A TRIBUTE TO M.S. NARASIMHAN

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My interest in vector bundles on curves was triggered by the work of Narasimhan and Ramanan. In the early 80's I was interested in the Schottky problem. Recall that one associates to a curve C of genus g a complex torus J of dimension g, the *Jacobian variety*, which one can view as parameterizing line bundles L of degree $g-1$ on C (by choosing one of these as the origin). Then the locus Θ of those line bundles which admit a nonzero section is a hypersurface in J, the *Theta divisor*. The pair (J,Θ) is what we call a *principally polarized abelian variety* – p.p.a.v. for short.

As soon as $g \geq 4$, the p.p.a.v.'s depend on more parameters than the Jacobians; the *Schottky problem* asks for a characterizations of Jacobians among all p.p.a.v.'s (A,Θ). There has been a flurry of activity around this in the early 80's; most approaches involve the linear system $|2\Theta|$ (that is, the projective space of divisors linearly equivalent to 2Θ). For $a \in A$, the divisor $\kappa(a) := (\Theta + a) + (\Theta - a)$ belongs to this linear system; the map $\kappa : A \to |2\Theta|$ embeds the *Kummer variety* $\mathrm{Km}(A) := A/i$ into $|2\Theta|$, where i is the involution $a \mapsto -a$ of A. One of the approaches to the Schottky problem characterizes Jacobians by the existence of trisecants to their Kummer variety in $|2\Theta|$.

I was studying these questions when I discovered that Narasimhan and Ramanan had found a remarkable connection between $|2\Theta|$ and rank 2 vector bundles. Let $\mathcal{M}$ be the moduli space of semi-stable, rank 2 vector bundles on C with trivial determinant. Given $E \in \mathcal{M}$, the locus of line bundles $L \in J$ such that $E \otimes L$ admits a nonzero section is an element $\theta(E)$ of $|2\Theta|$; we thus get a map $\theta : \mathcal{M} \to |2\Theta|$, which maps the singular locus of $\mathcal{M}$ exactly onto the Kummer variety. In the beautiful paper [N-R], Narasimhan and Ramanan work out completely the genus 3 case: θ is an embedding, and its image turns out to be the unique quartic hypersurface in $|2\Theta| \cong \mathbf{P}^7$ singular along $\mathrm{Km}(J)$. This hypersurface had been discovered by Coble long ago through algebraic manipulations; its geometric interpretation via vector bundles was entirely new.

This result inspired me to study the map $\theta : \mathcal{M} \to |2\Theta|$ in higher genus [B]. I noticed that there is a kind of duality between J and $\mathcal{M}$: given $L \in J$, one defines a divisor Θ_L on $\mathcal{M}$ as the locus of vector bundles E such that $E \otimes L$ has a nonzero section; the line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{M}}(\Theta_L)$ does not depend on L, and there is a natural isomorphism $|\mathcal{L}|^* \xrightarrow{\sim} |2\Theta|$ which identifies θ with the map $\varphi_{\mathcal{L}}$ defined by the global sections of $\mathcal{L}$.

For vector bundles of rank $r \geq 3$, the analogous statements make sense (with a rational map $\mathcal{M} \dashrightarrow |r\Theta|$), but the proof in rank 2 is not directly adaptable. After a few months I found a way to do it. Shortly after I met Narasimhan and Ramanan at the AMS Summer conference on theta functions (1987), and we realized we had had the same ideas – using the Hitchin fibration and the notion of very stable vector bundle introduced by Drinfeld. So we decided to write the joint paper [B-N-R]. Narasimhan invited me to the Tata Institute; I spent one month there in 1988. By that time we had essentially finished the paper, but I had many lively discussions with Narasimhan, both mathematical and non-mathematical – he was a very cultured man, with interesting points of view on a wide range of subjects.

The key ingedient in [B-N-R] was the computation of the dimension of the space of global sections $\Gamma(\mathcal{L})$ – often called "generalized theta functions". At about that time, word spread among mathematicians that physicists had a formula for the dimension of $\Gamma(\mathcal{L}^{\otimes k})$ for all k, in a much more general setting including for instance moduli spaces of G-bundles for all semi-simple groups G. This *Verlinde formula* soon became a challenge for algebraic geometers, and half a dozen proofs appeared in the following years, including one by Narasimhan with Kumar and Ramanathan [K-N-R] and one by Laszlo and myself [B-L]. Our (independent) proofs were actually quite close: both papers used infinite-dimensional algebraic geometry, with the language of infinite-dimensional manifolds in [K-N-R] and of algebraic stacks in [B-L].

I had no other opportunity to collaborate with Narasimhan. We met briefly in a few conferences; on one ocasion he came to Nice, we had a nice lunch with a very pleasant conversation. Narasimhan liked good food, art, and music. He was a great mathematician and a colleague of high human quality.

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