

Quotients of Jacobians

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Pour Enrico, après 50 ans d'amitié

Abstract: Let C be a curve of genus g , and let G be a finite group of automorphisms of C . The group G acts on the Jacobian J of C ; we prove that for $g \geq 21$ the quotient J/G has canonical singularities and Kodaira dimension 0. On the other hand we give examples with $g \leq 4$ for which J/G is uniruled.

1. Introduction

Let C be a (smooth, projective, complex) curve of genus g , and let G be a finite group of automorphisms of C . The group G acts on the Jacobian J of C . There are two possibilities for the quotient J/G [K-L]¹: either it has canonical singularities and Kodaira dimension 0, or it is uniruled. The aim of this note is to show that the latter case is rather exceptional – in fact, it does not occur for $g \geq 21$. This bound is rough, and can certainly be lowered. On the other hand we give examples of uniruled J/G for $g \leq 4$. When $g = 3$ this is an essential ingredient in our proof that the cycle $[C] - [(-1)^*C]$ on J is torsion modulo algebraic equivalence [B-S]; in fact this note grew out of the observation that already in genus 3, the case where J/G is uniruled is quite rare.

2. J/G canonical

Let C be a curve of genus g and J its Jacobian. Let G be a subgroup of $\text{Aut}(C)$, hence also of $\text{Aut}(J)$. By [K-L, Theorem 2], there are two possibilities:

arXiv: [2207.00884](https://arxiv.org/abs/2207.00884)

Received September 10, 2022.

2010 Mathematics Subject Classification: Primary 14H40; secondary 14L30, 14J10.

¹I am indebted to A. Horing for pointing out this reference.

- Either J/G has canonical singularities, and Kodaira dimension 0;
- or J/G is uniruled.

Proposition 1. *Assume $g \geq 21$. The quotient variety J/G has canonical singularities, and Kodaira dimension 0.*

Proof. We first observe that the fixed locus $\text{Fix}(\sigma) \subset J$ of any element $\sigma \neq 1$ of G has codimension ≥ 2 . Indeed the dimension of $\text{Fix}(\sigma)$ is the multiplicity of the eigenvalue 1 for the action of σ on $H^0(C, K_C)$, that is, the genus of $C/\langle\sigma\rangle$, which is $\leq g - 2$ as soon as $g \geq 4$ by the Hurwitz formula.

Let σ be an element of order r in G , and let $p \in J$ be a fixed point of σ . The action of σ on the tangent space $T_p(J)$ is isomorphic to the action on $T_0(J) = H^{0,1}(C)$. Let $\zeta = e^{\frac{2\pi i}{r}}$; we write the eigenvalues of σ acting on $H^{0,1}(C)$ in the form $\zeta^{a_1}, \dots, \zeta^{a_g}$, with $0 \leq a_i < r$. By Reid's criterion [R, Theorem 3.1], J/G has canonical singularities if and only if $\sum a_i \geq r$ for all σ in G .

The eigenvalues of σ acting on $H^1(C, \mathbb{C})$ are $\zeta^{a_1}, \dots, \zeta^{a_g}; \zeta^{-a_1}, \dots, \zeta^{-a_g}$. Thus $\text{Tr} \sigma^*_{|H^1(C, \mathbb{C})} = 2 \sum \cos \frac{2\pi a_i}{r}$. By the Lefschetz formula, this trace is equal to $2 - f$, where f is the number of fixed points of σ ; in particular, it is ≤ 2 . Using $\cos x \geq 1 - \frac{x^2}{2}$, we find

$$1 \geq \sum \cos \frac{2\pi a_i}{r} \geq g - \frac{2\pi^2}{r^2} \sum a_i^2,$$

hence

$$\left(\sum a_i \right)^2 \geq \sum a_i^2 \geq \frac{g-1}{2\pi^2} r^2.$$

Thus if $g \geq 1 + 2\pi^2 = 20.739\dots$, we get $\sum a_i > r$. □

Remark. The proposition does not extend to the case of an arbitrary abelian variety: indeed we give below examples where J/G is uniruled; then if A is any abelian variety, the quotient of $J \times A$ by G acting trivially on A is again uniruled.

3. J/G uniruled

We will now give examples of (low genus) curves C such that J/G is uniruled. The genus 2 case is quite particular (and probably well known). We put $\rho := e^{\frac{2\pi i}{3}}$. We assume $G \neq \{1\}$.

Proposition 2. *If $g(C) = 2$, J/G has Kodaira dimension 0 if and only if $G = \langle \sigma \rangle$ and σ is the hyperelliptic involution, or an automorphism of order 3 with eigenvalues (ρ, ρ^2) on $H^0(C, K_C)$, or an automorphism of order 6 with eigenvalues $(-\rho, -\rho^2)$ on $H^0(C, K_C)$.*

Proof. Let σ be an element of order r in G , and let ζ be a primitive r -th root of unity. The eigenvalues of σ acting on $H^0(C, K_C)$ are ζ^a, ζ^b , with $0 \leq a, b < r$. Suppose that J/G has canonical singularities. By Reid’s criterion we must have $a + b \geq r$; replacing ζ by ζ^{-1} gives $2r - a - b \geq r$, hence $a + b = r$. Since σ has order r , σ^k acts non-trivially on $H^0(C, K_C)$ for $0 < k < r$; therefore a and r are coprime. Replacing ζ by ζ^a we may assume that the eigenvalues of σ are ζ and ζ^{-1} .

Put $\zeta = e^{i\alpha}$; the trace of σ acting on $H^1(C, \mathbb{Z})$ is $2(\zeta + \zeta^{-1}) = 4 \cos \alpha$, and it is an integer. This implies $\cos \alpha = 0$ or $\pm \frac{1}{2}$, hence $\alpha \in \mathbb{Z} \cdot \frac{2\pi}{6}$ and $r \in \{2, 3, 6\}$. If $r = 2$ we must have $\zeta = \zeta^{-1} = -1$, hence σ is the hyperelliptic involution; if $r = 3$ or 6 we have $\{\zeta, \zeta^{-1}\} = \{\rho, \rho^2\}$ or $\{-\rho, -\rho^2\}$.

Thus G contains elements of order 3 and 6, and at most one element of order 2. Since the order of $\text{Aut}(C)$ is not divisible by 9 [B-L, 11.7], G is isomorphic to either $\mathbb{Z}/3$, or a central extension of $\mathbb{Z}/3$ by $\mathbb{Z}/2$ – that is, $\mathbb{Z}/6$. This proves the Proposition. □

The last two cases are realized by the curves C of the form $y^2 = x^6 + kx^3 + 1$, with the automorphisms $\sigma_1 : (x, y) \mapsto (\rho x, y)$ and $\sigma_2 : (x, y) \mapsto (\rho x, -y)$. The surfaces $S_i := J/\langle \sigma_i \rangle$ are (singular) K3 surfaces, each depending on one parameter. The surface S_1 is studied in detail in [B-V]: it has 9 singular points of type A_2 . The surface S_2 is the quotient of S_1 by the involution induced by $(-1)_J$; it has one singular point of type A_5 (the image of $0 \in J$), 4 points of type A_2 corresponding to pairs of singular points of S_1 , and 5 points of type A_1 corresponding to triples of points of order 2 of J .

It is more subtle to give examples with $g \geq 3$. By Reid’s criterion, we must exhibit an automorphism σ of C , of order r , such that the eigenvalues of σ acting on $H^0(C, K_C)$ are of the form $\zeta^{a_1}, \dots, \zeta^{a_g}$, with ζ a primitive r -th root of unity, $a_i \geq 0$ and $\sum a_i < r$. In the following tables we give the curve C (in affine coordinates), the order r of σ and ζ , the expression of σ , a basis of eigenvectors for σ acting on $H^0(C, K_C)$, the exponents a_1, \dots, a_g ; in the last column we check the inequality $\sum a_i < g$, which implies that $J/\langle \sigma \rangle$ is uniruled.

$$g = 3$$

curve	r	$\sigma(x, y) =$	basis of $H^0(K_C)$	a_1, \dots, a_3	$\sum a_i$
$y^2 = x^8 - 1$	8	$(\zeta x, y)$	$\frac{dx}{y}, \frac{xdx}{y}, \frac{x^2dx}{y}$	1, 2, 3	$6 < 8$
$y^2 = x(x^6 - 1)$	12	$(\zeta^2 x, \zeta y)$	$\frac{dx}{y}, \frac{xdx}{y}, \frac{x^2dx}{y}$	1, 3, 5	$9 < 12$
$y^2 = x(x^7 - 1)$	14	$(\zeta^2 x, \zeta y)$	$\frac{dx}{y}, \frac{xdx}{y}, \frac{x^2dx}{y}$	1, 3, 5	$9 < 14$
$y^3 = x(x^3 - 1)$	9	$(\zeta^3 x, \zeta y)$	$\frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{dx}{y}$	1, 4, 2	$7 < 9$
$y^3 = x^4 - 1$	12	$(\zeta^{-3} x, \zeta^{-4} y)$	$\frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{dx}{y}$	5, 2, 1	$8 < 12$

The fact that $J/\langle\sigma\rangle$ is uniruled for the fourth curve C is an important ingredient of the proof in [B-S] that the Ceresa cycle $[C] - [(-1)^*C]$ in J is torsion modulo algebraic equivalence. The same property for the last curve is observed in [L-S]. The second curve has been provided to me by D. Conti, A. Ghigi and R. Pignatelli, using the database [CGP].

$$g = 4$$

curve	r	$\sigma(x, y) =$	basis of $H^0(K_C)$	a_1, \dots, a_4	$\sum a_i$
$y^2 = x(x^9 - 1)$	18	$(\zeta^2 x, \zeta y)$	$\frac{dx}{y}, \frac{xdx}{y}, \frac{x^2dx}{y}, \frac{x^3dx}{y}$	1, 3, 5, 7	$16 < 18$
$y^3 = x^5 - 1$	15	$(\zeta^{-3} x, \zeta^{-5} y)$	$\frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{x^2dx}{y^2}, \frac{dx}{y}$	7, 4, 1, 2	$14 < 15$

In genus 5 there is no example:

Proposition 3. *Assume $g = 5$. The quotient variety J/G has canonical singularities, and Kodaira dimension 0.*

Proof. The paper [K-K] contains the list of possible automorphisms of genus 5 curves, and gives for each automorphism of order r its eigenvalues $\zeta^{a_1}, \dots, \zeta^{a_5}$ on $H^0(K)$, with $\zeta = e^{\frac{2\pi i}{r}}$ and $0 \leq a_i < 5$. We want to prove $\sum a_i \geq r$.

We first observe that if an eigenvalue and its inverse appear in the list, we have $\sum a_i \geq r$; this eliminates all cases with $r \leq 10$. For $r \geq 11$ the numbers a_1, \dots, a_5 are always distinct and nonzero; therefore $\sum a_i \geq 1 + 2 + \dots + 5 = 15$. The remaining cases are $r = 20$ and 22 ; one checks immediately that for all k coprime to r we have $\sum b_i \geq r$, where $b_i \equiv ka_i \pmod{r}$ and $0 < b_i < r$. This proves the Proposition. \square

Added in proof: R. Serova just proved that there are in fact no examples in genus ≥ 6 (*Quotients of Jacobians*, preprint [arXiv:2407.00860](https://arxiv.org/abs/2407.00860)).

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