The Ceresa cycle

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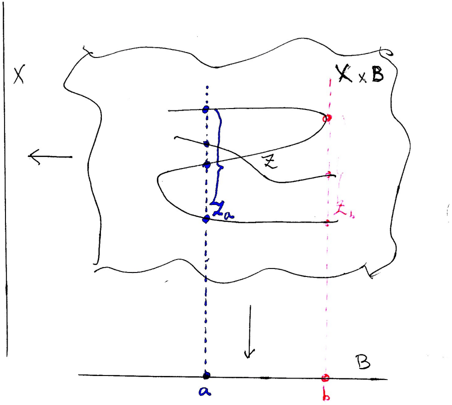
Algebraic equivalence of algebraic cycles

X smooth projective$_{|\mathbb{C}}$.

$Z^p(X) := \{\sum n_i Z_i \mid n_i \in \mathbb{Z}, Z_i \subset X \text{ closed irred., codim } Z_i = p\}$.

$A^p(X) := Z^p(X)/\sim_{\text{alg}}$.

$Z \sim_{\text{alg}} Z'$ if they appear in the same algebraic family:



$Z_a \sim_{\text{alg}} Z_b$

The cycle class map

With the intersection product, $A^*(X) := \bigoplus A^p(X)$ is a ring.

This notion goes back to the Italian geometers (Severi), and has been formalized by Weil in his Foundations (1946).

Despite its natural definition, $A^*(X)$ is still poorly understood.

Important tool: the cycle class map: $\mathrm{cl}^p : A^p(X) \to H^{2p}(X, \mathbb{Z})$.

Easy cases:

- $p = \dim X$: $\mathrm{cl}^p : A^p(X) \xrightarrow{\deg} H^{2p}(X, \mathbb{Z}) = \mathbb{Z}$ isomorphism.
- $p = 1$: $\mathrm{cl}^1 : A^1(X) \xrightarrow{\sim} \mathrm{NS}(X) \subset H^2(X, \mathbb{Z})$.

Question: Is cl^p always injective?

(Mentioned as an "interesting conjecture" by Samuel in his ICM talk (1958, Cambridge)).

Counter-examples

Griffiths, 1969: For "many" hypersurfaces in $\mathbb{P}^{2p}$, cl^p **not** injective.

Its kernel is called the **Griffiths group** $G^p(X)$.

From now on, Q-coefficients!

$$0 \to G^p(X) \to A^p(X) \xrightarrow{\mathrm{cl}^p} H^{2p}(X, \mathbb{Q})$$

For example: a general $X_5 \subset \mathbb{P}^4$ contains 2875 lines $\ell_1, \ldots, \ell_{2875}$; they are **linearly independent** in $G^2(X_5)$.

In fact, $\dim_{\mathbb{Q}} G^2(X_5) = \infty$ (Clemens, 1983: uses lines, conics, ...).

Same for general Calabi-Yau threefold (Voisin, 1994).

Another counter-example

A different, natural candidate for a counter-example:

C curve of genus $g \geqslant 3$, $J := \mathrm{Jac}(C)$. Choosing $p \in C$ gives an embedding $C \subset J$, $x \mapsto [x] - [p]$. We put

$$\mathfrak{z}_{\mathbf{C}} := [\mathbf{C}] - [-\mathbf{C}] \text{ in } \mathbf{A}^{\mathbf{g-1}}(\mathbf{J}).$$

Independent of the choice of p. Since (-1_J) acts as (-1) on $H^1(J)$, hence trivially on $H^2(J) = \bigwedge^2 H^1(J)$, $\mathfrak{z}_{\mathbf{C}} \in \mathbf{G}^{\mathbf{g-1}}(\mathbf{C})$.

Theorem (Ceresa, 1983).– *For C general,* $\mathfrak{z}_C \neq 0$.

But : For C hyperelliptic, $\mathfrak{z}_C = 0$!

Indeed: σ hyperelliptic involution, choose $p = \sigma p \rightsquigarrow C = -C$:

$$[p] - [x] = [\sigma x] - [p] \in C \subset J.$$

Question (Clemens): $\mathfrak{z}_C = 0 \implies C$ hyperelliptic?

A non-hyperelliptic curve with $\mathfrak{z}_C = 0$

Theorem (AB, Schoen, 2023). – $\mathfrak{z}_C = 0$ *for* $C : y^3 = x^4 + x$.

(C smooth, $g = 3$.)

Idea of proof : $G = \mathbb{Z}/9 \curvearrowright C$ by $(x, y) \mapsto (\zeta^3 x, \zeta y)$ $(\zeta = e^{\frac{2\pi i}{9}})$.

Key point : J/G is *uniruled*.

If J/G were smooth, $\pi : J \to J/G \rightsquigarrow \pi^* : G^2(J/G) \xrightarrow{\sim} G^2(J)^G$.

J/G uniruled $\Longrightarrow$ $G^2(J/G) = 0$, then $\mathfrak{z}_C \in G^2(J)^G \Longrightarrow \mathfrak{z}_C = 0$.

- Add some Hodge theory to take care of $\mathrm{Sing}(J/G)$. ■

Remark. – J/G uniruled is very rare (AB, Serova): only 3 non-hyperelliptic curves (two with $g = 3$ and one with $g = 4$).

After our paper, a number of other examples have appeared:

Other examples

- Lilienfeldt-Shnidman: $C : y^3 = x^4 + 1$ (same method).
- Laterveer: Humbert curves $\sum x_i^2 = \sum \alpha_i x_i^2 = \sum \beta_i x_i^2 = 0$ $(g = 5)$.
- Qiu-Zhang: various curves, including Bring curve $(g = 4)$ and Macbeath curve $(g = 7)$.
- Laga-Shnidman (2024): supersedes the above. In particular, **all** curves $y^3 = P_4(x)$.

I will explain the beautiful ideas of the latter article. But before,

Why do we need so many examples?

The property $\mathfrak{z}_C = 0$

It turns out that the vanishing of $\mathfrak{z}_C$ is equivalent to some other important properties.

The **modified diagonal cycle** (or Gross-Schoen cycle) is

$$\delta_C := \{(x,x,x)\} - \sum_{\circlearrowleft}\{(x,x,p)\} + \sum_{\circlearrowleft}\{(x,p,p)\} \in A^2(C^3)$$

One sees easily from the Künneth formula that $\boldsymbol{\delta_C} \in \mathbf{G^2}(\mathbf{C^3})$.

This cycle, and its analogue for any smooth projective variety, has been studied extensively (Moonen, O'Grady, Voisin, AB-Voisin, ...).

Proposition. – $\mathfrak{z}_C = 0 \iff \delta_C = 0$ (Zhang);

$\iff$ *C admits a MCK decomposition* (Fu-Laterveer-Vial).

We will see other equivalent properties later.

Remark.– Most of these results extend to the **Chow ring** CH^*. However the formulation is more complicated: for instance defining $\mathfrak{z}_C$ or δ_C requires choices.

A prequel to Laga-Shnidman

Back to $C : y^3 = x^4 + x$, and $G = \mathbb{Z}/9 \curvearrowright C$ by $(x, y) \mapsto (\zeta^3 x, \zeta y)$.

Proposition (AB, 2021). – $H^3(J, \mathbb{C})^G = 0$.

Motivation : Choosing $p = (0, 0)$, the *Abel-Jacobi* image of $\mathfrak{z}_C$ in the intermediate Jacobian of J is 0. This is a first approximation to $\mathfrak{z}_C = 0$, and it implies it modulo Bloch-Beilinson type conjectures.

Proof of Proposition : $H^3(J, \mathbb{C}) = \bigwedge^3 H^1(J, \mathbb{C})$,

$$H^1(J, \mathbb{C}) = V \oplus \overline{V}, \text{ with } V = H^0(C, K_C).$$

Basis of V : $(\frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{dx}{y})$. Eigenvalues of generator on $H^1(J, \mathbb{C})$:

$$(\zeta, \zeta^4, \zeta^2 ; \zeta^{-1}, \zeta^{-4}, \zeta^{-2}) .$$

Exercise: no product of 3 distinct elements is 1.

(Hint: $\pm 1 \pm 4 \pm 2$ is odd, hence $\not\equiv 0 \pmod 9$). ■

The Laga-Shnidman theorem

Theorem 1 (Laga-Shnidman): $G \curvearrowright C,\ H^3(J)^G = 0 \implies \mathfrak{z}_C = 0$.

This applies to all curves mentioned above. However the condition becomes very restrictive as g grows – the current record is $g = 7$.

Idea of proof: 3 ingredients

They are all very specific to abelian varieties. Let X be a g-dimensional abelian variety. For $k \in \mathbb{Z}$, let $\mathbf{k} : X \xrightarrow{\times k} X$.

① The endomorphisms $\mathbf{k}^*$ diagonalize simultaneously on $A^p(X)$:

$$A^p(X) = \bigoplus_{s=p-g}^{p-1} A^p_s, \text{ with } \mathbf{k}^* = \times k^{2p-s} \text{ on } A^p_s.$$

Conjecturally $A^p_s = 0$ for $s < 0$, $A^p(X) = A^p_0 \oplus A^p_1 \ldots \oplus A^p_{p-1}$ should correspond to the Bloch-Beilinson filtration.

Motivic decomposition

In particular, $A^{g-1}(J) = A_0^{g-1} \oplus A_1^{g-1} \oplus \ldots \oplus A_{g-2}^{g-1}$

$$[C] \; = [C]_0 \oplus [C]_1 \oplus \ldots \oplus [C]_{g-2}$$

Proposition. – $\mathfrak{z}_C = 0 \iff [C]_i = 0 \ \forall i \geqslant 1 \iff [C]_1 = 0.$

Note: $k = -1 \rightsquigarrow \mathfrak{z}_C = 0 \iff [C]_i = 0 \ \forall i$ odd.

So **we must prove :** $H^3(J)^G = 0 \implies [C]_1 = 0.$

② Using ①, Deninger-Murre obtained a decomposition

$$\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^1(X) \oplus \ldots \oplus \mathfrak{h}^{2g}(X) \qquad (\star)$$

where $\mathbf{k}^* = k^i$ on $\mathfrak{h}^i(X)$. This implies that $\mathbf{k}^*$ acts as $\times k^i$ on $H^*(\mathfrak{h}^i(X))$ and $A^*(\mathfrak{h}^i(X))$, hence

$$H^*(\mathfrak{h}^i(X)) = H^i(X) \quad \text{and} \quad A^p(\mathfrak{h}^i(X)) = A^p_{2p-i}\,.$$

(In particular, $(\star)$ is a *Chow-Künneth decomposition*).

End of proof

Back to (C, J): $[C]_1 \in (A_1^{g-1})^G = A^{g-1}(\mathfrak{h}^{2g-3}(J)^G)$.

Claim : $\mathfrak{h}^{2g-3}(J)^G = 0$.

③ $\mathfrak{h}(J)$ **finite-dimensional** (Kimura) $\Longrightarrow$ same for $\mathfrak{h}^{2g-3}(J)^G$.

For a finite-dimensional motive $\mathfrak{m}$, $H^*(\mathfrak{m}) = 0 \implies \mathfrak{m} = 0$.

Then $H^*(\mathfrak{h}^{2g-3}(J)^G) = H^{2g-3}(J)^G \cong H^3(J)^G = 0 \implies$
$\mathfrak{h}^{2g-3}(J)^G = 0 \implies [C]_1 = 0$. ■

The condition $H^3(J)^G = 0$ can be weakened for the **Picard curves**:

Theorem 2 (Laga-Shnidman). – $\mathfrak{z}_C = 0$ *for* $C : y^3 = P(x)$.

$$(P = \prod_{i=1}^{4}(x - \alpha_i),\ \alpha_i \text{ distinct; } C \text{ smooth, } g = 3.)$$

Proof of Theorem 2

Theorem 2. – $\mathfrak{z}_C = 0$ *for* $C : y^3 = P(x)$.

Idea : $G = \mathbb{Z}/3 \curvearrowright C$ by $(x,y) \mapsto (x, \rho y)$, $\rho = e^{\frac{2\pi i}{3}}$.

$\dim H^3(J,\mathbb{C})^G = 2$, but $\mathbf{H^0(J,\Omega^3_J)^G = 0}$ **:**

$H^0(\Omega^1_J) \cong H^0(C,K)$, basis $(\frac{dx}{y^2}, \frac{xdx}{y^2}, \frac{dx}{y})$, eigenvalues (ρ,ρ,ρ^2).

Thus $H^3(J,\mathbb{C})^G = V \oplus \overline{V}$, $\dim V = 1$. In fact:

$$H^3(J,\mathbb{Z})^G \underset{\text{HS}}{\xrightarrow{\sim}} H^1(E_\rho,\mathbb{Z})(-1), \ \text{with } E_\rho = \mathbb{C}/\mathbb{Z}[\rho].$$

This isomorphism is given by a Hodge class in $H^4(J \times E_\rho)$. But the **Hodge conjecture holds for** $J \times E_\rho$ (Schoen) $\Longrightarrow$

$$\mathfrak{h}^3(J)^G \xrightarrow{\sim} \mathfrak{h}^1(E_\rho(-1)), \text{ hence}$$

$[C]_1 \in A^2(J)^G_1 \cong A^2(\mathfrak{h}^3(J)^G) \cong A^1(\mathfrak{h}^1(E_\rho) \cong A^1(E_\rho)_1 = 0.$ ■

Questions

① All methods so far work only in low genus (record: $g = 7$). Are there examples in higher genus?

② Back to $\mathbb{Z}$-coefficients: for all the examples above, $\mathfrak{z}_C$ is torsion in $A^{g-1}(C)_{\mathbb{Z}}$. *It is not known whether* $\mathfrak{z}_C = 0$. Some new methods are needed!

The end

THE END

Happy birthday, Kapil!