MAXIMAL VARIATION OF LINEAR SYSTEMS

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ABSTRACT. Let X be a smooth projective variety over $\mathbb{C}$, and L a line bundle on X. We say that the linear system $|L|$ has *maximal variation* if its elements have the maximum number $\dim|L|$ of moduli. We discuss some cases where this situation is expected: hypersurfaces, double coverings of $\mathbb{P}^n$, K3 and hyperkähler manifolds, and abelian varieties.

INTRODUCTION

Let X be a smooth projective variety over $\mathbb{C}$, and let L be a line bundle on X. We assume that a general hypersurface S in the linear system $|L|$ is smooth and connected. We say that $|L|$ has *maximal variation* if there are only finitely many S' in $|L|$ isomorphic to S. If there is a reasonable moduli space $\mathscr{M}$ for S (this will be the case for most of our examples), this is equivalent to say that the natural rational map $|L| \dashrightarrow \mathscr{M}$ is generically finite, or equivalently that its image has dimension $\dim|L|$.

There is a simple differential criterion for this to happen, namely $H^0(T_{X|S}) = 0$ (§1). However this turns out to be surprisingly difficult to check. We will discuss some cases where maximal variation is expected: hypersurfaces (§2), double coverings of $\mathbb{P}^n$ (§3), K3 surfaces and hyperkähler manifolds (§4), and abelian varieties (§5). We show indeed that the variation is maximal in the last three cases, while we have only partial results in the two first cases. Finally we discuss some more examples and counter-examples (§6 and 7).

1. A DIFFERENTIAL CRITERION

Proposition 1. 1) *If* $H^0(S, T_{X|S}) = 0$, $|L|$ *has maximal variation.*

2) *If* $H^1(X, \mathcal{O}_X) = 0$ *and* $H^0(S, T_S) = 0$, *the converse holds.*

Proof : Let S be a general element of $|L|$, and let $k : \mathscr{S} \to B$ be a Kuranishi family for S. There is an open neighborhood U of $[S]$ in $|L|$ and a map $\mu : U \to B$ such that the pull back of k by μ is the family of hypersurfaces parameterized by U. $|L|$ has maximal variation if and only if μ is a local immersion at $[S]$, that is, the tangent map $T\mu$ is injective.

The exact sequence $0\to T_S\to T_{X|S}\to L_{|S}\to 0$ gives a cohomology exact sequence

$$0\to H^0(T_S)\to H^0(T_{X|S})\to H^0(L_{|S})\xrightarrow{\partial} H^1(T_S)\,.$$

Let s be a section of L with zero locus S. The exact sequence $0\to\mathcal{O}_X\xrightarrow{s} L\to L_{|S}\to 0$ gives a cohomology exact sequence

$$0\to H^0(L)/(\mathbb{C}\cdot s)\xrightarrow{r} H^0(L_{|S})\to H^1(\mathcal{O}_X)\,.$$

The tangent space to $|L|$ at S is $H^0(L)/(\mathbb{C}\cdot s)$, and the tangent space to B at $\mu(S)$ is $H^1(T_S)$; by [K-S, (12.4)], the tangent map to μ at $[S]$ is the composition

$$T\mu : H^0(L)/(\mathbb{C}\cdot s)\xrightarrow{r} H^0(L_{|S})\xrightarrow{\partial} H^1(T_S)\,.$$

If $H^0(T_{X|S})=0$, ∂ is injective; since r is injective, $T\mu$ is injective. Conversely if $T\mu$ is injective, under the hypotheses of 2) r is an isomorphism, so ∂ is injective and $H^0(T_{X|S})=0$ since $H^0(T_S)=0$. ■

2. Hypersurfaces

Let $X\subset\mathbb{P}^n$ be a smooth hypersurface of degree $d\geq 3$, defined by an equation $F=0$. We assume $n\geq 3$, and $d\geq 4$ if $n=3$. The *Jacobian ring* of X is the graded artinian ring $R:=\mathbb{C}[x_0,\ldots,x_n]/(F'_0,\ldots,F'_n)$, where we write $F'_i=\dfrac{\partial F}{\partial x_i}\cdot$

Proposition 2. *The linear system $|\mathcal{O}_X(1)|$ has maximal variation if and only if the multiplication map $\times\ell : R_{d-1}\to R_d$ is injective for ℓ general in R_1.*

The property that $\times\ell : R_{p-1}\to R_p$ is of maximal rank for ℓ general in R_1 and for all p is called the *weak Lefschetz property*; it is conjectured that it holds for all rings of the form $\mathbb{C}[x_0,\ldots,x_n]/(P_0,\ldots,P_n)$, with $P_0,\ldots,P_n$ homogeneous with no common zeroes in $\mathbb{P}^n$ (see for instance [J-M]). This would imply that $|\mathcal{O}_X(1)|$ has always maximal variation. We have only a weaker result:

Corollary 1. *$|\mathcal{O}_X(1)|$ has maximal variation in the following cases:*

1) $d\geq n+2$;
2) X *is a surface in* $\mathbb{P}^3$ *(with* $d\geq 4$*);*
3) X *is a cubic threefold;*
4) X *is general* (among hypersurfaces of degree d in $\mathbb{P}^n$).

Proof : 1) is proved in [B-M], improving by 1 the bound in [P-R-T]. This implies 2) for $d\geq 5$; for $d=4$ and $n=3$, the weak Lefschetz property is proved in [B-M-M-N].

3) is proved in [B2], as a consequence of [A-R].

4) The Fermat hypersurface $\sum X_i^d = 0$ in $\mathbb{P}^n$ satisfies the weak Lefschetz property (in all degrees), because its Jacobian ring is isomorphic to the cohomology ring $H^*((\mathbb{P}^{d-2})^n, \mathbb{C})$. Therefore the same holds for X in a Zariski open subset of $|\mathcal{O}_{\mathbb{P}^n}(d)|$. ■

Proof of the Proposition : We essentially repeat the proof of [B2], with one more detail for the case $n = 3$.

Let S be a smooth hyperplane section of X; we choose the coordinates so that it is defined by $x_0 = 0$. From the exact sequence

$$0 \to T_{X|S} \to T_{\mathbb{P}^n|S} \to \mathcal{O}_S(d) \to 0$$

we see that $H^0(T_{X|S})$ is the kernel of the homomorphism $\varphi : H^0(T_{\mathbb{P}^n|S}) \to H^0(\mathcal{O}_S(d))$.

From the commutative diagram of Euler exact sequences

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n}) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}) & \longrightarrow & H^0(T_{\mathbb{P}^n}) & \longrightarrow & 0 & \\
 & \downarrow \wr & & \downarrow & & \downarrow & & & \\
0 \longrightarrow & H^0(\mathcal{O}_S) & \longrightarrow & H^0(\mathcal{O}_S(1)^{n+1}) & \longrightarrow & H^0(T_{\mathbb{P}^n|S}) & \longrightarrow & H^1(\mathcal{O}_S) & \xrightarrow{u} H^1(\mathcal{O}_S(1)^{n+1})
\end{array}$$

we get an exact sequence

$$H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) \xrightarrow{r} H^0(S, T_{\mathbb{P}^n|S}) \to H^1(S, \mathcal{O}_S) \xrightarrow{u} H^1(S, \mathcal{O}_S(1)^{n+1})\,.$$

If $n \geq 4$ we have $H^1(S, \mathcal{O}_S) = 0$; if $n = 3$ (so that S is a curve), the homomorphism u is the transpose of the map $H^0(S, \mathcal{O}_S(d-4))^4 \xrightarrow{(x_0,\ldots,x_3)} H^0(S, \mathcal{O}_S(d-3))$, which is surjective since $d \geq 4$; thus u is injective. In each case we conclude that the restriction map r is surjective. It follows that $H^0(S, T_{\mathbb{P}^n|S})$ is generated by the elements $x_i \dfrac{\partial}{\partial x_j}$ for $i \geq 1$, $j \geq 0$, with the Euler relation $\sum_{i\geq 1} x_i \dfrac{\partial}{\partial x_i} = 0$.

Assume that $\times x_0 : R_{d-1} \to R_d$ is injective. Let $V \in H^0(S, T_{\mathbb{P}^n|S})$; we can write $V = \sum_i L_i \dfrac{\partial}{\partial x_i}$, where the L_i are linear forms in $x_1, \ldots, x_n$. The homomorphism $\varphi : H^0(T_{\mathbb{P}^n|S}) \to H^0(\mathcal{O}_S(d))$ maps V to $\sum L_i F'_i$. If $\varphi(V) = 0$, there exists $G \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(d-1))$ and $a \in \mathbb{C}$ such that $\sum L_i F'_i = x_0 G + aF$. Since $dF = \sum x_i F'_i$, it follows that $x_0 G$ is zero in R_d. By hypothesis this implies $G = 0$ in R_{d-1}, that is, $G = \sum a_i F'_i$ for some constants a_i. Therefore

$$\sum_{i=0}^{n} (L_i - a_i x_0 - b x_i) F'_i = 0 \qquad \text{with } b := a/d\,.$$

Since $(F'_0, \ldots, F'_n)$ is a regular sequence in $\mathbb{C}[x_0, \ldots, x_n]$, this implies $L_i - a_i x_0 - b x_i = 0$ for each i. Since x_0 does not appear in L_i, we get :

- For $i \geq 1$, $a_i = 0$ and $L_i = b x_i$; • for $i = 0$, $L_0 = 0$.

Hence $V = b\sum_{i\geq 1} x_i \dfrac{\partial}{\partial x_i} = 0$.

Conversely, suppose $H^0(S, T_{\mathbb{P}^n|S}) = 0$. Let $G \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(d-1))$ such that $x_0G = 0$ in R_d; there exist linear forms L_i such that $x_0G = \sum L_iF'_i$. Replacing G by $G - \sum a_iF'_i$, where a_i is the coefficient of x_0 in L_i, we can assume that the L_i are linear forms in $x_1, \ldots, x_n$. Then the vector field $V = \sum L_i \dfrac{\partial}{\partial x_i}$ satisfies $\varphi(V) = 0$, hence $V = 0$. This means that there exists $c \in \mathbb{C}$ such that $L_i = cx_i$ for $i > 0$ and $L_0 = 0$. Then $x_0G = c\sum_{i>0} x_iF'_i = c(dF - x_0F'_0)$, hence $cdF = x_0(G + cF'_0)$. Since F is irreducible, this implies $c = 0$, hence $G = 0$. ■

We now consider the linear system $|\mathcal{O}_X(e)|$ for $e \geq 2$.

Proposition 3. *$|\mathcal{O}_X(e)|$ has maximal variation in the following cases :*
1) *$|\mathcal{O}_X(1)|$ has maximal variation;*
2) *$e \geq d$.*

(In particular, each of the conditions 1) to 3) of Corollary 1 implies that $|\mathcal{O}_X(e)|$ has maximal variation for all $e \geq 1$).

Proof : The proof of Proposition 2 works identically for the linear system $|\mathcal{O}_X(e)|$, replacing x_0 by a general form h in $H^0(\mathcal{O}_{\mathbb{P}^n}(e))$. We find that $|\mathcal{O}_X(e)|$ has maximal variation if $\times h : R_{d-e} \to R_d$ is injective for h general in R_e.

1) Let ℓ be a general element of R_1. If $|\mathcal{O}_X(1)|$ has maximal variation, $\times\ell : R_{d-1} \to R_d$ is injective (Proposition 2). Then $\times\ell : R_{p-1} \to R_p$ is injective for $p \leq d$ (if $\ell x = 0$ for $x \in R_{p-1}$, we have $x \cdot R_{d-p} = 0$, hence $x = 0$). Therefore $\times\ell^e : R_{d-e} \to R_d$ is injective, and so is $\times h$ for h general in R_e.

2) If $e > d$, we have $R_{d-e} = 0$ so there is nothing to prove. If $e = d$, we just observe that $R_d \neq 0$, since the socle of R is in degree $(n+1)(d-2) \geq d$. ■

3. Double coverings of $\mathbb{P}^n$

Let $\pi : X \to \mathbb{P}^n$ a double covering, branched along a smooth hypersurface $B \subset \mathbb{P}^n$ of (even) degree d. We assume $n \geq 2$, $d \geq 4$, and $d \geq 6$ if $n = 2$. Let $F = 0$ be an equation of B, and let $R = \mathbb{C}[x_0, \ldots, x_n]/(F'_0, \ldots, F'_n)$ be the Jacobian ring of F.

Proposition 4. *The linear system $|\pi^*\mathcal{O}_{\mathbb{P}}(1)|$ has maximal variation if and only if the multiplication map $\times\ell : R_{d-1} \to R_d$ is injective for ℓ general in R_1.*

Using again [B-M], we get:

Corollary 2. *If $d \geq n+2$ (and $d \geq 6$ if $n = 2$), $|\pi^*\mathcal{O}_{\mathbb{P}}(1)|$ has maximal variation.*

Proof of the Proposition : The proof is inspired by [D-H, Proposition 5.2], which treats the case $n = 2$, $d = 6$.

Put $\delta := d/2$, and $\mathbb{P} = \mathbb{P}^n$. We have $\pi_*\Omega^1_X = \Omega^1_{\mathbb{P}} \oplus \Omega^1_{\mathbb{P}}\langle B\rangle(-\delta)$ [E-V, Lemma 3.16 (d)]. Using Grothendieck duality this gives

$$\pi_* T_X = T_{\mathbb{P}}(-\delta) \oplus T_{\mathbb{P}}\langle B\rangle$$

where $T_{\mathbb{P}}\langle B\rangle$ is the sheaf of vector fields on $\mathbb{P}$ tangent to B.

Let H be a general hyperplane in $\mathbb{P}$, and $S := \pi^* H$. We have

$$\pi_* T_{X|S} = (\pi_* T_X)_{|H} = T_{\mathbb{P}}(-\delta)_{|H} \oplus T_{\mathbb{P}}\langle B\rangle_{|H}\ ,$$

hence the vanishing of $H^0(T_{X|S})$ is equivalent to $H^0(T_{\mathbb{P}}(-\delta)_{|H}) = H^0(T_{\mathbb{P}}\langle B\rangle_{|H}) = 0$.

1) Consider the Euler exact sequence restricted to H:

$$0 \to \mathcal{O}_H(-\delta) \to \mathcal{O}_H(-\delta+1)^{n+1} \to T_{\mathbb{P}}(-\delta)_{|H} \to 0\,.$$

If $n \geq 3$, we have $H^1(\mathcal{O}_H(-\delta)) = 0$, hence $H^0(T_{\mathbb{P}}(-\delta)_{|H}) = 0$. If $n = 2$ (so $H \cong \mathbb{P}^1$), the homomorphism $H^1(\mathcal{O}_H(-\delta)) \to H^1(\mathcal{O}_H(-\delta+1)^3)$ is the transpose of the map $H^0(\mathcal{O}_H(\delta-3)^3) \xrightarrow{(x_0,x_1,x_2)} H^0(\mathcal{O}_H(\delta-2))$, which is surjective since $\delta \geq 3$; hence again $H^0(T_{\mathbb{P}}(-\delta)_{|H}) = 0$.

2) We have an exact sequence

$$0 \to T_{\mathbb{P}}\langle B\rangle \to \mathcal{O}_{\mathbb{P}}(1)^{n+1} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}}(d) \to 0\ , \quad \text{where } \varphi(L_0,\ldots,L_n) = \sum L_i F'_i\,.$$

Therefore we have $H^0(T_{\mathbb{P}}\langle B\rangle_{|H}) = 0$ if and only if $\varphi_{|H} : H^0(\mathcal{O}_H(1)^{n+1}) \to H^0(\mathcal{O}_H(d))$ is injective. Choose the coordinates so that H is defined by $x_0 = 0$. Let $L_0,\ldots,L_n$ be linear forms in $x_1,\ldots,x_n$ such that $\varphi_{|H}(L_0,\ldots,L_n) = 0$. This means that $\sum L_i F'_i = x_0 G$ for some $G \in H^0(\mathcal{O}_{\mathbb{P}}(d-1))$. If $\times x_0 : R_{d-1} \to R_d$ is injective, we get $G = \sum a_i F'_i$ for some scalars a_i, thus $\sum (L_i - a_i x_0) F'_i = 0$. As before this implies $L_i = a_i x_0$, and since x_0 does not appear in L_i, $L_i = 0$.

Conversely, suppose that $\varphi_{|H}$ is injective. Let $G \in R_{d-1}$ such that $x_0 G = 0$ in R_d: there exist linear forms $L_0,\ldots,L_n$ such that $x_0 G = \sum L_i F'_i$. This implies that the restrictions $\bar{L}_i$ of L_i to H satisfy $\varphi_{|H}(\bar{L}_0,\ldots,\bar{L}_n) = 0$, hence $\bar{L}_i = 0$ for all i. Therefore $L_i = b_i x_0$ for some scalars b_i, so that $G = \sum b_i F'_i$ is zero in R_{d-1}. ■

Remarks.– 1) Propositions 4 and 2 have the following curious consequence. The hypersurface B embeds into X; under the hypotheses of Proposition 4, *the linear system* $|\pi^*\mathcal{O}_{\mathbb{P}}(1)|$ *has maximal variation if and only if its restriction to* B *has maximal variation*. I don't know a direct proof of this equivalence.

2) As in §2, the proof extends to the case of the linear system $|\pi^*\mathcal{O}_{\mathbb{P}}(e)|$ for $e > 1$: we get that $|\pi^*\mathcal{O}_{\mathbb{P}}(e)|$ *has maximal variation when* $|\pi^*\mathcal{O}_{\mathbb{P}}(1)|$ *does, or when* $e \geq d$.

4. K3 AND HYPERKÄHLER

Proposition 5. *Any ample linear system on a K3 surface has maximal variation.*

This is [Ba, Corollary 2] (which uses [D-H, Proposition 5.2]). ■

The case of higher-dimensional hyperkähler manifolds turns out to be easier:

Proposition 6. *Let X be a hyperkähler manifold of dimension > 2. For any ample line bundle L on X, the linear system $|L|$ has maximal variation.*

Proof : Let $S \in |L|$. The exact sequence $0 \to T_X \otimes L^{-1} \to T_X \to T_{X|S} \to 0$ gives rise to a cohomology exact sequence $H^0(T_X) \to H^0(T_{X|S}) \to H^1(T_X \otimes L^{-1})$. Since $T_X \cong \Omega^1_X$, we have $H^0(T_X) = 0$ and $H^1(T_X \otimes L^{-1}) = 0$ by the Akizuki-Nakano vanishing theorem, hence $H^0(T_{X|S}) = 0$. ■

5. ABELIAN VARIETIES

Proposition 7. *Let X be an abelian variety, and L an ample line bundle on X. The linear system $|L|$ has maximal variation.*

Proof : As in Proposition 1, we need to prove that the map $T\mu : H^0(L)/(\mathbb{C}\cdot s) \to H^1(T_S)$ is injective. Let s be a section of L with zero locus S. The exact sequences

$$0 \to T_S \to T_{X|S} \to L_{|S} \to 0 \qquad \text{and} \qquad 0 \to \mathcal{O}_X \xrightarrow{s} L \to L_{|S} \to 0$$

give rise to a diagram

$$\begin{array}{ccccc} & & H^0(L)/(\mathbb{C}\cdot s) & & \\ & & \downarrow r & \searrow^{T\mu} & \\ H^0(T_{X|S}) & \xrightarrow{\varphi} & H^0(L_{|S}) & \longrightarrow & H^1(T_S) \\ & \searrow_{u} & \downarrow \partial & & \\ & & H^1(\mathcal{O}_X) & & \end{array}$$

where the horizontal and vertical lines are exact.

We claim that u is an isomorphism; this implies that $\operatorname{Im}\varphi \cap \operatorname{Im} r = (0)$, hence that $T\mu$ is injective. Since the vector bundle T_X is trivial, it is equivalent to consider the composition $\tilde{u} : H^0(T_X) \xrightarrow{\sim} H^0(T_{X|S}) \xrightarrow{u} H^1(\mathcal{O}_X)$.

Lemma 1. $\tilde{u} : H^0(T_X) \to H^1(\mathcal{O}_X)$ *is the cup-product with the Chern class* $c_1(L) \in H^1(\Omega^1_X)$.

Proof : We can find a finite covering $(U_\alpha)_{\alpha\in I}$ of X such that $L_{|U_\alpha}$ is free, generated by a section e_α; on $U_\alpha \cap U_\beta$ we have $e_\alpha = g_{\alpha\beta}e_\beta$ for an invertible function $g_{\alpha\beta}$. We write $s_{|U_\alpha} = s_\alpha e_\alpha$ for a function s_α on U_α; then $s_\beta = g_{\alpha\beta}s_\alpha$.

Let $V \in H^0(T_X)$. Then $\varphi(V)$ is given on $U_\alpha \cap S$ by $(V\cdot s_\alpha)e_\alpha$. This expression lifts to U_α, and $u(V) = \partial\varphi(V)$ is the class of the Čech 1-cocycle $(\alpha,\beta)\mapsto c_{\alpha\beta}$ such that :

$$c_{\alpha\beta}\, s = (V\cdot s_\beta)e_\beta - (V\cdot s_\alpha)e_\alpha = (V\cdot g_{\alpha\beta})g_{\alpha\beta}^{-1}\, s \ \text{ in } U_\alpha\cap U_\beta\,,\ \text{ hence }\ c_{\alpha\beta} = \langle V, g_{\alpha\beta}^{-1}\, dg_{\alpha\beta}\rangle\,.$$

The class of the cocycle $(\alpha,\beta)\mapsto g_{\alpha\beta}^{-1}\, dg_{\alpha\beta}$ in $H^1(X,\Omega^1_X)$ is the Chern class $c_1(L)$, hence the lemma. ∎

Lemma 2. *$\tilde{u}$ is an isomorphism.*

Proof : We put $g = \dim X$, and choose an isomorphism $\Omega^g_X \xrightarrow{\sim} \mathcal{O}_X$. This gives an isomorphism $T_X \xrightarrow{\sim} \Omega^{g-1}_X$; for $\alpha \in H^0(\Omega^1_X)$, the map $\langle -,\alpha\rangle : T_X \to \mathcal{O}_X$ is identified with the exterior product $\wedge\alpha : \Omega^{g-1}_X \to \Omega^g_X$. Therefore $\tilde{u}$ is identified with the cup-product $\cup c_1(L) : H^0(\Omega^{g-1}_X) \to H^1(\Omega^g_X)$. By the Lefschetz theorem the cup-product $\cup c_1(L) : H^{g-1}(X,\mathbb{C}) \to H^{g+1}(X,\mathbb{C})$ is an isomorphism, and it induces an isomorphism from $H^{g-1,0}$ onto $H^{g,1}$, hence our assertion. ∎

6. Some more examples

We collect here some other examples of linear systems with maximal variation. We always assume that the linear systems we consider contain a smooth connected curve.

• If $H^0(X,T_X) = 0$ and L is ample, $|L^m|$ has maximal variation for $m \gg 0$. Indeed for any $S \in |L^m|$ there is an exact sequence $H^0(T_X) \to H^0(T_{X|S}) \to H^1(T_X \otimes L^{-m})$, and the right hand term vanishes for $m \gg 0$ since L is ample.

• If Ω^1_X is ample, every linear system $|L|$ on X has maximal variation. Indeed for any $S \in |L|$ the quotient $\Omega^1_{X|S}$ is ample, hence does not admit a nonzero map to $\mathcal{O}_S$.

For surfaces we have a more precise result:

Proposition 8. *Let X be a surface of general type such that Ω^1_X is generated by its global sections. Then any very ample linear system $|L|$ has maximal variation.*

Proof : For a general 1-form $\omega \in H^0(\Omega^1_X)$, the zero locus $Z(\omega)$ is finite, reduced and nonempty (since $c_2 > 0$). Let $p \in Z(\omega)$. Since L is very ample, we can find a smooth curve $C \in |L|$ such that $C \cap Z(\omega) = \{p\}$. We have an exact sequence

$$0 \to K^{-1}_{X|C}(p) \to T_{X|C} \xrightarrow{\ \omega\ } \mathcal{O}_C(-p) \to 0\,;$$

it suffices to prove $\deg(K^{-1}_{X|C}(p)) < 0$, that is, $K_X\cdot C > 1$.

We observe that X is a minimal surface: if R is a smooth curve in X, Ω^1_R is a quotient of Ω^1_X, hence it must be globally generated, which implies $g(R)\geq 1$. Then by the index theorem we have $(K_X\cdot C)^2 \geq K_X^2\cdot C^2 > 1$, hence $K_X\cdot C > 1$ as required. ∎

Proposition 9. *Let X be a surface of general type such that $\mathrm{Pic}(X)=\mathbb{Z}\cdot[K_X]$ and that the Chern classes of X satisfy $c_1^2>c_2$. The linear systems $|rK_X|$, for $r\geq 1$, have maximal variation.*

Proof : The proof is borrowed from [B1]. Let C be a smooth curve in $|rK_X|$. Suppose that $T_{X|C}$ has a nonzero section; this gives a surjective homomorphism $\Omega^1_X\rightarrow \mathcal{O}_C(-E)$, where E is an effective divisor. The kernel F of this homomorphism is a rank 2 vector bundle with

$$c_1(F)=(r-1)c_1\ ,\quad c_2(F)=c_2-rc_1^2-\deg(E)\,,$$

hence
$$c_1^2(F)-4c_2(F)\geq (r+1)^2c_1^2-4c_2\geq 4(c_1^2-c_2)>0\,.$$

By [Bo, §10] F is unstable: there is an exact sequence

$$0\rightarrow K_X^a\rightarrow F\rightarrow \mathscr{I}_Z K_X^b\rightarrow 0$$

with $a\geq b$, and Z a finite subscheme of X.

This gives $c_1(F)=-(a+b)c_1$, $c_2(F)=\deg(Z)+abc_1^2$, hence

$$c_1^2(F)-4c_2(F)=(a-b)^2c_1^2-4\deg(Z)\,.$$

Comparing with the previous expression gives

$$(a-b)^2c_1^2\geq c_1(F)^2-4c_2(F)\geq (r+1)^2c_1^2-4c_2>(r^2+2r-3)c_1^2\geq (r-1)^2c_1^2$$

Since $a\geq b$, we conclude $a-b>r-1$, hence $a\geq 1$. This gives a nonzero homomorphism $K_X\rightarrow\Omega^1_X$, hence a nonzero section of T_X, which is impossible. ∎

Proposition 10. *Let $X=X_1\times X_2$, where X_1 and X_2 are smooth projective varieties of dimension ≥ 2, and $H^0(X_i,T_{X_i})=0$. Then any ample line bundle on X has maximal variation.*

Proof : Let p_i be the projection of X onto X_i. The restriction $\bar{p}_i:S\rightarrow X_i$ of p_i to an ample divisor $S\subset X$ is surjective with connected fibers: indeed for $x\in X_i$, $\bar{p}_i^{-1}(x)=S\cap(\{x_i\}\times X_{1-i})$ is an ample divisor in $\{x_i\}\times X_{1-i}$, hence nonempty and connected. Therefore we have $(\bar{p}_i)_*\mathcal{O}_S=\mathcal{O}_{X_i}$, and

$$H^0((p_i^*T_{X_i})_{|S})=H^0(\bar{p}_i^*T_{X_i})=H^0(T_{X_i})=0\ ,\ \text{ hence }\ H^0(T_{X|S})=0\,.\quad ∎$$

7. Some counter-examples

• Suppose that a connected linear group G acts non trivially on X. Then G acts trivially on $\mathrm{Pic}(X)$, hence acts on each linear system $|L|$ on X. If this action is nontrivial, $|L|$ does not have maximal variation. This is the case for instance if L is very ample, since one can find an element of $|L|$ passing through a point of X and not containing its orbit.

• Let C be a smooth projective curve, E a rank 2 very ample vector bundle on C (that is, $\mathcal{O}_{\mathbb{P}(E)}(1)$ is very ample), $X = \mathbb{P}(E)$, $L = \mathcal{O}_{\mathbb{P}(E)}(1)$. A general element of $|L|$ is isomorphic to C, so $|L|$ has no moduli.

• There are varieties of general type admitting a base point free pencil with no moduli. Take for instance the Fermat hypersurface $F : \sum_{i=0}^{n} x_i^d = 0$ in $\mathbb{P}^n$. Consider the rational map $p : F \dashrightarrow \mathbb{P}^1$ given by $p(x_0, \ldots, x_n) = (x_0, x_1)$. Let $\hat{F} \to F$ be the blowing up of the codimension 2 subvariety of F given by $x_0 = x_1 = 0$. Then p extends to a morphism $\hat{p} : \hat{F} \to \mathbb{P}^1$ whose general fiber is isomorphic to the Fermat hypersurface of degree d in $\mathbb{P}^{n-1}$. The pencil $|\hat{p}^*\mathcal{O}_{\mathbb{P}^1}(1)|$ has no moduli.

These examples lead to the following :

Question.– *If X is not uniruled, does every ample linear system $|L|$ have maximal variation*?

It is perhaps overoptimistic to expect a positive answer, but it seems not obvious to find a counter-example.

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