Castelnuovo and the Lüroth problem

Arnaud Beauville

Université de Nice

Venice, May 2015

Lüroth 1875

Theorem (Lüroth, 1875)

C plane curve, defined by polynomial $f(x,y)=0$, which can be parametrized by *rational* *functions :*

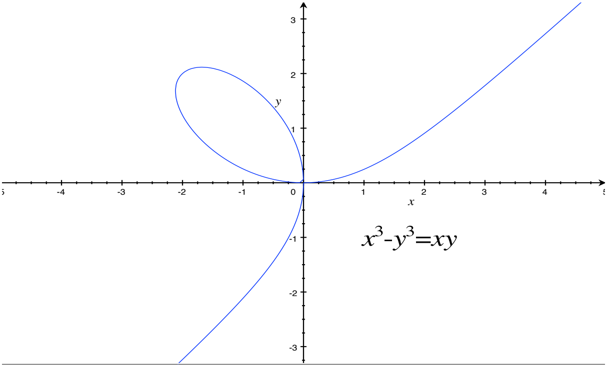
$$t \mapsto (x(t),y(t)) \ : \ f(x(t),y(t))=0 \ .$$

Then there is another parametrization $u \mapsto (x(u),y(u))$ such that $u \in \mathbb{C} \xleftrightarrow{1:1} (x,y) \in C$, with finitely many exceptions.

In other words :

$\exists$ rational dominant map $\mathbb{C} \dashrightarrow C \Longrightarrow \exists$ birational map $\mathbb{C} \overset{\sim}{\dashrightarrow} C$.

An example



Strophoid : parametrized by $x(t) = \dfrac{t^2}{1-t^6}$, $y(t) = \dfrac{t^4}{1-t^6}$.

Then t and $-t$ give same point; putting $u = t^2$ gives

$$x(u) = \frac{u}{1-u^3} \ , \ y(u) = \frac{u^2}{1-u^3} \qquad u \longleftrightarrow (x,y) \quad (u \neq 1) \ .$$

About the proof

Lüroth uses a 2-pages ingenious algebraic argument.

"Modern" proof, using Riemann surfaces :

($\mathbb{P}^1$ and $\overline{C}$ are *compact Riemann surfaces*)

$$\begin{tikzcd}
\mathbb{C} \arrow[r, hook] \arrow[d, dashed] & \mathbb{C} \cup \{\infty\} = \mathbb{P}^1_{\mathbb{C}} \arrow[d, "f"] \\
C_{\mathrm{reg}} \arrow[r, hook] & \overline{C}
\end{tikzcd}$$

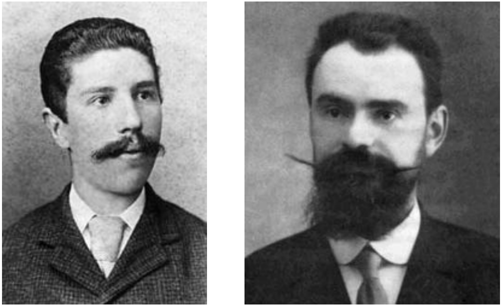
If ω holomorphic 1-form on $\overline{C}$ (locally, $\omega = f(z)dz$)

Then $f^*\omega$ holomorphic 1-form on $\mathbb{P}^1 \Rightarrow\ f^*\omega = 0\ \Rightarrow\ \omega = 0$

$\Rightarrow\ \overline{C} \cong \mathbb{P}^1$ (Riemann).

Castelnuovo-Enriques

In the decade 1890-1900, Castelnuovo and Enriques build the theory of algebraic surfaces.



Starting from a rather primitive stage, they obtain in a few years a rich harvest of results, culminating with an elaborate classification – now called the Enriques classification of surfaces.

Castelnuovo's theorem

One of the first questions Castelnuovo attacks is the Lüroth problem for surfaces :

Theorem (Castelnuovo, 1893)

S algebraic surface, $\exists\ \mathbb{C}^2 \dashrightarrow\!\!\!\!\rightarrow S \implies \exists\ \mathbb{C}^2 \overset{\sim}{\dashrightarrow} S$.

or : $$S \textit{ unirational} \implies S \textit{ rational} .$$

Castelnuovo's first proof was rather technical. Second proof :

S unirational $\Rightarrow$ S has no holomorphic 1-form or 2-form

(locally, $p(x,y)dx + q(x,y)dy$ or $r(x,y)dx \wedge dy$).

[*Footnote* : This is the modern formulation. Castelnuovo used the (equivalent) vanishing of the "geometric" and "numerical" genera.]

The Enriques surface

At first Castelnuovo tried to prove that this property characterizes rational surfaces, but he could not eliminate one particular type of surfaces. He asked Enriques, who found a non-rational surface of this type, now called the Enriques surface :

" *Guarda un po' se fosse tale una superficie del* 6^{o} *ordine avente como doppi i* 6 *spigoli d'un tetraedro (se esiste)?* "

These surfaces play an important role in the Enriques classification.

Enriques surface(s)

Castelnuovo's rationality criterion

Then Castelnuovo found the correct characterization (again, in modern terms) :

Theorem

S rational $\iff$ no (holomorphic) 1-form and quadratic 2-form ($\text{locally} = f(x,y)\,(dx \wedge dy)^2$).

A unirational surface has this property, hence is rational.

This is a major step in the classification of surfaces; even today, with our powerful modern methods, it is still a difficult theorem.

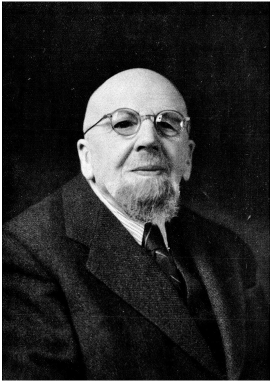
Higher dimension

Does "unirational $\Longrightarrow$ rational" hold in dimension ≥ 3?

In 1912, Enriques claims to give a counter-example :

a 3-fold in 5-space given by 2 general equations of degree 2 and 3.

Actually Enriques proves that they are unirational, and relies on an earlier paper of Fano (1908) for the non-rationality.



Fano's attempts

But Fano's analysis is incomplete. The geometry in dimension ≥ 3 is much more complicated than for surfaces; the intuitive methods of the Italian geometers were insufficient.

Fano made various other attempts (1915, 1947); in the last one he claims that a general cubic hypersurface in 4-space is not rational, a longstanding conjecture.

But none of these attempts are acceptable by modern standards. His work seems to have been widely accepted by the Italian school: Enriques 1912 paper is quoted without questioning its validity (Castelnuovo, ICM Bologna, 1928; Severi, *Serie...*, 1942; Conforto, review of Fano's paper, 1951; Terracini, Fano's obituary, 1952).

Doubts

Criticism appears in the 1955 book "*Algebraic threefolds, with special regard to problems of rationality*" by the British mathematician Leonard Roth.

Roth makes a detailed criticism of Fano's attempts; he concludes that none of these can be considered as correct.

He goes on giving a counter-example of his own, by mimicking in dimension 3 the construction of Enriques' surface.
He shows that it is unirational, and not simply-connected – hence not rational, because a rational (smooth, projective) variety is simply-connected.

Alas, 4 years later Serre showed that a *unirational* variety is simply-connected, so Roth also was wrong...

The modern era

Starting in the 50's new methods (sheaves, cohomology...) revolutionized algebraic geometry.

At first the development was rather abstract, and oriented towards arithmetic aspects, but progressively appeared a trend to use these methods to revisit classical problems, particularly in the US and the Soviet Union.

In 1971 appeared almost simultaneously 3 indisputable examples of unirational, non rational varieties :

Authors	**Example**	**Method**
Clemens-Griffiths	cubic $\subset \mathbb{P}^4$	Hodge theory
Iskovskikh-Manin	some quartics $\subset \mathbb{P}^4$	Fano's idea
Artin-Mumford	specific	Algebra (Brauer group)

Comments

- The 3 papers have been very influential: many other examples have been worked out.
- They are still (essentially) the only methods known to prove non-rationality.
- The 3 methods are very different, and apply to different kinds of varieties.
- Only the last one (Artin-Mumford) gives examples in dimension > 3; they are of the form $V \times \mathbb{C}^n$, where V is a quite particular threefold.

This situation changed last year when Claire Voisin improved the Artin-Mumford method to obtain :

New results

Theorem (Voisin)

The hypersurface $V \subset \mathbb{C}^4$ defined by $t^2 = f(x,y,z)$, where f is a general degree 4 polynomial, is unirational, but $V \times \mathbb{C}^n$ is never rational.

Her new idea is to use the Chow group of 0-cycles, a notion which had been defined and extensively studied by Severi.

Her method gives a number of new examples of unirational varieties V such that $V \times \mathbb{C}^n$ is never rational. For instance :

- The hypersurfaces $t^2 = f(x,y,z,u)$ in $\mathbb{C}^5$, or $t^2 = f(x,y,z,u,v)$ in $\mathbb{C}^6$, where f is a general degree 4 polynomial (AB);
- The variety $V \subset \mathbb{C}^6$ defined by 3 general quadratic equations (Hassett-Kresch-Tschinkel).

Conclusion

This is still a very active subject, with many open problems.

Let me conclude with one famous example. A (general) cubic threefold is not rational – a classical conjecture, solved by Clemens and Griffiths. The higher-dimensional case is not known :

Conjecture

A general cubic hypersurface (of dimension ≥ 4) is not rational.

Thank you!