Lefschetz property and maximal variation of hyperplane sections

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Introduction

$X \subset \mathbb{P}^n$. The family $\{X \cap H \mid H \in (\mathbb{P}^n)^*\}$ has **maximal variation** if its image in the moduli space has maximal dimension n.

Conjecture : *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $\geqslant 3$. Then the family $\{X \cap H \mid H \in (\mathbb{P}^n)^*\}$ has maximal variation.*

I will show that this conjecture follows from a much studied conjecture in commutative algebra, the **weak Lefschetz property**. This gives our conjecture in a high number of cases.

Plan:

① The weak Lefschetz property.

② The theorem and its corollaries.

③ The proof.

The weak Lefschetz property

X^n smooth projective$_{|\mathbb{C}}$, k : field of char.0. $R_p := H^{2p}(X,k)$.

$R := k \oplus R_1 \oplus \ldots \oplus R_n$ graded artinian k-algebra.

Poincaré duality: $R_p \times R_{n-p} \to R_n \cong k$ perfect pairing.

$\iff$ R **Gorenstein**. Moreover, by Lefschetz:

SLP : For ℓ in R_1 general, $\times \ell^i : R_p \to R_{p+i}$ is of maximal rank (i.e. injective or surjective): true for ℓ ample, hence for ℓ general.

WLP : For ℓ in R_1 general, $\times \ell : R_p \to R_{p+1}$ is of maximal rank.

Some Gorenstein k-algebras do not satisfy WLP (Stanley).

Conjecture : $R = k[x_0, \ldots, x_n]/(P_0, \ldots, P_n)$, P_i *homogeneous*, $\{P_0 = \cdots = P_n = 0\} = \varnothing$ in $\mathbb{P}^n$ $\Longrightarrow$ R *satisfies* WLP.

What is known about the conjecture

$R = k[x_0, \ldots, x_n]/(P_0, \ldots, P_n)$. Conjecture :

WLP : For ℓ in R_1 general, $\times\ell : R_p \to R_{p+1}$ is of maximal rank.

Holds for:

- P_i general with $\deg(P_i) = d_i$: because for $P_i = x_i^{d_i}$,
$$R = H^*(\mathbb{P}^{d_0-1} \times \ldots \times \mathbb{P}^{d_n-1}).$$
- $n = 1$: any $k[x_0, x_1]/I$ with I homogeneous (easy but clever).
- $n = 2$ (Harima-Migliore-Nagel-Watanabe; proof next slide).
- $n = 3$, $\deg(P_i) = d \leqslant 5$ (Boij, Migliore, Miró-Roig, Nagel)
- $n = 4$, $\deg(P_i) = 2$ (Alzati-Re).

Case $n = 2$

Theorem (Harima-Migliore-Nagel-Watanabe): WLP *for* $n = 2$.

Proof : To simplify notation, $\deg(P_i) = d$.

$$0 \to E \to \mathcal{O}_{\mathbb{P}^2}(-d)^3 \xrightarrow{(P_i)} \mathcal{O}_{\mathbb{P}^2} \to 0$$

$$\rightsquigarrow \qquad H^0(\mathcal{O}_{\mathbb{P}^2}(p-d))^3 \xrightarrow{(P_i)} H^0(\mathcal{O}_{\mathbb{P}^2}(p)) \to H^1(E(p)) \to 0$$

$$\rightsquigarrow \qquad R_p \xrightarrow{\sim} H^1(E(p)).$$

$\ell \in R_1$, $L := \{\ell = 0\}$. $\quad 0 \to E(p-1) \xrightarrow{\times \ell} E(p) \to E(p)_{|L} \to 0.$

$$\rightsquigarrow \qquad H^0(E(p)_{|L}) \to R_{p-1} \xrightarrow{\times \ell} R_p \to H^1(E(p)_{|L})$$

Grauert-Mülich: ℓ general, $E_{|L} = \mathcal{O}(a) \oplus \mathcal{O}(a)$ or $\mathcal{O}(a) \oplus \mathcal{O}(a+1)$.

$\Longrightarrow$ $H^0(E(p)_{|L}) = 0$ or $H^1(E(p)_{|L}) = 0$. ■

Geometry

$$X_d : \{F = 0\} \subset \mathbb{P}^n \text{ smooth, } d, n \geqslant 3 \ \ (\text{exclude } d = n = 3).$$

$$H \in (\mathbb{P}^n)^* \ \rightsquigarrow \ S := X \cap H, \text{ smooth if } H \notin X^*,$$

$$\text{hence} \ \ s : (\mathbb{P}^n)^* \smallsetminus X^* \longrightarrow \mathcal{M}_S.$$

$$R := \mathbb{C}[x_0, \ldots, x_n]/(F'_0, \ldots, F'_n) \text{ where } F'_i := \frac{\partial F}{\partial x_i} \ (\textit{Jacobian ring}).$$

Theorem : *$\times\ell : R_{d-1} \longrightarrow R_d$ injective for ℓ general $\Longleftrightarrow$*

s generically finite $\Longleftrightarrow$ $\dim \operatorname{Im} s = n$ ("maximal variation").

Remark. – $\times\ell : R_{p-1} \longrightarrow R_p$ always injective for $p < d$ (ℓ general).

Corollaries

Corollary 1 : X *general* $\Longrightarrow$ *maximal variation.*

Corollary 2 : For $X_3 \subset \mathbb{P}^4$, *s generically surjective* $(\dim M_S = 4)$.

Example. – $X_3 : \sum x_i^3 = 0 \rightsquigarrow$ a general cubic surface is $\cong X_3 \cap H$

$\Longrightarrow F(x_0, \ldots, x_3)$ general cubic form $= \ell_1^3 + \ldots + \ell_5^3$ (Sylvester).

(intersect $x_0^3 + \ldots + x_4^3 = 0$ with $x_0 = \ell(x_1, \ldots, x_4)$.)

Corollary 3 (Beorchia–Miró-Roig): $X_d \subset \mathbb{P}^n$, $d \geqslant n+2 \Longrightarrow$ $\{X \cap H\}_H$ *has maximal variation.*

Proof :

$$0 \to E \to \mathcal{O}_{\mathbb{P}^n}(-(d-1))^{n+1} \xrightarrow{(F_i')} \mathcal{O}_{\mathbb{P}^n} \to 0 \qquad H := \{\ell = 0\}$$

$R_p \xrightarrow{\sim} H^1(E(p))$, then $0 \to H^0(E(d)_{|H}) \to R_{d-1} \xrightarrow{\times \ell} R_d$.

E stable (not difficult) $\Longrightarrow$ $E_{|H}$ semi-stable (Flenner).

$\det E(d) = \mathcal{O}(n+1-d) \Longrightarrow H^0(E(d)_{|H}) = 0$ for $d > n+1$. ■

Corollary 4 : $X \subset \mathbb{P}^3 \Longrightarrow \{X \cap H\}_H$ *has maximal variation.*

Proof of the Theorem

$s : \mathbb{P}^* \smallsetminus X^* \to \mathcal{M}_S$, $\ell \in R_1$ with $S := X \cap H$ smooth.

The Theorem follows from :

Proposition: $H : \{\ell = 0\} \in (\mathbb{P}^n)^* \smallsetminus X^*$.

$$T_H(s) \textit{ injective} \iff \times\ell : R_{d-1} \longrightarrow R_d \textit{ injective.}$$

Proof : Exact sequence $0 \to T_S \to T_{X|S} \to \mathcal{O}_S(1) \to 0$ gives

$$\begin{tikzcd}
0 \arrow[r] & H^0(T_{X|S}) \arrow[r] & H^0(\mathcal{O}_S(1)) \arrow[r, "\partial"] \arrow[d, equal, "\wr"] & H^1(T_S) \arrow[d, equal, "\wr"] \\
 & & T_H(\mathbb{P}^*) \arrow[r, "T(s)"] & T_S(\mathcal{M}_S)
\end{tikzcd}$$

So $T_H(s)$ injective $\iff$ $H^0(T_{X|S}) = 0$.

$$0 \to T_{X|S} \to T_{\mathbb{P}|S} \to \mathcal{O}_S(d) \to 0.$$

So need $\varphi : H^0(T_{\mathbb{P}|S}) \to H^0(\mathcal{O}_S(d))$ injective.

Proof of the Theorem (continued)

We want $\varphi : H^0(T_{\mathbb{P}|S}) \to H^0(\mathcal{O}_S(d))$ injective. Choose $x_0 = \ell$.

$H^0(T_{\mathbb{P}|S})$ generated by $x_i\,\partial_j$, $i \geqslant 1$, with $\sum\limits_{i\geqslant 1} x_i\,\partial_i = 0$ $\left(\partial_i := \frac{\partial}{\partial x_i}\right)$.

$V \in H^0(T_{\mathbb{P}|S})$: $V = \sum L_i(x_1, \ldots, x_n)\partial_i$, $\quad \varphi(V) = \sum L_i F_i'$.

$\varphi(V) = 0 \implies \sum L_i F_i' = x_0 G + c\,F \implies x_0 G \in (F_0', \ldots, F_n')$.

WLP $\implies$ $G = \sum a_i F_i'$, hence $\sum (L_i - a_i x_0 - a x_i) F_i' = 0$. $\;a := \frac{c}{d}$.

But $\sum M_i F_i' = 0 \implies M_i = 0$

(Use e.g. that $(P_0, \ldots, P_n)$ is a regular sequence in $k[x_0, \ldots, x_n]$.)

Therefore $L_i(x_1, \ldots, x_n) - a_i x_0 - a x_i = 0 \implies$

- for $i > 0$, $a_i = 0$, $L_i = a x_i$;
- for $i = 0$, $a_i + a = 0$, $L_0 = 0$.

Hence $V = a \sum\limits_{i\geqslant 1} x_i\,\partial_i = 0$. ■

The end

THE END