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Symmetric tensors on the intersection of two quadrics and Lagrangian fibration

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Abstract

Let X be an n-dimensional (smooth) intersection of two quadrics, and let T^*X be its cotangent bundle. We show that the algebra of symmetric tensors on X is a polynomial algebra in n variables. The corresponding map $\Phi : T^*X \to \mathbb{C}^n$ is a Lagrangian fibration, which admits an explicit geometric description; its general fiber is a Zariski open subset of an abelian variety, which is a quotient of a hyperelliptic Jacobian by a 2-torsion subgroup. In dimension 3, Φ is the Hitchin fibration of the moduli space of rank 2 bundles with fixed determinant on a curve of genus 2.

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1. Introduction

Let $X \subset \mathbb{P}^{n+2}_{\mathbb{C}}$ be a smooth n-dimensional complete intersection of two quadrics, with $n \geq 2$, and let T^*X be its cotangent bundle. The $\mathbb{C}$-algebra $H^0(T^*X, \mathscr{O}_{T^*X})$ is canonically isomorphic to the algebra of symmetric tensors $H^0(X, \mathsf{S}^\bullet T_X)$. Recall that T^*X carries a canonical symplectic structure. Our main result is the following theorem:

Theorem 1.1.
(a) The vector space $W := H^0(X, \mathsf{S}^2 T_X)$ has dimension n, and the natural map $\mathsf{S}^\bullet W \to H^0(X, \mathsf{S}^\bullet T_X)$ is an isomorphism.
(b) The corresponding map, $\Phi : T^*X \to W^* \cong \mathbb{C}^n$, is a Lagrangian fibration.
(c) When X is general, the general fiber of Φ is of the form $A \smallsetminus Z$, where A is an abelian variety and codim $Z \geq 2$.

We will give a precise geometric description of the map Φ and of the abelian variety A in Sections 4 and 5.

1.1 Comments

(1) For $n = 2$, (a) follows from Theorem 5.1 in [DOL19], while (b) and (c) are proved in [KL22]. The proof is based on the isomorphism $T_X \cong \Omega^1_X(1)$. The theorem also follows from the fact that X is a moduli space for parabolic rank 2 bundles on $\mathbb{P}^1$ [Cas15], so $\Phi : T^*X \to \mathbb{C}^2$ is identified to the *Hitchin fibration* (see [BHK10]).

For $n = 3$, X is isomorphic to the moduli space of vector bundles of rank 2 and fixed determinant of odd degree [New68]; again, the theorem follows from the properties of the Hitchin fibration (see Section 2). It would be interesting to have a modular interpretation of Φ for $n \geq 4$. Note that the Hitchin map for G-bundles is homogeneous quadratic only when G is SL(2) or a product of copies of SL(2), so this limits the possibilities of using it.

(2) The map Φ is an example of an algebraically completely integrable system; see Remark 5.1. There is an abundant literature on such systems; see, for instance, [A96].

A classical example, the geodesic flow on an ellipsoid, is discussed in detail in [K80]. The corresponding Lagrangian fibration takes place on the cotangent bundle of *one* quadric; it is not related to our Φ. However, some of the tools we use in Sections 4 and 5, in particular the variety $\mathscr{X}$ and the family of planes $\mathscr{F}$, appear already in [K80] (with a different purpose).

(3) Such a situation is rather exceptional: Most varieties do not admit nonzero symmetric tensors (for instance, hypersurfaces of degree ≥ 3 [HLS22]); when they do, even for varieties as simple

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as quadrics, the algebra of symmetric tensors is fairly complicated (see, for instance, [BLi24]). We do not have a conceptual explanation for the particularly simple behavior in our case.

(4) For $n = 2$ or 3, the generality assumption on X in (c) is unnecessary. It seems likely that this is the case for all n, but our method does not allow us to make that conclusion.

1.2 Strategy

We will first treat the case $n = 3$, which is independent of the rest of this article (Section 2). For the general case, we will develop two different approaches. In the first one we exhibit a natural n-dimensional subspace $W \subset H^0(X, \mathsf{S}^2 T_X)$, from which we deduce a map $\Phi : T^*X \to W^* \cong \mathbb{C}^n$ (Section 3). We then show that Φ has the required properties, which implies (a), (b) and (c) for general X (5.1). In the second approach (Section 7), we directly prove (a) for all smooth X , by realizing X as a double covering of a quadric.

1.3 Notations

Throughout this article, X will be a smooth complete intersection of two quadrics in $\mathbb{P}^{n+2}$, with $n \geq 2$. We denote by T^*X its cotangent bundle and by $\mathbb{P}T^*X$ its projectivisation in the geometric sense (not in the Grothendieck sense). If V is a vector space, we denote by $\mathbb{P}(V)$ the associated projective space $V \smallsetminus \{0\}/\mathbb{C}^*$ parametrising 1-dimensional subspaces of V.

2. The case n=3

In this section we show how our general results can be obtained in the case $n = 3$ by interpreting X as a moduli space.

As in Section 4.1 below, we associate to X a genus 2 curve C such that the variety of lines in X is isomorphic to JC. Let us fix a line bundle N on C of degree 1; then X is isomorphic to the moduli space $\mathscr{M}$ of rank 2 stable vector bundles on C with determinant N [New68]. The cotangent bundle $T^*\mathscr{M}$ is naturally identified with the moduli space of *Higgs bundles*; that is, pairs (E, u) with $E \in \mathscr{M}$ and $u : E \to E \otimes K_C$ a homomorphism with $\mathrm{Tr}\, u = 0$. The *Hitchin map* $\Phi : T^*\mathscr{M} \to H^0(K_C^2)$ associates to a pair (E, u) the section $\det u$ of K_C^2. It is a Lagrangian fibration [Hit87].

Let $\omega \in H^0(K_C^2)$. *We assume in what follows that ω vanishes at 4 distinct points.* Let C_ω be the curve in the cotangent bundle T^*C defined by $z^2 = \omega$. The projection $\pi : C_\omega \to C$ is a double covering branched along $\mathrm{div}(\omega)$, and C_ω is a smooth curve of genus 5. Let P be the Prym variety associated to π, that is, the kernel of the norm map $\mathrm{Nm} : JC_\omega \to JC$; it is a 3-dimensional abelian variety.

Proposition 2.1. *The fibre $\Phi^{-1}(\omega)$ is isomorphic to the complement of a curve in P.*

Proof. Recall that the map $L \mapsto \pi_* L$ establishes a bijective correspondence between line bundles on C_ω and rank 2 vector bundles E on C endowed with a homomorphism $u : E \to E \otimes K_C$ such that $u^2 = \omega \cdot \mathrm{Id}_E$ or, equivalently, $\mathrm{Tr}\, u = 0$ and $\det u = \omega$ (see, for instance, [BNR89]). To get (E, u) in $\Phi^{-1}(\omega)$, we have to impose $\det E = N$ and E stable. Since $\det \pi_* L = \mathrm{Nm}(L) \otimes K_C^{-1}$, the first condition means that L belongs to the translate $P_N := \mathrm{Nm}^{-1}(K_C \otimes N)$ of P.

Then the vector bundle $\pi_* L$ is unstable if and only if it contains an invertible subsheaf M of degree 1; this is equivalent to saying that there is a nonzero map $\pi^* M \to L$; that is, $L = \pi^* M(p)$ for some point $p \in C_\omega$. The condition $L \in P_N$ means that $M^2(\pi(p)) = K_C \otimes N$, so

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M is determined by p up to the 2-torsion of JC. Thus the locus of line bundles $L \in P_N$ such that $\pi_* L$ is unstable is a curve. $\square$

Let $\rho : C \to \mathbb{P}^1$ be the canonical double covering, with $B \subset \mathbb{P}^1$ its branch locus. Since the homomorphism $\mathsf{S}^2 H^0(K_C) \to H^0(K_C^2)$ is surjective, the divisor of ω is of the form $\rho^*(p+q)$, for some $p, q \in \mathbb{P}^1$; by assumption, we have $p \neq q$ and $p, q \notin B$.

Proposition 2.2. *Let Γ be the double covering of $\mathbb{P}^1$ branched along $B \cup \{p, q\}$. There is an exact sequence*

$$0 \to \mathbb{Z}/2 \to J\Gamma \to P \to 0\,.$$

Proof. Let $\chi : \mathbb{P}^1 \to \mathbb{P}^1$ be the double covering branched along $\{p, q\}$. Since $\operatorname{div}(\omega) = \rho^*(p+q)$, there is a cartesian diagram of double coverings

$$\begin{array}{ccc} C_\omega & \xrightarrow{\ \xi\ } & \mathbb{P}^1 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \chi} \\ C & \xrightarrow[\ \rho\ ]{} & \mathbb{P}^1 \end{array}$$

which gives rise to two commuting involutions σ, τ of C_ω, exchanging the two sheets of π and ξ, respectively. The field of rational functions on C_ω is

$$\mathbb{C}(x, y, z) \ \text{ with } \ y^2 = f(x), z^2 = g(x),$$

where f and g are polynomials with $\operatorname{div} f = B$ and $\operatorname{div} g = \{p, q\}$. Then σ and τ change the sign of y and z, respectively.

The involution $\sigma\tau$ is fixed-point free, so the quotient $\Gamma := C_\omega / \langle \sigma\tau \rangle$ has genus 3; its field of functions is $\mathbb{C}(x, w)$, with $w = yz$ and $w^2 = f(x)g(x)$. We have again a cartesian square

$$\begin{array}{ccc} C_\omega & \xrightarrow{\ \varphi\ } & \Gamma \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \psi} \\ C & \xrightarrow[\ \rho\ ]{} & \mathbb{P}^1. \end{array}$$

Let $\alpha \in J\Gamma$. We have $\mathrm{Nm}_\pi \varphi^* \alpha = \rho^* \mathrm{Nm}_\psi \alpha = 0$; hence, φ^* maps $J\Gamma$ into $P \subset JC_\omega$. Since φ is étale, we have $\operatorname{Ker} \varphi^* = \mathbb{Z}/2$; since $\dim J\Gamma = \dim P = 3$, φ^* is surjective. $\square$

3. Definition of Φ

Let Y be a smooth degree d hypersurface in $\mathbb{P}^N$, defined by an equation $f = 0$. Recall that one associates to f a section h_f of $\mathsf{S}^2 \Omega^1_Y(d)$, the *hessian* or *second fundamental form* of f [GH79]: at a point y of Y, the intersection of Y with the tangent hyperplane H to Y at y is a hypersurface in H singular at y, and $h_f(y)$ is the degree 2 term in the Taylor expansion of $f_{|H}$ at y.

Now let $X \subset \mathbb{P}^{n+r}$ be a smooth complete intersection of r hypersurfaces of degree d; let

$$V \subset H^0(\mathbb{P}^{n+r}, \mathscr{O}_{\mathbb{P}}(d))$$

be the r-dimensional subspace of degree d polynomials vanishing on X. By restricting h_f, for $f \in V$, to X, we get a linear map

$$V\otimes\mathscr{O}_X\longrightarrow \mathsf{S}^2\Omega^1_X(d),$$

which gives at each point $x\in X$ a linear space of quadratic forms on the tangent space $T_x(X)$. Note that when $d=2$, the corresponding quadrics in $\mathbb{P}(T_x(X))$ can be viewed geometrically as follows: The projective space $\mathbb{P}(T_x(X))$ can be identified with the space of lines in $\mathbb{P}^{n+r}$ passing through x and tangent to X; then for each $q\in V$, the quadric defined by $h_q(x)$ parameterises the lines passing through x and contained in the quadric $\{q=0\}$.

Now we want to consider the 'inverse' of the quadratic form $h_f(x)$ on $T_x(X)$; that is, the form on $T^*_x(X)$ given in coordinates by the cofactor matrix. Intrinsically, each $f\in V$ gives a twisted symmetric morphism

$$h_f: T_X\longrightarrow \Omega^1_X(d),$$

which induces a twisted symmetric morphism on $(n-1)$-th exterior powers, namely,

$$\wedge^{n-1}h_f:\bigwedge^{n-1}T_X\longrightarrow\bigwedge^{n-1}\Omega^1_X((n-1)d)\,.$$

We now observe that $K_X=\mathscr{O}_X(-n-1-r+dr)$; hence

$$\bigwedge^{n-1}T_X\cong\Omega^1_X(n+1-r(d-1))\ \text{and}\quad \bigwedge^{n-1}\Omega^1_X\cong T_X(-n-1+r(d-1))\,,$$

so $\wedge^{n-1}h_f$ induces a symmetric morphism from $\Omega^1_X(n+1-r(d-1))$ to $T_X((n-1)d-n-1+r(d-1))$, hence provides a section

$$\wedge^{n-1}h_f\in H^0(X,\mathsf{S}^2T_X(d(n+2r-1)-2(n+r+1))).$$

Being locally given by the cofactor matrix, $\wedge^{n-1}h_f$ is homogeneous of degree $n-1$ in f. Hence, we have constructed a linear map

$$\alpha:\mathsf{S}^{n-1}V\longrightarrow H^0(X,\mathsf{S}^2T_X(d(n+2r-1)-2(n+r+1)))\quad\text{such that}\quad \alpha(f^{n-1})=\wedge^{n-1}h_f\,.$$

From now on, we restrict to the case $d=2$, $r=2$, so X is the complete intersection of two quadrics, and the previous construction gives a linear map

$$\alpha:\mathsf{S}^{n-1}V\longrightarrow H^0(X,\mathsf{S}^2T_X)\,.$$

Using the canonical isomorphism $H^0(T^*X,\mathscr{O}_{T^*X})=H^0(X,\mathsf{S}^\bullet T_X)$, we deduce from α a morphism

$$\Phi:T^*X\longrightarrow\mathsf{S}^{n-1}V^*\cong\mathbb{C}^n\,.$$

We have $\Phi(\lambda v)=\lambda^2\Phi(v)$ for $v\in T^*X$, $\lambda\in\mathbb{C}$, so Φ induces a rational map

$$\varphi:\mathbb{P}T^*X\dashrightarrow\mathbb{P}^{n-1},$$

whose indeterminacy locus Z is the image of $\Phi^{-1}(0)$.

Proposition 3.1.

(1) *α is injective.*
(2) *Φ is surjective.*
(3) *The image of Z by the structure map $p:\mathbb{P}T^*X\to X$ is a proper subvariety of X.*

Proof. Let x be a general point of X. We claim that the base locus in $\mathbb{P}(T_x(X))$ of the pencil of quadratic forms $\{h_q(x)\}_{q\in V}$ is smooth. Indeed, this locus can be viewed as the variety F_x of lines in X passing through x. Let F be the Fano variety of lines contained in X, and let

$$G\subset F\times X=\{(\ell,y)\mid y\in\ell\}\,.$$

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Then F and therefore G are smooth [Reid72, Theorem 2.6], hence F_x, which is the fibre above x of the projection $G \to X$, is smooth since x is general. It follows that in an appropriate system of coordinates $(k_1, \dots, k_n)$ of $T_x(X)$, the forms $\{h_q(x)\}$ can be written as

$$t\sum k_i^2 + \sum \alpha_i k_i^2 \quad \text{with } \alpha_i \text{ distinct in } \mathbb{C},\ t \in \mathbb{C}\,.$$

Then $\wedge^{n-1}h_q(x)$ is given by the diagonal matrix with entries $\beta_i := \prod_{j\neq i}(t+\alpha_j)$ $(i = 1, \dots, n)$. These polynomials in t are linearly independent; hence, they generate the space of quadratic forms on $T_x^*(X)$, which are diagonal in the basis (k_i). This linear system has dimension n, so α is injective; it has no base point, so φ induces a finite, surjective morphism $\mathbb{P}(T_x^*(X)) \to \mathbb{P}^{n-1}$. Thus, Φ is surjective, and $Z \cap \mathbb{P}(T_x^*(X)) = \varnothing$, which gives (2) and (3). ☐

We want to give a geometric construction of the rational map $\varphi : \mathbb{P}T^*X \dashrightarrow \mathbb{P}^{n-1}$. A point of $\mathbb{P}T^*X$ is a pair (x, H), where $x \in X$ and H is a hyperplane in $T_x(X)$. Restricting the pencil $\{h_q(x)\}_{q\in V}$ to H gives a pencil of quadrics on H, which for general(x, H) contains $n-1$ singular quadrics $q_1, \dots, q_{n-1}$. The subset $\{q_1, \dots, q_{n-1}\}$ of $\mathbb{P}(V)$ corresponds to a point $\varphi_{x,H}$ of $\mathbb{P}(\mathsf{S}^{n-1}V^*)$; namely, the hyperplane in $\mathsf{S}^{n-1}V$ spanned by $q_1^{n-1}, \dots, q_{n-1}^{n-1}$.

Proposition 3.2. $\varphi(x, H) = \varphi_{x,H}$.

Proof. We can assume that x is general. We have seen that the restriction of φ to $\mathbb{P}(T_x^*X)$ is the morphism given by the linear system of quadratic forms $W \cong \mathsf{S}^{n-1}V$ spanned by the forms $\wedge^{n-1}h_q(x)$, for $q \in V$; in other words, φ maps the point H of $\mathbb{P}(T_x^*(X))$ to the hyperplane of forms in W vanishing at H.

On the other hand, $\varphi_{x,H}$ is the hyperplane of $\mathsf{S}^{n-1}V$ spanned by the q^{n-1} for those $q \in V$ such that $h_q(x)_{|H}$ is singular; this condition is equivalent to saying that the form $\wedge^{n-1}h_q(x)$ on T_x^*X vanishes at H. Therefore, $\varphi_{x,H}$ is spanned by quadratic forms vanishing at H, hence coincides with $\varphi(x, H)$. ☐

Corollary 3.3. $\operatorname{codim} Z \geq 2$.

Proof. Suppose that Z contains a component Z_0 of codimension 1; since $p(Z) \neq X$, we have $Z_0 = p^{-1}(p(Z_0))$. We claim that this is impossible; in fact, Z cannot contain a fibre $p^{-1}(x)$. Indeed, its doing this would mean that for $q \in V$, the form $h_q(x)$ is singular along all hyperplanes $H \subset T_x(X)$; that is, $h_q(x)$ has rank $\leq n-2$. But the rank of $h_q(x)$ is the rank of the restriction of q to the projective tangent subspace to X at x. Restricting a quadratic form to a hyperplane lowers its rank by up to two. Since a general q in V has rank $n+3$, its restriction to a codimension 2 subspace has rank $\geq n-1$. ☐

4. Fibers of φ

In an appropriate system of coordinates $(x_0, \dots, x_{n+2})$, our variety X is defined by the equations $q_1 = q_2 = 0$, with

$$q_1 = \sum x_i^2 \quad q_2 = \sum \mu_i x_i^2 \qquad \text{, with } \mu_i \in \mathbb{C} \text{ distinct.}$$

Let $\Pi = \mathbb{P}(V)$ $(\cong \mathbb{P}^1)$ be the pencil of quadrics containing X. We choose a coordinate t on Π so that the quadrics of Π are given by $tq_1 - q_2 = 0$. Then the singular quadrics of Π correspond to the points $\mu_0, \dots, \mu_{n+2}$.

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The goal of this section is to describe the general fibre of the rational map $\varphi : \mathbb{P}T^*X \dashrightarrow \mathsf{S}^{n-1}\Pi$ $(\cong \mathbb{P}^{n-1})$. For $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in \mathsf{S}^{n-1}\Pi$, let $C_{\mu,\lambda}$ denote the hyperelliptic curve $y^2 = \prod(t-\mu_i)\prod(t-\lambda_j)$, of genus n. We will then prove the following:

Proposition 4.1. *For λ general in $\mathsf{S}^{n-1}\Pi$, the fibre $\varphi^{-1}(\lambda)$ is birational to the quotient of the Jacobian $JC_{\mu,\lambda}$ by the group $\Gamma := \{\pm 1_{JC}\} \times \Gamma^+$, where $\Gamma^+ \cong (\mathbb{Z}/2Z)^{n-2}$ is a group of translations by 2-torsion elements.*

4.1 Odd-dimensional intersection of 2 quadrics

We briefly recall here the results of Reid's thesis ([Reid72]; see also [DR76]). Let $Y \subset \mathbb{P}^{2g+1}$ be a smooth intersection of 2 quadrics, and let Ξ $(\cong \mathbb{P}^1)$ be the pencil of quadrics containing Y. Let $\Sigma \subset \Xi$ be the subset of $2g+2$ points corresponding to singular quadrics, and let C be the double covering of Ξ branched along Σ; this is a hyperelliptic curve of genus g. The intermediate Jacobian JY of Y is isomorphic to JC (as principally polarized abelian varieties). The variety F of $(g-1)$-planes contained in Y is also isomorphic to JC, but this isomorphism is not canonical.

In an appropriate system of coordinates, the equations of Y are of the form

$$\sum x_i^2 = \sum \alpha_i x_i^2 = 0, \qquad \text{with } \alpha_i \in \mathbb{C} \text{ distinct;}$$

then $\Sigma = \{\alpha_1, \dots, \alpha_{2g+2}\}$. The group $\Gamma := (\mathbb{Z}/2\mathbb{Z})^{2g+1}$ acts on Y (hence also on F) by changing the signs of the coordinates. Let $\Gamma^+ \subset \Gamma$ be the subgroup of elements that change an even number of coordinates. Choose an element $\gamma \in \Gamma \smallsetminus \Gamma^+$; there is an isomorphism $F \xrightarrow{\sim} JC$ such that γ corresponds to (-1_{JC}). Then the image of Γ^+ in $\mathrm{Aut}(JC)$ is the group T_2 of translations by 2-torsion elements of JC, and the image of Γ is $T_2 \times \{\pm 1_{JC}\}$ [DR76, Lemma 4.5].

4.2 An auxiliary construction

We consider the projective space $\mathbb{P}^{2n+1}$ equipped with the system of homogeneous coordinates

$$x_0, \dots, x_{n+2};\ y_1, \dots, y_{n-1}$$

and the affine space $\mathbb{A}^{n-1}$ equipped with the affine coordinates $\lambda_1, \dots, \lambda_{n-1}$. Let

$$\mathscr{X} \subset \mathbb{P}^{2n+1} \times \mathbb{A}^{n-1}$$

be the complete intersection of the two quadrics with equations

$$Q_1 = Q_2 = 0 \quad \text{with} \quad Q_1 = \sum_{i=0}^{n+2} x_i^2 + \sum_{j=1}^{n-1} y_j^2 \quad , \quad Q_2 = \sum_{i=0}^{n+2} \mu_i x_i^2 + \sum_{j=1}^{n-1} \lambda_j y_j^2 \,.$$

The second projection, $\mathscr{X} \to \mathbb{A}^{n-1}$, gives a family of complete intersections of two quadrics $\mathscr{X}_\lambda$ of dimension $2n-1$ parameterised by $\mathbb{A}^{n-1}$. Note that X is the intersection of $\mathscr{X}$ with the subspace $\mathbb{P}^{n+2} \subset \mathbb{P}^{2n+1}$ defined by $y_1 = \ldots = y_{n-1} = 0$.

Let $p : \mathscr{F} \to \mathbb{A}^{n-1}$ be the family of $(n-1)$-planes contained in the $\mathscr{X}_\lambda$; that is

$$\mathscr{F} = \{(P,\lambda) \mid \lambda \in \mathbb{A}^{n-1},\ P\ (n-1)\text{-plane} \subset \mathscr{X}_\lambda\}\,.$$

For λ general, the fibre $\mathscr{F}_\lambda$ is isomorphic to the Jacobian of the hyperelliptic curve $C_{\mu,\lambda}$ (4.1).

Let (P,λ) be a general point of $\mathscr{F}$. Then $P \cap \mathbb{P}^{n+2}$ is a point x of X. Let $\pi : \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^{n+2}$ be the projection $(x_i, y_j) \mapsto (x_i)$. Since the π_* differentials of Q_i and q_i coincide at x, the differential π_* maps $T_x(P) \subset T_x(\mathscr{X})$ into $T_x(X)$. Since P is general, $\pi_* T_x(P)$ is a hyperplane in $T_x(X)$; this will follow from the proof of Proposition 4.2, (1) below, where we explicitly construct pairs (P,λ) with this property.

Therefore, we have a rational map

$$\psi : \mathscr{F} \dashrightarrow \mathbb{P}T^*X \qquad (P,\lambda) \mapsto (x = P \cap \mathbb{P}^{n+2}\,,\ \pi_* T_x(P))\,.$$

The symmetric group $\mathfrak{S}_{n-1}$ acts on $\mathbb{P}^{2n+1}$ by permuting the y_j and acts on the group $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ by changing their signs; this gives an action of the semi-direct product $G := (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes \mathfrak{S}_{n-1}$. We make G act on $\mathbb{A}^{n-1}$ through its quotient $\mathfrak{S}_{n-1}$, by permutation of the λ_i. This induces an action of G on $\mathscr{X}$ and therefore on $\mathscr{F}$, which is compatible via p with the action on the base. The map ψ is invariant under this action; hence, it factors through the quotient $\mathscr{F}/G$. By passing to the quotient, we get a map $p^\sharp : \mathscr{F}/G \to \mathbb{A}^{n-1}/\mathfrak{S}_{n-1}$.

Proposition 4.2. (1) *ψ induces a birational map $\psi^\sharp : \mathscr{F}/G \dashrightarrow \mathbb{P}T^*X$.*

(2) *There is a commutative diagram*

$$\begin{array}{ccc} \mathscr{F}/G & \overset{\psi^\sharp}{\dashrightarrow} & \mathbb{P}T^*X \\ {\scriptstyle p^\sharp}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{A}^{n-1}/\mathfrak{S}_{n-1} & \underset{\sigma}{\xrightarrow{\sim}} & \mathbb{A}^{n-1} \subset \mathbb{P}^{n-1} \end{array}$$

where $p^\sharp$ is deduced from p and where *σ is the isomorphism given by symmetric functions.*

Proof. (1) Let $(x,H) \in \mathbb{P}T^*X$; we want to describe the pairs (P,λ) such that $P \cap \mathbb{P}^{n+2} = \{x\}$ and $\pi_* T_x(P) = H$. The latter condition says that via the decomposition

$$T_x(\mathbb{P}^{2n+1}) = T_x(\mathbb{P}^{n+2}) \oplus \operatorname{Ker} \pi_*\,,$$

$T_x(P)$ identifies with the graph of a linear map

$$\alpha : H \to \operatorname{Ker} \pi_*\,.$$

Using the basis $(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{n-1}})$ of $\operatorname{Ker}\pi_*$, we have $\alpha = (\alpha_1, \dots, \alpha_{n-1})$, where the α_i are linear forms on H. The condition $P \subset \mathscr{X}_\lambda$ implies that the hessians $h_{Q_1}(x)$ and $h_{Q_2}(x)$ vanish on $T_x(P)$, which gives

$$h_{q_1}(x)_{|H} = -\sum_i \alpha_i^2 \qquad h_{q_2}(x)_{|H} = -\sum_i \lambda_i \alpha_i^2\,. \tag{1}$$

This is a simultaneous diagonalisation of the quadratic forms $h_{q_1}(x)_{|H}$ and $h_{q_2}(x)_{|H}$; when they are in general position, this determines the λ_i up to permutation and the α_i up to sign and permutation, which proves (1).

(2) Let $(P,\lambda) \in \mathscr{F}$, and let $(x,H) := \psi(P,\lambda)$. According to Proposition 3.2, $\varphi(x,H)$ is given by the $(n-1)$-uple of quadrics $q \in \Pi$ such that the form $h_q(x)_{|H}$ is singular. Using $(\alpha_1, \dots, \alpha_{n-1})$ as coordinates on H, we see from (1) that this $(n-1)$-uple is given by $(\lambda_1, \dots, \lambda_{n-1})$, which proves (2). □

4.3 Proof of Proposition 4.1.

Let λ be a general element of $\mathbb{A}^{n-1}$. Let us denote by Γ the subgroup $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ of G. From Proposition 4.2 and the cartesian diagram

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$$\begin{array}{ccc} \mathscr{F}/\Gamma & \longrightarrow & \mathscr{F}/G \\ \big\downarrow p & & \big\downarrow p^\sharp \\ \mathbb{A}^{n-1} & \longrightarrow & \mathbb{A}^{n-1}/\mathfrak{S}_{n-1} \end{array}$$

we see that the fibre $\varphi^{-1}(\lambda)$ is birational to the quotient $\mathscr{F}_\lambda/\Gamma$. By (4.1) there is an isomorphism $\mathscr{F}_\lambda \xrightarrow{\sim} JC_{\mu,\lambda}$ such that Γ acts on $JC_{\mu,\lambda}$ as $\{\pm 1_J\} \times \Gamma^+$, where Γ^+ is a group of translations by 2-torsion elements. This proves the proposition.

5. Fibres of Φ

5.1 Results

We keep the settings of the previous section. Recall that our parameter λ lives in $\mathbb{A}^{n-1} \subset \mathsf{S}^{n-1}\Pi \cong \mathbb{P}^{n-1}$. For λ in $\mathbb{A}^{n-1}$, we denote by $\tilde{\lambda}$ a lift of λ in $\mathbb{C}^n$ for the quotient map $\mathbb{C}^n \smallsetminus \{0\} \to \mathbb{P}^{n-1}$.

Theorem 5.1. *Assume that X is general. For $\lambda \in \mathbb{A}^{n-1}$ general, the fibre $\Phi^{-1}(\tilde{\lambda})$ is isomorphic to $A \smallsetminus Z$, where:*

- *A is the abelian variety quotient of $JC_{\mu,\lambda}$ by a 2-torsion subgroup, isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-2}$;*
- *Z is a closed subvariety of codimension ≥ 2 in A.*

Corollary 5.2. *For every smooth complete intersection of two quadrics $X \subset \mathbb{P}^{n+2}$, the fibration $\Phi : T^*X \to \mathbb{C}^n$ is Lagrangian.*

Proof. Assume first that X is general. The symplectic form on T^*X is $d\eta$, where η is the Liouville form. By Theorem 5.1 and Hartogs' principle, the pull-back of η to a general fibre of Φ is the restriction of a 1-form on an abelian variety, hence is closed. This implies the result.

Let $p : \mathscr{X} \to B$ be a complete family of smooth intersection of two quadrics in $\mathbb{P}^{n+2}$. The constructions of §3 can be globalised over B: we have a rank 2 vector bundle $\mathscr{V}$ over B whose fibre at a point $b \in B$ is the space of quadratic forms vanishing on $\mathscr{X}_b$. We get a homomorphism $\mathsf{S}^{n-1}\mathscr{V} \to p_*T_{\mathscr{X}/B}$, which thus gives rise to a morphism $\boldsymbol{\Phi} : T^*(\mathscr{X}/B) \to \mathsf{S}^{n-1}\mathscr{V}^*$ over B which induces over each point $b \in B$ our map Φ. There is a natural Liouville form $\boldsymbol{\eta}$ on $T^*(\mathscr{X}/B)$: Since $d\boldsymbol{\eta}$ vanishes on a general fibre of $\boldsymbol{\Phi}$, it vanishes on all fibres. □

Corollary 5.3. *Assume that X is general. The multiplication map $\mathsf{S}^\bullet H^0(X, \mathsf{S}^2 T_X) \to H^0(X, \mathsf{S}^\bullet T_X)$ is an isomorphism.*

(We will give in Section 7 a proof that is valid with no generality assumption.)

Proof. Theorem 5.1 implies that every function on a general fibre of Φ is constant; hence, the pull-back $\Phi^* : H^0(\mathbb{C}^n, \mathscr{O}_{\mathbb{C}^n}) \to H^0(T^*X, \mathscr{O}_{T^*X})$ is an isomorphism. The right-hand space is canonically isomorphic to $H^0(X, \mathsf{S}^\bullet T_X)$; hence, we get an algebra isomorphism $\mathbb{C}[t_1, \ldots, t_n] \xrightarrow{\sim} H^0(X, \mathsf{S}^\bullet T_X)$. By construction, the t_i are mapped to elements of $H^0(X, \mathsf{S}^2 T_X)$, so the Corollary follows. □

Remark 5.4. Let $V_1, \ldots, V_n$ be the Hamiltonian vector fields on T^*X that are associated to the components of Φ. For λ general in $\mathbb{C}^n$, let us identify $\Phi^{-1}(\lambda)$ to $A \smallsetminus Z$ as in the theorem. Then by Hartogs' principle the V_i linearise on A; that is, they extend to a basis of $H^0(A, T_A)$. In principle, this allows to write explicit solutions of the Hamilton equations for Φ_i in terms of theta functions.

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5.2 Proof of the theorem: lemmas

We fix a general point $\lambda \in \mathbb{A}^{n-1}$. We denote by $\mathscr{F}^{\mathrm{o}}$ the open subset of $\mathscr{F}$ where the rational map ψ is well-defined and denote by $\mathscr{F}^{\mathrm{o}}_\lambda$ its intersection with the fibre $\mathscr{F}_\lambda$. Since λ is general, the complement of $\mathscr{F}^{\mathrm{o}}_\lambda$ in $\mathscr{F}_\lambda$ has codimension ≥ 2. The rational map ψ induces a morphism $\psi^{\mathrm{o}} : \mathscr{F}^{\mathrm{o}} \to \mathbb{P}T^*X$; we denote by $\psi^{\mathrm{o}}_\lambda$ its restriction to $\mathscr{F}^{\mathrm{o}}_\lambda$. Let $Z \subset \mathbb{P}T^*X$ be the indeterminacy locus of φ (§ 3), and let $\mathscr{F}^{\mathrm{bad}}_\lambda := (\psi^{\mathrm{o}}_\lambda)^{-1}(Z) \subset \mathscr{F}^{\mathrm{o}}_\lambda$.

Proposition 5.5. *$\mathscr{F}^{\mathrm{bad}}_\lambda$ has codimension ≥ 2 in $\mathscr{F}_\lambda$.*

We postpone the proof of Proposition 5.5 to the next section; here we show how it implies Theorem 5.1.

Let $0_X \subset T^*X$ be the zero section, and let $q : T^*X \smallsetminus 0_X \to \mathbb{P}T^*X$ be the quotient map. Let $\varphi^{\mathrm{o}} : \mathbb{P}T^*X \smallsetminus Z \to \mathbb{P}^{n-1}$ be the morphism induced by φ. We thus have $q(\Phi^{-1}(\tilde{\lambda})) = (\varphi^{\mathrm{o}})^{-1}(\lambda)$, and the restriction

$$q_\lambda : \Phi^{-1}(\tilde{\lambda}) \to (\varphi^{\mathrm{o}})^{-1}(\lambda)$$

is an étale double cover, with Galois involution ι induced by (-1_{T^*X}).

We put $\mathscr{F}^{\mathrm{oo}}_\lambda := \mathscr{F}^{\mathrm{o}}_\lambda \smallsetminus \mathscr{F}^{\mathrm{bad}}_\lambda$ and consider the restriction

$$\psi^{\mathrm{o}}_\lambda : \mathscr{F}^{\mathrm{oo}}_\lambda \to (\varphi^{\mathrm{o}})^{-1}(\lambda) \quad \text{of} \quad \psi^{\mathrm{o}}\,.$$

Lemma 5.6. *The fibre $\Phi^{-1}(\tilde{\lambda})$ is Lagrangian, and has a trivial tangent bundle.*

Proof. The étale double cover q_λ induces by fibred product an étale double cover

$$\pi : \widetilde{\mathscr{F}}^{\mathrm{oo}}_\lambda \to \mathscr{F}^{\mathrm{oo}}_\lambda$$

such that $\psi^{\mathrm{o}}_\lambda$ lifts to a morphism $\tilde{\psi}^{\mathrm{o}}_\lambda : \widetilde{\mathscr{F}}^{\mathrm{oo}}_\lambda \to \Phi^{-1}(\tilde{\lambda})$.

By Proposition 5.5, the complement of $\mathscr{F}^{\mathrm{oo}}_\lambda$ in $\mathscr{F}_\lambda$ has codimension ≥ 2, so π extends to an étale double cover $\widetilde{\mathscr{F}}_\lambda \to \mathscr{F}_\lambda$, where $\widetilde{\mathscr{F}}_\lambda$ is an abelian variety or the disjoint union of two abelian varieties. The morphism $\tilde{\psi}^{\mathrm{o}}_\lambda : \widetilde{\mathscr{F}}^{\mathrm{oo}}_\lambda \to \Phi^{-1}(\tilde{\lambda})$ is generically of maximal rank. Again by Proposition 5.5, the holomorphic 1-forms on $\widetilde{\mathscr{F}}^{\mathrm{oo}}_\lambda$ are closed; hence by pull-back, the same holds for the holomorphic 1-forms on $\Phi^{-1}(\tilde{\lambda})$. As in the proof of Corollary 5.2, this implies that $\Phi^{-1}(\tilde{\lambda})$ is Lagrangian. The second assertion is a basic property of Lagrangian fibres. □

Lemma 5.7 *The morphism $\psi^{\mathrm{o}}_\lambda$ lifts to a morphism $\tilde{\psi}^{\mathrm{o}}_\lambda : \mathscr{F}^{\mathrm{oo}}_\lambda \to \Phi^{-1}(\tilde{\lambda})$.*

Proof. It suffices to show that the double covering $\pi : \widetilde{\mathscr{F}}^{\mathrm{oo}}_\lambda \to \mathscr{F}^{\mathrm{oo}}_\lambda$ splits.

Assuming the contrary, $\widetilde{\mathscr{F}}_\lambda$ is an abelian variety. By Lemma 5.6 $H^0(\Phi^{-1}(\tilde{\lambda}), \Omega^1)$ has dimension n. It follows that the pull-back $(\tilde{\psi}^{\mathrm{o}}_\lambda)^* : H^0(\Phi^{-1}(\tilde{\lambda}), \Omega^1) \to H^0(\widetilde{\mathscr{F}}^{\mathrm{oo}}_\lambda, \Omega^1)$ is bijective. Since the Galois involution of the double covering π acts trivially on holomorphic 1-forms, the same holds for the Galois involution ι of the double covering $q_\lambda : \Phi^{-1}(\tilde{\lambda}) \to (\varphi^{\mathrm{o}})^{-1}(\lambda)$.

Now we observe that the 1-forms on $\Phi^{-1}(\tilde{\lambda})$ are 'pure'; that is, they extend to any smooth projective compactification of $\Phi^{-1}(\tilde{\lambda})$. This follows from the fact that this holds after pull-back to $\widetilde{\mathscr{F}}^{\mathrm{oo}}_\lambda$. But the quotient $\Phi^{-1}(\tilde{\lambda})/\iota$ is isomorphic to a Zariski open subset of $\varphi^{-1}(\lambda)$, which, by Proposition 4.1, has no nonzero holomorphic 1-forms, so any Zariski open subset has no nonzero closed pure holomorphic 1-forms. This contradiction proves the lemma. □

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5.3 Proof of Theorem 5.1

Lemma 5.7 gives a factorisation,

$$\psi^{\mathrm{o}}_\lambda : \mathscr{F}^{\mathrm{oo}}_\lambda \xrightarrow{\tilde\psi^{\mathrm{o}}_\lambda} \Phi^{-1}(\tilde\lambda) \xrightarrow{q_\lambda} (\varphi^{\mathrm{o}})^{-1}(\lambda)\,.$$

By Proposition 4.1, $\psi^{\mathrm{o}}_\lambda$ induces a birational morphism,

$$\psi^{\mathrm{o}}_{\lambda,\Gamma} : \mathscr{F}^{\mathrm{oo}}_\lambda/\Gamma \longrightarrow (\varphi^{\mathrm{o}})^{-1}(\lambda)$$

it follows that for some subgroup $\Gamma' \subset \Gamma$ of index 2, the morphism $\tilde\psi^{\mathrm{o}}_\lambda : \mathscr{F}^{\mathrm{oo}}_\lambda \to \Phi^{-1}(\tilde\lambda)$ factors through a birational morphism,

$$\tilde\psi^{\mathrm{o}}_{\lambda,\Gamma'} : \mathscr{F}^{\mathrm{oo}}_\lambda/\Gamma' \longrightarrow \Phi^{-1}(\tilde\lambda)\,.$$

By Lemma 5.6, the cotangent bundle of $\Phi^{-1}(\tilde\lambda)$ is trivial. Therefore, the cotangent bundle of $\mathscr{F}^{\mathrm{oo}}_\lambda/\Gamma'$ is generically generated by its global sections. This implies that Γ' acts trivially on holomorphic 1-forms and, hence, is the subgroup Γ^+ of Γ generated by translations, isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-2}$; thus $\mathscr{F}_\lambda/\Gamma'$ is an abelian variety A.

To simplify notation, we write $A^{\mathrm{o}} := \mathscr{F}^{\mathrm{oo}}_\lambda/\Gamma'$ and $u := \tilde\psi^{\mathrm{o}}_{\lambda,\Gamma'}$. The rational map $u^{-1} : \Phi^{-1}(\tilde\lambda) \dashrightarrow A$ is everywhere defined (e.g. [BL92, Theorem 4.9.4]), so we have two morphisms

$$A^{\mathrm{o}} \xrightarrow{u} \Phi^{-1}(\tilde\lambda) \xrightarrow{u^{-1}} A$$

whose composition is the inclusion $A^{\mathrm{o}} \hookrightarrow A$. Since the tangent bundles of A and $\Phi^{-1}(\tilde\lambda)$ are trivial, the determinant of $Tu : T_{A^{\mathrm{o}}} \to u^*T_{\Phi^{-1}(\tilde\lambda)}$ is a function on A^{o}, hence constant by Proposition 5.5. Therefore, u is étale and birational, hence an open embedding. This implies that every function on $\Phi^{-1}(\tilde\lambda)$ is constant (because its restriction to A^{o} is constant). Then the previous argument shows that u^{-1} is also an open embedding, hence $\Phi^{-1}(\tilde\lambda)$ is isomorphic to an open subset of A containing A^{o}. This proves the theorem.

6. Proof of Proposition 5.5

We keep the notations of Section 4.2. Recall that we have coordinates $(x_0, \ldots, x_{n+2}; y_1, \ldots, y_{n-1})$ on $\mathbb{P}^{2n+1}$ and subspaces $\mathbb{P}^{n+2}$ and $\mathbb{P}^{n-2}$ in $\mathbb{P}^{2n+1}$ defined by $y = 0$ and $x = 0$.

Let $q_1(x) = q_2(x) = 0$ be the equations defining X in $\mathbb{P}^{n+2}$, and let R be the vector space of quadratic forms in $y = (y_1, \ldots, y_{n-1})$. We define an extended family $\mathscr{X}^e \subset \mathbb{P}^{2n+1} \times R^2$ as

$$\mathscr{X}^e = \{((x,y);(r_1,r_2)) \in \mathbb{P}^{2n+1} \times R^2 \mid q_1(x) + r_1(y) = q_2(x) + r_2(y) = 0\}\,.$$

The fibre $\mathscr{X}^e_r$ at a point $r = (r_1, r_2)$ of R^2 is the intersection in $\mathbb{P}^{2n+1}$ of the two quadrics $q_1(x) + r_1(y) = q_2(x) + r_2(y) = 0$. Let $\mathbb{G}$ be the Grassmannian of $(n-1)$-planes in $\mathbb{P}^{2n+1}$; we define as before

$$\mathscr{F}^e := \{(P,r) \in \mathbb{G} \times R^2 \mid P \subset \mathscr{X}^e_r\}$$

and the extended rational map $\psi^e : \mathscr{F}^e \dashrightarrow \mathbb{P}T^*X$, which maps a general $P \subset \mathscr{X}^e_r$ to the pair (x, H), with $\{x\} = P \cap \mathbb{P}^{n+2}$ and $H = \pi_* T_x(P)$.

We observe that a general pair $r = (r_1, r_2)$ of R^2 is simultaneously diagonalisable, so the restriction of ψ^e to $\mathscr{F}^e_r$ coincides, for an appropriate choice of the coordinates (y_i), with the map ψ_λ that we want to study.

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Proposition 6.1. *Assume that X is general.*

1. *Let $\Gamma \subset \mathscr{F}^e$ be the locus of points (P,r) such that either $\dim P \cap \mathbb{P}^{n+2} > 0$, or $P \cap \mathbb{P}^{n-2} \neq \varnothing$. Then Γ has codimension ≥ 2 in $\mathscr{F}^e$.*
2. *There exists no divisor in $\mathscr{F}^e \smallsetminus \Gamma$ that dominates R^2 and that is mapped to the base locus $Z \subset \mathbb{P}T^*X$ by ψ_e.*

We claim that this implies Proposition 5.5. Indeed, as just explained above, it suffices to prove the analogue of Proposition 5.5 for ψ^e. Next, it is clear that the indeterminacy locus of ψ^e is contained in Γ, so ψ^e is well-defined on $\mathcal{F}^e \smallsetminus \Gamma$. By Proposition 6.1, (1), it now suffices to prove the analogue of Proposition 5.5 for the restriction of ψ^e to $\mathcal{F}^e \smallsetminus \Gamma$. This is exactly the statement of Proposition 6.1, (2).

Proof of Proposition 6.1: (1) Let $\mathcal{Q}$ be the vector space of quadratic forms on $\mathbb{P}^{2n+1}$ of the form $q(x)+r(y)$ for some quadratic forms q and r. For each pair of integers (k,l) with $k \geq 0$, $l \geq -1$, let $\mathbb{G}_{k,l}$ be the locally closed subvariety of $(n-1)$-planes $P \in \mathbb{G}$ such that

$$\dim(P \cap \mathbb{P}^{n+2}) = k \qquad \dim(P \cap \mathbb{P}^{n-2}) = l\,.$$

(We put, by convention, $l=-1$ if $P \cap \mathbb{P}^{n-2} = \varnothing$.) Let

$$\mathscr{F}^{\mathcal{Q}} := \{(P,(Q_1,Q_2)) \in \mathbb{G} \times \mathcal{Q}^2 \mid Q_{1|P} = Q_{2|P} = 0\}$$

and

$$\mathscr{F}^{\mathcal{Q}}_{k,l} := \mathscr{F}^{\mathcal{Q}} \cap (\mathbb{G}_{k,l} \times \mathcal{Q}^2)\,.$$

The general fibre of the projection $\mathscr{F}^{\mathcal{Q}} \to \mathcal{Q}^2$ is an abelian variety, and we recover $\mathscr{F}^e$ by restricting $\mathscr{F}^{\mathcal{Q}}$ to pairs of quadratic forms of the form $(q_1(x)+r_1(y), q_2(x)+r_2(y))$, with $(q_1(x),q_2(x))$ fixed. Because we assume X general, the pair $(q_1(x),q_2(x))$ is general in $\mathcal{Q}^2$. It thus suffices to prove the result for the larger family $\mathscr{F}^{\mathcal{Q}}$; that is, to show that $\mathscr{F}^{\mathcal{Q}}_{k,l}$ has codimension ≥ 2 in $\mathscr{F}^{\mathcal{Q}}$.

This is done by a dimension count. For $P \in \mathbb{G}$, let φ_P be the restriction map $\mathcal{Q} \to H^0(P,\mathscr{O}_P(2))$. The fibre of the projection $\mathscr{F}^{\mathcal{Q}} \to \mathbb{G}$ is the vector space $(\mathrm{Ker}\varphi_P)^{\oplus 2}$. For P general, φ_P is surjective: This is the case, for instance, if P is contained in the $(n+2)$-plane in $\mathbb{P}^{2n+1}$ defined by $y_i = x_i$ $(i=1,\ldots,n-1)$. However, φ_P is not surjective for $P \in \mathbb{G}_{k,l}$ because the forms $r(y)_{|P}$ are singular along $P \cap \mathbb{P}^{n+2}$ and the forms $q(x)_{|P}$ are singular along $P \cap \mathbb{P}^{n-2}$; this implies that the subspaces $P \cap \mathbb{P}^{n+2}$ and $P \cap \mathbb{P}^{n-2}$ are apolar for all forms in $\mathrm{Im}\varphi_P$. Therefore, the corank of φ_P is $\geq (k+1)(l+1)$, and there is equality when P is contained in the subspace defined by $x_0 = \ldots = x_{n+1-k} = y_1 = \ldots = y_{n-2-l} = 0$, hence for P general in $\mathbb{G}_{k,l}$. Thus our assertion follows from:

$$\begin{aligned}\mathrm{codim}(\mathscr{F}^{\mathcal{Q}}_{k,l},\mathscr{F}^{\mathcal{Q}}) &= \mathrm{codim}(\mathbb{G}_{k,l},\mathbb{G}) - 2(k+1)(l+1)\\ &= k(k+1) + (l+1)(l+4) - 2(k+1)(l+1)\\ &= (k-l)(k-l-1) + 2(l+1)\\ &\geq 2 \quad \text{if} \quad k \geq 1 \quad \text{or} \quad l \geq 0\,.\end{aligned}$$

(2) The base locus $Z \subset \mathbb{P}T^*X$ has codimension ≥ 2 (Corollary 3.3). Note that ψ^e is well-defined in $\mathscr{F}^e \smallsetminus \Gamma$. If $\mathscr{D}$ is a codimension 1 subvariety in $\mathscr{F}^e \smallsetminus \Gamma$, with $\psi^e(\mathscr{D}) \subset Z$, the map ψ^e does not have maximal rank along $\mathscr{D}$. This contradicts the following lemma:

Lemma 6.2. *ψ^e has maximal rank on $\mathscr{F}^e \smallsetminus \Gamma$.*

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Proof. Let (x,H) be a point of $\mathbb{P}T^*X$; we view H as a hyperplane in the projective tangent space to x at X. The fibre of $\psi^e : \mathscr{F}^e \smallsetminus \Gamma \to \mathbb{P}T^*X$ at (x,H) is the locus

$$(\psi^e)^{-1}(x,H) = \{(P,r_1,r_2) \in \mathbb{G}\times R^2 \mid P\cap \mathbb{P}^{n+2} = \{x\}\,,\ P\cap\mathbb{P}^{n-2} = \varnothing\,,\ \pi(P) = H, \tag{2}$$

$$(q_i + r_i)_{|P} = 0 \quad (i=1,2)\}\,. \tag{3}$$

The equations (2) define a smooth, locally closed subvariety $\mathbb{G}_{x,H}$ of $\mathbb{G}$. Let $P\in\mathbb{G}_{x,H}$, and let $\chi_P : R\to H^0(P,\mathscr{O}_P(2))$ be the restriction map. We will show below that the image of χ_P is the space of quadratic forms on P that are singular at x. Since the forms $q_{i|P}$ are singular at x, this implies that the solutions of (3) form an affine space over $(\mathrm{Ker}\chi_P)^{\oplus 2}$. Therefore, $(\psi^e)^{-1}(x,H)$ admits an affine fibration over $\mathbb{G}_{x,H}$, hence is smooth.

Clearly the quadrics in $\mathrm{Im}\chi_P$ are singular at x. To prove the opposite inclusion, choose the coordinates (x_i) so that $x=(1,0,\ldots,0)$. Since $P\cap\mathbb{P}^{n+2}=\{x\}$, there exist linear forms $\ell_1,\ldots,\ell_{n+2}$ in the y_j so that P is defined by $x_i=\ell_i(y)$ for $i=1,\ldots,n+2$. Then a quadratic form on $\mathbb{P}^{2n+1}$ singular at x can be written as a form in $x_1,\ldots,x_{n+2};y_1,\ldots,y_{n-1}$; hence, its restriction to P is in $\mathrm{Im}\chi_P$. This proves the lemma and, hence, also the proposition. □

7. Symmetric tensors: second approach

7.1 The cotangent bundle of a smooth quadric

We consider a smooth quadric $Q\subset\mathbb{P}^{n+1}$ defined by an equation $q=0$. Its cotangent bundle $\mathbb{P}T^*Q$ parameterises pairs (x,P) with $x\in Q$ and P a $(n-1)$-plane tangent to Q at x. Thus, we get a morphism γ from $\mathbb{P}T^*Q$ to the Grassmannian $\mathbb{G}$ of $(n-1)$-planes in $\mathbb{P}^{n+1}$, which is the morphism defined by the linear system $|\mathscr{O}_{\mathbb{P}T^*Q}(1)|$. It is birational onto its image, but contracts the subvariety $\mathscr{C}\subset\mathbb{P}T^*Q$ that consists of pairs (x,P), such that P is tangent to Q along a line $\ell\subset Q$ with $x\in\ell$; then $\gamma^{-1}(P)$ consists of the pairs (x,P) with $x\in\ell$.

Let $h_q\in H^0(Q,\mathsf{S}^2\Omega^1_Q(2))$ be the hessian form of q (§3). Choosing coordinates (x_i) such that $q(x)=\sum x_i^2$, we have $h_q=\sum(dx_i)^2$ (note that this is, up to a scalar, the unique element of $H^0(Q,\mathsf{S}^2\Omega^1_Q(2))$ invariant under $\mathrm{Aut}(Q)$). Then $h_q(x)$ is non-degenerate at each point x of Q, so h_q induces an isomorphism $\Omega^1_Q(1)\xrightarrow{\sim}T_Q(-1)$, hence also $\mathsf{S}^2\Omega^1_Q(2)\xrightarrow{\sim}\mathsf{S}^2T_Q(-2)$. The image in $H^0(Q,\mathsf{S}^2T_Q(-2))$ of h_q by this isomorphism is $h'_q=\sum\partial_j^2$. We will view h'_q as an element of $H^0(\mathbb{P}T^*Q,\mathscr{O}_{\mathbb{P}T^*Q}(2)\otimes p^*\mathscr{O}_Q(-2))$, where $p:\mathbb{P}T^*Q\to Q$ is the projection.

Proposition 7.1. *The divisor of h'_q is $\mathscr{C}$. The projection $p_{|\mathscr{C}}:\mathscr{C}\to Q$ is a smooth quadric fibration, and $\mathscr{C}$ is a prime divisor for $n\geq 3$.*

Proof. Let $x\in Q$; the hyperplane tangent to Q at x cuts out a cone over the smooth quadric $Q_x\subset\mathbb{P}(T_x(Q))$ defined by $h_q(x)=0$ (Section 3). The isomorphism $T_x(Q)\xrightarrow{\sim}T^*_x(Q)$ given by $h_q(x)$ carries Q_x into the dual quadric Q^*_x in $\mathbb{P}(T^*_x(Q))$. On the other hand, a point $y\in p^{-1}(x)$ corresponds to a hyperplane $H_y\subset\mathbb{P}(T_x(Q))$, and y belongs to $\mathscr{C}$ if and only if H_y is tangent to Q_x; that is, if $y\in Q^*_x$. This proves the equality $\mathscr{C}=\mathrm{div}(h'_q)$ and thus, that the fiber of $p_{|\mathscr{C}}:\mathscr{C}\to Q$ at x is Q_x, which is smooth and connected if $n\geq 3$. □

Remark 7.2 The variety $\mathscr{C}$ is an example of a total dual VMRT [HLS22]. For the proof of the theorem, we will combine this tool with the birational transformation of $\mathbb{P}T^*X$ defined by a double cover. (Compare with [AH23]).

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We will have to consider the following situation: Let Q' be another quadric in $\mathbb{P}^{n+1}$, such that the intersection $B := Q \cap Q'$ is a smooth hypersurface in Q. The surjection $T_Q \to N_{B/Q}$ gives a section of $\mathbb{P}T^*Q$ over B, hence an embedding $s : B \hookrightarrow \mathbb{P}T^*Q$.

Lemma 7.3. *The image $s(B)$ is not contained in $\mathscr{C}$.*

Proof. Let $x \in B$. The point $s(x)$ in $\mathbb{P}(T_x^*(Q))$ corresponds to the hyperplane image of $T_x(B)$ in $T_x(Q)$; we must therefore show that this hyperplane is not tangent to the quadric $Q_x := h_q(x)$. In terms of projective space, this means that the projective tangent space to Q' at x is not tangent, at a smooth point y, to the cone cut out on Q by the projective tangent space to Q' at x.

Suppose that this is the case, with $y = (y_0, \ldots, y_{n+1})$. We can assume that Q' is defined by $\sum \alpha_i x_i^2 = 0$, with $\alpha_i \in \mathbb{C}$ distinct. Then the (projective) tangent space to Q' at x, given by $\sum (\alpha_i x_i)\xi_i = 0$, must coincide with the tangent space to Q at y, given by $\sum y_i \xi_i = 0$. This implies $y = (\alpha_0 x_0, \ldots, \alpha_{n+1} x_{n+1})$. Thus the point x must satisfy

$$\sum x_i^2 = \sum \alpha_i x_i^2 = \sum \alpha_i^2 x_i^2 = 0\,.$$

If these relations hold for all x in B, the quadric $\sum \alpha_i^2 x_i^2 = 0$ must belong to the pencil spanned by Q and Q'. This means that there exist scalars λ, μ, ν such that

$$\lambda \alpha_i^2 + \mu \alpha_i + \nu = 0 \quad \text{for all } i\,,$$

which is impossible since the α_i are distinct. Therefore, there exists $x \in B$ such that $s(x) \notin \mathscr{C}$. ☐

7.2 Explicit description of symmetric tensors

We keep the notation of the previous sections: $X \subset \mathbb{P} = \mathbb{P}^{n+2}$ is defined by $q_1 = q_2 = 0$, and with

$$q_1 = \sum_{i=0}^{n+2} x_i^2, \qquad q_2 = \sum_{i=0}^{n+2} \mu_i x_i^2 \qquad \text{with } \mu_i \in \mathbb{C} \text{ distinct.}$$

We put $\partial_i := \dfrac{\partial}{\partial x_i}\cdot$ We have an exact sequence

$$0 \to T_X \to T_{\mathbb{P}|X} \xrightarrow{(dq_1, dq_2)} \mathscr{O}_X(2)^2 \to 0\,,$$

where dq_i maps the restriction of a vector field V on $\mathbb{P}$ to $V \cdot q_i$. This gives the exact sequence of symmetric tensors

$$0 \to \mathsf{S}^2 T_X \to \mathsf{S}^2 T_{\mathbb{P}|X} \xrightarrow{(dq_1, dq_2)} T_{\mathbb{P}|X}(2)^2\,, \tag{4}$$

where $dq_i(V_1 V_2) = (V_1 \cdot q_i) V_2 + (V_2 \cdot q_i) V_1$ for V_1, V_2 in $H^0(X, T_{\mathbb{P}|X})$.

Proposition 7.4. *The quadratic vector fields $s_i := \sum_{j \neq i} \dfrac{(x_i \partial_j - x_j \partial_i)^2}{\mu_j - \mu_i}$ in $H^0(X, \mathsf{S}^2 T_{\mathbb{P}|X})$ belong to the image of $H^0(X, \mathsf{S}^2 T_X)$.*

Proof. According to the exact sequence (4), we have to prove $dq_1(s_i) = dq_2(s_i) = 0$.

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We have $(x_i\partial_j - x_j\partial_i)\cdot q_1 = 0$; hence, $dq_1(s_i) = 0$ and $dq_2(x_i\partial_j - x_j\partial_i, x_i\partial_j - x_j\partial_i) = 4(\mu_j - \mu_i)x_ix_j(x_i\partial_j - x_j\partial_i)$. Hence, using $\sum x_j\partial_j = 0$ and $q_{1|X} = 0$,

$$dq_2(s_i) = 4x_i^2\sum_{j\neq i} x_j\partial_j - 4x_i(\sum_{j\neq i} x_j^2)\partial_i = 0\,, \quad \text{which proves the proposition.}$$

□

In the rest of this article, we will consider the s_i to be elements of $H^0(X, \mathsf{S}^2T_X)$.

7.3 The double cover

Let $p_0 : \mathbb{P}^{n+2} \dashrightarrow \mathbb{P}^{n+1}$ be the projection $(x_0, \ldots, x_{n+2}) \mapsto (x_1, \ldots, x_{n+2})$. The image $p_0(X)$ is the smooth quadric Q in $\mathbb{P}^{n+1}$ defined by

$$\sum_{i=1}^{n+2}(\mu_i - \mu_0)x_i^2 = 0\,.$$

The restriction $\pi : X \to Q$ of p_0 is a double covering that is branched along the subvariety $B \subset Q$ defined by

$$\sum_{i=1}^{n+2} x_i^2 = \sum_{i=1}^{n+2}\mu_i x_i^2 = 0\,.$$

It is a smooth complete intersection of 2 quadrics in $\mathbb{P}^{n+1}$. The ramification locus $R \subset X$ of π (isomorphic to B) is the hyperplane section $x_0 = 0$ of X.

The tangent map of $\pi : X \to Q$ gives a morphism,

$$\tau : T_X \to \pi^*T_Q,$$

which is an isomorphism outside of R. Consider the normal exact sequence

$$0 \to T_R \to T_{X|R} \to N_{R/X} \to 0\,.$$

The involution $\iota : (x_0, \ldots, x_{n+2}) \mapsto (-x_0, x_1, \ldots, x_{n+2})$ acts on $T_{X|R}$; this splits the exact sequence, giving a decomposition

$$T_{X|R} = T_R \oplus N_{R/X}$$

into eigenspaces for the eigenvalues $+1$ and -1. Let $\rho : T_{X|R} \to T_R$ be the projection on the first summand. We deduce from ρ a sequence of homomorphisms

$$h^k : H^0(X, \mathsf{S}^kT_X) \longrightarrow H^0(X, \mathsf{S}^kT_{X|R}) \xrightarrow{\mathsf{S}^k\rho} H^0(R, \mathsf{S}^kT_R)\,.$$

Since $\iota_*\partial_0 = -\partial_0$ and $\iota_*\partial_j = \partial_j$ for $j > 0$, we have

$$h^2(s_0) = 0 \quad \text{and} \quad h^2(s_i) = \sum_{\substack{j>0\\ j\neq i}} \frac{(x_i\partial_j - x_j\partial_i)^2}{\mu_j - \mu_i} \quad \text{for } i > 0 \tag{5}$$

in other words, h^2 maps $s_1, \ldots, s_{n+2}$ to the elements $\hat{s}_1, \ldots, \hat{s}_{n+2}$ of $H^0(R, \mathsf{S}^2T_R)$ constructed in Proposition 7.2.1 applied to R.

Let $\pi^*\mathbb{P}T^*Q$ be the pull-back under π of the projective bundle $\mathbb{P}T^*Q \to Q$. The homomorphism $\tau : T_X \to \pi^*T_Q$ gives rise to the birational map $g : \pi^*\mathbb{P}T^*Q \dashrightarrow \mathbb{P}T^*X$. Following the geometric description of the tangent map as an elementary transformation of vector bundles in the sense of Maruyama in [Mar72] and [Mar73, Corollary 1.1.1], one has a commutative diagram

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$$
\begin{array}{ccc}
 & \Gamma & \\
 {\scriptstyle\mu}\swarrow & & \searrow{\scriptstyle\nu} \\
\pi^*\mathbb{P}T^*Q & \overset{g}{\dashrightarrow} & \mathbb{P}T^*X \\
 {\scriptstyle p}\searrow & & \swarrow{\scriptstyle q} \\
 & X &
\end{array}
\tag{6}
$$

where p and q are the canonical projections; $\nu:\Gamma\to\mathbb{P}T^*X$ is the blow-up along the subspace $\mathbb{P}T^*R\subset\mathbb{P}T^*X$ defined by the projection ρ; and $\mu:\Gamma\to\pi^*\mathbb{P}T^*Q$ is the blow-up of the image B' of the embedding $B\hookrightarrow\pi^*\mathbb{P}T^*Q$ deduced from the surjective homomorphism $\pi^*T_Q\to\pi^*N_{B/X}$.

Let E_μ be the exceptional divisor of μ. By [Mar73, Theorem 1.1], there is an isomorphism

$$\mu^*\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(1)\otimes\mathscr{O}_\Gamma(-E_\mu)\cong\nu^*\mathscr{O}_{\mathbb{P}T^*X}(1) \tag{7}$$

as well as the equality

$$\nu_*E_\mu=q^*R\,. \tag{8}$$

7.4 The divisor of s_0

We now consider the divisor $\mathscr{C}\subset\mathbb{P}T^*Q$ defined in (7.1) and the cartesian diagram

$$
\begin{array}{ccc}
\pi^*\mathbb{P}T^*Q & \xrightarrow{\pi'} & \mathbb{P}T^*Q \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi} & Q\,.
\end{array}
$$

Set $\mathscr{C}':={\pi'}^{-1}(\mathscr{C})$. The projection $\mathscr{C}'\to X$ is again a smooth quadric fibration, so $\mathscr{C}'$ is smooth and connected for $n\geq 3$.

Recall that we have defined the element $s_0:=\sum_{j=1}^{n+2}\frac{(x_0\partial_j-x_j\partial_0)^2}{\mu_j-\mu_0}\in H^0(X,\mathsf{S}^2T_X)$ (7.2). We will now view s_0 as an element of $H^0(\mathbb{P}T^*X,\mathscr{O}(2))$.

Proposition 7.5 *Assume $n\geq 3$. We have $g_*\mathscr{C}'=\operatorname{div}(s_0)$.*

Proof. We first show that $g_*\mathscr{C}'\in|\mathscr{O}_{\mathbb{P}T^*X}(2)|$. By Proposition 7.1 we have $\mathscr{C}'\in|\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(2)\otimes p^*\mathscr{O}_X(-2)|$. Using (7), (8) and the projection formula, we get the linear equivalences

$$\nu_*\mu^*\mathscr{C}'\sim 2\nu_*\mu^*(c_1(\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(1)-p^*R))\sim 2(c_1(\mathscr{O}_{\mathbb{P}T^*X}(1))+q^*R)-2q^*R=c_1(\mathscr{O}_{\mathbb{P}T^*X}(2))\,.$$

Thus, it is enough to prove that $\nu_*\mu^*\mathscr{C}'$ is irreducible. Since $\mathscr{C}'$ is irreducible and μ is the blow-up along $B'\subset\pi^*\mathbb{P}T^*Q$, it suffices to show that B' is not contained in $\mathscr{C}'$. If this is the case, then we have $\pi'(B')\subset\pi'(\mathscr{C}')=\mathscr{C}$. But $\pi'(B')=s(B)$, where $s:B\hookrightarrow\mathbb{P}T^*Q$ is the embedding defined by the surjective homomorphism $T_Q\to N_{B/Q}$. Then the result follows from Lemma 7.3.

Since $g_*\mathscr{C}'$ and $\operatorname{div}(s_0)$ are linearly equivalent effective divisors and $g_*\mathscr{C}'$ is irreducible, it suffices to show that their restrictions to $\mathbb{P}T^*_x(X)$ coincide at a general point $x\in X$.

Fix a point $x=(x_0,\ldots,x_{n+2})\in X\smallsetminus R$ so that $x_0\neq 0$. Then the tangent map $T\pi(x):T_x(X)\to T_{\pi(x)}(Q)$ is an isomorphism; in diagram (6), the maps μ,ν and g restricted over the fibres at x are all isomorphisms. Let us show that $\mathscr{C}'$ and $T\pi(\operatorname{div}(s_0))$ define the same quadric in $\mathbb{P}(T_{\pi(x)}(Q))$.

Note that $\mathscr{C}' \cap \mathbb{P}(T^*_x(X)) = \mathscr{C} \cap \mathbb{P}(T^*_{\pi(x)}(Q))$ is the quadric defined by the element h'_q of (7.1). In the coordinates (z_i) defined by $z_i = (\mu_i - \mu_0)^{1/2} x_i$, the equation of Q is $\sum_{j=1}^{n+2} z_j^2 = 0$, so

$$h'_q = \sum_{j=1}^{n+2} \Big(\frac{\partial}{\partial z_j}\Big)^2 = \sum_{j=1}^{n+2} \frac{\partial_j^2}{\mu_j - \mu_0} \;.$$

On the other hand, since $\pi(x_0, \ldots, x_{n+2}) = (x_1, \ldots, x_{n+2})$, we have $T\pi(\partial_0) = 0$ and $T\pi(\partial_j) = \partial_j$ for $j > 0$; hence,

$$T\pi(s_0) = x_0^2 \sum_{j=1}^{n+2} \frac{\partial_j^2}{\mu_j - \mu_0} \;.$$

Since $x_0 \neq 0$, this proves the proposition. □

7.5 Proof of part (a) of the theorem

Suppose now that $n \geq 3$. Consider the double cover $\pi : X \to Q$ and the ramification divisor $R \subset X$. The restriction maps h^k defined in Section 7.3 yield a homomorphism of graded $\mathbb{C}$-algebras

$$h : S(X) := H^0(X, \mathsf{S}^\bullet T_X) \longrightarrow H^0(R, \mathsf{S}^\bullet T_R) =: S(R).$$

Proposition 7.6 *The kernel $\mathscr{I}$ of h is the ideal generated by s_0.*

Proof. Since $\mathscr{I}$ is a homogeneous ideal, it suffices to prove that every homogeneous element $s \in \mathscr{I}$ can be written as $s = s' s_0$ for some element $s' \in S(X)$.

Choose an element $s \in \mathscr{I}$ of degree k. This element corresponds to an effective Cartier divisor G in the linear system $|\mathscr{O}_{\mathbb{P}T^*X}(k)|$. Recall the commutative diagram (6):

$$\begin{array}{ccccc} & & \Gamma & & \\ & \swarrow^{\mu} & & \searrow^{\nu} & \\ \pi^*\mathbb{P}T^*Q & & \overset{g}{\dashrightarrow} & & \mathbb{P}T^*X \\ & \searrow_{p} & & \swarrow_{q} & \\ & & X & & \end{array}$$

Choose $\hat{G} := \mu_* \nu^* G \subset \pi^* \mathbb{P}T^*Q$. By (7), $\hat{G}$ belongs to the linear system $|\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(k)|$.

Here is the key observation: Since $s \in \mathscr{I}$, the divisor $\hat{G} \subset \pi^*\mathbb{P}T^*Q$ contains p^*R. Indeed, since $(\pi^*T_Q)_{|R}$ is invariant under ι, the homomorphism $\tau_{|R}$ factors as

$$\tau_{|R} : T_{X|R} \xrightarrow{\rho} T_R \longrightarrow (\pi^* T_Q)_{|R} \,.$$

Therefore, we have a commutative diagram,

$$\begin{array}{ccc} H^0(X, \mathsf{S}^k T_X) & \xrightarrow{h^k} & H^0(R, \mathsf{S}^k T_R) \\ {\scriptstyle \mathsf{S}^k\tau}\downarrow & & \downarrow \\ H^0(X, \mathsf{S}^k \pi^* T_Q) & \longrightarrow & H^0(R, \mathsf{S}^k (\pi^* T_Q)_{|R}) \end{array}$$

and $\mathsf{S}^k\tau(s)$ vanishes on R. But $\hat{G}$ is the divisor of $\mathsf{S}^k\tau(s)$, viewed as a section of $\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(k)$; hence, $\hat{G}$ contains p^*R.

Now we want to show that the divisor $\mathscr{C}' \subset \pi^*\mathbb{P}T^*Q$ is a component of $\hat{G} - p^*R$. Recall from (7.1) that $\mathscr{C}$ is the union of the lines ℓ that are contracted by the morphism $\gamma : \mathbb{P}T^*Q \to \mathbb{G}$ and that $c_1(\mathscr{O}_{\mathbb{P}T^*Q}(1)) \cdot \ell = 0$. Thus the curves $\ell' := {\pi'}^*\ell$ cover $\mathscr{C}'$ and satisfy $c_1(\mathscr{O}_{\pi^*\mathbb{P}T^*Q}(1)) \cdot \ell' = 0$. On the other hand, the divisor $R \subset X$ is a hyperplane section, so $p^*R \cdot \ell' = R \cdot p_*\ell' > 0$. Therefore,

$$(\hat{G} - p^*R) \cdot \ell' < 0\,,$$

so $\mathscr{C}'$ is a component of $\hat{G}$. Thus, $g_*\mathscr{C}'$ is a component of G. Since $g_*\mathscr{C}' = \operatorname{div}(s_0)$ by Proposition 7.5, this proves the proposition. □

The following proposition implies part (a) of our main theorem:

Proposition 7.7. *Assume $n \geq 2$. For any choice of indices $0 \leq i_1 < \ldots < i_n \leq n+2$, the homomorphism $\mathbb{C}[t_1, \ldots, t_n] \to S(X)$, which maps t_j to s_{i_j}, with $\deg(t_i) = 2$, is an isomorphism of graded $\mathbb{C}$-algebras.*

Proof. We argue by induction on n. The statement for $n = 2$ follows from [DOL19, Theorem 5.1], except for the fact that any two of the s_i generate $H^0(X, \mathsf{S}^2T_X)$. Up to the permuting of the coordinates, it suffices to prove that s_0 and s_1 are linearly independent. But $h^2 : H^0(X, \mathsf{S}^2T_X) \to H^0(R, \mathsf{S}^2T_R)$ maps s_0 to zero and maps s_i (for $i > 0$) to the corresponding elements $\hat{s}_i$ of $H^0(R, \mathsf{S}^2T_R)$; this implies our assertion.

Assume $n \geq 3$. By the induction hypothesis, the homomorphism $\mathbb{C}[t_1, \ldots, t_{n-1}] \to S(R)$, which maps t_i to $\hat{s}_i$, is an isomorphism of graded $\mathbb{C}$-algebras (with $\deg(t_i) = 2$). It follows that h is surjective and that $(s_0, \ldots, s_{n-1})$ form a basis of $H^0(X, \mathsf{S}^2T_X)$ and generate the $\mathbb{C}$-algebra $S(X)$. Thus we have a surjective homomorphism $u : \mathbb{C}[t_0, \ldots, t_{n-1}] \to S(X)$, with $u(t_i) = s_i$.

In particular, the Krull dimension of $S(X)$ is at most n. On the other hand, the ring $S(X)$ is a domain, and s_0 is neither zero nor a unit. Thus, by Krull's Hauptidealsatz, the Krull dimension of $S(X)$ is equal to n; hence, u is an isomorphism. By permutation of the coordinates, we get the same result for any choice of n elements in $\{s_0, \ldots, s_{n+2}\}$. This proves the proposition. □

Conflicts of Interest

None.

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