The Chow ring of hyperkähler manifolds

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The Chow ring

X complex projective manifold $\rightsquigarrow$ graded ring $CH^\bullet(X)$, analogous to cohomology ring, with a purely algebraic definition:

$$CH^p(X) = \{n_1 Z_1 + \ldots + n_k Z_k\}/\sim \qquad CH_p := CH^{n-p}$$

Z_i irreducible of codimension p, $\sim$ = rational equivalence (generalizes linear equivalence of divisors).

Product given by intersection.

$CH^0(X) = \mathbb{Z}$, $CH^1(X) = \mathrm{Pic}(X)$, $CH^2(X) =???$

Unlike cohomology, the Chow ring is poorly understood.

It is usually very large: if X has a nontrivial holomorphic form, $CH_0(X)$ cannot be parametrized by an algebraic variety (Roitman).

The Chow ring of K3 surfaces

S = projective K3 surface (e.g. $S_4 \subset \mathbb{P}^3$).

$\mathrm{Pic}(S) \cong \mathbb{Z}^\rho$, $1 \leqslant \rho \leqslant 20$ ("Picard number"); $CH_0(S)$ very large.

Theorem 1 (Voisin-AB, 2004)

1. All points of S lying on a rational (singular) curve have the same class c_S in $CH_0(S)$.
2. $\mathrm{Pic}(S) \otimes \mathrm{Pic}(S) \xrightarrow{\mu} \mathbb{Z} \cdot c_S \subset CH_0(S)$.
3. $c_2(S) = 24 c_S$.

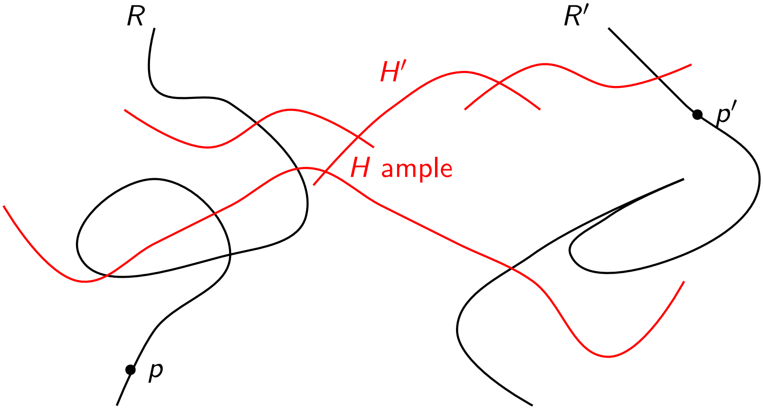
Proof of ① **and** ② : easy consequence of:

Theorem (Mumford-Bogomolov, Mori-Mukai)

Any curve $C \subset S$ is linearly equivalent to a sum of rational curves.

(Intuitive reason: by Riemann-Roch, $\dim |C| = g(C)$.)

Proof of ① and ②



Thus p and p' are linked by a chain of rational curves

$\Rightarrow$ $[p] = [p']$ in $CH_0(S)$.

Proof of ② : $C \cdot C' \sim \sum C \cdot R_i \sim \sum x_{ij}$ with $x_{ij} \in R_i$.

Remarks

③ is much more involved. We deduce it from the vanishing of the *modified diagonal cycle* in $CH_2(S \times S \times S)$ (choosing some $r \in R$):

$$\{(x,x,x)\} - (\{(r,x,x)\} + \text{permutations}) + (\{(r,r,x)\} + \text{permutations})$$

This is proved by using the fact that S is covered by (singular) elliptic curves.

Remarks

- The vanishing of the modified diagonal cycle has been studied recently by O'Grady, Voisin, Moonen-Yin, with interesting results and conjectures.
- Theorem 1 is quite particular to K3 surfaces: O'Grady has examples of $S_d \subset \mathbb{P}^3$ with $\mathrm{rk}(\mathrm{Im}\,\mu) \geqslant [\dfrac{d-1}{3}]$.

A reformulation

Consider the graded ideal $CH^\bullet_{hom}(X)$ of $CH^\bullet(X)$:

$$0 \to CH^p_{hom}(X) \to CH^p(X) \to H^{2p}(X, \mathbb{Z})$$

$\rightsquigarrow$ one-step filtration of $CH(X)$: $F^0 = CH(X)$, $F^1 = CH(X)_{hom}$.

① and ② $\Longleftrightarrow$ **Multiplicative splitting** of this filtration:

$$CH = CH_{(0)} \oplus CH_{hom}\ ,\quad CH_{(0)} \text{ stable under multiplication.}$$

For S K3:

$$CH^1(S) = \mathrm{Pic}(S) \oplus (0)$$
$$CH^2(S) = \mathbb{Z} \cdot c_S \oplus CH^2(S)_{hom}$$

Question : For which other varieties do we have such a splitting?

Abelian varieties

A abelian variety: natural splitting

$\mathrm{Pic}(A) \otimes \mathbb{Q} = \mathrm{Pic}^+(A) \oplus \mathrm{Pic}^-(A)$ of (± 1)-eigenspaces for $(-1_A)^*$,

with $\mathrm{Pic}^+(A) \cong H^2(A, \mathbb{Q})_{alg}$ and $\mathrm{Pic}^-(A) = \mathrm{Pic}^{\mathrm{o}}(A) \otimes \mathbb{Q}$.

Already necessary to invert 2, hence

Convention : From now on, CH means $CH \otimes \mathbb{Q}$.

Theorem (O'Sullivan, 2011)

$\exists$ multiplicative splitting $CH(A) = CH(A)_{(0)} \oplus CH(A)_{hom}$,
extending the previous one for CH^1.

$CH(A)_{(0)}$ is the space of "symmetrically distinguished cycles". The construction is quite involved (80 pages).

The property (WSP)

Proving the existence of a multiplicative splitting is quite difficult, already for abelian varieties. However, assuming $b_1(X) = 0$ (hence $CH^1(X)_{hom} = 0$), it implies the following weaker property:

(WSP) Let $DCH(X)$ be the subalgebra of $CH(X)$ spanned by divisors. The cycle class map $DCH(X) \to H(X, \mathbb{Q})$ is injective.

Or equivalently:

Any polynomial relation $P(D_1, \ldots, D_k) = 0$ between divisor classes in $H(X, \mathbb{Q})$ already holds in $CH(X)$.

Voisin has refined (WSP) to incorporate part ③ of Theorem 1 :

(WSP$^+$) The cycle class map is injective on the subalgebra of $CH(X)$ spanned by divisors and the Chern classes of X.

For which varieties does (WSP) or (WSP$^+$) hold?

Claim : (WSP) does **not** hold for all Calabi-Yau varieties.

Lemma

$X \to Y$ surjective, (WSP) for X $\Rightarrow$ (WSP) for Y.

Proof :

$$\begin{array}{ccc} DCH(X) & \hookrightarrow & H(X,\mathbb{Q}) \\ \uparrow & & \uparrow \\ DCH(Y) & \longrightarrow & H(Y,\mathbb{Q}) \end{array} .$$

∎

Example : $b : Y \to \mathbb{P}^3$ blow up of $C \subset \mathbb{P}^3$ of genus 2, degree 5; E exceptional divisor. Then $\mathrm{Pic}(Y) = \langle b^*H, E\rangle$. For general C, $b^*H^2, b^*H \cdot E, E^2$ linearly independent in $DCH^2(Y)$, but $b_4(Y) = b_2(Y) = 2$, so $DCH^2(Y) \not\hookrightarrow H^4(Y)$.

However...

Then $X :=$ double covering of Y branched along $D \in |-2K_Y|$ is a Calabi-Yau threefold, $DCH^2(X) \not\to H^4(X)$. ■

However, for a Calabi-Yau **hypersurface** X of dimension n:

$$CH^p(X) \otimes CH^{n-p}(X) \xrightarrow{\mu} \mathbb{Q} \cdot h^n \subset CH^n(X) \qquad (1)$$

(Voisin); this was extended to complete intersections by Lie Fu.

The key point of the proof is to express the modified diagonal cycle in $CH^{2n}(X \times X \times X)$ in terms of the lines contained in X.

Question : Is there a larger (natural) class of Calabi-Yau manifolds for which (1) holds?

Hyperkähler manifolds

Conjecture

(WSP$^+$) holds for projective hyperkähler manifolds.

Here hyperkähler = irreducible holomorphic symplectic (IHS) = simply-connected + $H^0(X, \Omega^2_X) = \mathbb{C}\sigma$, σ symplectic 2-form.

Recall : Many interesting properties, but very few examples.

Up to deformation, only two series in each (even) dimension:

1. for S K3, $S^{[n]}$:= Hilbert scheme $= \{Z \subset S \mid \text{length}(Z) = n\}$ = desingularization of the symmetric product $Sym^n S$.
2. K_n ("generalized Kummer varieties"): analogous construction starting from S = abelian surface.

+ 2 sporadic examples in dimension 6 and 10 (O'Grady).

Deformations

Recall : For each g, one 19-dimensional moduli space $\mathcal{F}_g$ of K3 surfaces $S \subset \mathbb{P}^g$ ($S_4 \subset \mathbb{P}^3$, $S_{2,3} \subset \mathbb{P}^4$, etc.)

The $S^{[n]}$ for $S \in \mathcal{F}_g$ form only a **hypersurface** in the deformation space of $S^{[n]}$, which has dimension 20.

We say that X is *of type* $K3^{[n]}$ if it is deformation equivalent to $S^{[n]}$ for some K3 surface S; same for type K_n.

Challenge : Describe explicitely complete families of projective varieties of type $K3^{[n]}$.

Example (Donagi-AB): The variety $F(V_3)$ of lines contained in a smooth cubic fourfold $V_3 \subset \mathbb{P}^5$ is of type $K3^{[2]}$, and has 20 moduli.

Other examples: O'Grady, Iliev-Ranestad, Debarre-Voisin ($n = 2$); Iliev-Kapustka^2-Ranestad ($n = 3$), Lehn^2-Sorger-v. Straten ($n = 4$).

Results

No example known for type K_n.

Proposition (Voisin)

S K3, $\tau := \operatorname{rk} H^2(S)_{tr} = 22 - \operatorname{rk}\operatorname{Pic}(S)$. Then (WSP$^+$) holds for $S^{[n]}$ for $n \leqslant 2\tau + 4$, in particular for $n \leqslant 8$.

Idea : Using de Cataldo-Migliorini, reduce to analogous statement for S^n: for $n \leqslant 2\tau + 1$, $DDCH(S^n) \hookrightarrow H(S^n)$, where $DDCH(S^n) :=$ subalgebra of $CH(S^n)$ spanned by pull back of divisors in S and the diagonal in $S \times S$.

Then write down complete list of relations between these generators of $DDCH(S^n)$, and check that they hold already in $CH(S^n)$. ∎

Results (continued)

Remark (Q. Yin): Can we go one step further, namely prove $DDCH(S^n) \hookrightarrow H(S^n)$ for $n = 2\tau + 2\,?\Longleftrightarrow$ "$\bigwedge^{\tau+1} H^2(S)_{tr} = 0$" in the sense of Chow motives, i.e. **the Chow motive of S is finite-dimensional** in the sense of Kimura. This is probably very hard to prove ...

- Maulik-Voisin: (WSP) holds for $S^{[n]}$ for every n (uses the action of Nakajima's Heisenberg-type algebra).
- (WSP$^+$) holds for K_n $\forall n$ (Fu), for $F(V_3)$ (Voisin), for a general double EPW-sextic (Ferretti).

Beware that the Chow group is not stable under deformation! So (WSP) for $S^{[n]}$ implies nothing for type $K3^{[n]}$.

Riess' theorem

Proposition (U. Riess)

(WSP) holds for every X of type $K3^{[n]}$ or K_n with $\rho(X) \geqslant 5$.

Idea : Recall: for any X HK of dimension $2n$, $\exists$ quadratic form $q : H^2(X,\mathbb{Z}) \to \mathbb{Z}$ such that $\alpha^{2n} = c \cdot q(\alpha)^n$ for all $\alpha \in H^2(X,\mathbb{Z})$.

Easy part: $\operatorname{Ker}\Big(S^\bullet \operatorname{Pic}(X) \otimes \mathbb{C} \to H^\bullet(X,\mathbb{C})\Big)$ = ideal spanned by classes D^{n+1} for $D \in \operatorname{Pic}(X) \otimes \mathbb{C}$, $q(D) = 0$ (Bogomolov).

Thus (WSP) $\Leftarrow$ for these classes, $D^{n+1} = 0$ in $CH(X) \otimes \mathbb{C}$.

If $\rho \geqslant 5$, the quadric $q = 0$ in $\mathbb{P}(\operatorname{Pic}(X) \otimes \mathbb{C})$ has a $\mathbb{Q}$-point $\Rightarrow$

(WSP) $\Leftarrow$ $\forall$ $D \in$ **Pic(X)** with $q(D) = 0$, $D^{n+1} = 0$ in $CH(X)$.

Riess' theorem: hard part

Using work of Markman, Matsushita, Bayer-Macrì, ..., reduce to prove $D^{n+1} = 0$ in $CH(X)$ for $D = f^*H$, where:

$f : X \overset{\varphi}{\dashrightarrow} X' \xrightarrow{p} \mathbb{P}^n$, X' HK, φ birational, p Lagrangian fibration.

$p^*H^{n+1} = 0$ in $CH(X') \Rightarrow D^{n+1} = \varphi^* p^* H^{n+1} = 0$ in $CH(X)$. ∎

The Bloch-Beilinson filtration

The one-step filtration $CH^\bullet_{hom} \subset CH^\bullet$ should extend:

Conjecture (Bloch-Beilinson)

For every X smooth projective, $\exists$ filtration $F^\bullet$ on $CH(X)$:

$$CH^p = F^0 \supset F^1 = CH^p_{hom} \supset \ldots \supset F^{p+1} = 0$$

which is functorial (both for f^* and f_*) and multiplicative.

Hope : For hyperkähler manifolds, the B-B filtration admits a **multiplicative splitting**, i.e. comes from a graded ring structure:

$$CH^p(X) = CH^p_{(0)} \oplus \ldots \oplus \underbrace{CH^p_{(i)} \oplus \ldots \oplus CH^p_{(p)}}_{F^i} \ .$$

Recent work of Voisin gives some evidence in the case of CH_0 :

The opposite filtration

For any projective X, $\mathrm{gr}^p_{F^\bullet} CH_0(X)$ should be controlled by $H^0(X, \Omega^p_X)$; thus for X HK of dimension $2n$, $F^{2p-1} = F^{2p}$ and

$$CH_0(X) = F^0 \supset F^2 \supset \ldots \supset F^{2n} \ .$$

Voisin defines another filtration $S^\bullet$ of $CH_0(X)$ which should be *opposite* to $F^\bullet$.

For $x \in X$, put $\qquad O_x := \{y \in X \mid y \sim_{rat} x\}$.

O_x is a countable union of closed subvarieties Z which are *isotropic* – i.e. $\sigma_{|Z} = 0$. In particular $\dim O_x \leqslant n$.

The conjectural splitting

Definition : $S^i(X) := \{x \in X \mid \dim O_x \geqslant i\}$

Stratification of $X = S^0(X) \supset S^1(X) \supset \ldots \supset S^n(X)$.

$\rightsquigarrow$ filtration $S^i CH_0(X) = \langle S^i(X) \rangle$:

$$CH_0(X) = S^0 \supset S^1 \supset \ldots \supset S^n \supset S^{n+1} = 0 \ .$$

Example : For S K3, $S^1(S) = \{x \in S \mid [x] = c_S \text{ in } CH_0(S)\}$,
$S^1 CH_0(S) = \mathbb{Q} \cdot c_S$.

Conjecture (Voisin): The filtration $F^\bullet$ and $S^\bullet$ are opposite; i.e., if
$CH_{(j)} := S^{n-j} \cap F^{2j}$:

$$CH_0(X) = \overbrace{CH_{(0)} \oplus \ldots \oplus CH_{(2i)}}^{S^{n-i}} \oplus \ldots \oplus \underbrace{CH_{(2j)} \oplus \ldots \oplus CH_{(2n)}}_{F^{2j}} \ .$$

Some evidence (Voisin)

Proposition

The conjecture holds for $S^{[n]}$ and $F(V_3)$.

The proof rests on a more explicit description of $S^\bullet$ in these cases.

Towards the general case : Consider the stratification

$$X = S^0(X) \supset S^1(X) \supset \ldots \supset S^n(X)\ .$$

$S^n(X) = \{x \in X \mid \dim O_x = n\}$ has dimension n.

Conjecture : $\dim S^i(X) = 2n - i$.

Proposition

$\dim S^i(X) = 2n - i \ \Rightarrow\ CH_0(X) = S^{n-i} + F^{2i+2}\,.$

Ingredients of the proof

The proof rests on symplectic geometry:

Proposition

$Z \subset S^i(X)$ irreducible of dimension $2n-i \;\Rightarrow\; Z$ **coisotropic** $(T_Z^{\perp} \subset T_Z)$ and $\exists\, f : Z \dashrightarrow B$, fibers of f = orbits.

$\Rightarrow\;\; \sigma_{|B} = f^*\sigma_B$, σ_B symplectic.

$\Rightarrow\;\; \sigma_{|B}^{n-i} \neq 0 \;\Rightarrow\; H^0(X, \Omega_X^p) \hookrightarrow H^0(Z, \Omega_Z^p)$ for $0 \leqslant p \leqslant n-i$.

By expected properties of B-B filtration,

$\Rightarrow\;\; S^{n-i}CH_0(X) \twoheadrightarrow CH_0(X)/F^{2i+2}$. ∎

THE END

Happy birthday, Ron!