overline{\mathcal{M}}_c$ does not contain the exceptional divisor, that is, that a general sheaf E of type a) (7.1) satisfies $H^0(X, \mathcal{I}_c E(1)) = 0$, where \mathcal{I}_c is the ideal sheaf of c in X . Let us pick a conic $d \subset X$ such that the plane spanned by d does not meet c ; let L be the positive generator of $\text{Pic}(d)$, and let E be the kernel of the natural map $H^0(d, L) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow L$. Taking tensor product with $\mathcal{I}_c(1)$ we get an exact sequence

$$0 \rightarrow \mathcal{I}_c E(1) \rightarrow H^0(d, L) \otimes_{\mathbb{C}} \mathcal{I}_c(1) \rightarrow L(1) \rightarrow 0.$$

Let V be the image of $H^0(X, \mathcal{I}_c(1))$ in $H^0(d, \mathcal{O}_d(1))$; by hypothesis it is a base-point free 2-dimensional subspace, and $H^0(X, \mathcal{I}_c E(1))$ can be identified with the kernel of the canonical map $H^0(d, L) \otimes V \rightarrow H^0(d, L(1))$; the Koszul exact sequence

$$0 \rightarrow L(-1) \rightarrow V \otimes_{\mathbf{C}} L \rightarrow L(1) \rightarrow 0$$

gives isomorphisms $H^0(X, \mathcal{I}_c E(1)) \cong H^0(d, L(-1)) = 0$. \square

9. Application: a completely integrable system

9.1. In the introduction of [T3], Tyurin made the following beautiful observation. Let X be a Fano threefold, and S a smooth anticanonical divisor in X (so that S is a K3 surface). Let \mathcal{M}_X be the moduli space of stable vector bundles on X with fixed rank r and Chern classes c_1, c_2, c_3 , and let \mathcal{M}_S be the moduli space of stable vector bundles on S with rank r and Chern classes $c_{1|S}, c_{2|S}$. According to Mukai [Mu], \mathcal{M}_S is smooth and carries a symplectic structure: at a point $[F] \in \mathcal{M}_S$, the symplectic form on $T_F(\mathcal{M}_S) = H^1(S, \mathcal{E}nd(F))$ is given by the cup-product

$$H^1(S, \mathcal{E}nd(F)) \otimes H^1(S, \mathcal{E}nd(F)) \rightarrow H^2(S, \mathcal{E}nd(F)) \xrightarrow{\text{Tr}} H^2(S, \mathcal{O}_S) \xrightarrow{\sim} \mathbf{C}.$$

Let $[E] \in \mathcal{M}_X$; assume that

- a) The restriction of E to S is stable;
- b) $H^2(X, \mathcal{E}nd(E)) = 0$.

By a) the restriction map $r : \mathcal{M}_X \rightarrow \mathcal{M}_S$ is defined in a neighborhood of $[E]$. Using b) and Serre duality, the exact sequence $0 \rightarrow \mathcal{E}nd(E) \otimes K_X \rightarrow \mathcal{E}nd(E) \rightarrow \mathcal{E}nd(E)|_S \rightarrow 0$ gives rise to a cohomology exact sequence

$$0 \rightarrow H^1(X, \mathcal{E}nd(E)) \xrightarrow{r_*} H^1(S, \mathcal{E}nd(E)|_S) \xrightarrow{r^*} H^1(X, \mathcal{E}nd(E))^* \rightarrow 0,$$

where r_* is the tangent map to r at $[E]$ and r^* its transpose with respect to the symplectic form. Therefore r_* is injective and its image is a maximal isotropic subspace; thus r induces an isomorphism of some open neighborhood of $[E]$ in \mathcal{M}_X onto a Lagrangian submanifold of \mathcal{M}_S .

9.2. In practice, condition b) is usually difficult to check. We now turn back to our situation, taking for \mathcal{M}_X our moduli space \mathcal{M} of rank 2 vector bundles on the cubic threefold X with $c_1 = 0, c_2 = 2$. We will show using the previous results that \mathcal{M}_X embeds as a Lagrangian submanifold of the moduli space \mathcal{M}_S . We will even show that this submanifold varies in a Lagrangian fibration (equivalently, a completely integrable Hamiltonian system) defined on an open subset of \mathcal{M}_S .

We pick a quadric $Q \subset \mathbf{P}^4$ such that $S = X \cap Q$ is a smooth K3 surface with $\text{Pic}(S) = \mathbf{Z}$ [De]. We denote by \mathcal{M}_S the moduli space of stable rank 2 vector bundles on S with $c_1 = 0, c_2 = 4$.

Lemma 9.3. *The vector bundles F in \mathcal{M}_S admitting a resolution*

$$0 \rightarrow \mathcal{O}_Q(-2)^6 \xrightarrow{M} \mathcal{O}_Q(-1)^6 \xrightarrow{P} F \rightarrow 0,$$

where M is a skew-symmetric matrix with linear entries, form an open subset \mathcal{M}_S^0 of \mathcal{M}_S . For such a vector bundle, the matrix M is uniquely determined up to a transformation $M \mapsto AM^tA$, with $A \in GL(6, \mathbf{C})$.

PROOF. If F admits the above resolution, we have:

- a) For $j = 0$ or 1 , the map $H^0(S, \mathcal{O}_S(j))^6 \rightarrow H^0(S, F(j+1))$ deduced from p is surjective.
- b) $H^1(S, F(j)) = 0$ for $j = 0$ or 1 . Let N be the kernel of the natural map $H^0(S, F(1)) \otimes_{\mathbf{C}} \mathcal{O}_Q \rightarrow F(1)$. In our case it is isomorphic to $\mathcal{O}_Q(-1)^6$, so:
- c) The sheaf $N(1)$ is spanned by its global sections.

Conditions a), b) and c) define an open subset \mathcal{M}_S° of \mathcal{M}_S . Let $[F] \in \mathcal{M}_S^{\circ}$; by a), b) and Riemann-Roch we find $h^0(S, F(1)) = h^0(Q, N(1)) = 6$; hence N is a rank 6 vector bundle on Q , and the surjective map $\mathcal{O}_Q^6 \rightarrow N(1)$ must be an isomorphism.

Thus we have an exact sequence $0 \rightarrow \mathcal{O}_Q(-2)^6 \xrightarrow{M} \mathcal{O}_Q(-1)^6 \rightarrow F \rightarrow 0$. Reasoning as in [B3], Theorem B, we can choose bases of $\mathcal{O}_Q(-2)^6$ and $\mathcal{O}_Q(-1)^6$ such that M is skew-symmetric; then the matrix M is uniquely determined up to the action of $GL(6, \mathbf{C})$ ([B3], 2.3). \square

We now fix our K3 surface S , but allow X to vary in the linear system $\Pi (\cong \mathbf{P}^5)$ of cubics containing S . For $[F] \in \mathcal{M}_S^{\circ}$, the cubic pf $M = 0$ is well determined by $[F]$, and belongs to Π ; we get in this way a morphism $H : \mathcal{M}_S^{\circ} \rightarrow \Pi$.

Proposition 9.4. *H is a Lagrangian fibration³; the fibre of H at a smooth cubic $X \in \Pi$ is isomorphic to the moduli space \mathcal{M}_X (through the restriction map $r : \mathcal{M}_X \rightarrow \mathcal{M}_S$). In particular, r induces an isomorphism of \mathcal{M}_X onto a Lagrangian submanifold of \mathcal{M}_S .*

PROOF. Let $[E] \in \mathcal{M}_X$. From the exact sequence $0 \rightarrow E(-2) \rightarrow E \rightarrow E|_S \rightarrow 0$ and the vanishing of $H^1(X, E(-2))$ (2.1), we get $H^0(S, E|_S) = 0$; since $\text{Pic}(S) = \mathbf{Z}$ this means that $E|_S$ is stable.

Next let us compute $H^2(X, \mathcal{E}nd(E))$. Choosing a general section of E , we deduce from the exact sequence (2.3), taking tensor product with $E \cong E^*$, an exact sequence

$$0 \rightarrow E(-1) \rightarrow \mathcal{E}nd(E) \rightarrow E(1) \rightarrow E(1)|_{\Gamma} \rightarrow 0.$$

Because $H^i(X, E(j)) = 0$ for $i = 1, 2$, this gives an isomorphism $H^2(X, \mathcal{E}nd(E)) \xrightarrow{\sim} H^1(\Gamma, E(1)|_{\Gamma})$. Since $E(1)|_{\Gamma}$ is isomorphic to $N_{\Gamma/X}$ (2.3.c) and $H^1(\Gamma, N_{\Gamma/X}) = 0$ (3.3.a), our assertion follows.

Put $F = E|_S$. The resolution $0 \rightarrow \mathcal{O}_{\mathbf{P}^4}(-2)^6 \xrightarrow{M} \mathcal{O}_{\mathbf{P}^4}(-1)^6 \rightarrow E \rightarrow 0$ (2.2) gives by restriction to Q a resolution

$$0 \rightarrow \mathcal{O}_Q(-2)^6 \xrightarrow{M} \mathcal{O}_Q(-1)^6 \rightarrow F \rightarrow 0 \quad (9.4.a)$$

hence $E|_S$ belongs to $H^{-1}(X)$. Conversely, any $F \in H^{-1}(X)$ admits a resolution (9.4.a), with pf $M = 0$ on X ; the matrix M defines an injective map $\mathcal{O}_{\mathbf{P}^4}(-2)^6 \rightarrow \mathcal{O}_{\mathbf{P}^4}(-1)^6$, whose cokernel is a bundle $E \in \mathcal{M}_X$ such that $E|_S \cong F$. Thus r induces an isomorphism $\mathcal{M}_X \xrightarrow{\sim} H^{-1}(X)$.

Finally by (9.1) the smooth part of any fibre of H is Lagrangian. \square

Remarks 9.5. a) The moduli space \mathcal{M}_S of stable rank 2 bundles on a K3 surface S with $c_1 = 0$, $c_2 = 4$ is a very interesting object (by Riemann-Roch, $c_2 = 4$ is the smallest value for which the moduli space is not empty). O'Grady proved that it is irreducible, and admits a smooth compactification $\widehat{\mathcal{M}}_S$ which is holomorphic symplectic [O]. It seems that H does not extend to a Lagrangian

³This means, by definition, that the smooth part of each fibre of H is a Lagrangian submanifold.

fibration $\widehat{\mathcal{M}}_S \rightarrow \Pi$; however it is conceivable that it extends to some other compact holomorphic symplectic manifold birationally isomorphic to $\widehat{\mathcal{M}}_S$.

b) A Lagrangian fibration is the same thing as a completely integrable system: if $H = (H_1, \dots, H_5)$ in some local coordinates system on Π , the functions H_i Poisson commute and the Hamiltonian vector fields X_{H_i} span the tangent space to the fibre at each smooth point of H . Since the fibres are open subsets of abelian varieties, it seems likely that these vector fields *linearize* on each fibre, that is, come from global vector fields on the abelian variety (such a system is often called *algebraically completely integrable*). To prove this it would be enough to exhibit some partial compactification $H' : \mathcal{M}'_S \rightarrow \Pi$ of H such that $H'^{-1}(X)$, for X general in Π , is isomorphic to the complement of a subvariety of codimension ≥ 2 in $J^2(X)$.

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